Solitons for the cubic-quintic nonlinear Schrödinger equation with time and space modulated coefficients

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Abstract

In this paper, we construct, by means of similarity transformations, explicit solutions to the cubic quintic nonlinear Schrödinger equation with potentials and nonlinearities depending both on time and spatial coordinates. We present the general approach and use it to calculate bright and dark soliton solutions for nonlinearities and potentials of physical interest in applications to Bose-Einstein condensates and nonlinear optics.

Key words: Cubic-quintic nonlinear Schrödinger equations, transformations, bright and dark solitons, Bose-Einstein condensates.

1 Introduction

During the past several years, there have been a great deal of theoretical and experimental investigations in models based on the nonlinear Schrödinger equation (NLSE). The physical models of this type emerge in various nonlinear physical phenomena, such as nonlinear optics [1,2], Bose-Einstein condensates [3], biomolecular dynamics [4], and others [5,6].

One of the simplest extensions of the cubic NLSE is the so-called cubic-quintic nonlinear Schrödinger (CQNLS) model, which, in normalized units and 1D, is

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + G_1|\psi|^2\psi + G_2|\psi|^4\psi \tag{1}$$

The CQNLS equation is another universal mathematical model describing many situations of physical interest and approximating other more complicated ones. It arises in plasma physics [7], condensed matter physics [8], nuclear physics [9], Bose-Einstein condensates [10], nonlinear optics [11], etc. In the case of Bose-Einstein condensates, the cubic and quintic nonlinearity terms appear as a consequence of the two- and three-body interactions, respectively. Efimov resonances, which are responsible for three-body interactions have been observed in an ultra-cold gas of cesium atoms [12]. In optics, the CQNLS equation can describe the propagation of an electromagnetic wave in photorefractive materials as long as variable t represents the propagation coordinate of the wave. The cubic-quintic nonlinearity occurs due to an intrinsic nonlinear resonance in the material, which also gives rise to strong two-photon absorption [13].

In the past decade, techniques for managing nonlinearity [14] have attracted cosiderable attention. For instance, nonlinearity management arises in optics for transverse beam propagation in layered optical media [15], as well as in atomic physics for the Feshbach resonance of the scattering length of interatomic interactions in BECs [16]. In these situations, one has to deal with the governing equations with the nonlinearity coefficients being functions of time [16,17] , or equivalently, the variable representing the propagation distance [14,18–20]. In a recent reference [21], by using similarity transformations, the authors went beyond previous studies considering space and time dependent nonlinearities, and constructed explicit solutions of the cubic nonlinear Schrödinger equation with coefficients depending on time and on the spatial coordinates, which are experimentally feasible due to the flexible and precise control of the scattering length achievable in quasi-one-dimensional BECs with tunable interactions.

The aim of the present paper is to study explicit solutions of the cubic-quintic nonlinear Schrödinger equation with time and space dependent potentials and nonlinearities, which can be written in the following dimensionless form

$$iu_t = -u_{xx} + V(t,x)u + g_1(t,x)|u|^2u + g_2(t,x)|u|^4u \tag{2}$$

In the case of Bose-Einstein condensates, $u(t,x)$ represents the macroscopic wave function, $V(t,x)$ is an external potential, $g_1(t,x)$ and $g_2(t,x)$ are the cubic and quintic nonlinear coefficients, corresponding to the two-body and three-body interactions, respectively. The signs of $g_1(t,x)$ and $g_2(t,x)$ can be positive or negative, indicating that the interactions are repulsive or attractive, respectively. Thus, in this paper, we construct different types of explicit solutions such as bright and dark soliton solutions. In order to do that, we resort to the similarity transformation technique, and analyze a general class of potentials $V(t,x)$ and nonlinearity functions $g_1(t,x)$ and $g_2(t,x)$.

The paper is organized as follows. In Section 2, we introduce the general approach of similarity transformations. In Sections 3 and 4, we use the method for constructing explicit solutions of the time dependent cubic-quintic nonlinear Schrödinger equation. Section 5 contain a remark on the stability of the previous solutions and the conclusions of our results.

2 General Theory

We consider the cubic-quintic nonlinear Schrödinger equation with time and space dependent coefficients $g_1(t,x)$, $g_2(t,x)$, $V(t,x)$, Eq. (2).

Our first goal is to transform Eq. (2) into the stationary CQNLS equation

$$EU = -U_{XX} + G_1 |U|^2 U + G_2 |U|^4 U, \tag{3}$$

where both $U \equiv U(X)$ and $X \equiv X(t,x)$ are real functions, E denotes the eigenvalue of the nonlinear equation (which correspond to the chemical potential in the Bose–Einstein condensates framework and the propagation constant in nonlinear optics), and G_1 and G_2 are the (constant) nonlinearity parameters.

To connect solutions of Eq. (2) with those of Eq. (3) we will use the transformation

$$u(t,x) = r(t,x)U\left[X(t,x)\right], \tag{4}$$

requiring $U(X)$ to satisfy Eq. (3) and $u(t,x)$ to be a solution of Eq. (2). The potential $V(t,x)$ and the nonlinearities $g_1(t,x)$ and $g_2(t,x)$ are determined after the transformation is applied. Although the methodology of this transformation has been explained in reference [21] for the cubic nonlinear Schrödinger equation, for the sake of completeness we show here the method. Moreover, the CQNLSE presents an extra term with respect to the CNLSE, which implies extra complications in the treatment of the solutions. Other similarity transformations have been studied for NLS equations in different contexts [22,23].

In our calculations, it is convenient to introduce the polar form for the complex modulating function $r = r(t,x)$

$$r(t,x) = \rho(t,x)e^{i\varphi(t,x)}, \tag{5}$$

where ρ and φ are real functions (ρ is also non-negative). Upon substitution of Eq. (4) into Eq. (2), we impose the following system of conditions on g_1, g_2 and V:

$$
\begin{aligned}
g_1(t,x) &= G_1\,\frac{X_x^2}{\rho^2}, && (6)\\
g_2(t,x) &= G_2\,\frac{X_x^2}{\rho^4}, && (7)\\
V(t,x) &= \frac{\rho_{xx}}{\rho} - \varphi_t - \varphi_x^2 - EX_x^2, && (8)
\end{aligned}
$$

together with

$$
\begin{cases}
\rho\rho_t + \left(\rho^2\varphi_x\right)_x = 0,\\
\left(\rho^2 X_x\right)_x = 0,\\
X_t + 2\varphi_x X_x = 0,
\end{cases}
\tag{9}
$$

in order to arrive at a set of nontrivial solutions to Eq. (2). By solving (9) one gets

$$X_x(t,x) = \frac{\gamma^2(t)}{\rho^2(t,x)}, \tag{10}$$

($\gamma(t)$ is an arbitrary positive definite function of time) and

$$X_{tx}X_x - X_t X_{xx} = \frac{\gamma_t}{\gamma}\,X_x^2. \tag{11}$$

This last equation yields

$$\left(\frac{X_t}{X_x}\right)_x = \frac{\gamma_t}{\gamma}, \tag{12}$$

which, after integration, results in

$$X(t,x) = F(\xi), \qquad \xi(t,x) = \gamma(t)\,x + \delta(t), \tag{13}$$

where both $\gamma(t)$ and $\delta(t)$ are differentiable functions.

Thus, we have proven the following result: the substitution

$$u(t,x) = \rho(t,x)e^{i\varphi(t,x)}U\left[X(t,x)\right], \tag{14}$$

where

$$
\begin{aligned}
\rho(t,x) &= \sqrt{\frac{\gamma}{F'(\xi)}}, && (15)\\
\varphi(t,x) &= -\frac{\gamma_t}{4\gamma}x^2 - \frac{\delta_t}{2\gamma}x + \varepsilon, && (16)
\end{aligned}
$$

and $\varepsilon(t)$ is a differentiable arbitrary function, leads to Eq. (2), with

$$g_1(t,x) = G_1 \frac{\gamma^4}{\rho^6}, \tag{17}$$

$$g_2(t,x) = G_2 \frac{\gamma^4}{\rho^8}, \tag{18}$$

$$V(t,x) = \frac{\rho_{xx}}{\rho} - \varphi_t - \varphi_x^2 - E\frac{\gamma^4}{\rho^4} \tag{19}$$

where the function $\rho(x,t)$ must be sign definite and C^2 (i.e. twice continuously differentiable) for $g_1(t,x), g_2(t,x)$ and $V(t,x)$ to be properly defined. Therefore, choosing $\delta(t), \gamma(t)$ and $F(\xi)$ (or equivalently $\rho(x,t)$) we can generate $g_1(t,x), g_2(t,x)$ and $V(t,x)$ for which the solutions of Eq. (2) can be obtained from those of Eq. (3) via Eqs. (4). We will exploit this fact to construct soliton solutions exhibiting interesting nontrivial behavior.

3 Exact Solutions I

In this section, we first address solutions to Eq. (2) when the potential $V(t,x)$ depends both on time and space while the nonlinearities are of the form

$$g_1(t,x) = G_1\gamma(t) \tag{20}$$

$$g_2(t,x) = G_2 \tag{21}$$

These nonlinearities can be derived by taking

$$\rho(t,x) = \sqrt{\gamma(t)}, \tag{22}$$

which corresponds to the choice $F(\xi) = \xi$. From Eq. (19), one finds that the potential can be cast as

$$V(t,x) = f(t)x^2 + h(t)x - \left(\varepsilon_t + \frac{\delta_t^2}{4\gamma} + E\gamma^2\right), \tag{23}$$

where

$$f(t) = \frac{\gamma_{tt}\gamma - 2\gamma_t^2}{4\gamma^2}, \tag{24}$$

$$h(t) = \frac{\delta_{tt}\gamma - 2\delta_t\gamma_t}{2\gamma^2}. \tag{25}$$

Moreover, $X(t,x)$ becomes

$$X(t,x) = \int_0^\xi \frac{\gamma}{\rho^2} d\xi = \xi = \gamma(t)x + \delta(t). \tag{26}$$

On the other hand, many solutions of Eq. (3) are known. In this paper, we use the bright soliton, given by

$$U = \frac{\eta}{\left(\sqrt{1-\beta}\cosh(2\sqrt{-E}x) + 1\right)^{1/2}} \tag{27}$$

where

$$\eta = \sqrt{\frac{4E}{G_1}}, \quad \beta = \frac{-16EG_2}{3G_1^2} \tag{28}$$

Since we assume that $G_2 > 0$ and $E < 0$ we have that $G_1 < 0$. Thus, a restriction on β is that $0 \leq \beta < 1$. Another solution, representing a dark soliton is given by the following expression:

$$U = \sqrt{a_1(\mathrm{sech}(\mu X) - 1)} \tag{29}$$

with

$$\mu^2 = -\frac{4}{5}E, \tag{30}$$

and

$$G_1 = \frac{8E}{5a_1}, \quad G_2 = \frac{3E}{5a_1^2}, \tag{31}$$

and $a_1 < 0$ being an arbitrary constant. Thus G_1 is positive while G_2 is negative.

In what follows, we deal with different cases which turn out to be of physical relevance. It is worth remarking that the solutions which will be shown below cannot be calculated in the pure cubic equation, since in the cubic-quintic NLSE, nonlinearities compete each other when they have different signs. Thus, the cubic quintic NLS equation, in some sense, generalizes the cubic case.

3.1 Solutions in the free space with constant nonlinearities

We take

$$\gamma(t) = 1, \quad \delta(t) = \upsilon t \tag{32}$$

with $\upsilon \in \mathbb{R}$ and $\varepsilon = -(\upsilon^2/4 + E)t$. Hence, the external potential and the nonlinearities become

$$V(t,x) = 0, \quad g_1(t,x) = G_1, \quad g_2(t,x) = G_2 \tag{33}$$

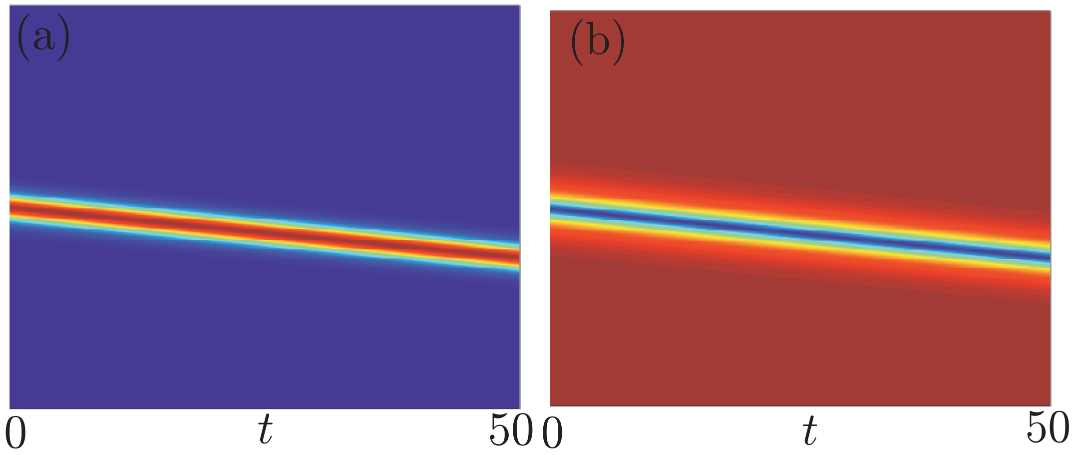

Fig. 1. [Color online] (a) Pseudocolor plot of $|u(t,x)|^2$ where $u(t,x)$ is the solution of Eq. (2), for $V(t,x) = 0$ and $g_1(t,x) = G_1, g_2(t,x) = G_2$, and for (a) bright soliton profiles ($E = -1, G_1 = -2, G_2 = 0.5$) and (b) dark soliton profiles ($E = -1, a_1 = -1$). In both cases $x \in [-20, 20]$ and $\upsilon = 0.1$.

The solution is then given by

$$u(t,x) = e^{i\varphi(t,x)}U[\xi(t,x)], \tag{34}$$

with $\varphi(t,x) = -(\upsilon/2)x - (\upsilon^2/4 + E)t$ and $\xi(t,x) = x + \upsilon t$, where U is given by Eq. (27), in the case of bright soliton, and Eq. (29), in the case of dark soliton. In Figs. 1(a) and (b), we plot these solutions corresponding to Eq. (34), for $\upsilon = 0.1$.

This case, which corresponds to a soliton moving in the free space with a velocity υ, reproduces the Galilean invariance of the system providing consequently a validation of the similarity transformation for the trivial case.

3.2 Solitons with linear potential and constant nonlinearities

If we choose

$$\gamma(t) = 1, \quad \delta(t) = \upsilon t^2 \tag{35}$$

the external potential and the nonlinearities become

$$V(t,x) = \upsilon x, \quad g_1(t,x) = G_1, \quad g_2(t,x) = G_2 \tag{36}$$

The solution is then given by

$$u(t,x) = e^{i\varphi(t,x)}U[\xi(t,x)], \tag{37}$$

with $\varphi(t,x) = -\upsilon t x - \upsilon^2 t^3/3 - Et$ and $\xi(t,x) = x + \upsilon t^2$, where U is given by Eq. (27), in the case of bright soliton, and Eq. (29), in the case of dark soliton.

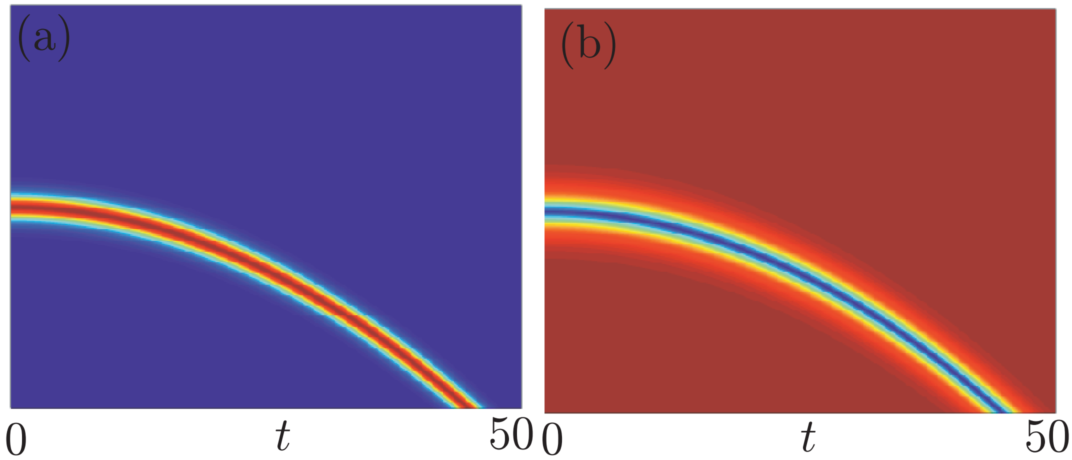

Fig. 2. [Color online] (a) Pseudocolor plot of $|u(t,x)|^2$ where $u(t,x)$ is the solution of Eq. (2), for potential and nonlinearities given by Eqs. (36), and for (a) bright soliton profiles ($E=-1, G_1=-2, G_2=0.5$) and (b) dark soliton profiles ($E=-1, a_1=-1$). In both cases $x\in[-20,20]$ and $\upsilon=0.01$.

In Figs. 2(a) and (b), we plot these solutions corresponding to Eq. (37), for $\upsilon=0.01$.

3.3 Solitons with a linear potential modulated by a time sinusoidal function and constant nonlinearities

Taking $\gamma(t)=1$ and $\delta(t)=-\cos(\Omega t+\beta_0)$, we obtain

$$v(t,x)=\frac{\Omega^2}{2}\cos(\Omega t+\beta_0)x,\quad g_1(t,x)=G_1,\quad g_2(t,x)=G_2. \tag{38}$$

With this choice, the solution is given by

$$u(t,x)=e^{i\varphi(t,x)}U[\xi(t,x)], \tag{39}$$

where

$$\varphi(t,x)=-\frac{\Omega}{2}\sin(\Omega t+\beta_0)x-\frac{\Omega^2}{8}t+\frac{\Omega}{16}\sin(2\Omega t+2\beta_0)-Et \tag{40}$$

and U is given by Eq. (27), for the bright soliton, and Eq. (29), for the dark soliton. In Figs. 3(a) and (b), we plot these solutions corresponding to Eq. (39).

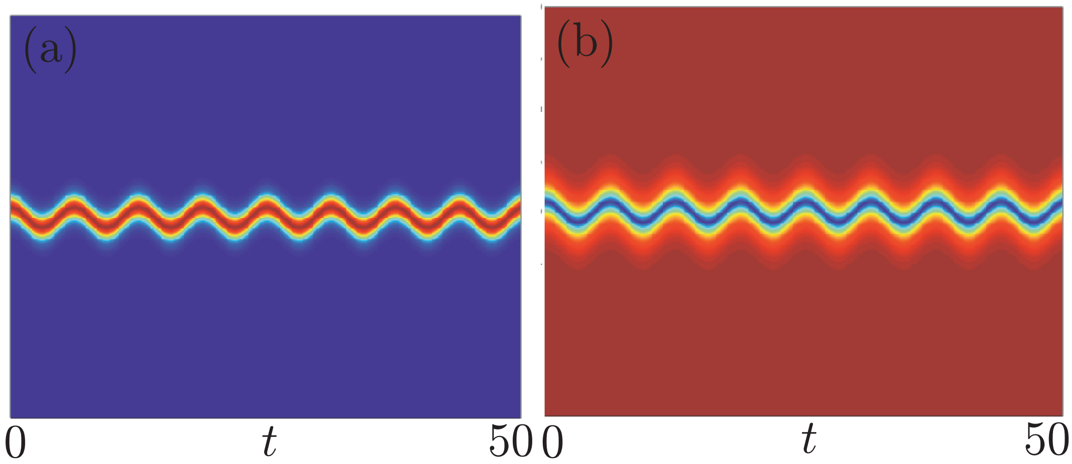

Fig. 3. [Color online] (a) Pseudocolor plot of $|u(t,x)|^2$ where $u(t,x)$ is the solution of Eq. (2), for potential and nonlinearities given by Eqs. (38), and for (a) bright soliton profiles ($E=-1, G_1=-2, G_2=0.5$) and (b) dark soliton profiles ($E=-1, a_1=-1$). In both cases $x \in [-20, 20]$.

4 Exact solutions II

In this section, we show two extra examples among the vast amount of inhomogeneous models. The first one is the case of localized nonlinearities in space and time with no potential. In the second case, we calculate solutions for Eq. (2) with both periodic potential and nonlinearities. We think that our results can be applied to Bose–Einstein condensates with optically controlled interactions [24].

4.1 Localized Nonlinearities in space and time

In this subsection, we address to nonlinearities localized in space and time. We focus on the localized nonlinearities:

$$g_1(t,x) = G_1\gamma(t)(\mathrm{sech}(\xi))^3, \quad g_2(t,x) = G_2(\mathrm{sech}(\xi))^4 \tag{41}$$

being the external potential

$$V(t,x) = \frac{\gamma^2(t)}{4}\left(1 + (1-4E)(\mathrm{sech}(\xi))^2\right) + f(t)x^2 + h(t)x + m(t) \tag{42}$$

where $f(t)$ and $h(t)$ are given by Eqs. (24) and

$$m(t) = -\left(\varepsilon_t + \frac{\delta_t^2}{4\gamma^2(t)}\right) \tag{43}$$

Our choice corresponds to

$$\rho(t,x) = \sqrt{\gamma(t)\cosh(\xi)} \tag{44}$$

The potential $V(t,x)$ vanishes by taking $E = 1/4$, $\gamma(t) = 1$, $\delta(t) = 0$ and $\varepsilon(t) = t/4$. A solution of Eq. (3) with $E = 1/4$ is given by

$$U = \frac{\mathrm{sn}(\mu X, k)}{\sqrt{a_0 + a_1\,\mathrm{dn}^2(\mu X, k)}} \tag{45}$$

with the requirement $a_0 > |a_1|$, k being the modulus of the Jacobian elliptic functions,

$$\mu = \sqrt{\frac{-E(a_0 + a_1)}{2k^2 a_1 - k^2 a_0 - a_0 - a_1}} \tag{46}$$

and

$$G_1 = \frac{2Ek^2(a_0^2 - 2a_1 a_0 k^2 - a_1^2 + a_1^2 k^2)}{a_0 k^2 + a_0 - 2k^2 a_1 + a_1} \tag{47}$$

$$G_2 = \frac{3E a_1 a_0 k^4 (a_0 + a_1 - k^2 a_1)}{k^2 a_0 + a_0 - 2k^2 a_1 + a_1} \tag{48}$$

We demonstrate below that Eq. (2) in the free space with localized nonlinearities given by Eq. (41), can support bound states with an arbitrary number of solitons, resembling the results of Ref. [21].

In this particular case, as

$$X(t,x) = \int_0^{\xi} \frac{\gamma(t)}{\rho^2(t,x)} d\xi \tag{49}$$

one can obtain $\cos X = -\tan(\xi)$, thus $0 < X < \pi$. In order to meet the boundary conditions $\psi(\pm\infty) = 0$ one has to impose $U(0) = U(\pi) = 0$. Evidently, $U(0) = 0$ is satisfied for the solution (45) and in order to meet $U(\pi) = 0$, the condition $\mu\pi = 2nK(k)$ where $K(k)$ is the elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2(x)}} \tag{50}$$

must hold. Hence for every n, we can find a value of k leading to the following family of solutions

$$\psi(t,x) = \sqrt{\cosh(\xi)}\, e^{it/4} U(X(t,x)) \tag{51}$$

where $X(t,x)$ is given by Eq. (49) and U is given by Eq. (45). The solutions corresponding to $n = 1, 2, 3$ are depicted in Fig. 4. It is observed that ψ has exactly $n - 1$ zeros.

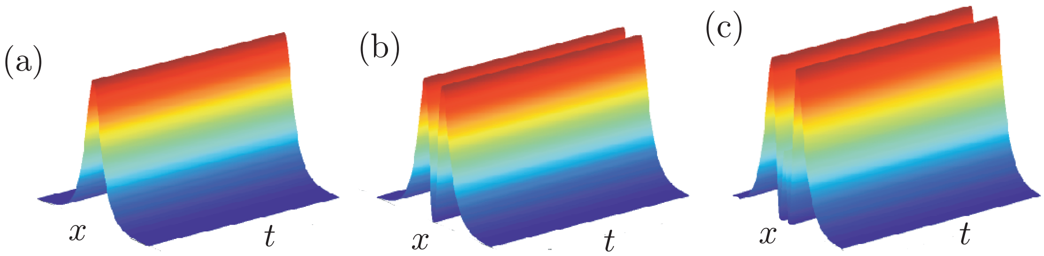

Fig. 4. [Color online] Plots of $|u(t,x)|^2$ for solutions of Eq. (2), given by Eq. (51), corresponding to (a) n=1 (b) n=2 and (c) n=3. In all cases, $x \in [-7,7]$ and $t \in [0,50]$.

4.2 Periodic potential and nonlinearities

Finally, as a last application of our method, we take $\rho(t,x) = \sqrt{\gamma(t)(1+\alpha\cos(\omega\xi))}$. In this way, nonlinearities $g_1(t,x)$ and $g_2(t,x)$ are given by

$$g_1(t,x) = G_1\gamma(t)(1+\alpha\cos(\omega\xi))^{-3} \tag{52}$$
$$g_2(t,x) = G_2(1+\alpha\cos(\omega\xi))^{-4} \tag{53}$$

If, moreover, we take $E = -\alpha\omega/2$, with $0 < \alpha < 1$ and $\omega > 0$, the external potential is given by

$$V(t,x) = -\frac{\alpha\omega^2\gamma^2}{2}\frac{2+\alpha\cos(\omega\xi)}{(1+\alpha\cos(\omega\xi))^2} + f(t)x^2 + h(t)x - \varepsilon_t - \frac{\delta_t^2}{4\gamma} \tag{54}$$

Taking $\gamma = 1, \delta = \upsilon t$, for $\upsilon \in \mathbb{R}$ and $\varepsilon = -\upsilon^2 t/4$, we get the periodic potential

$$V(t,x) = -\frac{\alpha\omega^2}{2}\frac{2+\alpha\cos(\omega\xi)}{(1+\alpha\cos(\omega\xi))^2} \tag{55}$$

and the nonlinearities finally are given by

$$g_1(t,x) = G_1(1+\alpha\cos(\omega\xi))^{-3} \tag{56}$$
$$g_2(t,x) = G_2(1+\alpha\cos(\omega\xi))^{-4} \tag{57}$$

where $\xi = x + \upsilon t$. For small α these nonlinearities are approximately harmonic

$$g_1(x) \simeq G_1(1-3\alpha\cos(\omega\xi)), \quad g_2(x) \simeq G_2(1-4\alpha\cos(\omega\xi)), \quad \alpha \ll 1 \tag{58}$$

We can construct our canonical transformation by using Eq. (49) and obtain

$$\tan(\frac{\omega}{2}\sqrt{1-\alpha^2}X(t,x)) = \frac{\sqrt{1-\alpha}}{1+\alpha}\tan(\frac{\omega\xi}{2}) \tag{59}$$

Again, we use the Eqs. (27) and (29) as solutions of Eq. (3).

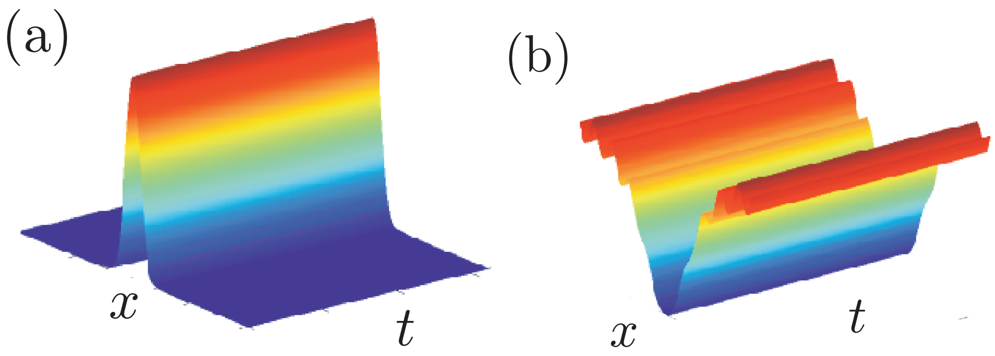

Fig. 5. [Color online] Plots of $|u(t,x)|^2$ for different solutions of Eq. (2), given by Eq. (60), corresponding to $\upsilon = 0$, $\omega = 1$ for (a) bright soliton solution, with $\alpha = 0.5$ and (b) dark soliton solution, with $\alpha = 0.1$. In both cases, $x \in [-15, 15]$ and $t \in [0, 120]$.

Thus, solutions of Eq. (2) are given by

$$\psi(t,x) = \sqrt{1 + \alpha \cos(\omega\xi)}\, e^{i\varphi(t,x)} U(X) \tag{60}$$

where $\varphi(t,x) = -\upsilon x/2 + \varepsilon$ and $U(X)$ is given by Eq. (27) or (29).

In Fig. 5, we have draw different solutions of Eq. (2) corresponding to bright and dark soliton solutions for $\upsilon = 0$.

5 Conclusions

In this paper, we have used similarity transformations to find exact solutions of the cubic-quintic nonlinear Schrödinger equation with time and space modulated nonlinearities and potentials. We have explicitly calculated bright and dark soliton solutions of Eq. (2), with nonlinearities and potentials of physical interest in applications to Bose-Einstein condensates and nonlinear optics. The ideas contained in this paper can also be extended to study multicomponent systems, higher-dimensional profiles, etc. We hope that this paper will stimulate further research on those topics and help to understand the behaviour of nonlinear waves in systems where, not only the potentials, but the nonlinearities are inhomogeneous in space and time.

We have also tried to study the stability of these exact solutions by means of a direct simulation, taking as initial condition $u(0,x)$. Those simulations were performed by means of a finite-difference discretization of the spatial derivatives together with a 5th order Dormand–Prince integrator (i.e. a Runge–Kutta scheme with stepsize control). Simulations prove the stability of the bright and dark solitons in the free space (34) and with a linear potential

(37). However, simulations for time modulated potentials [that is, (39) and those of Section 4] are cumbersome as they need a special choice of boundary conditions. We are currently working in this issue and in the orbital stablity of solitons in time periodic potentials. The results of these studies will be the aim of further publications.

Acknowledgements

This work has been supported by grant: PCI08-0093 (Consejería de Educación y Ciencia de la Junta de Comunidades de Castilla-La Mancha, Spain) and FIS2006-04190 (Ministerio de Educación y Ciencia, Spain).

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