EXISTENCE OF PULLBACK ATTRACTOR FOR A REACTION-DIFFUSION EQUATION IN SOME UNBOUNDED DOMAINS WITH NON-AUTONOMOUS FORCING TERM IN H^{-1}

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To the Memory of Professor Valery S. Melnik

The existence of a pullback attractor in $L^2(\Omega)$ for the following non-autonomous reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t), \text{ in } \Omega \times (\tau, +\infty), \\ u = 0, \quad \text{on } \ \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

is proved in this paper, when the domain Ω is not necessarily bounded but satisfying the Poincaré inequality, and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. The main concept used in the proof is the asymptotic compactness of the process generated by the problem.

Keywords: pullback attractor, asymptotic compactness, evolution process, non-autonomous reaction-diffusion equation.

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1. Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be an open set, not necessarily bounded and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2\, dx \le \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2\, dx, \quad \forall u \in H_0^1(\Omega). \tag{2}$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t), \text{ in } \Omega \times (\tau, +\infty), \\ u = 0, \quad \text{on } \ \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{3}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f \in C(\mathbb{R})$ satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, $l \ge 0$, and $p > 2$ such that

$$-\alpha_1 |s|^p \le f(s)s \le -\alpha_2 |s|^p, \tag{4}$$

$$(f(s) - f(r))(s - r) \le l(s - r)^2 \quad \forall r, s \in \mathbb{R}. \tag{5}$$

Using (4), it follows that

$$|f(s)| \leq \alpha_1 |s|^{p-1} \quad \forall s \in \mathbb{R}. \tag{6}$$

The aim of this paper is to show the existence of a pullback attractor in the phase space $L^2(\Omega)$ for the problem (3) in the case of open domains not necessarily bounded but satisfying the Poincaré inequality. This, and the fact that the non-autonomous h belongs to the space $L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, are the main novelties of our problem.

The lack of compactness of the injection $H^1_0(\Omega) \subset L^2(\Omega)$ (in the case of unbounded domains) implies that the standard techniques previously used, particularly the one involving the so-called flatenning property (see [Kloeden & Langa, 2007], [Li & Zhong, 2007], [Song & Wu, 2007], [Wang & Zhong, 2008], amongst others), which have been successfully used when Ω is bounded and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, do not work in our case.

Instead, we will use the asymptotic compactness already used in the case of non-autonomous 2D-Navier-Stokes (see [Caraballo *et al.*, 2006] and [Caraballo *et al.*, 2006b]), and which was previously used in [Rosa, 1998] for the autonomous case. We would like to emphasize that this technique seems to be the only one which allows to prove the main result of this paper (namely Theorem 4.4) concerning the existence of pullback attractor for our problem.

It is also worth mentioning that our problem has received much attention over the last years in the case of a bounded domain or for a less general term h, as we will recall now.

In [Caraballo *et al.*, 2003] it is proved the existence of pullback attractor in the space $L^2(\Omega)$ (and that it possesses finite Hausdorff dimension) when the domain is bounded and h is unbounded but with polynomial growth, i.e

$$\|h(t)\|_{L^2(\Omega)} \leq k_1 |t|^{\alpha} + k_2$$

where k_1, k_2 and α are nonnegative constants.

When Ω is bounded and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and is translation bounded, i.e.

$$\sup_{m \in \mathbb{R}} \int_m^{m+1} \|h(s)\|_{L^2(\Omega)} \, ds < \infty, \tag{7}$$

the existence of a pullback attractor in the space $H^1_0(\Omega)$ is proved in [Song & Wu, 2007], while in [Li & Zhong, 2007] the translation bounded condition (7) is weakened to

$$\|h(s)\|^2_{L^2(\Omega)} \leq M e^{\alpha |s|},$$

where $0 \leq \alpha \leq \lambda_1$, and λ_1 denotes the first eigenvalue of the Laplacian.

In [Wang & Zhong, 2008], the existence of pullback attractor in $H^1_0(\Omega)$ is shown for a bounded domain and for a $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ such that

$$\int_{-\infty}^{t} e^{\sigma s} \left(\|h(s)\|^2_{L^2(\Omega)} + \|h'(s)\|^2_{L^2(\Omega)} \right) ds < +\infty$$

for all $t \in \mathbb{R}$ and certain $\sigma \geq 0$.

For a bounded domain Ω, and a translation bounded function $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, the existence of a uniform attractor in $L^p(\Omega)$ is demonstrated in [Song & Zhong, 2008].

Finally, the reader can find similar results for several variants of our model in the references [Wang *et al.*, 2007], [Prizzi, 2003], [Morillas & Valero, 2005], [Sun & Zhong, 2005], among others.

We will provide in this paper a sufficient condition ensuring the existence of pullback attractor in $L^2(\Omega)$ when the domain is not necessarily bounded and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. A case that has not been considered in the literature yet, as far as we know.

2. Existence and uniqueness of solution

We state in this section a result on the existence and uniqueness of solution of problem (3). Instead of working directly with our equation, we will establish a general result which, in particular, can be applied to handle our problem.

2.1. An abstract result

Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Let V_i, $i = 1, ..., m$, be $m \geq 1$ reflexive and separable Banach spaces such that $\bigcup_{i=1}^{m} V_i \subset H$, $\bigcap_{i=1}^{m} V_i$ is dense in H, and V_i, $i = 1, ..., m$ is included in H with continuous injection.

By $\|\cdot\|_i$ we denote the norm in V_i, by $\|\cdot\|_{*i}$ the norm in V_i', $i = 1, ..., m$ and, by V the space

$$V = \bigcap_{i=1}^{m} V_i,$$

with the norm

$$\|v\| = \sum_{i=1}^{m} \|v\|_i, \quad \forall v \in V.$$

It is easy to see that V is a separable Banach space.

We will use $\langle \cdot, \cdot \rangle$ to denote the duality product between V_i' and V_i, for each $i = 1, ..., m$.

We identify H with its dual H' using the Riesz Theorem, but if V_i is a Hilbert space, we do not identify V_i with V_i'. Let $u \in H$, we identify u with $T_u \in \cap_{i=1}^{m} V_i'$ such that

$$\langle T_u, v \rangle = (u, v), \quad \forall v \in V_i, \ \forall i = 1, ..., m.$$

Let $\tau \in \mathbb{R}$ be an initial time, and let $A_i : (\tau, \infty) \times V_i \to V_i'$, $i = 1, ..., m$, be m operators, in general nonlinear, such that

A1) For each $v \in V_i$, the function $t \in (\tau, \infty) \longmapsto A_i(t, v) \in V_i'$ is Lebesgue measurable.

A2) Each operator A_i is hemicontinuous, i.e, for all $t \in (\tau, \infty)$ and $u, v, w \in V_i$, the function $\theta \in \mathbb{R} \longmapsto \langle A_i(t, u + \theta v), w \rangle \in \mathbb{R}$ is continuous.

Suppose that there exist $2 \leq p_i < +\infty$, $i = 1, ..., m$, and there exist constants $c > 0$, $\alpha > 0$ and $\lambda \geq 0$, and a nonnegative function $C(t) \in L^1(\tau, T)$, for all $T > \tau$, such that for each $i = 1, ..., m$,

A3) (Boundedness)

$$\|A_i(t, v)\|_{*_i} \leq c(1 + \|v\|_i^{p_i - 1}),$$

for all $t \in (\tau, \infty)$, $v \in V_i$.

A4) (Monotonicity)

$$\langle A_i(t, v) - A_i(t, w), v - w \rangle + \lambda |v - w|^2 \geq 0,$$

for all $t \in (\tau, \infty)$ and for all $v, w \in V_i$.

A5) (Coercivity)

For each i there exists a seminorm $[\cdot]_i$ in V_i, such that there exists $\lambda_i \geq 0$ for which $[v]_i + \lambda_i |v|$ is another norm in V_i. Moreover, $[v]_i + \lambda_i |v|$ and $\|\cdot\|_i$ are equivalent, and

$$\langle A_i(t, v), v \rangle + \lambda |v|^2 + C(t) \geq \alpha [v]_i^{p_i},$$

for all $t \in (\tau, \infty)$ and for all $v \in V_i$.

Consider m functions

$$h_i(t) \in L^{p_i'}(\tau, T; V_i'), \ \ \forall T > \tau, i = 1, ..., m, \quad (8)$$

and the initial condition

$$u_\tau \in H. \quad (9)$$

If we set

$$A(t, v) = \sum_{i=1}^{m} A_i(t, v), \qquad h(t) = \sum_{i=1}^{m} h_i(t),$$

we can consider the following problem

$$\begin{cases} u \in \bigcap_{i=1}^{m} L^{p_i}(\tau, T; V_i) \quad \forall T > \tau, \\ u'(t) + A(t, u(t)) = h(t), \ \text{in } \mathcal{D}'(\tau, \infty; V'), \\ u(\tau) = u_\tau. \end{cases} \quad (10)$$

The proof of the following result is similar to that of Theorem 1.4, Chapter 2 in [Lions, 1969].

Theorem 2.1. *Assume A1)-A5), (8) and (9). Then, there exists a unique solution u of (10), such that*

$$u \in C([\tau, \infty); H), \qquad u' \in \sum_{i=1}^{m} L^{p_i'}(\tau, T; V_i'), \quad (11)$$

for all $T > \tau$.

2.2. Existence and uniqueness of solution of problem (3)

We use Theorem 2.1 to show the existence and uniqueness of solution of (3).

Consider $m = 2$, $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$ and $V_2 = L^p(\Omega) \cap L^2(\Omega)$ with $p > 2$, and denote $V_1' = H^{-1}(\Omega)$, $V_2' = L^{p'}(\Omega) + L^2(\Omega)$.

Recall that $|\cdot|$ denotes the norm in H, by $\|\cdot\|_1 = |\nabla \cdot|$ we will denote the norm in V_1, and by $\|\cdot\|_2 = \|\cdot\|_{L^p(\Omega)} + |\cdot|$ the norm in V_2.

If we set

$$A_1(t, u) = -\Delta u,$$

$$A_2(t, u) = -f(u),$$

and

$$h_1(t) = h(t), \quad h_2(t) = 0,$$

then, it is not difficult to apply Theorem 2.1 with $p_1 = 2$ and $p_2 = p$, and we obtain

Theorem 2.2. *Assume that $f \in C(\mathbb{R})$ satisfies (4) and (5), and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Then, for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, there exists a unique solution $u(t) = u(t; \tau, u_\tau)$ of (3) such that*

$$u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \quad \forall T > \tau,$$
$$\frac{d}{dt}(u(t), v) - \langle \Delta u(t), v\rangle = \langle f(u(t)), v\rangle + \langle h(t), v\rangle, \text{ in } \mathcal{D}'(\tau, \infty), \ \forall v \in H_0^1(\Omega) \cap L^p(\Omega),$$
$$u(\tau) = u_\tau.$$

Moreover,

$$u \in C([\tau, \infty); L^2(\Omega)),$$

and u satisfies the energy equation,

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |\nabla u(t)|^2 = \langle f(u(t)), u(t)\rangle + \langle h(t), u(t)\rangle \quad \text{in } \mathcal{D}'(\tau, \infty). \tag{12}$$

3. Preliminaries on the theory of pullback attractors

Now, we will recall the main points from the theory of pullback attractors which will be needed in order to prove our objective (see [Caraballo *et al.*, 2006] and [Caraballo *et al.*, 2006b] for more details).

Let us consider a process (also called a two-parameter semigroup) U on a metric space X, i.e., a family $\{U(t,\tau);\ -\infty < \tau \le t < +\infty\}$ of continuous mappings $U(t,\tau) : X \to X$, such that $U(\tau,\tau)x = x$, and

$$U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau \le r \le t. \tag{13}$$

Suppose $\mathcal{D}$ is a nonempty class of parameterized sets $\widehat{D} = \{D(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 3.1. The process $U(\cdot,\cdot)$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact (i.e. pre-compact) in X.

Definition 3.2. It is said that $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $U(\cdot,\cdot)$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \le t$ such that

$$U(t,\tau)D(\tau) \subset B(t) \quad \textit{for all } \tau \le \tau_0(t,\widehat{D}).$$

Definition 3.3. The family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ if

1. $A(t)$ is compact for all $t \in \mathbb{R}$,
2. $\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,
$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0,$$
for all $\widehat{D} \in \mathcal{D}$, and all $t \in \mathbb{R}$,
3. $\widehat{A}$ is invariant, i.e.,
$$U(t,\tau)A(\tau) = A(t), \quad \textit{for } -\infty < \tau \le t < +\infty.$$

We have the following result.

Theorem 3.4. *Suppose that the process $U(\cdot,\cdot)$ is pullback $\mathcal{D}$-asymptotically compact and that $\widehat{B} \in \mathcal{D}$ is a family of pullback $\mathcal{D}$-absorbing sets for $U(\cdot,\cdot)$.*

Then, the family $\widehat{A} = \{A(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by $A(t) = \Lambda(\widehat{B}, t)$, $t \in \mathbb{R}$, where for each $\widehat{D} \in \mathcal{D}$

$$\Lambda(\widehat{D}, t) = \bigcap_{s\le t}\left(\overline{\bigcup_{\tau\le s} U(t,\tau)D(\tau)}\right),$$

is a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ which satisfies in addition that $A(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}$, for $t \in \mathbb{R}$. Furthemore, $\widehat{A}$ is minimal in the sense that if $\widehat{C} = \{C(t);\ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that $\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$, then $A(t) \subset C(t)$.

4. Existence of the pullback attractor

Now, we can prove our main aim in this paper. First, we need a continuity result which is established in the next subsection.

4.1. Weak Continuity

Let $f \in C(\mathbb{R})$ be a function, and suppose that f satisfies (4) and (5), and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

Thanks to Theorem 2.2, we can define a process $\{U(t,\tau),\ \tau \le t\}$ in $L^2(\Omega)$, as

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau)\ \forall u_\tau \in L^2(\Omega),\ \forall \tau \le t. \tag{14}$$

From the uniqueness of solution to problem (3), it follows that (14) defines a process in $L^2(\Omega)$. In addition, it can be proved that the process defined by (14) is continuous in $L^2(\Omega)$.

Moreover, U is weakly continuous, and more exactly the following result holds true. We will denote by "$\rightharpoonup$" the weak convergence in the corresponding indicated space, while "$\to$" will denote the strong convergence, as usual.

Proposition 4.1. *Let $\{u_{\tau_n}\} \subset L^2(\Omega)$ be a sequence converging weakly in $L^2(\Omega)$ to an element $u_\tau \in L^2(\Omega)$. Then, for all $T > \tau$, it follows*

$$U(t,\tau)\,u_{\tau_n} \rightharpoonup U(t,\tau)\,u_\tau \text{ in } L^2(\Omega)\ \ \forall t \geq \tau, \tag{15}$$

$$U(\cdot,\tau)\,u_{\tau_n} \rightharpoonup U(\cdot,\tau)\,u_\tau \text{ in } L^2(\tau,T;H_0^1(\Omega)), \tag{16}$$

$$U(\cdot,\tau)\,u_{\tau_n} \rightharpoonup U(\cdot,\tau)\,u_\tau \text{ in } L^p(\tau,T;L^p(\Omega)), \tag{17}$$

$$f(U(\cdot,\tau)\,u_{\tau_n}) \rightharpoonup f(U(\cdot,\tau)\,u_\tau) \text{ in } L^{p'}(\tau,T;L^{p'}(\Omega)). \tag{18}$$

If Ω is a bounded set, then

$$U(\cdot,\tau)\,u_{\tau_n} \longrightarrow U(\cdot,\tau)\,u_\tau \text{ in } L^2(\tau,T;L^2(\Omega)). \tag{19}$$

Proof. From the assumptions, we deduce that there exists a positive constant C such that

$$|u_{\tau_n}| \leq C \quad \forall\, n \geq 1. \tag{20}$$

Fix $\tau \in \mathbb{R}$, and set

$$u_n(t) = U(t,\tau)\,u_{\tau_n}, \quad u(t) = U(t,\tau)\,u_\tau. \tag{21}$$

Using (4), it follows

$$\frac{d}{dt}|u_n(t)|^2 + 2|\nabla u_n(t)|^2 \leq -2\alpha_2 \|u_n(t)\|^p_{L^p(\Omega)} + 2\langle h(t), u_n\rangle.$$

Integrating between τ and t, we obtain

$$\begin{aligned} &|u_n(t)|^2 + 2\int_\tau^t |\nabla u_n(s)|^2\,ds \\ &\quad + 2\alpha_2 \int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)}\,ds \\ &\leq |u_{\tau_n}|^2 + 2\int_\tau^t \langle h(s), u_n\rangle\,ds \quad \forall\, t \geq \tau. \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} \int_\tau^t \langle h(s), u_n\rangle\,ds &\leq \int_\tau^t \|h\|_{H^{-1}(\Omega)}|\nabla u_n|\,ds \\ &\leq \frac{1}{2}\int_\tau^T \|h\|^2_{H^{-1}(\Omega)}\,ds \\ &\quad + \frac{1}{2}\int_\tau^t |\nabla u_n|^2\,ds, \end{aligned}$$

which, jointly with (20) and (22) imply

$$\begin{aligned} &|u_n(t)|^2 + \int_\tau^t |\nabla u_n(s)|^2\,ds + 2\alpha_2\int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)}\,ds \\ &\leq C^2 + \int_\tau^t \|h\|^2_{H^{-1}(\Omega)}\,ds \quad \forall\, t > \tau. \end{aligned}$$

We deduce that $\{u_n\}$ is bounded in

$$L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega)) \cap \mathcal{C}([\tau,T];L^2(\Omega)), \tag{23}$$

for all $T > \tau$.

Fix $T > \tau$. In particular, $\{u_n(T)\}$ is bounded in $L^2(\Omega)$. On the other hand, from (6) we have

$$\|f(u_n(t))\|^{p'}_{L^{p'}(\Omega)} \leq \alpha_1^{p'}\|u_n(t)\|^p_{L^p(\Omega)},$$

whence $f(u_n)$ is bounded in $L^{p'}(\tau,T;L^{p'}(\Omega))$.

Then, there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \overset{*}{\rightharpoonup} v \text{ weak-star in } L^\infty(\tau,T;L^2(\Omega)),$$

$$u_\mu \rightharpoonup v \text{ in } L^p(\tau,T;L^p(\Omega)), \tag{24}$$

$$u_\mu(T) \rightharpoonup \xi \text{ in } L^2(\Omega), \tag{25}$$

$$u_\mu \rightharpoonup v \text{ in } L^2(\tau,T;H_0^1(\Omega)), \tag{26}$$

and

$$f(u_\mu) \rightharpoonup \chi \text{ in } L^{p'}(\tau,T;L^{p'}(\Omega)). \tag{27}$$

Now (26) imply that

$$\Delta u_\mu \rightharpoonup \Delta v \text{ in } L^2(\tau,T;H^{-1}(\Omega)).$$

From (26), (27), and thanks to the equation

$$u_\mu'(t) = \Delta u_\mu(t) + f(u_\mu(t)) + h(t), \tag{28}$$

it is a standard matter to prove that we can pick an element in the equivalence class of v satisfying

$$v(t) = u_\tau + \int_\tau^t (\Delta v(s) + \chi(s) + h(s))ds, \tag{29}$$

for all $t \in [\tau,T]$.

We are now in position to show that $\xi = v(T)$ and $\chi(t) = f(v(t))$.

Let $w \in H_0^1(\Omega) \cap L^p(\Omega)$. Integrating (28) between τ and T, we obtain

$$\begin{aligned} (u_\mu(T), w) &= (u_{\tau_\mu}, w) \\ &\quad + \int_\tau^T \langle \Delta u_\mu(s) + f(u_\mu(s)) + h(s), w\rangle\,ds, \end{aligned}$$

and thus

$$(\xi, w) = (u_\tau, w) + \int_\tau^T \langle \Delta v(s) + \chi(s) + h(s), w\rangle \, ds,$$

when $\mu \longrightarrow +\infty$. By density and using (29), it follows

$$v(T) = \xi. \tag{30}$$

To prove that $\chi(t) = f(v(t))$, we argue similarly to [Rosa, 1998]. Integrating the equality

$$\begin{aligned} \frac{d}{ds}(u_\mu(s), w) &= -(\nabla u_\mu(s), \nabla w) \\ &\quad + \langle f(u_\mu(s)), w\rangle + \langle h(s), w\rangle, \end{aligned}$$

between t and $t+a$, with $a \in (0, T-\tau)$, $t \in (\tau, T-a)$, and using the Hölder inequality, we obtain

$$\begin{aligned} &(u_\mu(t+a) - u_\mu(t), w) \\ &\quad \le \int_t^{t+a} |\nabla u_\mu(s)| \, |\nabla w| \, ds \\ &\qquad + \int_t^{t+a} \|f(u_\mu(s)\|_{L^{p'}(\Omega)} \|w\|_{L^p(\Omega)} \, ds \\ &\qquad + \int_t^{t+a} \|h(s)\|_{H^{-1}(\Omega)} |\nabla w| \, ds \\ &\quad \le |\nabla w| \, a^{1/2} \|u_\mu\|_{L^2(\tau,T;H_0^1(\Omega))} \\ &\qquad + \|w\|_{L^p(\Omega)} a^{1/p} \|f(u_\mu)\|_{L^{p'}(\tau,T;L^{p'}(\Omega))} \\ &\qquad + |\nabla w| \, a^{1/2} \|h\|_{L^2(\tau,T;H^{-1}(\Omega))}, \end{aligned}$$

and thanks to (23), we deduce that there exists a constant $C^{(1)}$ such that

$$\begin{aligned} &(u_\mu(t+a) - u_\mu(t), w) \\ &\quad \le C^{(1)}(a^{1/2} + a^{1/p})(|\nabla w| + \|w\|_{L^p(\Omega)}). \end{aligned}$$

If we take in the last inequality $w = u_\mu(t+a) - u_\mu(t) \in H_0^1(\Omega) \cap L^p(\Omega)$ a.e. $t \in (\tau, T-a)$, we obtain

$$\begin{aligned} &|u_\mu(t+a) - u_\mu(t)|^2 \\ &\le C^{(1)}(a^{1/2} + a^{1/p}) |\nabla u_\mu(t+a) - \nabla u_\mu(t)| \\ &\quad + C^{(1)}(a^{1/2} + a^{1/p}) \|u_\mu(t+a) - u_\mu(t)\|_{L^p(\Omega)}, \end{aligned}$$

a.e. $t \in (\tau, T-a)$.

Integrating between τ and $T-a$,

$$\begin{aligned} &\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt \\ &\le C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} |\nabla u_\mu(t+a)| \, dt \\ &+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} |\nabla u_\mu(t)| \, dt \\ &+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} \|u_\mu(t+a)\|_{L^p(\Omega)} \, dt \\ &+ C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T-a} \|u_\mu(t)\|_{L^p(\Omega)} \, dt, \end{aligned}$$

it follows

$$\begin{aligned} &\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt \\ &\le 2C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T} |\nabla u_\mu(s)| \, ds \\ &+ 2C^{(1)}(a^{1/2} + a^{1/p}) \int_\tau^{T} \|u_\mu(s)\|_{L^p(\Omega)} \, ds, \end{aligned}$$

and using the Hölder inequality, we obtain

$$\begin{aligned} &\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt \\ &\le 2C^{(1)}(a^{1/2} + a^{1/p})(T-\tau)^{1/2} \|u_\mu\|_{L^2(\tau,T;H_0^1(\Omega))} \\ &+ 2C^{(1)}(a^{1/2} + a^{1/p})(T-\tau)^{1/p'} \|u_\mu\|_{L^p(\tau,T;L^p(\Omega))}. \end{aligned}$$

Thanks to (23) we deduce that there exists a constant $\widetilde{C}_T$ such that

$$\int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt \le \widetilde{C}_T(a^{1/2} + a^{1/p}),$$

for all μ, and all $a \in (0, T-\tau)$, and thus

$$\lim_{a \to 0} \left(\sup_\mu \int_\tau^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 \, dt \right) = 0. \tag{31}$$

Now, for all $m \in \mathbb{Z}$, $m \ge 1$, we denote

$$\Omega_m = \Omega \cap \left\{ x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m \right\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$.

Let $\phi \in C^1([0, +\infty))$ be a function such that $0 \le \phi(s) \le 1$, $\phi(s) = 1 \ \forall s \in [0,1]$, and $\phi(s) = 0 \ \forall s \ge 2$.

For each μ and $m \ge 1$, we define

$$v_{\mu,m}(x,t) = \phi\left(\frac{|x|^2_{\mathbb{R}^N}}{m^2} \right) u_\mu(x,t) \quad .$$

From (23), for all $m \geq 1$, we obtain that $\{v_{\mu,m}\}_{\mu\geq 1}$ is bounded in $L^2(\tau,T;H_0^1(\Omega_{2m})) \cap L^p(\tau,T;L^p(\Omega_{2m})) \cap L^\infty(\tau,T;L^2(\Omega_{2m}))$.
In particular,

$$\lim_{a\to 0}\sup_{\mu}\left(\int_{\tau}^{\tau+a}|v_{\mu,m}(t)|^2_{L^2(\Omega_{2m})}\,dt + \right.$$
$$\left. + \int_{T-a}^{T}|v_{\mu,m}(t)|^2_{L^2(\Omega_{2m})}\,dt\right) = 0. \quad (32)$$

On the other hand, from (31) we deduce that for all $m \geq 1$,

$$\lim_{a\to 0}\sup_{\mu}\int_{\tau}^{T-a}|v_{\mu,m}(t+a) - v_{\mu,m}(t)|^2_{L^2(\Omega_{2m})}\,dt = 0. \quad (33)$$

Moreover, as Ω_{2m} is a bounded set, then $H_0^1(\Omega_{2m})$ is included in $L^2(\Omega_{2m})$ with compact injection.

Then, by the compactness Theorem 13.3 of [Temam, 1983] with $X = L^2(\Omega_{2m})$, $Y = H_0^1(\Omega_{2m})$, $r = 2$ and $\mathcal{G} = \{v_{\mu,m}\}_{\mu\geq 1}$, we obtain that $\{v_{\mu,m}\}_{\mu\geq 1}$ is relatively compact in $L^2\left(\tau,T;L^2(\Omega_{2m})\right)$, and thus, taking into account that $v_{\mu,m}(x,t) = u_\mu(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$

$$\left\{u_{\mu|\Omega_m}\right\}_{\mu\geq 1} \text{ is pre-compact in } L^2\left(\tau,T;L^2(\Omega_m)\right). \quad (34)$$

It is not difficult to conclude from (34), (26) and (2), via a diagonal procedure, the existence of a subsequence $\{u_\mu^\mu\}_{\mu\geq 1} \subset \{u_\mu\}_{\mu\geq 1}$ such that

$$u_\mu^\mu \to v \quad \text{a.e. in } \Omega_m \times (\tau,T) \text{ as } \mu \longrightarrow \infty \quad \forall m \geq 1.$$

Then, as f is continuous,

$$f(u_\mu^\mu) \to f(v) \text{ a.e. in } \Omega_m \times (\tau,T),$$

and as $\{f(u_\mu^\mu)\}$ is bounded in $L^{p'}(\Omega_m \times (\tau,T))$, by Lemma 1.3, Chapter 1 in [Lions, 1969], we obtain

$$f(u_\mu^\mu) \rightharpoonup f(v) \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega_m)\right).$$

From (27)

$$f(u_\mu) \rightharpoonup \chi_{|\Omega_m\times(\tau,T)} \text{ weakly in } L^{p'}\left(\tau,T;L^{p'}(\Omega_m)\right).$$

By the uniqueness of the weak limit, we have

$$\chi = f(v) \text{ a.e. in } \Omega_m \times (\tau,T) \quad \forall m \geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^{\infty}\Omega_m = \Omega$, we obtain

$$\chi = f(v) \text{ a.e. in } \Omega \times (\tau,T). \quad (35)$$

From (29) and (35), and by the uniqueness of solutions we have $v(t) = u(t)$ for all $t \in [\tau,T]$. And then, if we consider (30) in (26) and (35) in (27), we have

$$u_\mu \rightharpoonup u \text{ in } L^2(\tau,T;H_0^1(\Omega)),$$

$$u_\mu \rightharpoonup u \text{ in } L^p(\tau,T;L^p(\Omega)),$$

$$u_\mu(T) \rightharpoonup u(T) \text{ in } L^2(\Omega),$$

$$f(u_\mu) \rightharpoonup f(u) \text{ in } L^{p'}(\tau,T;L^{p'}(\Omega)).$$

Then, by a contradiction argument we deduce

$$u_n \rightharpoonup u \text{ in } L^2(\tau,T;H_0^1(\Omega)),$$

$$u_n \rightharpoonup u \text{ in } L^p(\tau,T;L^p(\Omega)),$$

$$u_n(T) \rightharpoonup u(T) \text{ in } L^2(\Omega),$$

$$f(u_n) \rightharpoonup f(u) \text{ in } L^{p'}(\tau,T;L^{p'}(\Omega)),$$

and, as $T > \tau$ has been taken arbitrarily, the first part of the proof is finished.

Now, if Ω is bounded, we deduce from (34) that

$$\left\{u_{\mu|\Omega}\right\}_{\mu\geq 1} \text{ is pre-compact in } L^2\left(\tau,T;L^2(\Omega)\right). \quad (36)$$

Finally, (19) follows from (36). ∎

Remark 4.2. From the proof, it is clear that for any $m \geq 1$,

$$U(\cdot,\tau)u_{\tau_n} \to U(\cdot,\tau)u_\tau \text{ in } L^2(\tau,T;L^2(\Omega_m)).$$

Moreover, it is possible to prove that

$$U(t,\tau)u_{\tau_n} \to U(t,\tau)u_\tau \quad \text{in } L^2(\Omega_m),$$

for all $t > \tau$.

Remark 4.3. Notice that all the results obtained in the previous analysis hold true for a general non-empty open subset $\Omega \subset \mathbb{R}^N$.

4.2. The existence of the global pullback attractor

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r:\mathbb{R}\to(0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{\lambda_1 t} r^2(t)=0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subset\mathcal{P}(L^2(\Omega))$ such that $D(t)\subset\overline{B}(0,r_{\widehat{D}}(t))$, for some $r_{\widehat{D}}\in\mathcal{R}_{\lambda_1}$, where $\overline{B}(0,r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$.

Now, we can prove the following result.

Theorem 4.4. *Suppose that Ω satisfies (2), and suppose that $f\in C(\mathbb{R})$ satisfies (4) and (5) with $l=0$. Let $h\in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$ such that*

$$\int_{-\infty}^t e^{\lambda_1 s}\|h(s)\|^2_{H^{-1}(\Omega)}\,ds<+\infty\quad\forall t\in\mathbb{R}.$$

Then, there exists a unique global pullback $\mathcal{D}_{\lambda_1}$-attractor for the process U, which belongs to $\mathcal{D}_{\lambda_1}$, and is defined by (14).

Proof. Let $\tau\in\mathbb{R}$, and $u_\tau\in L^2(\Omega)$ be fixed, and denote

$$u(t)=u(t;\tau,u_\tau)=U(t,\tau)u_\tau\quad\forall t\geq\tau.$$

Taking into account (4) and the energy equality,

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right)+2e^{\lambda_1 t}|\nabla u(t)|^2\\
&\quad=\lambda_1 e^{\lambda_1 t}|u(t)|^2+2e^{\lambda_1 t}\langle f(u(t)),u(t)\rangle\\
&\qquad+2e^{\lambda_1 t}\langle h(t),u(t)\rangle \qquad (37)\\
&\quad\leq\lambda_1 e^{\lambda_1 t}|u(t)|^2\\
&\qquad+e^{\lambda_1 t}|\nabla u(t)|^2+e^{\lambda_1 t}\|h(t)\|^2_{H^{-1}(\Omega)},
\end{aligned}$$

and thus, from (2), we obtain

$$\frac{d}{dt}\left(e^{\lambda_1 t}|u(t)|^2\right)\leq e^{\lambda_1 t}\|h(t)\|^2_{H^{-1}(\Omega)}.$$

Integrating between τ and t, it follows

$$\begin{aligned}
e^{\lambda_1 t}|u(t)|^2&\leq\int_\tau^t e^{\lambda_1 s}\|h(s)\|^2_{H^{-1}(\Omega)}\,ds+e^{\lambda_1\tau}|u_\tau|^2\\
&\leq\int_{-\infty}^t e^{\lambda_1 s}\|h(s)\|^2_{H^{-1}(\Omega)}\,ds+e^{\lambda_1\tau}|u_\tau|^2.
\end{aligned}$$

Let $\widehat{D}\in\mathcal{D}_{\lambda_1}$ be given. Then

$$\begin{aligned}
|U(t,\tau)u_\tau|^2\ &\leq\ e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}\|h(s)\|^2_{H^{-1}(\Omega)}\,ds\\
&\qquad+e^{\lambda_1(\tau-t)}r^2_D(\tau), \qquad (38)
\end{aligned}$$

for all $u_\tau\in D(\tau)$ and for all $t\geq\tau$.

Denote by $R_{\lambda_1}(t)$ the nonnegative number given for each $t\in\mathbb{R}$ by

$$R^2_{\lambda_1}(t)=e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}\|h(s)\|^2_{H^{-1}(\Omega)}\,ds+1. \qquad (39)$$

Observe that

$$\lim_{t\to-\infty}e^{\lambda_1 t}R^2_{\lambda_1}(t)=0,$$

and, consequently,

$$R_{\lambda_1}\in\mathcal{R}_{\lambda_1}.$$

Now, consider the family $\widehat{B}_{\lambda_1}$ of closed balls in $L^2(\Omega)$

$$\widehat{B}_{\lambda_1}=\{B_{\lambda_1}(t):t\in\mathbb{R}\},$$

defined by

$$B_{\lambda_1}(t)=\left\{v\in L^2(\Omega):|v|\leq R_{\lambda_1}(t)\right\}.$$

It is straightforward to check that

$$\widehat{B}_{\lambda_1}\in\mathcal{D}_{\lambda_1},$$

and moreover, by (38), the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process U.

According to Theorem 3.4, to finish the proof of the theorem we only have to prove that U is pullback $\mathcal{D}_{\lambda_1}$-asymptotically compact.

Let us fix $\widehat{D}\in\mathcal{D}_{\lambda_1}$, a sequence $\tau_n\to-\infty$, a sequence $u_{\tau_n}\in D(\tau_n)$, and $t\in\mathbb{R}$. We have to prove that from the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ we can extract a subsequence that converges in $L^2(\Omega)$.

As the family $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, for each integer $k\geq0$, there exists a $\tau_D(k)\leq t-k$ such that

$$U(t-k,\tau)D(\tau)\subset B_{\lambda_1}(t-k)\quad\forall\tau\leq\tau_D(k). \qquad (40)$$

Again, by a diagonal procedure, it is not difficult to conclude from (40), that there exist a subsequence $\left\{\left(\tau_{n'},u_{\tau_{n'}}\right)\right\}\subset\{(\tau_n,u_{\tau_n})\}$, and a sequence $\{w_k;k\geq0\}\subset L^2(\Omega)$ such that for all $k\geq0$, and $w_k\in B_{\lambda_1}(t-k)$,

$$U(t-k,\tau_{n'})u_{\tau_{n'}}\rightharpoonup w_k\ \text{ in } L^2(\Omega). \qquad (41)$$

Observe that, by Proposition 4.1.

$$\begin{aligned} w_0 &= weak - \lim_{n'\to\infty} U(t,\tau_{n'})u_{\tau_{n'}} \\ &= weak - \lim_{n'\to\infty} U(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \\ &= U(t,t-k)\left(weak - \lim_{n'\to\infty} U(t-k,\tau_{n'})u_{\tau_{n'}} \right). \end{aligned}$$

i.e.,

$$U(t,t-k)w_k = w_0 \quad \forall k \geq 0. \tag{42}$$

Then, by the lower semi-continuity of the norm, using (41) we obtain

$$|w_0| \leq \liminf_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right|. \tag{43}$$

If we now prove that also

$$\limsup_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right| \leq |w_0|, \tag{44}$$

then we will have

$$\lim_{n'\to\infty} \left|U(t,\tau_{n'})u_{\tau_{n'}}\right| = |w_0|.$$

And this, together with the weak convergence, will imply the strong convergence in $L^2(\Omega)$ of $U(t,\tau_{n'})u_{\tau_{n'}}$ to w_0.
In order to prove (44), consider

$$[u] := |\nabla u|^2 - \frac{\lambda_1}{2}|u|^2 - \langle f(u), u\rangle. \tag{45}$$

From (37), and integrating between τ and t,

$$\begin{aligned} e^{\lambda_1 t}|u(t)|^2 - e^{\lambda_1 \tau}|u_\tau|^2 = -2\int_\tau^t e^{\lambda_1 s}[u(s)]\,ds \\ +2\int_\tau^t e^{\lambda_1 s}\langle h(s), u(s)\rangle\,ds, \end{aligned}$$

i.e.,

$$\begin{aligned} &|U(t,\tau)u_\tau|^2 = e^{\lambda_1(\tau-t)}|u_\tau|^2 \qquad (46) \\ &+2\int_\tau^t e^{\lambda_1(s-t)}\left(\langle h(s), U(s,\tau)u_\tau\rangle - [U(s,\tau)u_\tau]\right)ds. \end{aligned}$$

From (46) it is immediate that for all $k \geq 0$ and all $\tau_{n'} \leq t-k$,

$$\begin{aligned} &\left|U(t,\tau_{n'})u_{\tau_{n'}}\right|^2 \\ &= \left|U(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right|^2 \qquad (47) \\ &= \left|U(t-k,\tau_{n'})u_{\tau_{n'}}\right|^2 e^{-\lambda_1 k} \\ &+2\int_{t-k}^t e^{\lambda_1(s-t)}\left\langle h(s), U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right\rangle ds \\ &-2\int_{t-k}^t e^{\lambda_1(s-t)}\left[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right]ds. \end{aligned}$$

As, thanks to (40),

$$U(t-k,\tau_{n'})u_{\tau_{n'}} \in B_{\lambda_1}(t-k) \quad \forall \tau_{n'} \leq \tau_D(k), \;\; k \geq 0,$$

we have

$$\begin{aligned} \limsup_{n'\to\infty}\left(\left|U(t-k,\tau_{n'})u_{\tau_{n'}}\right|^2 e^{-\lambda_1 k}\right) \\ \leq R^2_{\lambda_1}(t-k)e^{-\lambda_1 k} \quad \forall\, k \geq 0. \end{aligned} \tag{48}$$

On the other hand, from (41) and Proposition 4.1 we deduce that

$$U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \rightharpoonup U(\cdot,t-k)w_k \tag{49}$$

in $L^2(t-k,t;H_0^1(\Omega))$.
Taking into account that, in particular,

$$e^{\lambda_1(s-t)}h(s) \in L^2(t-k,t;H^{-1}(\Omega)),$$

we obtain from (49),

$$\begin{aligned} &\lim_{n'\to\infty}\int_{t-k}^t e^{\lambda_1(s-t)}\left\langle h(s), U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right\rangle ds \\ &= \int_{t-k}^t e^{\lambda_1(s-t)}\langle h(s), U(s,t-k)w_k\rangle\,ds. \qquad (50) \end{aligned}$$

Now we will prove that

$$\begin{aligned} &\int_{t-k}^t e^{\lambda_1(s-t)}[U(s,t-k)w_k]\,ds \qquad (51) \\ &\leq \liminf_{n'\to\infty}\int_{t-k}^t e^{\lambda_1(s-t)}\left[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\right]ds. \end{aligned}$$

Denote

$$J_k(v) = J_k^{(1)}(v) + J_k^{(2)}(v),$$

where

$$J_k^{(1)}(v) = \int_{t-k}^t e^{\lambda_1(s-t)}\left(|\nabla v(s)|^2 - \frac{\lambda_1}{2}|v(s)|^2\right)ds,$$

and

$$J_k^{(2)}(v) = -\int_{t-k}^t e^{\lambda_1(s-t)}\langle f(v), v\rangle\,ds,$$

for all $v \in L^2(t-k,t;H_0^1(\Omega)) \cap L^p(t-k,t;L^p(\Omega))$.
Then, we want to prove

$$\begin{aligned} &J_k(U(\cdot,t-k)w_k) \\ &\leq \liminf_{n'\to\infty} J_k(U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}), \end{aligned}$$

what will be done if we prove that

$$J_k^{(i)}(U(\cdot,t-k)w_k) \le \liminf_{n'\to\infty} J_k^{(i)}(U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}) \quad i=1,2.$$

As, thanks to (2), $|\nabla v|^2 - \frac{\lambda_1}{2}|v|^2$ defines a norm in $H_0^1(\Omega)$, which is equivalent to the usual one, we also obtain from (41) and using Proposition 4.1

$$\liminf_{n'\to\infty}(J_k^{(1)}(U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}) \ge J_k^{(1)}(U(\cdot,t-k)w_k). \quad (52)$$

Now denote

$$A_{k,n'}(s) := U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}},$$

and

$$B_k(s) := U(s,t-k)w_k.$$

We easily obtain

$$\begin{aligned}&\liminf_{n'\to\infty}(J_k^{(2)}(A_{k,n'}(\cdot))\\ &= \liminf_{n'\to\infty}\left\{\int_{t-k}^t e^{\lambda_1(s-t)}\times\right.\\ &\times\langle f(B_k(s)) - f(A_{k,n'}(s)), A_{k,n'}(s) - B_k(s)\rangle ds\\ &- \int_{t-k}^t e^{\lambda_1(s-t)}\langle f(B_k(s)), A_{k,n'}(s)\rangle ds\\ &+ \int_{t-k}^t e^{\lambda_1(s-t)}\langle f(B_k(s)), B_k(s)\rangle ds\\ &\left.- \int_{t-k}^t e^{\lambda_1(s-t)}\langle f(A_{k,n'}(s)), B_k(s)\rangle ds\right\}.\end{aligned}$$

Using (5) with $l=0$, it follows

$$\begin{aligned}&\liminf_{n'\to\infty}\left(J_k^{(2)}(A_{k,n'}(\cdot))\right) \quad (53)\\ &\ge \liminf_{n'\to\infty}\left(-\int_{t-k}^t e^{\lambda_1(s-t)}\langle f(B_k(s)), A_{k,n'}(s)\rangle ds\right)\\ &+ \int_{t-k}^t e^{\lambda_1(s-t)}\langle f(B_k(s)), B_k(s)\rangle ds\\ &+ \liminf_{n'\to\infty}\left(-\int_{t-k}^t e^{\lambda_1(s-t)}\langle f(A_{k,n'}(s)), B_k(s)\rangle ds\right).\end{aligned}$$

From (49), we have

$$\begin{aligned}&\liminf_{n'\to\infty}(J_k^{(2)}(A_{k,n'}(\cdot))) \quad (54)\\ &\ge \liminf_{n'\to\infty}\left(-\int_{t-k}^t e^{\lambda_1(s-t)}\langle f(A_{k,n'}(s)), B_k(s)\rangle ds\right).\end{aligned}$$

From (41) and Proposition 4.1 we obtain

$$f(A_{k,n'}(\cdot)) \rightharpoonup f(B_k(\cdot)) \text{ in } L^{p'}\left(t-k,t;L^{p'}(\Omega)\right)$$

as $n'\to\infty$. Then,

$$\begin{aligned}\lim_{n'\to\infty}&\int_{t-k}^t e^{\lambda_1(s-t)}\langle f(A_{k,n'}(s)), B_k(s)\rangle ds\\ &= \int_{t-k}^t e^{\lambda_1(s-t)}\langle f(B_k(s)), B_k(s)\rangle ds,\end{aligned}$$

which, jointly with (54), yield that

$$\begin{aligned}&\liminf_{n'\to\infty}\left(J_k^{(2)}(U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})\right)\\ &\ge J_k^{(2)}(U(\cdot,t-k)w_k).\end{aligned}$$

Therefore (51) is easily obtained from the last inequality and (52).
Then, (47), (48), (50) and (51) imply

$$\begin{aligned}&\limsup_{n'\to\infty}\left|U(t,\tau_{n'})u_{\tau_{n'}}\right|^2 \quad (55)\\ &\le R_{\lambda_1}^2(t-k)e^{-\lambda_1 k}\\ &\quad + 2\int_{t-k}^t e^{\lambda_1(s-t)}\langle h(s), U(s,t-k)w_k\rangle ds\\ &\quad - 2\int_{t-k}^t e^{\lambda_1(s-t)}\left[U(s,t-k)w_k\right] ds,\end{aligned}$$

for all $k\ge 1$. Now, from (42) and (46),

$$\begin{aligned}|w_0|^2 &= |U(t,t-k)w_k|^2 = e^{-\lambda_1 k}|w_k|^2 \quad (56)\\ &+ 2\int_{t-k}^t e^{\lambda_1(s-t)}\left(\langle h(s), U(s,t-k)w_k\rangle\right.\\ &\left.- [U(s,t-k)w_k]\right) ds.\end{aligned}$$

From (55) and (56), we have

$$\begin{aligned}&\limsup_{n'\to\infty}\left|U(t,\tau_{n'})u_{\tau_{n'}}\right|^2\\ &\le R_{\lambda_1}^2(t-k)e^{-\lambda_1 k} + |w_0|^2 - e^{-\lambda_1 k}|w_k|^2\\ &\le R_{\lambda_1}^2(t-k)e^{-\lambda_1 k} + |w_0|^2,\end{aligned}$$

for all $k\ge 1$. Taking into account (39), we easily obtain

$$R_{\lambda_1}^2(t-k)e^{-\lambda_1 k} \to 0 \text{ when } k\to\infty,$$

and

$$\limsup_{n'\to\infty}\left|U(t,\tau_{n'})u_{\tau_{n'}}\right|^2 \le |w_0|^2.$$

■

Remark 4.5. Observe that the universe $\mathcal{D}_{\lambda_1}$ contains the families of fixed bounded sets (i.e. for any bounded $C \subset L^2(\Omega)$ it follows that $\widehat{C} = \{C(t) \equiv C, t \in \mathbb{R}\} \in \mathcal{D}_{\lambda_1}$). Thus, Theorem 4.4 implies that, the global pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ (whose existence is guaranteed by this theorem) is formed by a family of compact subsets of $L^2(\Omega)$ which pullback attracts the bounded subsets of $L^2(\Omega)$, what implies the existence of the pullback attractor $\widehat{A}_0$ in the sense of Crauel et al. [Crauel *et al.*, 1995] (recall that $\widehat{A}_0$ is a family of compact sets, invariant and pullback attracting the bounded subsets of X), and is given by

$$A_0(t) = \overline{\bigcup_{\substack{C \subset X \\ C \text{ bounded}}} \Lambda(C,t)}.$$

Furthermore, by the minimality of $\widehat{A}_0$ it follows that

$$A_0(t) \subset A(t) \quad \text{for any } t \in \mathbb{R}.$$

In fact, it can be proved (see [Marín-Rubio & Real, 2008]) that if there exists a value $T \in \mathbb{R}$ such that

$$\sup_{t \le T} R_{\lambda_1}(t) < +\infty, \tag{57}$$

where R_{λ_1} is the function defined in (39), then

$$A_0(t) = A(t) \qquad \forall t \le T.$$

A sufficient condition for (57) is that $h \in L^\infty(-\infty, T; H^{-1}(\Omega))$.

Remark 4.6. Theorem 4.4 also holds if, instead of assuming that $l = 0$, we impose that the function $sf(s)$ is concave.

Remark 4.7. If Ω is a bounded set, thanks to (19) we easily obtain Theorem 4.4 for all $l \ge 0$. Moreover we can replace (4) by

$$-\alpha_1 |s|^p - \beta \le f(s)s \le -\alpha_2 |s|^p + \beta,$$

with $\beta \ge 0$.

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