A GRADIENT-LIKE NON-AUTONOMOUS EVOLUTION PROCESS

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To the Memory of Professor Valery S. Melnik

In this paper we consider a dissipative damped wave equation with non-autonomous damping of the form

$$u_{tt} + \beta(t)u_t = \Delta u + f(u) \tag{1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions, where f is a dissipative smooth nonlinearity and the damping $\beta : \mathbb{R} \to (0, \infty)$ is a suitable function. We prove, if (1) has finitely many equilibria, that all global bounded solutions of (1) are backwards and forwards asymptotic to equilibria. Thus, we give a class of examples of non-autonomous evolution processes for which the structure of the pullback attractors is well understood. That complements the results of [Carvalho & Langa, 2009] on characterization of attractors, where it was shown that a small non-autonomous perturbation of an autonomous gradient-like evolution process is also gradient-like. Note that the evolution process associated to (1) is not a small non-autonomous perturbation of any autonomous gradient-like evolution processes. Moreover, we are also able to prove that the pullback attractor for (1) is also a forwards attractor and that the rate of attraction is exponential.

Keywords: pullback attractor, asymptotic compactness, evolution process, non-autonomous damped wave equation.

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1. Introduction

Consider the following non-autonomous damped wave equation

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f(u) \text{ in } \Omega \\ u(x,t) = 0 \text{ in } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Assume that $f \in C^2(\mathbb{R})$ satisfies the following growth and dissipativeness conditions

$$|f'(s)| \leqslant c(1+|s|^{p-1}), \quad \limsup_{|s|\to\infty} \frac{f(s)}{s} \leqslant 0, \tag{3}$$

with $c > 0$ and $p < \frac{n}{n-2}$.

Assume that $\beta : \mathbb{R} \to \mathbb{R}$ is bounded, globally Lipschitz, and that

$$\beta_0 \leqslant \beta(t) \leqslant \beta_1 \text{ for some } \beta_0, \beta_1 \in (0,\infty). \tag{4}$$

Let $X = H^1_0(\Omega) \times L^2(\Omega)$, for $u_t = v$ and $w = \binom{u}{v}$, we rewrite (2) as

$$\begin{aligned} &w_t = C(t)w + F(w),\ t > \tau, \\ &w(\tau) = w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in X, \end{aligned} \tag{5}$$

where

$$C(t) = \begin{pmatrix} 0 & I \\ -A & -\beta(t)I \end{pmatrix}, \ F(w) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}, \tag{6}$$

and $A = -\Delta$ with Dirichlet boundary condition.

Note that, if

$$\mathcal{L}(\varphi,\phi) = \frac{1}{2}|\varphi|^2_{H^1_0(\Omega)} + \frac{1}{2}|\phi|^2_{L^2(\Omega)} - \int_\Omega G(\varphi), \tag{7}$$

with $(\varphi,\phi) \in X$, $G(r) = \int_0^r f(\theta)d\theta$, and $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$ a regular solution of (5), then

$$\frac{d}{dt}\mathcal{L}(u,u_t) = -\beta(t)|u_t|^2_{L^2(\Omega)}.$$

Hence $\mathcal{L} : X \to \mathbb{R}$ is a continuous function which is decreasing along solutions of (5). In addition, if $t \mapsto \mathcal{L}(w(t))$ is constant in a non-trivial interval of $\mathbb{R}$, then $w(t)$ is an equilibrium.

This means that $\mathcal{L}$ is a Lyapunov function for (5). Nonetheless, we cannot say that the solutions of (5) have properties similar to those of gradient autonomous evolution processes (e.g. are backwards and forwards asymptotic to equilibria, [Hale, 1988; Lemma 3.8.2], [Ladyzhenskaya, 1991], [Robinson, 2001],[Sell & You, 2002] or [Temam, 1988]) since the usual proofs are strongly tied to the properties of the autonomous evolution processes.

The aim of this paper is to show that, under certain assumptions, the solutions of (5) are backwards and forwards asymptotic to equilibria and homoclinic structures are not present; that is, the evolution process associated to (5) is gradient-like in the sense of [Carvalho & Langa, 2009], concluding that the associated pullback attractor is characterized as the union of the unstable manifolds of equilibria. We also show that the pullback attractor associated to (5) is an exponential forwards attractor. Note that, in general, there is no relationship between pullback and forwards attraction (see [Cheban *et al.*, 2002] or [Langa *et al.*, 2007]).

2. Gradient-like evolution processes

Let X be a Banach space. A nonlinear evolution process is a family of continuous maps $\{T(t,\tau) : t \geqslant \tau\}$ from X into itself such that

1) $T(\tau,\tau) = I$,

2) $T(t,\sigma)T(\sigma,\tau) = T(t,\tau)$, for each $t \geqslant \sigma \geqslant \tau$, and

3) $(t,\tau) \mapsto T(t,\tau)z_0$ is continuous for $t \geqslant \tau$, $z_0 \in X$.

A continuous function $z : \mathbb{R} \to X$ is a global solution for $\{T(t,\tau) : t \geqslant \tau\}$ if $T(t,\tau)z(\tau) = z(t)$, for all $t \geqslant \tau$. If a global solution is constant, it is called an equilibrium and the set of equilibria is denoted by $\mathcal{E}$.

A semigroup is a family $\{S(t) : t \geq 0\}$ with the property that $\{T(t,\tau) = S(t-\tau) : t \geqslant \tau \in \mathbb{R}\}$ is an evolution process. In this case $\{T(t,\tau) : t \geqslant \tau\}$ is called an autonomous evolution process, otherwise it is called a non-autonomous process. Recall that a set A is invariant under a semigroup $\{S(t) : t \geqslant 0\}$ if $S(t)A = A$ for all $t \geqslant 0$.

Since a fixed set A in X will not, in general, be fixed by a non-autonomous process, *invariance* for a process is defined as:

- A family $\{\mathcal{A}(t) \subset X : t \in [\sigma,\infty)\}$ is invariant under $\{T(t,\tau) : t \geqslant \tau\}$ if $T(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geqslant \tau$.

Definition 2.1. A family of compact sets $\{\mathcal{A}(t) \subset X : t \in \mathbb{R}\}$, with $\cup_{s \leqslant t}\mathcal{A}(t)$ bounded for each $t \in \mathbb{R}$, is a *pullback attractor* for $\{T(t,\tau):\ t \geqslant \tau\}$ if it is invariant and attracts all bounded subsets of X 'in the pullback sense', that is,

$$\lim_{\tau\to-\infty} \text{dist}(T(t,\tau)B, \mathcal{A}(t)) = 0,\ \forall t \in \mathbb{R}.$$

Furthermore, for each $t \in \mathbb{R}$, $\mathcal{A}(t)$ is characterized by

$$\begin{array}{l}\mathcal{A}(t) = \{\xi(t) \in X : \xi \in C(\mathbb{R}, X) \text{ is a global} \\ \text{bounded solution for } \{T(t,\tau) : t \geqslant \tau\}\}.\end{array} \tag{8}$$

For an autonomous evolution process, the concept of a pullback attractor coincides with the concept of global attractor. The characterization in (8) shows that, in a certain sense, this notion is a 'natural' generalization of the notion of global attractors to processes. We observe that the pullback attractor will not necessarily posses any kind of forwards attraction.

Definition 2.2. The unstable set of an equilibrium $y_0^* \in \mathcal{E}$ for a semigroup $\{S(t) : t \geqslant 0\}$ is defined by

$$\begin{array}{l}W^{\mathrm{u}}(y_0^*) = \{z \in X :\ \text{there is a global solution } y(t) \\ \ \text{for } \{S(t) : t \geqslant 0\} \text{ satisfying } y(0) = z \text{ and} \\ \text{such that } \lim_{t\to-\infty} \|y(t) - y_0^*\|_X = 0\}.\end{array}$$

Definition 2.3. A semigroup $\{S(t) : t \geqslant 0\}$ is *gradient* if there exists a continuous function $V : X \to \mathbb{R}$ (a Lyapunov function) such that

- $t \mapsto V(S(t)z) : [0,\infty) \to X$ is non-increasing for each $z \in X$.
- If $\xi(\cdot) : \mathbb{R} \to X$ is a global solution and $V(\xi(t)) = V(\xi(0))$ for all $t \geq 0$ or for all $t \leq 0$, then ξ is an equilibrium.

The following characterization result is well known (see [Hale, 1988]):

Theorem 2.4. *If $\{S(t) : t \geqslant 0\}$ is a gradient semigroup with a global attractor $\mathcal{A}$ and a finite set of equilibria $\mathcal{E} = \{y_1^*, \cdots, y_{\mathfrak{p}}^*\}$, then*

$$\mathcal{A} = \bigcup_{i=1}^{\mathfrak{p}} W^{\mathrm{u}}(y_i^*).$$

Furthermore, if $V : X \to \mathbb{R}$ is the Lyapunov function associated to $\{S(t) : t \geqslant 0\}$, for each global solution $y(\cdot) : \mathbb{R} \to X$ in $\mathcal{A}$, there are $1 \leqslant i, j \leqslant \mathfrak{p}$ with $V(y_i^*) < V(y_j^*)$, such that

$$\lim_{t\to-\infty} y(t) = y_j^* \ \textit{and} \ \lim_{t\to+\infty} y(t) = y_i^*.$$

The extension of the notion of gradient semigroups to processes is done in [Carvalho & Langa, 2009], where the notion of gradient-like evolution processes is introduced. That extension corresponds, in the case of semigroups, to the decomposition of the flow as a gradient part and a chain recurrent part (Morse Decomposition). To that end we need the following definitions.

Definition 2.5. Let $\{T(t,\tau) : t \geqslant \tau\}$ be a process and $\{\Xi(t) : t \in \mathbb{R}\}$ be an invariant family for it. The set $\Gamma = \cup\{\Xi(t) : t \in \mathbb{R}\}$ is called trace of Ξ. If there exists $\delta > 0$ such that any global solution $\xi : \mathbb{R} \to X$ with $\xi(t) \in \mathcal{O}_\delta(\Gamma) := \{z \in X : \text{dist}(z,\Gamma) < \delta\}$ for all $t \in \mathbb{R}$, must satisfy $\xi(t) \in \Xi(t)$, for all $t \in \mathbb{R}$, then we say that $\{\Xi(t) : t \in \mathbb{R}\}$ is an *isolated invariant family*. $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$ is said a *set of isolated invariant families* if each Ξ_i^* is an isolated invariant family and there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Gamma_i^*) \cap \mathcal{O}_\delta(\Gamma_j^*) = \emptyset$, $1 \leqslant i < j \leqslant n$, where Γ_i^* is the trace of Ξ_i^*.

Let $\{T(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ be a nonlinear evolution process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ which contains a finite set of isolated invariant families $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$.

Definition 2.6. A homoclinic structure in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a sequence $\{\Xi_{\ell_i}^* : 1 \leqslant i \leqslant \mathfrak{p}\}$ in $\mathcal{S}$ and a sequence of global solutions $\{\xi_i : 1 \leqslant i \leqslant \mathfrak{p}\}$ such that $\xi_i \stackrel{t\to-\infty}{\longrightarrow} \Gamma_{\ell_i}^*$, $\xi_i \stackrel{t\to\infty}{\longrightarrow} \Gamma_{\ell_{i+1}}^*$ and $\Xi_{\ell_1}^* = \Xi_{\ell_{\mathfrak{p}+1}}^* = \Xi^*$, where $\Gamma_{\ell_i}^*$ is the trace of $\Xi_{\ell_i}^*$.

Definition 2.7. Let X be a Banach space and $\{T(t,\tau) : t \geqslant \tau\}$ be a nonlinear evolution process in X with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. We say that $\{T(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is a gradient-like process if the following two hypotheses are satisfied:

(H1) There is a finite set $\mathcal{S} = \{\Xi_i^* : \mathbb{R} \to X : 1 \leqslant i \leqslant n\}$ of isolated invariant families in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ with the property that any global solution $\xi : \mathbb{R} \to X$ in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$

satisfies

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t),\Gamma_i^*)=0$$

and

$$\lim_{t\to\infty} \operatorname{dist}(\xi(t),\Gamma_j^*)=0,$$

for some $1\leqslant i,j\leqslant n$.

(H2) $\mathcal{S}=\{\Xi_1^*,\cdots,\Xi_n^*\}$ does not contain any homoclinic structure.

Definition 2.8. Let $\{T(t,\tau): t\geqslant\tau\in\mathbb{R}\}$ be an evolution process. The unstable set of an isolated invariant family Ξ^* with trace Γ^* is the set

$$W^{\mathrm{u}}(\Xi^*)=\{\,(\tau,\zeta)\in\mathbb{R}\times X: \text{ there is a global solution } \xi:\mathbb{R}\to X \text{ such that } \xi(\tau)=\zeta \text{ and } \lim_{t\to-\infty}\operatorname{dist}(\xi(t),\Gamma^*)=0\}.$$

Also, $W^{\mathrm{u}}(\Xi^*)(\tau):=\{\,\zeta\in X:(\tau,\zeta)\in W^{\mathrm{u}}(\Xi^*)\}$.

It is observed in [Carvalho & Langa, 2009] that, for an autonomous evolution process, the above definition of unstable set coincides with the usual definition of an unstable set of an invariant set. That may not be the case for non-autonomous evolution processes. Nonetheless, they coincide if the following condition, which is automatically satisfied for autonomous evolution processes, holds

- If a solution $\xi(t)$ stays inside a suitably small neighborhood of Γ_i^* for all t in an interval of the form $(-\infty,t_0]$ (respectively, of the form $[t_0,\infty)$), then $\operatorname{dist}(\xi(t),\Xi(t))\overset{t\to-\infty}{\longrightarrow}0$ (respectively, $\operatorname{dist}(\xi(t),\Xi(t))\overset{t\to\infty}{\longrightarrow}0$).

With these definitions, the following result is proved in [Carvalho & Langa, 2009]:

Theorem 2.9. *Let $\{T_\eta(t,\tau): t\geqslant\tau\in\mathbb{R}\}$ be an evolution process in X with pullback attractor $\{\mathcal{A}_\eta(t): t\in\mathbb{R}\}$, $\eta\in[0,1]$. Assume that*

a) $\overline{\cup_{\eta\in[0,1]}\cup_{t\in\mathbb{R}}\mathcal{A}_\eta(t)}$ is compact.

b) $T_0(t,\tau)=S(t-\tau)$, $t\geqslant\tau$ and $\{S(t): t\geqslant 0\}$ is a gradient-like semigroup with isolated invariant sets $\{\Gamma_{1,0}^*,\cdots\Gamma_{n,0}^*\}$.

c) $\{T_\eta(t,\tau): t\geqslant\tau\}$ has a finite set of isolated invariant families $\mathcal{S}_\eta=\{\Xi_{1,\eta}^*,\cdots,\Xi_{n,\eta}^*\}$ with traces $\{\Gamma_{1,\eta}^*,\cdots,\Gamma_{n,\eta}^*\}$, $\eta\in[0,1]$ such that

$$\sup_{1\leqslant i\leqslant n}\,[\operatorname{dist}(\Gamma_{i,\eta}^*,\Gamma_{i,0}^*)+\operatorname{dist}(\Gamma_{i,0}^*,\Gamma_{i,\eta}^*)]\overset{\eta\to0}{\longrightarrow}0.$$

d) $\|T_\eta(t+\tau,\tau)u-T_0(t+\tau,\tau)u\|_X\overset{\eta\to0}{\longrightarrow}0$ uniformly for $\tau\in\mathbb{R}$, (t,u) in compact subsets of $[0,\infty)\times X$.

e) there are $\delta>0$ and $\eta_0\in(0,1]$ such that, if $\eta<\eta_0$, $\xi_\eta:\mathbb{R}\to X$ is a global solution in $\{\mathcal{A}_\eta(t): t\in\mathbb{R}\}$, $t_0\in\mathbb{R}$ and $\operatorname{dist}(\xi_\eta(t),\Gamma_{i,\eta}^*)<\delta$ for all $t\leqslant t_0$ ($t\geqslant t_0$), then $\operatorname{dist}(\xi_\eta(t),\Xi_{i,\eta}^*(t))\overset{t\to-\infty}{\longrightarrow}0$ ($\operatorname{dist}(\xi_\eta(t),\Xi_{i,\eta}^*(t))\overset{t\to+\infty}{\longrightarrow}0$).

Then, there exists $\eta_0>0$ such that, for all $\eta\leqslant\eta_0$, $\{T_\eta(t,\tau): t\geqslant\tau\in\mathbb{R}\}$ is a gradient-like nonlinear evolution process. Consequently, there exists $\eta_0>0$ such that

$$\mathcal{A}_\eta(t)=\bigcup_{i=1}^n W^u(\Xi_{i,\eta}^*)(t), \tag{9}$$

for all $t\in\mathbb{R}$ and for all $\eta\leqslant\eta_0$.

This result shows that any non-autonomous perturbation of a gradient-like nonlinear semigroup becomes a gradient-like evolution process. Thus, it gives a natural way to construct examples of non-autonomous gradient-like evolution processes as small non-autonomous perturbations of gradient-like semigroups with all equilibria being hyperbolic and the isolated global solutions will be hyperbolic bounded global solutions.

Theorem 2.9 is also interesting even in the purely autonomous cases and yet, it provides examples of non-autonomous gradient-like evolution processes.

The important question to address next is the possible existence of gradient-like evolution processes which are not given as a small non-autonomous perturbations of a gradient-like semigroup.

3. On the characterization of the pullback attractor

The existence of the pullback attractor for (2) has been recently proved [Caraballo *et al.*, preprint]. In this Section we want to describe in detail the geometrical structure of the pullback attractor associated to (2). To this end, we firstly need some auxiliary results on the regularity of this family of compact sets.

3.1. Regularity of the pullback attractor

Now we prove that the pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ is such that $\cup_{t\in\mathbb{R}} A(t)$ is a bounded subset of $X^1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. To that end, let $\xi : \mathbb{R} \to X$ be a global bounded solution of (2). Then, the set $\{\xi(t) : t \in \mathbb{R}\}$ is a bounded subset of $X = H_0^1(\Omega) \times L^2(\Omega)$.

For each initial value $w_0 \in X$ and each initial time $s \in \mathbb{R}$, system (5) possesses a unique solution which can be written as

$$S(t,s)w_0 = L(t,s)w_0 + U(t,s)w_0 = \begin{pmatrix} u(t,s,w_0) \\ u_t(t,s,w_0) \end{pmatrix}, \quad (10)$$

where $L(t,s)$ is the solution operator for $w_t = C(t)w$, and

$$U(t,s)w_0 = \int_s^t L(t,\tau)F(S(\tau,s)w_0)d\tau. \quad (11)$$

It is proved in [Caraballo *et al.*, preprint] that the non-autonomous process associated to (2) has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ with

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \ \text{ bounded in } X. \quad (12)$$

It is also proved in [Caraballo *et al.*, preprint] that

Theorem 3.1. *$\{U(t,s) : t \geqslant s\}$ is compact and there are constants $K > 0$, $\alpha > 0$ such that*

$$\|L(t,s)\| \leqslant Ke^{-\alpha(t-s)},\ t \geqslant s.$$

The constants K and α depend on β_0 and β_1 but are independent of the choice of the function β satisfying (4).

Hence, if $\xi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} : \mathbb{R} \to X$ is such that $\xi(t) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) = L(t,s)\xi(s) + \int_s^t L(t,\theta)F(\xi(\theta))d\theta,$$

and, using Theorem 3.1, we have that it can be written as

$$\xi(t) = \int_{-\infty}^t L(t,\theta)F(\xi(\theta))d\theta. \quad (13)$$

Consider, for $w_0 = \xi(s)$,

$$U(t,s)w_0 = \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} = \int_s^\tau L(\tau,\theta)F(S(\theta,s)w_0)d\theta$$

and note that,

$$\begin{cases} w_{tt} + \beta(t)w_t = \Delta w + f(u(t,s;w_0)), \\ w(s) = w_t(s) = 0. \end{cases} \quad (14)$$

To estimate the solution of (14), for w_0 in a bounded subset B of X, we consider, for $b > 0$, the energy functional for $(\varphi,\phi) \in X$

$$V(\varphi,\phi) = \frac{1}{2}|\nabla\varphi|_{L^2}^2 + 2b\langle\varphi,\phi\rangle_{L^2} + \frac{1}{2}|\phi|_{L^2}^2, \quad (15)$$

to obtain that

$$\begin{aligned} \frac{d}{dt}&V(w(t),w_t(t)) \\ &= -(\beta(t)-2b)\|w_t\|_{L^2}^2 - 2b\|\nabla w\|_{L^2}^2 \\ &+ 2b\int_\Omega wf(u) - 2b\beta(t)\int_\Omega ww_t + \int_\Omega w_t f(u) \\ &\leqslant -\frac{\beta_0}{2}\|w_t\|_{L^2}^2 - b\|\nabla w\|_{L^2}^2 + C, \end{aligned}$$

where we have used (12), the fact that f takes bounded subsets of $H_0^1(\Omega)$ into bounded subsets of $L^2(\Omega)$ (from the growth condition in (3)) and a chosen value of b suitable small. From this we obtain that

$$\bigcup_{s\leqslant\tau\leqslant t} U(\tau,s)B \ \text{ is a bounded subset of } X. \quad (16)$$

Hence, if $v = w_t$,

$$\begin{cases} v_{tt} + \beta(t)v_t = \Delta v - \beta'(t)v \\ \qquad\qquad + f'(u(t,s;u_0))u_t(t,s;u_0) \\ v(s) = 0,\ v_t(s) = f(w_0). \end{cases} \quad (17)$$

Proceeding as in [Babin & Vishik, 1992] we define, for $\epsilon > 0$, $Y^\epsilon = D((-\Delta)^{\frac{\epsilon}{2}})$ with the graph norm and $Y^{-\epsilon} = (Y^\epsilon)'$. Now, to estimate the solution of (17) we consider, for $b > 0$, the following energy functional for $(\varphi,\phi) \in Y^{1-\epsilon} \times Y^{-\epsilon}$

$$V_\epsilon(\varphi,\phi) = \frac{1}{2}|\varphi|_{Y^{1-\epsilon}}^2 + 2b\langle\varphi,\phi\rangle_{Y^{-\epsilon}} + \frac{1}{2}|\phi|_{Y^{-\epsilon}}^2. \quad (18)$$

Using (12) and (16) we have that, for $\epsilon_1 = \frac{(p-1)(n-2)}{2} < 1$ and for some constant $K > 0$,

$$\begin{aligned} \|f'(u)u_t\|_{Y^{-\epsilon_1}} &\leqslant c\|f'(u)u_t\|_{L^{\frac{2n}{n+2\epsilon_1}}} \\ &\leqslant \|u_t\|_{L^2}\|f'(u)\|_{L^{\frac{n}{\epsilon_1}}} \\ &\leqslant c\|u_t\|_{L^2}(1+\|u\|^{p-1}_{L^{\frac{2n}{n-2}}}) \\ &\leqslant K. \end{aligned} \tag{19}$$

Also

$$\begin{aligned} &\frac{d}{dt}V_{\epsilon_1}(v(t),v_t(t)) \\ &= -(\beta(t)-2b)\|v_t\|^2_{Y^{-\epsilon_1}} - 2b\|v\|^2_{Y^{1-\epsilon_1}} \\ &- (2b\beta(t)+\beta'(t))\langle v,v_t\rangle_{Y^{-\epsilon_1}} - 2b\beta'(t)\|v\|^2_{Y^{-\epsilon_1}} \\ &+ 2b\langle v, f'(u)u_t\rangle_{Y^{-\epsilon_1}} + \langle v_t, f'(u)u_t\rangle_{Y^{-\epsilon_1}} \\ &\leqslant -(\beta_0-2b)\|v_t\|^2_{Y^{-\epsilon_1}} - 2b\|v\|^2_{Y^{1-\epsilon_1}} \\ &+ (2b\beta_1+L)\|v\|_{Y^{-\epsilon_1}}\|v_t\|_{Y^{-\epsilon_1}} + 2bL\|v\|^2_{Y^{-\epsilon_1}} \\ &+2b\|v\|_{Y^{-\epsilon_1}}\|f'(u)u_t\|_{Y^{-\epsilon_1}}+\|v_t\|_{Y^{-\epsilon_1}}\|f'(u)u_t\|_{Y^{-\epsilon_1}} \\ &\leqslant -\frac{\beta_0}{2}\|v_t\|^2_{Y^{-\epsilon_1}} - b\|v\|^2_{Y^{1-\epsilon_1}} + C, \end{aligned}$$

where we used (19), (16), (12) and chosen b suitable small. From this, from (13) and from (8) we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } Y^{2-\epsilon_1}\times Y^{1-\epsilon_1}. \tag{20}$$

Using (20) and reestarting from (19) with $\epsilon_2 = (p+1)\epsilon_1 - p$ we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } Y^{2-\epsilon_2}\times Y^{1-\epsilon_2}. \tag{21}$$

Iterating this procedure a finite number of times, we obtain that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } Y^{2}\times Y^{1}. \tag{22}$$

and (22) implies that

$$\sup_{\xi\in\mathcal{A}}\sup_{t\in\mathbb{R}}\{\|\xi(t)\|_X, \|\xi(t)\|_{X^1}, \|\xi_t(t)\|_X\} < \infty, \tag{23}$$

where $\mathcal{A}$ is the set of global bounded solutions for (2).

3.2. *Structure of the attractor*

Let $\{\mathcal{A}(t) : t\in\mathbb{R}\}$ be the pullback attractor for (2), and assume that there are only finitely many solutions $\{u_1^*,\cdots,u_p^*\}$ of

$$\begin{cases} \Delta u + f(u) = 0, \ x\in\Omega, \\ u = 0, \ x\in\partial\Omega. \end{cases} \tag{24}$$

Denote by $\mathcal{E} = \{e_1^*,\cdots,e_p^*\}$ where $e_i^* = \binom{u_i^*}{0}$. Under this assumption, we prove in this section that the evolution process $\{S(t,s) : t\geqslant s\}$ associated to (2) is gradient-like; that is, conditions $(H1)$ and $(H2)$ in Definition 2.7 are satisfied. As a consequence, we will get that

$$\mathcal{A}(t) = \bigcup_{i=1}^{p} W^u(e_i^*)(t), \text{ for all } t\in\mathbb{R}. \tag{25}$$

We first observe that the function in (7) is such that, given a solution $\xi : [0,\infty)\to X$ of (2), then

$$[0,\infty)\ni t\mapsto \mathcal{L}(\xi(t))\in\mathbb{R}$$

is decreasing. In addition, if $\mathcal{L}(\xi(t))$ is constant in a nontrivial interval of $\mathbb{R}$, then ξ must be an equilibrium.

These considerations imply that $\mathcal{L} : X\to\mathbb{R}$ is a Lyapunov function for (2) and that, in $\mathcal{E}$, there is no homoclinic structure. The remaining of the paper is dedicated to show that all solutions in the pullback attractor of (2) are forwards and backwards asymptotic to equilibria. These two conditions ensure that $\{S(t,s) : t\geqslant s\}$ is a gradient-like evolution process.

Clearly, if β is a positive constant or, as a consequence of Theorem 2.9, if β is uniformly close to a positive constant, $\{S(t,s) : t\geqslant s\}$ is gradient-like. Our goal is to show that even if β is not uniformly close to a constant, the process associated to (2) is still gradient-like and, therefore, the pullback attractor is still given by (25).

Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For each $n\in\mathbb{N}$, let $\beta_n : \mathbb{R}\to\mathbb{R}$ be the function defined by $\beta_n(t) = \beta(t_n+t)$. Under these assumptions, the family $\{\beta_n\}_{n\in\mathbb{N}}$ is uniformly bounded and uniformly equicontinuous. Consequently, it has a subsequence (which we denote the same) and a globally Lipschitz and bounded function $\gamma : \mathbb{R}\to[0,\infty)$ such that $\beta_n(t) \stackrel{n\to\infty}{\longrightarrow} \gamma(t)$ uniformly in compact subsets of $\mathbb{R}$.

Now consider the sequence of linear problems

$$\begin{cases} u_{tt} + \beta(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega),\ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{26}$$

$$\begin{cases} u_{tt} + \beta_n(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega),\ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{27}$$

and

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega, \\ u(s) = u_0 \in H_0^1(\Omega),\ u_t(s) = v_0 \in L^2(\Omega). \end{cases} \tag{28}$$

Denote by $L(t,s)$, $L_n(t,s)$ and $L_\infty(t,s)$ the processes associated to (26), (27) and (28) in $X = H_0^1(\Omega) \times L^2(\Omega)$, respectively.

Clearly, from Theorem 3.1, there are constants $M \geqslant 1$ and $\omega > 0$ such that

$$\begin{aligned} &\|L(t,s)\|_{\mathcal{L}(X)} \leqslant Me^{-\omega(t-s)},\ t \geqslant s, \\ &\|L_n(t,s)\|_{\mathcal{L}(X)} \leqslant Me^{-\omega(t-s)},\ t \geqslant s, \\ &\|L_\infty(t,s)\|_{\mathcal{L}(X)} \leqslant Me^{-\omega(t-s)},\ t \geqslant s. \end{aligned}$$

Also, $L(t_n+t, t_n+s) = L_n(t,s)$. In fact, (26) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} 0 \\ \beta(t)u_t \end{pmatrix}, \quad \begin{pmatrix} u \\ u_t \end{pmatrix}(s) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

and writing

$$L(t,s)U_0 = \begin{pmatrix} \ell_1(t,s)U_0 \\ \ell_2(t,s)U_0 \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

we have, by the variation of constants formula, that

$$\begin{aligned} &L(t_n+t, t_n+s)U_0 \\ &= e^{C(t-s)}U_0 \\ &- \int_{t_n+s}^{t_n+t} e^{C(t+t_n-\theta)} \begin{pmatrix} 0 \\ \beta(\theta)\ell_2(\theta, t_n+s)U_0 \end{pmatrix} d\theta \\ &= e^{C(t-s)}U_0 \\ &- \int_s^t e^{C(t-\theta)} \begin{pmatrix} 0 \\ \beta_n(\theta)\ell_2(t_n+\theta, t_n+s)U_0 \end{pmatrix} d\theta \\ &= L_n(t,s)U_0 \end{aligned}$$

Now,

$$\begin{aligned} &[L_n(t,s) - L_\infty(t,s)]U_0 \\ &= \int_s^t e^{C(t-\theta)}\beta_n(\theta) \begin{pmatrix} 0 \\ (\ell_2)_n(\theta,s)U_0 - (L_2)_\infty(\theta,s)U_0 \end{pmatrix} d\theta \\ &+ \int_s^t e^{C(t-\theta)}[\beta_n(\theta) - \gamma(\theta)] \begin{pmatrix} 0 \\ (\ell_2)_\infty(\theta,s)U_0 \end{pmatrix} d\theta, \end{aligned}$$

and a simple application of Gronwall's inequality yields that, for each $T > 0$

$$\sup_{t-T\leqslant s\leqslant t} \|L_n(t,s) - L_\infty(t,s)\|_{\mathcal{L}(X)} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{29}$$

Now, let $\xi : \mathbb{R} \to X$ be a global bounded solution of (5) and recall that, from (23),

$$\sup_{t\in\mathbb{R}}\{\|\xi(t)\|_X, \|\xi(t)\|_{X^1}, \|\xi_t(t)\|_X\} < \infty.$$

Thus, by the Arzelà-Ascoli Theorem, we have that the sequence ξ_n in $C(\mathbb{R}, X)$ defined by $\xi_n(t) = \xi(t_n+t)$ has a subsequence which converges uniformly in compact subsets of $\mathbb{R}$ to a continuous function $\zeta : \mathbb{R} \to X$.

Now, as

$$\begin{aligned} \xi(t) &= \begin{pmatrix} \xi_1(t) \\ (\xi_1)_t(t) \end{pmatrix} \\ &= L(t,s)\xi(s) + \int_s^t L(t,\theta) \begin{pmatrix} 0 \\ f(\xi_1(\theta)) \end{pmatrix} d\theta \end{aligned} \tag{30}$$

we also have that

$$\xi(t) = \int_{-\infty}^t L(t,\theta) \begin{pmatrix} 0 \\ f(\xi_1(\theta)) \end{pmatrix} d\theta,$$

and, consequently,

$$\begin{aligned} \xi(t+t_n) &= \int_{-\infty}^{t+t_n} L(t+t_n,\theta) \begin{pmatrix} 0 \\ f(\xi_1(\theta)) \end{pmatrix} d\theta \\ &= \int_{-\infty}^t L(t_n+t, t_n+\theta) \begin{pmatrix} 0 \\ f(\xi_1(\theta+t_n)) \end{pmatrix} d\theta \\ &= \int_{-\infty}^t L_n(t,\theta) \begin{pmatrix} 0 \\ f(\xi_1(\theta+t_n)) \end{pmatrix} d\theta. \end{aligned}$$

From this and (29), it is not difficult to see that

$$\zeta(t) = \int_{-\infty}^{t} L_\infty(t,\theta) \begin{pmatrix} 0 \\ f(\zeta(\theta)) \end{pmatrix} d\theta$$

and, in particular, $\zeta : \mathbb{R} \to X$ is a global bounded solution of

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = f(u) \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega. \end{cases} \tag{31}$$

To that end, we consider the Lyapunov function in (7). Then $\mathbb{R} \ni t \mapsto \mathcal{L}(\xi(t)) \in \mathbb{R}$ is non-increasing and the only global solutions ξ where V is constant are the equilibria in $\mathcal{E}$. Since $\{\xi(t) : t \in \mathbb{R}\}$ lies in a compact set in X, there are real numbers ς_i and ς_j such that

$$\varsigma_i \overset{t\to-\infty}{\longleftarrow} \mathcal{L}(\xi(t+r)) \overset{t\to\infty}{\longrightarrow} \varsigma_j$$

for all $r \in \mathbb{R}$.

If $t_n \overset{n\to\infty}{\longrightarrow} \infty$, taking subsequences, if necessary, $\beta(t_n + r) \overset{n\to\infty}{\longrightarrow} \gamma(r)$ uniformly in compact subsets of $\mathbb{R}$, $\xi(t_n + r) \overset{n\to\infty}{\longrightarrow} \zeta(r)$ in X, uniformly for r in compact subsets of $\mathbb{R}$, and $(\zeta(t), \zeta_t(t))$ is a global solution of the problem

$$\begin{cases} u_{tt} + \gamma(t)u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \tag{32}$$

with the property that $\mathcal{L}(\zeta(t), \zeta_t(t)) = \varsigma_j$, for all $t \in \mathbb{R}$. Hence $\begin{pmatrix} \zeta(t) \\ \zeta_t(t) \end{pmatrix} = e_j^*$. Taking $\tilde{t}_n \overset{n\to\infty}{\longrightarrow} -\infty$ we obtain an analogous result.

Suppose that there are sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{\bar{t}_n\}_{n\in\mathbb{N}}$ with $t_{n+1} > \bar{t}_n > t_n$, $n \in \mathbb{N}$, such that $\xi(t_n) \overset{n\to\infty}{\longrightarrow} e_k^*$ and $\xi(\bar{t}_n) \overset{n\to\infty}{\longrightarrow} \bar{e}_k^*$. Now, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $V(\xi(t)) \in (\varsigma_j - \epsilon, \varsigma_j + \epsilon)$ for all $t \in [t_n, \bar{t}_n]$. If $\tau_n \in (t_n, \bar{t}_n)$, $\tau_n \overset{n\to\infty}{\longrightarrow} \infty$ and (taking subsequences if necessary), $\beta(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{\gamma}(r)$. We have that $\xi(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{\zeta}(t)$, which is a solution of

$$\begin{cases} u_{tt} + \bar{\gamma}(t)u_t - \Delta u = f(u), \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases} \tag{33}$$

with $\mathcal{L}(\bar{\zeta}(t), \bar{\zeta}_t(t)) = \varsigma_j$ for all $t \in \mathbb{R}$, and, consequently, $\bar{\zeta}(t) \equiv e_m^*$ with $\mathcal{L}(e_m^*) = \varsigma_j$. That leads to a contradiction with the fact that there are only finitely many equilibria.

Form the fact that the evolution process $\{S(t,s) : s \leqslant t\}$ associated to (2) has a Lyapunov function (see (7)), $(H2)$ is automatically satisfied. We can resume all our previous analysis in the following theorem,

Theorem 3.2. *Suppose that there are only finitely many solutions $\{u_1^*, \cdots, u_p^*\}$ of (24), (with isolated invariant families $\{\Xi_j^* \equiv e_j^*\}$). Then the evolution process $\{S(t,s) : t \geqslant s\}$ associated to (2) is gradient-like and, as a consequence, we can write the pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ as in (25).*

4. Exponential Regular Pullback Attractors

In this section we consider the situation in which (2) possesses an exponential pullback attractor. As we will see, the characterization of the pullback attractor and the ideas in Sec.3.2 play a fundamental role in the proof of the exponential attraction.

Hypothesis 1. *Let $f \in C^2(\mathbb{R})$ and $\beta \in C^1(\mathbb{R})$ be such that conditions (3) and (4) are satisfied, and assume that (24) has a finite number of solutions $\mathcal{E} = \{e_1^*, \cdots, e_p^*\}$ where $e_i^* = \begin{pmatrix} u_i^* \\ 0 \end{pmatrix}$.*

From the results of Sec.3.2 we have that the process associated to (2) has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ which is characterized by (25).

Consider the linear process $\{L_i(t,s) : t \geqslant s\}$ associated to

$$\begin{cases} u_{tt} + \beta(t)u_t = \Delta u + f'(u_i^*)u, \ t > 0, \ x \in \Omega, \\ u(t,x) = 0, \ t > 0, \ x \in \partial\Omega \\ u(0,\cdot) = u_0 \in H_0^1(\Omega), \ u_t(0,\cdot) = v_0 \in L^2(\Omega). \end{cases} \tag{34}$$

Definition 4.1. We say that the linear evolution process $\{L_i(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has exponential dichotomy with exponent ω and constant M if there is a family of bounded linear projections $\{Q_i(t) : t \in \mathbb{R}\}$ in X such that

1. $Q_i(t)L_i(t,s) = L_i(t,s)Q_i(s)$, for all $t \geqslant s$.
2. The restriction $L_i(t,s)_{|_{R(Q_i(s))}}$, $t \geqslant s$ is an isomorphism from $R(Q_i(s))$ into $R(Q_i(t))$; we denote its inverse by $L_i(s,t) : R(Q_i(t)) \to R(Q_i(s))$.
3. There are constants $\omega > 0$ and $M \geqslant 1$ such that

$$\begin{aligned} \|L_i(t,s)(I - Q_i(s))\|_{L(X)} &\leqslant Me^{-\omega(t-s)} \quad t \geqslant s \\ \|L_i(t,s)Q_i(s)\|_{L(X)} &\leqslant Me^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{35}$$

When $\{L_i(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has exponential dichotomy, we say that e_i^* is a hyperbolic equilibrium point.

Hypothesis 2. *Assume that all equilibria in $\mathcal{E}$ are hyperbolic.*

Remark 4.2. We remark that, when β is independent of t, an equilibrium e^* of (2) is hyperbolic if and only if zero is not an eigenvalue of $A(A = \Delta + f'(e^*)I$ with homogeneous Dirichlet boundary conditions). Unfortunately, the case when β is time dependent is much more difficult and cannot be easily obtained from the knowledge of the spectrum of A. We conjecture that, under our assumptions, e^* is hyperbolic if and only if $0 \notin \sigma(A)$.

The following lemma can be seen in [Carvalho & Langa, 2007], for example.

Lemma 4.3. *Assume that Hypothesis 1 and Hypothesis 2 are satisfied. If $\{Q_i(t) : t \in \mathbb{R}\}$ is the family of projections given in Definition 4.1, for each $1 \leqslant i \leqslant p$, there is a neighborhood $\mathcal{V}_i$ of e_i^* and a function $\Sigma_i : R(Q_i(t)) \to Ker(Q_i(t))$ such that*

$$W^u(e_i^*)(t) \cap \mathcal{V}_i = \{e_i^* + Q_i(t)u + \Sigma_i(Q_i(t)u) : u \in X\} \cap \mathcal{V}_i,$$

(recall that $\{S(t,s) : s \leqslant t\}$ is a gradient-like evolution process, so the above intersection is the local unstable manifold) and there exists $\gamma > 0$ such that, for any $u_0 \in V_i$, and as long as $S(t+s,s)u_0 \in V_i$,

$$\sup_{s\in\mathbb{R}} \|(I - Q_i(t+s))S(t+s,s)u_0 - \Sigma_i^u((Q_i(t+s)S(t+s,s)u_0))\|_X \leqslant Me^{-\gamma t}. \quad (36)$$

It is easy to see that $\{S(t,s) : t \geqslant 0\}$ is Lipschitz continuous, that is, given a bounded subset B of X, there are constants $c = c(B)$ and $L = L(B) > 0$ such that, for all $u, v \in B$

$$\sup_{s\in\mathbb{R}} \|S(t+s,s)u - S(t+s,s)v\| \leqslant ce^{Lt}\|u-v\|. \quad (37)$$

In what follows, based on the results of [Carvalho & Langa, 2009], [Babin & Vishik, 1992] or [Vishik & Zelik, preprint], we show that the pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ of the evolution process associated to (2) under Hypothesis 1 and Hypothesis 2 is also an exponential pullback attractor.

Theorem 4.4. *There exists $\gamma > 0$ and, for each bounded subset $B \subset X$, there exists a constant $c(B) > 0$ such that, for all $u_0 \in B$*

$$\sup_{s\in\mathbb{R}} \sup_{u_0\in B} \text{dist}(S(t+s,s)u_0, \mathcal{A}(t+s)) \leqslant c(B)e^{-\gamma t}. \quad (38)$$

To prove this theorem we need the following important lemmas (which extend the corresponding results in [Carvalho & Langa, 2009] where they are proved for processes which are a small perturbation of autonomous evolution processes).

Lemma 4.5. *Assume that Hypothesis 1 and Hypothesis 2 are satisfied. If $\{S(t,s) : t \geqslant s\}$ is the evolution process associated to (2), given $\delta < \frac{1}{2}\min\{\|e_i^* - e_j^*\|_X : 1 \leqslant i,j \leqslant k,\ i \neq j\}$ and a bounded set $B \subset X$, there is a positive number $T = T(\delta, B)$ such that $\{S(t+s,s)u_0 : 0 \leqslant t \leqslant T\} \cap \cup_{i=1}^n B_\delta(e_i^*) \neq \emptyset$ for all $u_0 \in B$ and for all $s \in \mathbb{R}$.*

Proof: We argue by contradiction. Assume that there is a sequence u_k in B, and a sequence of positive numbers $t_k \overset{k\to\infty}{\longrightarrow} \infty$, and a sequence of real numbers s_k such that $\{S(t+s_k,s_k)u_k : 0 \leqslant t \leqslant t_k\} \cap \cup_{i=1}^n B_\delta(e_i^*) = \emptyset$. Extracting subsequences we have that there is a function $\gamma : \mathbb{R} \to [\beta_0, \beta_1]$ and a global solution $\xi : \mathbb{R} \to X$ of (31) such that $S(t + \frac{t_k}{2} + s_k, s_k)u_k \to \xi(t)$ uniformly in compact subsets of $\mathbb{R}$. Clearly, from its construction, $\xi(t) \notin \cup_{i=1}^n B_\delta(e_i^*)$ for all $t \in \mathbb{R}$ and this contradicts (25).□

Lemma 4.6. *Assume that Hypothesis 1 and Hypothesis 2 are satisfied. If $\{S(t,s) : t \geqslant s\}$ is the evolution process associated to (2), given $0 < \delta < \frac{1}{2}\min\{\|e_i^* - e_j^*\|_X : 1 \leqslant i,j \leqslant k,\ i \neq j\}$, there is a $\delta' > 0$ such that, if for some $1 \leqslant i \leqslant n$, $\|u_0 - e_i^*\|_X < \delta'$ and, for some $t_1 > 0$, $\|S(t_1+s,s)u_0 - e_i^*\|_X \geqslant \delta$, then $\|S(t+s,s)u_0 - e_i^*\|_X > \delta'$ for all $t \geqslant t_1$ and for all $s \in \mathbb{R}$.*

Proof: Assume that, for some $1 \leqslant i \leqslant n$, there is a sequence u_k in X with $\|u_k - e_i^*\|_X < \frac{1}{k}$ and sequences $s_k \in \mathbb{R}$, and $0 < t_k < \tau_k$ such that $\|S(t_k+s_k,s_k)u_k - e_i^*\|_X \geqslant \delta$ and $\|S(\tau_k+s_k,s_k)u_k - e_i^*\|_X < \frac{1}{k}$. Clearly t_k is bounded from below and that contradicts the fact that $\mathcal{E}$ does not contain any homoclinic structure.□

Proof of Theorem 4.4: To prove (38) we first choose $\delta < \delta_0$ such that $B_\delta(e_i^*) \subset V_i$ and V_i is the neighborhood given in Lemma 4.3 for e_i^*. From Lemma 4.6, for all suitably small δ, there exists $\delta' = \delta'(\delta) < \delta$ such that, if $u_0 \in B_{\delta'}(e_i^*)$ and for some $t_1 > 0$

$$S(t_1 + s, s)u_0 \notin B_\delta(e_i^*),$$

then

$$S(t + s, s)u_0 \notin B_{\delta'}(e_i^*), \text{ for all } t \geqslant t_1.$$

Now, let B be a bounded subset of X and B_0 be a closed ball centered at $u = 0$ that contains B and $\cup\{B_\delta(u) : u \in A(t),\ t \in \mathbb{R}\}$. From Lemma 4.5, there exists $T = T(\delta', B_0)$ such that, for all $u_0 \in B_0$

$$S(t + s, s)u_0 \in \mathcal{O}_{\delta'} = \bigcup_{i=1}^{n} B_{\delta'}(e_i^*)$$

for some $t \leqslant T$ and $\forall s \in \mathbb{R}$.

Thus, given $u_0 \in B_0$, there are sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M \leqslant n$ and $\{e_i^*\}_{i=1}^M$ such that

$$t_-^0 \leqslant T, \qquad t_-^i - t_+^{i-1} \leqslant T,\ 1 \leqslant i \leqslant M \qquad t_+^M = +\infty$$

for which $S(t+s, s)u_0 \in \mathcal{O}_\delta(e_i^*)$, for all $t \in [t_-^i, t_+^i]$, $s \in \mathbb{R}$, and $i \in \{1, \ldots, M\}$. Then, by Lemma 4.3,

$$\sup_{s\in\mathbb{R}} \mathrm{dist}(S(t + s, s)u_0, \mathcal{A}(t + s)) \leqslant c_0(B_0)e^{-\gamma t},$$

for all $t \in [t_-^i, t_+^i]$. On the other hand, for $t \in [t_+^{i-1}, t_-^i]$, $t = \sigma + t_+^{i-1}$, for some $\sigma \leqslant T$, and using (37) we have that

$$\begin{aligned}
&\mathrm{dist}(S(t + s, s)u_0, \mathcal{A}(t + s)) \\
&= \mathrm{dist}(S(\sigma + t_+^{i-1} + s, s)u_0, \mathcal{A}(t + s)) \\
&= \mathrm{dist}(S(\sigma + t_+^{i-1}, t_+^{i-1})S(t_+^{i-1} + s, s)u_0, \\
&S(\sigma + t_+^{i-1}, t_+^{i-1})\mathcal{A}(t + s)) \\
&\leqslant c_1(B_0)e^{kT}\mathrm{dist}(S(t_+^{i-1} + s, s)u_0, \mathcal{A}(t + s)) \\
&\leqslant c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_+^{i-1}} \\
&= c(B_0)e^{-\gamma t}. \square
\end{aligned}$$

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References

Babin, A. B. & Vishik, M.I. [1992] *Attractors of evolution equations*. North Holland.

Caraballo, T., Carvalho, A. N., Langa, J. A. & Rivero, F. [preprint] "Existence of pullback attractor in pullback asymptotically compact process".

Carvalho, A. N. & Langa, J. A. [2007] "The existence and continuity of stable and unstable manifolds for semilinear problems under non-autonomous perturbation in Banach spaces". *J. Differential Equations* **233** 622–653.

Carvalho, A. N. & Langa, J. A. [2009] "An extension of the concept of gradient systems which is stable under perturbation". *J. Differential Equations, doi:10.1016/j.jde.2009.01.007.*

Cheban, D. N., Kloeden, P. E. & Schmalfuß, B. [2002] "The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems". *Nonlinear Dyn. Syst. Theory* **2** 125–144.

Hale, J. K. [1988] *Asymptotic Behavior of Dissipative System.* American Mathematical Society.

Ladyzhenskaya, O. [1991] *Attractors for semigroups and evolution equations.* Lezioni Lincee. Cambridge University Press, Cambridge.

Langa, J. A., Robinson, J. C., Rodríguez-Bernal, A., Suárez, A. & Vidal-López, A. [2007] "Existence and nonexistence of unbounded forwards attractor for a class of non-autonomous reaction diffusion equations". *Discrete Contin. Dyn. Syst.* **18** (2-3) 483–497.

Robinson, J. C. [2001] *Infinite-Dimensional Dynamical System. An introduction to dissipative parabolic PDEs and the theory of global attractors.* Cambridge Text in Applied Mathematics.

Sell, G. R. & You, Y. [2002] *Dynamics of evolutionary equations.* Applied Mathematical Sciences (143). Springer-Verlag.

Temam, R. [1988] *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag.

Vishik, M. I. & Zelik, S. V. [preprint] "Regular attractors and their nonautonomous perturbations".