Estudio del comportamiento asintótico de las ecuaciones de Navier–Stokes no autónomas y algunas de sus variantes

Study of the asymptotic behaviour of the non-autonomous Navier–Stokes equations and some of their variations

Memoria escrita por

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Para optar al grado de doctor del

Programa Oficial de Doctorado Matemáticas

Universidad de Sevilla

Vº Bº El Director del Trabajo

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Universidad de Sevilla

8 de abril de 2013

A mis padres, por su apoyo incondicional,
y a Pepe Real, por su infinita generosidad.

Agradecimientos

Quisiera comenzar expresando mi más profundo agradecimiento a todas aquellas personas que me han ayudado en la elaboración y consecución de esta Memoria Doctoral.

En primer lugar, me gustaría agradecer a mis padres todo su gran esfuerzo y entrega, que con tanto amor y cariño han sabido siempre llevar a cabo, y gracias a los que me ha sido posible realizar mis estudios y, aún más importante, formarme como persona. Son un ejemplo de generosidad, humildad y constancia, y a ellos les debo todo lo que soy hoy. Gracias también a mis hermanos por su paciencia y comprensión, a toda mi familia por animarme siempre tanto y a Beli por preocuparse e interesarse por mí.

Me resulta casi imposible describir con palabras y en pocas líneas la enorme gratitud que siento hacia mis Directores de Tesis, los Profesores Pedro Marín Rubio y José Real Anguas. Muchas gracias por confiar en mí, por ayudarme y apoyarme siempre que lo he necesitado, y por dedicarme tantas horas. Ha sido un placer y un orgullo trabajar con vosotros y poder compartir tantos buenos momentos juntos. Para mí hubiese sido una gran satisfacción haber podido entregar personalmente un ejemplar de esta Tesis al Profesor José Real Anguas, sin el que esta Memoria hubiese sido imposible realizar. Una vez más, gracias por tu infinita generosidad, por enseñarme tanto de Matemáticas y de la vida, y por hacerme mejor persona. Siempre estarás en mi corazón.

También quisiera mostrar mi más sincero agradecimiento al Profesor James C. Robinson y a mi amigo Alejandro Vidal López, por acogerme con tanto cariño y tratarme con tanta hospitalidad en mis dos estancias en la Universidad de Warwick. Ale, muchas gracias por hacerme sentir tan a gusto allí y por ayudarme en momentos cruciales.

Asimismo, me gustaría dar las gracias al resto de los miembros del grupo de investigación, María José Garrido, José Antonio Langa y, muy especialmente, Tomás Caraballo, a quien agradezco enormemente sus ánimos, ayuda y paciencia en todo momento.

Finalmente, también quiero dar las gracias a todos los miembros del Departamento de Ecuaciones Diferenciales y Análisis Numérico de la Universidad de Sevilla por el afectuoso trato que me han dispensado durante estos cuatro años y, de forma especial, a mis compañeros becarios por compartir conmigo tantas anécdotas y vivencias.

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Introduction

The mathematical theory of fluid dynamics began in the seventeenth century with the work of Isaac Newton, who was the first to apply the mechanical laws to flow movements. Afterwards, Leonhard Euler wrote for the first time (1755) the differential equations that govern the motion of an ideal fluid, that is, in absence of dissipation due to the interaction between molecules. And finally, Claude-Louis Navier (1822) and, independently, George Gabriel Stokes (1845) added into the model the viscosity term and reached the so-called 'Navier–Stokes equations'.

The Navier–Stokes equations are the fundamental partial differential equations that describe the motion of incompressible fluids (or equivalently, divergence-free fluids). They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. Moreover, these equations in their full and simplified forms help with the design of aircraft and cars, the design of power stations, the analysis of pollution, and many other things. Actually, they are also of great interest in a purely mathematical sense. As we will take up again in the next paragraph, although mathematicians have already achieved many results on the study of the solutions to these equations, somewhat surprisingly, given their wide range of practical uses, the existence and regularity of global in time *classical solutions* (which are required to satisfy the equations in a pointwise sense), when the space dimension is three, are long standing open problems of fluid dynamics. These are called the *Navier–Stokes existence and smoothness* problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a one million dollar prize for a solution or a counterexample.

The first contribution to the mathematical study of the initial value problem for the non-stationary Navier–Stokes equations is contained in a series of remarkable papers of Jean Leray published in 1933 and 1934 (c.f. [57, 58, 59]). Once he obtained the existence of classical solutions for regular data during a short interval of time $(0, T)$, Leray found himself with a problem: he was not able to control *a priori* the increase of the velocity and its derivatives as time goes on, which ruined any chance of coming up with a global solution. Facing this difficulty, he decided on the procedure already followed by Hilbert while dealing with the Dirichlet problem for the Laplacian operator, and set out the problem within the frame of the so-called *weak* or *turbulent solutions* (that satisfy the Navier–Stokes equations in an average sense) in the Sobolev spaces. In fact, Leray was able to show the global existence of classical (in dimension two) and weak solutions, but the uniqueness of weak solutions in dimension three remains as an open problem. Since the first paper by Leray was published, a number of authors have again become interested in these equations and have obtained numerous results with the aid of various methods from modern functional analysis (see Constantin and Foias [21], Foias *et al.* [25], Lions [61], Temam [87], and the references therein).

On the other hand, the asymptotic behaviour of dynamical systems is an interesting and challenging problem, since it can provide useful information on the future evolution of the system. To this respect, much attention has been paid over the last few decades to the theory of attractors, with the aim of going further in the analysis of complex dynamical systems and of dealing with some open problems as the understanding of turbulence. Actually, much significant information can be obtained with this theory, such as finite fractal and Hausdorff dimensions, determining modes and nodes, inertial manifolds and finite-dimensionality behaviour, among others (see [87, 25, 78] and the references therein).

Although the concept of global attractor has become a powerful tool in the asymptotic analysis of autonomous dynamical systems, the appearance of more complex and realistic models that aimed to deal with terms depending non-trivially on time involved substantial changes and additional difficulties. Namely, this is the case when the model is non-autonomous. The theory of global attractors for autonomous partial differential equations is deeply developed in works of many mathematicians with books like Hale [39], Temam [87], Ladyzhenskaya [54], Babin and Vishik [4], Vishik [88], Robinson [78] or Sell and You [83]. Roughly speaking, a global attractor is an invariant compact set in some metric space, and which attracts all the trajectories of the dynamical system, uniformly on bounded sets. To show the existence of the global attractor one usually needs to verify that there exists a bounded attracting set, that is, a bounded set such that the distance from any orbit to this set tends to zero when the time goes to ∞. Conditions for the existence of such global attractors and examples can be found in [4, 39, 87, 88].

However, as it was pointed out above, most cases need of a non-autonomous model to describe the system and, consequently, a non-autonomous technique is necessary to handle the problem. A first and natural approach to extend the notion of global attractor to the non-autonomous case was that of uniform attractor defined by Chepyzhov and Vishik (see [15, 16] and the references therein). Nevertheless, this kind of attractor is only valid for some situations and it need not be invariant unlike the global attractor for autonomous systems. Afterwards, other different approaches appeared to allow unbounded time-depending terms and processes, as random or stochastic models.

Being possible various options to deal with the problem of attractors for non-autonomous systems (kernel sections [16], skew-product formalism [82], etc.), the notions of pullback and forward attractors seem to be general ways to extend results in this direction (see [14] for a comparison of these two last concepts). In this work we have chosen that of pullback (or cocycle) attractor (see Schmalfuß [80, 81], Crauel *et al.* [23], Kloeden and Schmalfuß [52, 53], Langa and Schmalfuß [56]) since it allows to handle more general non-autonomous terms, and it works under the presence of random environments as well (e.g. see [24]).

In connection with the above, it is clear from applications that in order to establish the existence of pullback attractors of bounded sets for the dynamical system associated to one problem, it is sometimes useful to go up to a bigger framework, replacing the usual universe of autonomous bounded sets by a universe of families of time-depending sets, that we will denote by $\mathcal{D}$. This more general concept of $\mathcal{D}$-attractor is better adapted to different situations (e.g. see Chueshov [19] and the references therein), and easier to obtain, even when the existence of the usual attractor is unclear. In examples, this $\mathcal{D}$-attractor is usually related to a tempered universe, that is, where the families of time-depending sets are given by a tempered condition on their growth in time. Actually, it usually happens that the universe of autonomous bounded sets is a subset of the tempered universe. More precisely, as it was studied in Marín-Rubio and Real [70], the existence of the pullback $\mathcal{D}$-attractor provides a sufficient condition that ensures the existence of the pullback attractor for fixed bounded sets. Although the cases of an ODE or a PDE in a bounded domain do not usually require this way of proceeding, in other situations where compactness of the (semi-)process does not hold or is unknown, this approach is helpful. Even in the random case, where the relation between both objects is well known (they coincide, or at least in a probability sense, cf. [22]), it is sometimes useful to study the previous existence of a random $\mathcal{D}$-attractor in order to obtain a sufficient condition that ensures the existence of the random attractor in the sense of Crauel, Debussche, and Flandoli (for fixed bounded sets). In the non-random case, which will be treated in Chapter 1, in particular we will obtain sufficient conditions that guarantee that these two objects are in fact the same. Therefore, we can claim that the attractor for fixed bounded sets, previously considered only the attractor of bounded sets, attracts in fact more objects.

This work is structured in six chapters. The first chapter is devoted to present the abstract theory of pullback attractors for non-autonomous dynamical systems within the framework of universes. In the rest of the chapters, this theory will be applied to several models based on the incompressible Navier–Stokes equations, in order to analyze the existence and relationships among different families of minimal pullback attractors for them.

Chapter 1 is divided into two sections. In Section 1.1 we recall some abstract results in order to construct pullback attractors for a dynamical system or process associated to a problem via the solution operator. Actually, in Theorem 1.11 and Corollary 1.13, we provide results on the existence of minimal pullback attractors for the two possible choices of the attracted universes introduced before, namely, the standard one of fixed bounded sets, and secondly, one given by a tempered condition. Moreover, based on the paper by Marín-Rubio and Real [70], these two notions of attractors are related in Corollary 1.13, and in Remark 1.14 we show that under a simple additional assumption they generate in fact the same object. Finally, in Theorem 1.15 we establish a result comparing two families of attractors associated to the same process but with different phase spaces and universes.

On the other hand, as we will also see in Section 1.1, one of the ingredients necessary for obtaining the pullback attractor is the asymptotic compactness of the corresponding process. To verify this property one can either proceed directly, or make use of a splitting of the solutions into high and low components. Such a splitting is a very common technique in the study of the qualitative behaviour of solutions for PDEs problems, in particular when considering the long-time behaviour of dynamics, as in the construction of invariant manifolds [17,41] and inertial manifolds [18,28], the squeezing property [26,87], the notion of 'determining modes' [27,42], and the theory of attractors [63]. In the context of proofs of the existence of attractors it was formalized by Ma, Wang, and Zhong [63] as their celebrated 'condition (C)'. A more descriptive terminology, 'the flattening property', was coined by Kloeden and Langa [48], and we will adopt this terminology here. However, it is worth making the observation that this is not so much a 'property' as a (powerful) technique for obtaining the asymptotic compactness of a flow, be it autonomous or non-autonomous. With regard to this, in Section 1.2 we will introduce the concept of the flattening property in a Banach space and, in Proposition 1.18, we will prove that it implies the asymptotic compactness of the corresponding process.

In Chapter 2 we consider the incompressible two-dimensional Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) \qquad \nabla\cdot u = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. The existence of minimal pullback attractors in H (essentially L^2) and in V (essentially H^1) is established for both different kinds of universes cited above, and when the forcing term has the minimal regularity required to obtain solutions that evolve continuously in these phase spaces, namely, $f \in L^2_{\rm loc}(\mathbb{R}; V')$ and $f \in L^2_{\rm loc}(\mathbb{R}; H)$ respectively. In this chapter, the existence of such attractors will be shown by proving the pullback asymptotic compactness of the process associated to our problem, via an energy method which relies on the continuity of the solutions. Nevertheless, since the asymptotic behaviour in H was already established in Caraballo *et al.* [7], in Section 2.2.1 we only summarize the main facts in this phase space. Actually, in [7] the authors deal with a more general case in which the domain is unbounded. However, to our knowledge, a richer structure on the asymptotic behaviour of the solutions to this non-autonomous problem when the initial datum also belongs to V has not previously been studied. In Section 2.2.2 we prove the existence of pullback attractors in V norm and, thanks to regularity properties, the relations between these families of attractors and the corresponding in H are successfully obtained. Finally, in Sections 2.3 and 2.4, we also study some regularity properties for the attractors, such as the H^2-boundedness and the tempered behaviour in V and H^2, when time goes to $-\infty$.

Chapter 3 is also devoted to analyze the existence of minimal pullback attractors in H and in V for the same non-autonomous 2D Navier–Stokes model stated in Chapter 2, but by verifying the flattening property.

On the one hand, in Section 3.1 we show that when $f \in L^2_{\rm loc}(\mathbb{R}; V')$ – which is the minimum regularity of f consistent with weak solutions that have $u \in L^2_{\rm loc}(\mathbb{R}; V)$ and $du/dt \in L^2_{\rm loc}(\mathbb{R}; V')$ – the process is pullback asymptotically compact. We do this using again the same *energy continuous method* developed in the previous chapter, and also show as a consequence that the process satisfies 'Condition (C)'. With only a little more regularity of f, namely $f \in L^p_{\rm loc}(\mathbb{R}; V')$ for some $p > 2$, in Section 3.1.1 we are able to show, using the semigroup approach of Fujita and Kato [29] and ideas from the ϵ-regularity theory developed by Arrieta and Carvalho [2], that in fact there is a compact pullback absorbing family in H. In particular, in the autonomous case it follows that for $f \in V'$ there is a compact absorbing set.

On the other hand, in Section 3.2 we treat attractors in V when $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, which is significantly more straightforward. One can seek to prove asymptotic compactness directly, as in Lemma 2.14, but this is little easier than the analysis we present here for the phase space H. In fact in this case use of the Fourier splitting technique ('the flattening property') makes the analysis significantly simpler, and the argument is much shorter than that in Chapter 2. Finally, in Section 3.2.1, with a little extra regularity (again, $f \in L^p_{\text{loc}}(\mathbb{R}; H)$ for some $p > 2$) the semigroup approach yields – in this case very quickly – the existence of a compact pullback absorbing family in V.

In Chapters 4 and 5 we consider the incompressible two-dimensional Navier–Stokes equations including a finite delay term:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) + g(t, u_t) \qquad \nabla \cdot u = 0,$$

where we denote by u_t the function defined on $(-h, 0)$ by the relation $u_t(s) = u(t+s)$, $s \in (-h, 0)$, and h is the time of memory effect. Observe that in this problem the external force term g contains some hereditary features. These situations may appear when we want to control the system (in a certain sense) by applying a force which takes into account not only the present state of the system but also the history of the solution.

The importance of physical models for fluid mechanics problems including delay terms is related, for instance, to real applications where devices to control properties of fluids (temperature, velocity, etc.) are inserted in domains and make a local influence on the behaviour of the system (e.g. cf. [64] for a wind-tunnel model).

On the other hand, the study of Navier–Stokes models including delay terms – existence, uniqueness, stationary solutions, exponential decay, and other asymptotic properties such as the existence of attractors – was initiated by Caraballo and Real in the references [9, 10, 11], and after that, many different questions, as dealing with unbounded domains, and models (for instance in three dimensions for modified terms) have been addressed (e.g. cf. [38, 65, 66, 67, 69, 71, 72, 76] among others).

However, to our knowledge, in all finite delay frameworks the assumptions for the delay terms used to involve estimates in L^2 spaces, which in turn means some restrictive conditions on the operators and on the function driving the delayed time. As long as the solution for the problem (without delay) in dimension two is continuous in time, it seems natural to develop a theory just considering a phase space only requiring continuity in time. In Chapter 4 we treat a relaxation on the assumptions for the delay operator involved, removing a condition on square integrable control of the memory terms, which allows us to consider a bigger class of delay terms (for instance, just under a measurability condition on the delay function leading the delayed time). More precisely, in Section 4.1 we obtain a result of existence and uniqueness of solution to our model under less restrictive assumptions than in [9, Theorem 2.1]. Our method to prove existence of solution in this new framework requires more technicalities than in previous papers, namely, an energy method for continuous functions. Moreover, in Sections 4.2 and 4.3, we deal with dynamical systems in suitable phase spaces within two metrics, the L^2 norm and the H^1 norm, respectively. Actually, we prove that, under suitable assumptions, pullback attractors not only of fixed bounded sets but also of a set of tempered universes do exist. Finally, from comparison results of attractors we establish relations among them and, under suitable additional assumptions, we conclude that these families of attractors are in fact the same object.

Our goal in Section 5 is to keep all usual conditions for the delay operator and to compare both kind of attractors, for both possibilities of phase spaces (continuous in time, or just square integrable in time). Observe that in the autonomous framework this issue would be almost immediate since one inclusion is clear by continuous embedding, and the other is obtained after an elapsed time as long as the memory effect. However, in the non-autonomous case (that we will treat in this chapter) this is not the case at all. Using the theory of attraction for universes studied in Chapter 1, we deal with different families and again under two different metrics, namely, the L^2 norm and the H^1 norm. Furthermore, we also improve some results previously obtained in Caraballo and Real [11], since we can consider the phase space $V \times L^2(-h,0;V)$ and not only $H \times L^2(-h,0;H)$. Finally, and as in the previous chapter, we establish that all these families of attractors coincide.

To conclude, in Chapter 6 we analyze the asymptotic behaviour of the solutions for the incompressible three-dimensional Navier–Stokes–Voigt equations

$$\frac{\partial}{\partial t}(u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(t) \qquad \nabla \cdot u = 0,$$

when the initial datum belongs to both V and $D(A)$ phase spaces.

The Navier–Stokes–Voigt (NSV) model of viscoelastic incompressible fluid, introduced by Oskolkov in [74], gives an approximate description of the Kelvin–Voigt fluid (see [75, 55]), and recently was proposed as a regularization of the three-dimensional Navier–Stokes equations for the purpose of direct numerical simulations in [5].

The extra regularizing term $-\alpha^2 \Delta \frac{\partial u}{\partial t}$ changes the parabolic character of the equation, which makes it so that in 3D the problem is well-posed (forward and backward), but one does not observe any immediate smoothing of the solution, as expected in parabolic PDEs. Moreover, the generated semigroup is only asymptotically compact, similarly to damped hyperbolic systems.

One of the studied topics about the problem is the inviscid question in some different senses. It is also worth observing that when $\nu = 0$, the inviscid equation that one recovers is the simplified Bardina subgrid scale model of turbulence. The relationship between the original and inviscid models was also addressed in [5]. On other hand, some questions on the inviscid regularization have been recently used for the study of a 2D surface quasi-geostrophic model (cf. [47]).

The long-time dynamics of the autonomous model was studied by Kalantarov [43] and Kalantarov and Titi [45]. Namely, the existence of global compact attractor was proved, and estimates on its fractal and Hausdorff dimensions, and upper bounds on the number of determining modes were given. Other related results are the Gévrey regularity of the global attractor (again for the autonomous model) when the force term is analytic of Gévrey type, and the establishment of similar statistical properties (and invariant measures) as the three-dimensional Navier–Stokes equations (cf. [44, 60, 77]).

On the other hand, the difference of this model in comparison with the *standard* two-dimensional Navier–Stokes (NS) model is that there exists a regularizing effect in the Navier–Stokes model (in 2D), while not here. For NS a continuous energy method can be applied thanks to the extra estimates that holds in higher norms (e.g. cf. [71]), which does not seem to hold for the Navier–Stokes–Voigt model. Some of the proofs in the previously cited references about NSV (e.g. cf. [45]) rely on splitting the problem in two, one with exponential decay, and the other with good asymptotic properties in the domain of a suitable fractional power of the Stokes operator. However, in this chapter we will provide a simpler proof, which does not require the above-mentioned technicalities, but a sharp use of the energy equality, and the energy method used by Rosa in [79]. Moreover, it is worth pointing out that our results in Section 6.2 do not use the regularity assumption on the boundary of the domain at all, and the force term may take values in V' instead of in L^2 as appears in [45].

We may also cite in this non-autonomous framework the paper [89], where the existence of uniform attractor for a Navier–Stokes–Voigt model is studied. However, there appears the same treatment with the fractional powers of the Stokes operator, and they require more regularity in the non-autonomous case that we need here.

Our main goal in this chapter is to obtain sufficient conditions such that the minimal pullback attractors for the process associated to our Navier–Stokes–Voigt problem do exist. As we pointed out above, in order to prove the asymptotic compactness of this process we will apply an energy method used by Rosa in [79]. As a second goal, we analyze some additional properties of the obtained attractors. Namely, in Section 6.3, extra regularity is deduced by using a bootstrapping argument, which now does rely on fractional powers of the Stokes operator, similarly as done in [45] for the autonomous case. Finally, the attraction in $D(A)$ norm is also proved in Section 6.4 by using the same energy method as before and previous results on strong solutions.

Spanish Summary

Una de las ramas de la Física más interesantes y complicadas de investigar es la Mecánica de Fluidos, que estudia el comportamiento de los fluidos en reposo (Estática de Fluidos) o en movimiento (Dinámica de Fluidos), así como las aplicaciones y mecanismos de ingeniería que utilizan fluidos. La Mecánica de Fluidos es primordial en campos tan diversos como la aeronáutica, la ingeniería química, civil e industrial, la meteorología, las construcciones navales y la oceanografía.

Las ecuaciones fundamentales de la Dinámica de Fluidos son las ecuaciones de Navier–Stokes, las cuales describen el movimiento de fluidos incompresibles. En el último siglo y medio, estas ecuaciones han sido aplicadas por físicos e ingenieros con apreciable éxito en muy variados campos, entre ellos la hidráulica, la meteorología y la aeronáutica, y sin ellas resultaría matemáticamente imposible describir, por ejemplo, los flujos de aire turbulento o los remolinos que se forman cuando el agua discurre por una tubería.

Por otra parte, en las últimas décadas también se ha prestado una considerable atención a la teoría de atractores, la cual se ha convertido en una interesante y eficaz herramienta en el estudio del comportamiento asintótico de los sistemas dinámicos, tanto autónomos como no autónomos. Por ejemplo, dado un problema diferencial autónomo para el que tengamos asegurada, para cada dato inicial, unicidad de solución definida para todo instante futuro, cabe preguntarse cómo evoluciona en el tiempo dicha solución. La teoría de atractores para sistemas dinámicos autónomos nos permite garantizar, bajo ciertas condiciones mínimas, la existencia de atractor global, que de forma muy general viene a ser un conjunto compacto e invariante, tal que atrae todas las trayectorias del sistema dinámico, uniformemente en conjuntos acotados.

No obstante, la aplicación práctica de modelos basados en ecuaciones diferenciales, tanto ordinarias como en derivadas parciales, revela que las fuerzas externas utilizadas para su modelización han de depender explícitamente del tiempo. Mientras que dicho factor no supone un cambio esencial en el estudio en intervalos finitos de tiempo, sí ocurre que aparecen diferencias sustanciales al examinar la evolución del modelo para todo tiempo. De esta manera, los sistemas dinámicos no autónomos han generalizado el tipo de respuestas posibles, dando lugar a nuevos conceptos tales como el de atractor *pullback* (desde atrás). La teoría de atractores pullback ha progresado profundamente en los últimos años y ha sido aplicada para intentar dar solución a una amplia variedad de problemas provenientes de distintas ramas de la Ciencia como Química, Física y Biología, y por supuesto para abordar varios modelos relacionados con las ecuaciones de Navier–Stokes.

Este trabajo está dividido en seis capítulos. En el Capítulo 1 presentamos algunos resultados abstractos que garantizan la existencia de atractores pullback minimales para sistemas dinámicos no autónomos en un marco teórico que puede depender de distintos universos como 'campos de fase'. Esta teoría será aplicada en el resto de los capítulos a diversos modelos basados en las ecuaciones de Navier–Stokes no autónomas, con la finalidad de obtener la existencia de distintas familias de atractores pullback para dichos modelos.

En el Capítulo 1 generalizamos la teoría sobre atractores globales para sistemas dinámicos autónomos al marco no autónomo. Más concretamente, en la Sección 1.1 definimos algunos conceptos básicos y estudiamos varios resultados abstractos relativos a la teoría de atractores pullback, los cuales nos permitirán garantizar, bajo ciertas hipótesis mínimas, la existencia de dichos atractores para un proceso evolutivo asociado a un determinado problema y para dos tipos diferentes de universos, el de los conjuntos acotados fijos y otro universo formado por familias parametrizadas en tiempo y definido en términos de una condición temperada. Dicho resultado sobre la existencia de atractores pullback minimales corresponde al Teorema 1.11. Además, en el Teorema 1.15 establecemos también un resultado que nos permitirá comparar atractores pullback en distintos espacios de fase y universos. Finalmente, en la Sección 1.2 definimos el concepto de propiedad flattening y probamos que la compacidad asintótica del correspondiente proceso evolutivo, necesaria para la existencia de atractor pullback, puede obtenerse a partir de esta propiedad (véase la Proposición 1.18).

En el Capítulo 2 consideramos las ecuaciones de Navier–Stokes bidimensionales y no autónomas en un dominio acotado y estudiamos el comportamiento asintótico de las soluciones cuando el dato inicial del problema pertenece a distintos espacios de fase. Haciendo uso de la teoría desarrollada en el capítulo anterior, en la Sección 2.2 probamos la existencia de atractores pullback minimales en los espacios H y V, para varios universos de los dos tipos citados anteriormente y bajo condiciones mínimas de regularidad sobre el campo de fuerzas f, que aseguren la existencia de soluciones débiles y fuertes respectivamente. Cabe señalar que la compacidad asintótica en V del proceso asociado a nuestro modelo se obtiene mediante un método de energía basado en la continuidad de las soluciones y en ciertas funciones no crecientes (véase el Lema 2.14). Asimismo, aplicando de nuevo la teoría dada en el Capítulo 1, establecemos algunas relaciones entre los atractores pullback definidos en H y en V. Finalmente, en las Secciones 2.3 y 2.4, analizamos algunas propiedades de regularidad para dichos atractores, tales como la acotación en H^2 y su comportamiento temperado en los espacios V y H^2.

A continuación, en el Capítulo 3 establecemos nuevamente la existencia de atractores pullback minimales en H y en V para el mismo modelo considerado en el Capítulo 2, pero mediante la propiedad *flattening* (aplastamiento). Cabe destacar que, mientras que en el caso del espacio de fase H, una prueba directa de la compacidad asintótica del proceso conlleva el mismo esfuerzo que verificar la propiedad flattening, en el caso del espacio V, es notablemente más inmediato obtener la compacidad asintótica a partir de la propiedad flattening que mediante el método de energía utilizado en el Lema 2.14. Además, en la Sección 2.2.1, exigiendo más regularidad sobre el campo de fuerzas (en concreto, suponiendo que $f \in L^p_{\mathrm{loc}}(\mathbb{R}; V')$ para algún $p > 2$), y basándonos en el enfoque sobre la teoría de semigrupos llevado a cabo por Fujita y Kato [29] y en algunas ideas sobre la teoría de la ϵ-regularidad desarrollada por Arrieta y Carvalho [2], obtenemos la existencia de una familia compacta y pullback absorbente en H. Análogamente, en la Sección 2.2.2, también probamos la existencia de una familia compacta y pullback absorbente en V, imponiendo que $f \in L^p_{\mathrm{loc}}(\mathbb{R}; H)$ para algún $p > 2$.

Por otro lado, en los Capítulos 4 y 5 consideramos las ecuaciones de Navier–Stokes no autónomas cuando el problema contiene términos con retardo finito, en un dominio acotado de $\mathbb{R}^2$. En ambos capítulos obtenemos resultados similares relativos a la existencia de atractores pullback minimales y algunas relaciones entre ellos. No obstante, las hipótesis que suponemos sobre el término que contiene el retardo son distintas y es por ese motivo por el que llevamos a cabo el estudio de este modelo en dos capítulos por separado.

En el Teorema 4.5 de la Sección 4.1 probamos un resultado sobre existencia, unicidad y regularidad de solución para el modelo considerado, pero bajo hipótesis menos restrictivas sobre el operador con retardo que en Capítulo 5. Más concretamente, eliminamos hipótesis relacionadas con estimaciones en la norma del espacio L^2 sobre el término que contiene el retardo. No obstante, esto nos obliga a restringirnos al espacio de fase de las funciones continuas en tiempo. En contraposición, como una ventaja a tener en cuenta, podemos considerar una mayor clase de operadores con retardo. De hecho, en esta sección exponemos también un ejemplo en el que sólo exigimos medibilidad sobre el término con retado, sin necesidad de que sea de clase C^1 con derivada acotada, tal y como se impone en algunos artículos previos que tratan el mismo problema. Por otro lado, y de nuevo haciendo uso de la teoría desarrollada en el Capítulo 1, en las Secciones 4.2 y 4.3 probamos la existencia de atractores pullback minimales en los espacios H y V respectivamente, para el universo de los conjuntos acotados fijos y para diversos universos temperados (véanse los Teoremas 4.14 y 4.25). Para ello, en la Proposición 4.13 y en el Lema 4.24, probaremos previamente la compacidad asintótica del correspondiente proceso en ambos espacios de fase, de nuevo mediante el mismo método de energía ya usado en el Lema 2.14. Por último, en la Sección 4.3 establecemos también algunas propiedades de regularidad para dichos atractores y, bajo determinadas hipótesis, veremos que todos ellos coinciden.

En el Capítulo 5 mantenemos todas las condiciones usuales sobre el término con retardo, incluidas aquellas relativas a estimaciones en norma L^2. En las Secciones 5.2 y 5.3 establecemos de nuevo la existencia de atractores pullback minimales en H y en V respectivamente, tanto para el espacio de fase de las funciones continuas en tiempo como para el de las funciones de cuadrado integrable en tiempo (véanse los Teoremas 5.10, 5.19 y 5.21). Además, bajo determinadas hipótesis, obtenemos algunos resultados de regularidad para estos atractores, tales como la atracción en el espacio de las funciones continuas (en tiempo) con valores en V, y analizamos las relaciones existentes entre dichas familias de atractores, lo que nos conduce finalmente (véase el Teorema 5.23) a relaciones con los atractores obtenidos en el Capítulo 4.

Por último, en el Capítulo 6 consideramos el modelo de Navier-Stokes-Voigt para fluidos viscoelásticos e incompresibles, el cual fue introducido por Oskolkov en [74] para describir de manera aproximada un fluido del tipo Kelvin–Voigt. Más concretamente, analizamos el comportamiento asintótico de las soluciones de un problema no autónomo para las ecuaciones de Navier-Stokes-Voigt en un dominio acotado de $\mathbb{R}^3$, cuando el dato inicial pertenece a distintos espacios de fase. En la Sección 6.1 probamos un resultado sobre existencia, unicidad y regularidad de solución, haciendo uso de las aproximaciones de Galerkin y algunos resultados de compacidad. Una vez más, el principal objetivo de este capítulo es obtener condiciones suficientes con las que podamos garantizar la existencia de atractores pullback minimales para el proceso evolutivo asociado a nuestro modelo. Estos resultados serán expuestos en las Secciones 6.2 y 6.4. Dicho análisis se llevará a cabo en el espacio V y en el dominio del operador de Stokes $D(A)$, de nuevo para dos tipos de universos distintos. Debido a que dicho modelo no posee efecto regularizante (al contrario que en las ecuaciones de Navier–Stokes bidimensionales), para probar la compacidad asintótica del proceso no podemos utilizar el método de energía aplicado en los capítulos anteriores, pues necesitaríamos mejores estimaciones para las soluciones del problema. En su lugar hacemos uso de otro método de energía, desarrollado por Rosa en [79] (véanse los Lemas 6.14 y 6.26). Finalmente, en el Teorema 6.20 de la Sección 6.3 analizamos algunas propiedades de regularidad para estos atractores usando un argumento de bootstrapping, el cual se basa en las potencias fraccionarias del operador de Stokes.

Chapter 1

Abstract Results on Minimal Pullback Attractors. Pullback Flattening Property

The theory of pullback attractors for non-autonomous dynamical systems has been extensively developed in the last years in a vast range of problems (e.g. cf. [20, 50]). This approach studies under minimal requirements not only the future of the dynamical system but what are the current attracting sections when the initial data come from $-\infty$. Namely, it has been applied in many different situations as for instance those coming from chemical, physical, and biological motivations, and also for several models related to the Navier–Stokes system (e.g. cf. [23, 24, 36, 69, 70, 73]).

Recent advances in the theory of non-autonomous dynamical systems include the consideration of universes of initial data changing in time (usually in terms of a tempered growth condition), instead of the universe of autonomous bounded sets, accordingly to the intrinsically non-autonomous model (e.g. cf. [8, 19]). Nevertheless, it usually happens that the universe of autonomous bounded sets is a subset of the tempered universe. However, many questions remained open in this direction, as for instance a proper comparison between pullback attractors in the classical sense and the so-called pullback $\mathcal{D}$-attractors (this problem was addressed in [70]).

In this chapter we present some definitions and abstract results in order to ensure the existence of minimal pullback attractors in a general universe. Moreover, we point out some relations between two possible families of attractors, each of them associated with the two cited types of universes, that of fixed bounded sets, and another one given by a tempered condition. An abstract result comparing two families of attractors associated to the same process but with different phase spaces and/or universes will be also established. Finally, we will recall the concept of the flattening property in a Banach space, and we will show that the asymptotic compactness of a dynamical process can be obtained via the Fourier splitting method, that is, by verifying the flattening property.

All the above results can be found in [31, 35, 70].

1.1 Existence and comparison of minimal pullback attractors

The results in this section are a slight modification and generalization of those presented in [70] (see also [8] and [7]). In particular, we consider the process U being closed (cf. [62], see below Definition 1.1). The proofs are not difficult, but some of them are given explicitly for the sake of completeness.

Consider given a metric space (X, d_X), and let us define $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t,r)U(r,\tau)x = U(t,\tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

Definition 1.1. *Let U be a process on X.*

a) U is said to be continuous if for any pair $\tau \leq t$, the mapping $U(t,\tau) : X \to X$ is continuous.

b) U is said to be strong-weak continuous if for any pair $\tau \leq t$, the mapping $U(t,\tau) : X \to X$ transforms sequences converging in the strong topology into sequences converging in the weak topology.

c) U is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Remark 1.2. *It is clear that every continuous process is strong-weak continuous, and that every strong-weak continuous process is closed.*

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ (observe that we do not require any additional condition on these sets as compactness or boundedness).

Definition 1.3. *We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Denote the omega-limit set of $\widehat{D}_0$ by

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau) D_0(\tau)}^X \quad \forall\, t \in \mathbb{R}, \tag{1.1}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x,y).$$

The following result is standard, and it does not use any continuity assumption on U (e.g. cf. [8, 70]).

Proposition 1.4. *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact, then for any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}_0, t)$ given by (1.1) is a nonempty compact subset of X, and*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0. \tag{1.2}$$

Moreover, the family $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), C(t)) = 0,$$

then $\Lambda(\widehat{D}_0, t) \subset C(t)$.

Proof. Let us fix $t \in \mathbb{R}$, and consider two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n. Since the process U is pullback $\widehat{D}_0$-asymptotically compact, there exist two subsequences $\{\tau_{n'}\}$ and $\{x_{n'}\}$, and an element $y \in X$, such that $U(t, \tau_{n'})x_{n'}$ converges to y in X. Then, $y \in \Lambda(\widehat{D}_0, t)$ and therefore $\Lambda(\widehat{D}_0, t)$ is a nonempty subset of X.

On the other hand, by construction it is evident that the set $\Lambda(\widehat{D}_0, t)$ is closed. Then, in order to prove that this set is compact, it is sufficient to prove that it is relatively compact in X. To this end, let us consider a sequence $\{y_n\} \subset \Lambda(\widehat{D}_0, t)$. We must prove that it is possible to extract a convergent subsequence of $\{y_n\}$ in X.

From the characterization of $\Lambda(\widehat{D}_0, t)$, for each integer n there exist $\tau_n \leq t - n$ and $x_n \in D_0(\tau_n)$, such that $d_X(y_n, U(t, \tau_n)x_n) \leq 1/n$. Again, since the process U is pullback $\widehat{D}_0$-asymptotically compact, from the sequence $\{U(t, \tau_n)x_n\}$ we can extract a convergent subsequence in X. Thus, it is clear that the corresponding subsequence of $\{y_n\}$ also converges to the same element in X.

Now, we prove (1.2) by a contradiction argument. Suppose that there exists $t \in \mathbb{R}$ such that (1.2) does not hold. Then, there exist $\varepsilon > 0$, and two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that

$$d_X(U(t, \tau_n)x_n, \Lambda(\widehat{D}_0, t)) \geq \varepsilon \quad \forall\, n \geq 1. \tag{1.3}$$

Since U is pullback $\widehat{D}_0$-asymptotically compact, from the sequence $\{U(t, \tau_n)x_n\}$ we can extract a subsequence that converges to an element $x \in \Lambda(\widehat{D}_0, t)$. And this is a contradiction at light of (1.3).

Finally, consider a family $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets such that

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), C(t)) = 0. \tag{1.4}$$

Let $x \in \Lambda(\widehat{D}_0, t)$ be given. Then, there exist sequences $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $x_n \in D_0(\tau_n)$ for all n, such that $U(t, \tau_n)x_n$ converges to x in X. By (1.4) we have $x \in \overline{C(t)} = C(t)$, and therefore $\Lambda(\widehat{D}_0, t) \subset C(t)$. ∎

Assuming also that U is closed, we obtain the invariance of the family of sets $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$.

Proposition 1.5. *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact and closed, then the family of sets $\{\Lambda(\widehat{D}_0,t) : t \in \mathbb{R}\}$, defined by (1.1), is invariant for U, that is*

$$\Lambda(\widehat{D}_0,t) = U(t,\tau)\Lambda(\widehat{D}_0,\tau) \quad \forall\, \tau \le t.$$

Proof. Consider $\tau < t$ and $y \in \Lambda(\widehat{D}_0,\tau)$. Then, there exist sequences $\{\tau_n\} \subset (-\infty,\tau]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(\tau,\tau_n)x_n \to y$.

On the one hand, from the pullback $\widehat{D}_0$-asymptotic compactness of the process U, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact, so there exists a subsequence $U(t,\tau_{n'})x_{n'} \to z \in \Lambda(\widehat{D}_0,t)$. Since $U(t,\tau_n) = U(t,\tau)U(\tau,\tau_n)$ for all n, from the fact that U is closed, we deduce that $z = U(t,\tau)y$. The inclusion $U(t,\tau)\Lambda(\widehat{D}_0,\tau) \subset \Lambda(\widehat{D}_0,t)$ is thus proved.

On the other hand, consider $z \in \Lambda(\widehat{D}_0,t)$, and $\{\tau_n\} \subset (-\infty,\tau]$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(t,\tau_n)x_n \to z$. By using the concatenation property of the process, it holds that $U(t,\tau_n) = U(t,\tau)U(\tau,\tau_n)$ for all n. Now, since the sequence $\{U(\tau,\tau_n)x_n\}$ is also relatively compact, for a subsequence we deduce that $U(\tau,\tau_{n'})x_{n'} \to y \in \Lambda(\widehat{D}_0,\tau)$. Again, since U is closed, it holds that $z = U(t,\tau)y$. Thus, we have proved the inclusion $\Lambda(\widehat{D}_0,t) \subset U(t,\tau)\Lambda(\widehat{D}_0,\tau)$. ■

Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 1.6. *It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(\widehat{D},t) \le t$ such that*

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall\, \tau \le \tau_0(\widehat{D},t).$$

Observe that, in the definition above, $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Proposition 1.7. *If $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U on X, then*

$$\Lambda(\widehat{D},t) \subset \Lambda(\widehat{D}_0,t) \quad \forall\, \widehat{D} \in \mathcal{D},\ t \in \mathbb{R}.$$

In addition, if $\widehat{D}_0 \in \mathcal{D}$, then

$$\Lambda(\widehat{D}_0,t) \subset \overline{D_0(t)}^X \quad \forall\, t \in \mathbb{R}.$$

Proof. Let $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$ be fixed. If $\Lambda(\widehat{D},t)$ is nonempty, for any $y \in \Lambda(\widehat{D},t)$ there exist two sequences $\{\tau_n\} \subset (-\infty,t]$ and $\{x_n\} \subset X$, with $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, such that $U(t,\tau_n)x_n \to y$.

Since $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U, for each integer $k \ge 1$ there exists $\tau_{n_k} \in \{\tau_n\}$ with $\tau_{n_k} \le t-k$, and $y_{n_k} = U(t-k,\tau_{n_k})x_{n_k} \in D_0(t-k)$. As $U(t,t-k)y_{n_k} = U(t,\tau_{n_k})x_{n_k} \to y$, then $y \in \Lambda(\widehat{D}_0,t)$.

Finally, let $t \in \mathbb{R}$ be given, and suppose that $\widehat{D}_0 \in \mathcal{D}$. If $\Lambda(\widehat{D}_0,t)$ is nonempty, observe that for any $y \in \Lambda(\widehat{D}_0,t)$, there exist sequences $\{\tau_n\} \subset (-\infty,t]$ and $\{x_n\} \subset X$, with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(t,\tau_n)x_n \to y$. Since $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U, from certain $n \in \mathbb{N}$, $U(t,\tau_n)x_n \in D_0(t)$. Thus, $y \in \overline{D_0(t)}^X$.
■

Definition 1.8. *A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.*

As a consequence of Propositions 1.4 and 1.5, we have the following result.

Proposition 1.9. *Assume that the process U on X is closed and pullback $\mathcal{D}$-asymptotically compact. Then, for each $\widehat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}, t)$ is a nonempty compact subset of X, invariant for U, and attracts $\widehat{D}$ in the pullback sense, that is*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t)) = 0. \tag{1.5}$$

Moreover, for each $\widehat{D} \in \mathcal{D}$, the family $\{\Lambda(\widehat{D}, t) : t \in \mathbb{R}\}$ is minimal amongst all the families of closed sets that satisfy (1.5).

Proposition 1.10. *Assume that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X, and U is pullback $\widehat{D}_0$-asymptotically compact. Then, the process U is also pullback $\mathcal{D}$-asymptotically compact.*

Proof. Consider fixed $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n. We must prove that from the sequence $\{U(t,\tau_n)x_n\}$ we can extract a subsequence converging in X.

Observing that $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U on X, we deduce that for any integer $k \geq 1$ there exists $\tau_{n_k} \in \{\tau_n\}$ such that $\tau_{n_k} \leq t-k$, and $y_{n_k} = U(t-k, \tau_{n_k})x_{n_k} \in D_0(t-k)$. As U is pullback $\widehat{D}_0$-asymptotically compact, from the sequence $\{U(t,t-k)y_{n_k}\}$ we can extract a subsequence $\{U(t,t-k')y_{n_{k'}}\}$ converging in X. But $U(t,t-k')y_{n_{k'}} = U(t,t-k')U(t-k',\tau_{n_{k'}})x_{n_{k'}} = U(t,\tau_{n_{k'}})x_{n_{k'}}$. This finishes the proof. ∎

With the above definitions and results, we obtain the main result of this section.

Theorem 1.11. *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ that is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^{X} \quad t \in \mathbb{R},$$

has the following properties:

(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$,

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e., $\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $(t,\tau) \in \mathbb{R}^2_d$,

(d) if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Proof. As $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for U, from Proposition 1.7 we know that $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t)$ for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, and if moreover $\widehat{D}_0 \in \mathcal{D}$, then $\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

Since U is pullback $\widehat{D}_0$-asymptotically compact, by Proposition 1.4, the set $\Lambda(\widehat{D}_0, t)$ is a nonempty compact set for any $t \in \mathbb{R}$.

By Proposition 1.10, U is also pullback $\mathcal{D}$-asymptotically compact. Thus, by Proposition 1.9, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, the set $\Lambda(\widehat{D}, t)$ is nonempty and compact.

These considerations prove (a) and (d).

Moreover, as evidently

$$\operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) \leq \operatorname{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t))$$

for any $\widehat{D} \in \mathcal{D}$, (b) is also a consequence of Proposition 1.9.

Now, in order to prove (c) we observe that by Proposition 1.5, it also holds

$$U(t,\tau)\Lambda(\widehat{D}, \tau) = \Lambda(\widehat{D}, t) \quad \forall \tau \leq t, \ \widehat{D} \in \mathcal{D}. \tag{1.6}$$

If $y \in \mathcal{A}_{\mathcal{D}}(t)$, there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}$ and $\{y_n\} \subset X$, such that $y_n \in \Lambda(\widehat{D}_n, t)$ and $y_n \to y$. But by (1.6), $y_n = U(t,\tau)x_n$, with $x_n \in \Lambda(\widehat{D}_n, \tau) \subset \mathcal{A}_{\mathcal{D}}(\tau)$. By the compactness of this last set, there exists a subsequence $\{x_{n'}\} \subset \{x_n\}$ such that $x_{n'} \to x \in \mathcal{A}_{\mathcal{D}}(\tau)$. But then, as U is closed, $y = U(t,\tau)x$, and this proves that $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$. The reverse inclusion can be proved analogously.

Finally, the minimality is also easy to obtain taking into account Proposition 1.9 and the definition of $\mathcal{A}_{\mathcal{D}}$. ∎

Remark 1.12. *A family $\mathcal{A}_{\mathcal{D}}$ that satisfies properties (a)–(c) in Theorem 1.11 is called a minimal pullback $\mathcal{D}$-attractor for the process U.*

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b) and (c).

Sufficient conditions in order to have $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ are

(i) $\widehat{D}_0 \in \mathcal{D}$,

(ii) the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and

(iii) the universe $\mathcal{D}$ is inclusion-closed, i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$.

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}_F(X)$, the corresponding minimal pullback $\mathcal{D}_F(X)$-attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli [23, Theorem 1.1, p. 311], and will be denoted by $\mathcal{A}_{\mathcal{D}_F(X)}$.

Now, it is easy to conclude the following result.

Corollary 1.13. *Under the assumptions of Theorem 1.11, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \forall\, t \in \mathbb{R}.$$

Remark 1.14. *It can be proved (see [70]) that, under the assumptions of the preceding corollary, if for some $T \in \mathbb{R}$, the set $\cup_{t \le T} D_0(t)$ is a bounded subset of X, then*

$$\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \forall\, t \le T.$$

Now, we establish an abstract result that allows us to compare two attractors for a process under appropriate assumptions.

Theorem 1.15. *Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be two metric spaces such that $X_1 \subset X_2$ with continuous injection, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that we have a map U that acts as a process in both cases, i.e., $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$ is a process.*

For each $t \in \mathbb{R}$, let us denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \qquad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence of the respective topology.

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

If in addition

(i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$, and

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and $t^*_{\widehat{D}_1} \le t$ (both possibly depending on t and $\widehat{D}_2$), such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \le t^*_{\widehat{D}_1}$ there exists $\tau_s \le s$ such that

$$U(s,\tau) D_2(\tau) \subset D_1(s) \quad \forall\, \tau \le \tau_s,$$

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Proof. Since the omega-limit set is characterized as

$$\Lambda_i(\widehat{D}_i,t)=\{x\in X_i:\exists\tau_n\to-\infty, x_n\in D_i(\tau_n), x=X_i-\lim_n U(t,\tau_n)x_n\},$$

by the continuous injection of X_1 into X_2 it holds that $\Lambda_1(\widehat{D}_1,t)\subset\Lambda_2(\widehat{D}_1,t)$, for all $\widehat{D}_1\in\mathcal{D}_1$ and any $t\in\mathbb{R}$. This implies that

$$\bigcup_{\widehat{D}_1\in\mathcal{D}_1}\Lambda_1(\widehat{D}_1,t)\subset\bigcup_{\widehat{D}_1\in\mathcal{D}_1}\Lambda_2(\widehat{D}_1,t)\subset\bigcup_{\widehat{D}_2\in\mathcal{D}_2}\Lambda_2(\widehat{D}_2,t).$$

Again from the continuous injection of X_1 into X_2, we obtain one inclusion:

$$\mathcal{A}_1(t)=\overline{\bigcup_{\widehat{D}_1\in\mathcal{D}_1}\Lambda_1(\widehat{D}_1,t)}^{X_1}\subset\overline{\bigcup_{\widehat{D}_2\in\mathcal{D}_2}\Lambda_2(\widehat{D}_2,t)}^{X_2}=\mathcal{A}_2(t).$$

For the opposite inclusion, assuming (i) and (ii), consider $\widehat{D}_2\in\mathcal{D}_2$ and $t\in\mathbb{R}$ given. For any $x\in\Lambda_2(\widehat{D}_2,t)$ there exist two sequences $\{\tau_n\}$ and $\{x_n\}$ with $\tau_n\le t$ for all n, satisfying $\tau_n\to-\infty$, $x_n\in D_2(\tau_n)$, and $x=X_2-\lim_n U(t,\tau_n)x_n$. By assumption (ii), there exist a $\widehat{D}_1\in\mathcal{D}_1$ and an integer $k_{\widehat{D}_1}\ge 1$ such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $k\ge k_{\widehat{D}_1}$ there exist $x_{n_k}\in\{x_n\}$ and $\tau_{n_k}\le t-k$ such that

$$y_{n_k}=U(t-k,\tau_{n_k})x_{n_k}\in D_1(t-k).$$

As U is pullback $\widehat{D}_1$-asymptotically compact, there exists a subsequence of the sequence $\{x_{n_k}\}$ (relabelled the same) such that

$$X_1-\lim_k U(t,t-k)y_{n_k}=z\in\Lambda_1(\widehat{D}_1,t).$$

But taking into account that $U(t,t-k)y_{n_k}=U(t,\tau_{n_k})x_{n_k}$, by the continuous injection of X_1 into X_2, we deduce that $z=x$. Thus, $x\in\Lambda_1(\widehat{D}_1,t)$.

Consequently,

$$\bigcup_{\widehat{D}_2\in\mathcal{D}_2}\Lambda_2(\widehat{D}_2,t)\subset\bigcup_{\widehat{D}_1\in\mathcal{D}_1}\Lambda_1(\widehat{D}_1,t)\subset\mathcal{A}_1(t).$$

As $\mathcal{A}_1(t)$ is compact in X_1, from the continuous injection, it is also compact in X_2, and in particular, closed. Taking closure in X_2 in the above inclusion, we conclude that $\mathcal{A}_2(t)\subset\mathcal{A}_1(t)$. The proof is finished. ∎

Remark 1.16. *In the preceding theorem, if instead of assumption (ii) we consider the following condition:*

(ii') for any $\widehat{D}_2\in\mathcal{D}_2$ and any sequence $\tau_n\to-\infty$, there exist another family $\widehat{D}_1\in\mathcal{D}_1$ and another sequence $\tau'_n\to-\infty$ with $\tau'_n\ge\tau_n$ for all n, such that U is pullback $\widehat{D}_1$-asymptotically compact, and

$$U(\tau'_n,\tau_n)D_2(\tau_n)\subset D_1(\tau'_n)\quad\forall\, n,$$

then, with a similar proof, one can obtain that the equality $\mathcal{A}_1(t)=\mathcal{A}_2(t)$ also holds for all $t\in\mathbb{R}$.

Observe that a sufficient condition for (ii') is that for each $t\in\mathbb{R}$, there exists $T=T(t)>0$ such that for any $\widehat{D}_2\in\mathcal{D}_2$, there exists a $\widehat{D}_1\in\mathcal{D}_1$ satisfying that U is pullback $\widehat{D}_1$-asymptotically compact, and $U(\tau+T,\tau)D_2(\tau)\subset D_1(\tau+T)$ for all $\tau<t-T$.

1.2 Pullback $\widehat{D}_0$-flattening property

Now, we introduce a notion which is a slight modification of Ma, Wang, and Zhong's "Condition (C)" [63] (renamed the "flattening property" by Kloeden and Langa [48]), after Definition 2.24 in the book by Carvalho, Langa, and Robinson [12], where P_ε need not be a projection operator.

Definition 1.17. *Assume that X is a Banach space with norm $\|\cdot\|_X$, and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a given family. We will say that the process U on X satisfies the pullback $\widehat{D}_0$-flattening property if for any $t \in \mathbb{R}$, and $\varepsilon > 0$, there exist $\tau_\varepsilon < t$, a finite dimensional subspace X_ε of X, and a mapping $P_\varepsilon : X \to X_\varepsilon$, all depending on $\widehat{D}_0$, t and ε, such that*

$$\{P_\varepsilon U(t,\tau)u^\tau : \tau \le \tau_\varepsilon,\ u^\tau \in D_0(\tau)\} \text{ is bounded in } X$$

and

$$\|(I - P_\varepsilon)U(t,\tau)u^\tau\|_X < \varepsilon \quad \text{for any } \tau \le \tau_\varepsilon,\ u^\tau \in D_0(\tau).$$

Similarly to the results in [63] and [48] (see also [12]) we will see that to show that a process U is pullback $\widehat{D}_0$-asymptotically compact, it is enough to verify the pullback $\widehat{D}_0$-flattening property given in the definition above.

Proposition 1.18. *Assume that X is a Banach space and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a given family such that the process U on X satisfies the pullback $\widehat{D}_0$-flattening property. Then the process U is pullback $\widehat{D}_0$-asymptotically compact.*

Proof. Let $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ such that $\tau_n \to -\infty$, and a sequence $\{x_n\} \subset X$ such that $x_n \in D_0(\tau_n)$ for all n, be fixed.

For a fixed integer $k \ge 1$, by the pullback $\widehat{D}_0$-flattening property, there exist $N_k \ge 1$, a finite dimensional subspace X_k of X, and a mapping $P_k : X \to X_k$, such that $\{P_k U(t,\tau_n)x_n : n \ge N_k\}$ is a bounded subset of X_k, and therefore a relatively compact subset of X, and $\|(I - P_k)U(t,\tau_n)x_n\|_X \le 1/(2k)$ for all $n \ge N_k$. Thus, $\{U(t,\tau_n)x_n : n \ge 1\}$ can be covered by a finite number of balls in X of radius $1/k$. As k is arbitrary, it is not difficult to check that $\{U(t,\tau_n)x_n : n \ge 1\}$ possesses a Cauchy subsequence in X. Since X is complete, this subsequence is convergent, whence $\{U(t,\tau_n)x_n : n \ge 1\}$ is relatively compact in X. ∎

Remark 1.19. *It can be proved (see [12, Theorem 2.25, p. 37] or [35]) that, reciprocally, when X is a uniformly convex Banach space, if the process U is pullback $\widehat{D}_0$-asymptotically compact, then it satisfies the pullback $\widehat{D}_0$-flattening property.*

Chapter 2

Pullback Attractors for Non-Autonomous 2D Navier–Stokes Equations

The Navier–Stokes equations have received very much attention over the last decades due to their importance in the understanding of fluids motion and turbulence. These equations have been the object of numerous works since the first paper by Leray was published in 1933 (e.g. cf. [21,25,51,61,87] and the references therein).

The main aim of this chapter is to analyze the asymptotic behaviour of the solutions to a non-autonomous 2D Navier–Stokes model in a bounded domain, when the initial datum belongs to H or V (defined below precisely). We will prove the existence of minimal pullback attractors in these two different phase spaces, when the non-autonomous forcing term is taken with the minimal regularity required for the existence of weak and strong solutions, namely, in $L^2_{\mathrm{loc}}(\mathbb{R};V')$ and $L^2_{\mathrm{loc}}(\mathbb{R};H)$ respectively. Actually, we will be able to obtain these pullback attractors not only of fixed bounded sets but also of a set of universes given by a tempered condition.

Moreover, we will present a study on the regularity of these different families of pullback attractors. On the one hand, in Section 2.3 we will show a general result about the $(H^2(\Omega))^2 \cap V$-boundedness of invariant sets for the associated evolution process. Then, as a consequence, we will deduce that, under suitable assumptions, the pullback attractors for our non-autonomous 2D Navier–Stokes problem are bounded not only in V but also in $(H^2(\Omega))^2$. On the other hand, in Section 2.4, two results about the tempered behaviour in V and $(H^2(\Omega))^2$ of the pullback attractors, when time goes to $-\infty$, will be obtained.

The results in this chapter can be found in [7,30,31,61,78,87].

2.1 Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth enough boundary $\partial\Omega$, and consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following Navier–Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f(t) \quad \text{in } \ \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \quad \text{in } \ \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \ \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u^\tau(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, u^τ is the initial velocity field, and f is the external force term depending on time.

To set our problem in the abstract framework, we consider the following usual function spaces in the variational theory of Navier–Stokes equations:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x)\, dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot,\cdot \rangle$ for the duality between V' and V. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v \rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and compact.

Now, define the operator $A : V \to V'$ as

$$\langle Au, v \rangle = ((u, v)) \quad \forall\, u, v \in V.$$

Let us denote by $D(A) = \{u \in V : Au \in H\}$ the domain of A. By the regularity of $\partial\Omega$, one has $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent (see [13] or [86]), and $D(A)$ is compactly and densely injected in V.

Let us also define

$$b(u,v,w)=\sum_{i,j=1}^{2}\int_{\Omega}u_i\frac{\partial v_j}{\partial x_i}w_j\,dx,$$

for all functions $u,v,w:\Omega\to\mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b makes sense for all $u,v,w\in V$, and is a continuous trilinear form on $V\times V\times V$.

Some useful properties concerning b that we will use throughout the following chapters are (see [78] or [87]):

$$\begin{aligned}
b(u,v,w) &= -b(u,w,v)\quad \forall\, u,v,w\in V, &(2.2)\\
b(u,v,v) &= 0\quad \forall\, u,v\in V, &(2.3)\\
|b(u,v,w)| &\le 2^{-1/2}|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}\quad \forall\, u,v,w\in V, &(2.4)
\end{aligned}$$

and there exists a constant $C_1>0$, depending only on Ω, such that

$$|b(u,v,w)|\le C_1|u|^{1/2}|Au|^{1/2}\|v\||w|\quad \forall\, u\in D(A),\ v\in V,\ w\in H, \tag{2.5}$$

and

$$|b(u,v,w)|\le C_1|Au|\|v\||w|\quad \forall\, u\in D(A),\ v\in V,\ w\in H. \tag{2.6}$$

In fact, (2.4) is a slight improvement of Lemma 3.3 in [86, p. 291] (see [36]).

For any u, $v\in V$, we will also denote by $B(u,v)$ the operator of V' given by

$$\langle B(u,v),w\rangle=b(u,v,w)\quad \forall\, w\in V$$

and $B(u)=B(u,u)$.

Assume that $u^\tau\in H$ and $f\in L^2_{\mathrm{loc}}(\mathbb{R};V')$.

Definition 2.1. *A weak solution to (2.1) is a function u that belongs to $L^2(\tau,T;V)\cap L^\infty(\tau,T;H)$ for all $T>\tau$, with $u(\tau)=u^\tau$, and such that for all $v\in V$,*

$$\frac{d}{dt}(u(t),v)+\nu\langle Au(t),v\rangle+b(u(t),u(t),v)=\langle f(t),v\rangle, \tag{2.7}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,\infty)$.

Note that for the right-hand side to be defined we certainly require $f(t)\in V'$ for almost every (*a.e.* for short) $t>\tau$; we choose $f\in L^2_{\mathrm{loc}}(\mathbb{R};V')$ so that we can interpret the initial condition and obtain an energy equality for solutions. Indeed, if u is a weak solution to (2.1) and $f\in L^2_{\mathrm{loc}}(\mathbb{R};V')$ then from (2.7) we deduce that for any $T>\tau$, one has $u'\in L^2(\tau,T;V')$, and so $u\in C([\tau,\infty);H)$, whence the initial datum has full sense. Moreover, in this case the following energy equality holds:

$$|u(t)|^2+2\nu\int_s^t\|u(r)\|^2\,dr=|u(s)|^2+2\int_s^t\langle f(r),u(r)\rangle\,dr\quad \forall\,\tau\le s\le t.$$

In Section 2.2.1 we will prove the existence of pullback attractors in H with this (minimal) regularity requirement on f, coupled with the boundedness condition

$$\int_{-\infty}^{0} e^{\mu s}\|f(s)\|_*^2\,ds < \infty \tag{2.8}$$

for some $\mu \in (0, 2\nu\lambda_1)$, where $\lambda_1 = \inf_{v\in V\setminus\{0\}} \|v\|^2/|v|^2 > 0$ is the first eigenvalue of the Stokes operator A.

A notion of more regular solution is also suitable for problem (2.1).

Definition 2.2. *A strong solution to problem (2.1) is a weak solution u to (2.1) such that u belongs to $L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.*

If $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ and u is a strong solution to (2.1), then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, \infty); V)$. In this case the following energy equality holds:

$$\begin{aligned} & \|u(t)\|^2 + 2\nu\int_s^t |Au(r)|^2\,dr + 2\int_s^t b(u(r), u(r), Au(r))\,dr \\ = \;& \|u(s)\|^2 + 2\int_s^t (f(r), Au(r))\,dr \quad \forall\, \tau \le s \le t. \end{aligned} \tag{2.9}$$

We study pullback attractors in the space V in Section 2.2.2, again taking the minimal regularity requirement on f for the existence of such solutions, along with a condition parallel to (2.8), namely

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\,ds < \infty \tag{2.10}$$

for some $\mu \in (0, 2\nu\lambda_1)$.

2.2 Existence of minimal pullback attractors

In this section we define a suitable process U associated to problem (2.1), and, by applying the abstract theory studied in Chapter 1, we are able to obtain the existence of minimal pullback attractors in both H and V spaces for several universes, under suitable assumptions. Furthermore, some relations among these attractors will be also analyzed.

2.2.1 Pullback attractors in H norm

Results concerning existence and uniqueness of weak solutions for problem (2.1), and continuity with respect to the initial condition, summarized in the following theorem and proposition, are well known (see [61, 78, 87], for example).

Theorem 2.3. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ be given. Then, for each $\tau \in \mathbb{R}$ and $u^\tau \in H$, there exists a unique weak solution $u(\cdot) = u(\cdot; \tau, u^\tau)$ to (2.1).*

Moreover, if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, then *(a)* $u \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ *for all* $T > \tau+\varepsilon > \tau$.

(b) If $u^\tau \in V$, u *is in fact a strong solution to (2.1).*

Therefore, when $f \in L^2_{\text{loc}}(\mathbb{R}; V')$, we can define a process $U : \mathbb{R}^2_d \times H \to H$ as

$$U(t,\tau)u^\tau = u(t;\tau,u^\tau) \quad \forall\, (t,\tau) \in \mathbb{R}^2_d,\ u^\tau \in H, \tag{2.11}$$

and if $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, the restriction of this process to $\mathbb{R}^2_d \times V$ is a process on V.

Proposition 2.4. *If* $f \in L^2_{\text{loc}}(\mathbb{R}; V')$*, for any pair* $(t,\tau) \in \mathbb{R}^2_d$*, the map* $U(t,\tau)$ *is continuous from* H *into* H*. Moreover, if* $f \in L^2_{\text{loc}}(\mathbb{R}; H)$*, then* $U(t,\tau)$ *is also continuous from* V *into* V.

The following result guarantees the existence of a pullback absorbing family for the process U on H.

Lemma 2.5. *Let* $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ *be given and consider any fixed* $\mu \in (0, 2\nu\lambda_1)$*. Then, for any* $\tau \in \mathbb{R}$*, and* $u^\tau \in H$*, the solution* $u(\cdot) = u(\cdot;\tau,u^\tau)$ *to (2.1) satisfies*

$$|u(t)|^2 \le e^{-\mu(t-\tau)}|u^\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu\theta}\|f(\theta)\|_*^2\, d\theta \quad \forall\, t \ge \tau. \tag{2.12}$$

Proof. By the energy equality we have

$$\frac{d}{d\theta}|u(\theta)|^2 + 2\nu\|u(\theta)\|^2 = 2\langle f(\theta), u(\theta)\rangle, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\frac{d}{d\theta}(e^{\mu\theta}|u(\theta)|^2) + 2\nu e^{\mu\theta}\|u(\theta)\|^2 = \mu e^{\mu\theta}|u(\theta)|^2 + 2e^{\mu\theta}\langle f(\theta), u(\theta)\rangle, \quad \text{a.e. } \theta > \tau.$$

Observing that by Young's inequality,

$$2|\langle f(\theta), u(\theta)\rangle| \le \frac{1}{2\nu - \mu\lambda_1^{-1}}\|f(\theta)\|_*^2 + (2\nu - \mu\lambda_1^{-1})\|u(\theta)\|^2,$$

from above we deduce

$$\frac{d}{d\theta}(e^{\mu\theta}|u(\theta)|^2) \le \frac{e^{\mu\theta}}{2\nu - \mu\lambda_1^{-1}}\|f(\theta)\|_*^2, \quad \text{a.e. } \theta > \tau,$$

and thus, integrating in time,

$$e^{\mu t}|u(t)|^2 \le e^{\mu\tau}|u^\tau|^2 + \frac{1}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu\theta}\|f(\theta)\|_*^2\, d\theta \quad \forall\, t \ge \tau.$$

So, from this last inequality we obtain (2.12). ■

Once the above estimate has been established, we introduce the following universe in $\mathcal{P}(H)$.

Definition 2.6. *For any $\mu > 0$, we will denote by $\mathcal{D}_\mu(H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0.$$

Accordingly to the notation introduced in the previous chapter, $\mathcal{D}_F(H)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Observe that for any $\mu > 0$, $\mathcal{D}_F(H) \subset \mathcal{D}_\mu(H)$, and that the universe $\mathcal{D}_\mu(H)$ is inclusion-closed.

Corollary 2.7. *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies*

$$\int_{-\infty}^{0} e^{\mu s} \|f(s)\|_*^2 \, ds < \infty \quad \text{for some } \mu \in (0, 2\nu\lambda_1). \tag{2.13}$$

Then, the family $\widehat{D}_{0,\mu} = \{D_{0,\mu}(t) : t \in \mathbb{R}\}$ defined by $D_{0,\mu}(t) = \overline{B}_H(0, R_H(t))$, the closed ball in H of center zero and radius $R_H(t)$, where

$$R_H^2(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s} \|f(s)\|_*^2 \, ds,$$

is pullback $\mathcal{D}_\mu(H)$-absorbing for the process U on H given by (2.11) (and thus pullback $\mathcal{D}_F(H)$-absorbing too), and $\widehat{D}_{0,\mu} \in \mathcal{D}_\mu(H)$.

The asymptotic behaviour in H is also well known, and again we only summarize the main facts (e.g. cf. [8, 7]). Actually, the results in this case can be obtained in an analogous way, but simpler, to that which we will use later for the asymptotic behaviour in V.

Lemma 2.8. *Under the assumptions of Corollary 2.7, the process U defined by (2.11) is pullback $\mathcal{D}_\mu(H)$-asymptotically compact.*

Combining all the above statements, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}^2_d \times H \to H$ defined by (2.11).

Theorem 2.9. *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies the condition (2.13). Then, there exist the minimal pullback $\mathcal{D}_F(H)$-attractor $\mathcal{A}_{\mathcal{D}_F(H)}$ and the minimal pullback $\mathcal{D}_\mu(H)$-attractor $\mathcal{A}_{\mathcal{D}_\mu(H)}$ for the process U on H given by (2.11). The family $\mathcal{A}_{\mathcal{D}_\mu(H)}$ belongs to $\mathcal{D}_\mu(H)$, and it holds that*

$$\mathcal{A}_{\mathcal{D}_F(H)}(t) \subset \mathcal{A}_{\mathcal{D}_\mu(H)}(t) \subset \overline{B}_H(0, R_H(t)) \quad \forall\, t \in \mathbb{R}. \tag{2.14}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_\mu(H)}$ and $\mathcal{A}_{\mathcal{D}_F(H)}$ is a direct consequence of the abstract results given in Theorem 1.11 and Corollary 1.13 respectively, since all the assumptions, closed process (continuous in fact, by Proposition 2.4), pullback absorbing family (Corollary 2.7) and pullback asymptotic compactness (Lemma 2.8), are satisfied.

Then, the claim that $\mathcal{A}_{\mathcal{D}_\mu(H)}$ belongs to $\mathcal{D}_\mu(H)$ follows from Theorem 1.11 and Remark 1.12, since the universe $\mathcal{D}_\mu(H)$ is inclusion-closed, the family $\widehat{D}_{0,\mu}$ belongs to $\mathcal{D}_\mu(H)$, and the set $D_{0,\mu}(t)$ is closed in H for all $t \in \mathbb{R}$.

Finally, the first inclusion in (2.14) is a consequence of Corollary 1.13, since $\mathcal{D}_F(H) \subset \mathcal{D}_\mu(H)$. The last inclusion in (2.14) follows again from Theorem 1.11. ■

2.2.2 Pullback attractors in V norm

The goal of this Section is to prove analogous results to those given above, but concerning to the map U defined as a process on V.

From now on we assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, and satisfies

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < \infty \quad \text{for some } \mu \in (0, 2\nu\lambda_1). \tag{2.15}$$

Now, we have the following result in which we establish several estimates in finite intervals of time when the initial time is sufficiently shifted in a pullback sense.

Lemma 2.10. *Suppose that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies the condition (2.15). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u^\tau \in D(\tau)$, it holds*

$$\left\{\begin{aligned} |u(r;\tau,u^\tau)|^2 &\leq \rho_1(t) \qquad \forall\, r \in [t-3, t],\\ \|u(r;\tau,u^\tau)\|^2 &\leq \rho_2(t) \qquad \forall\, r \in [t-2, t],\\ \int_{r-1}^{r} |Au(\theta;\tau,u^\tau)|^2\, d\theta &\leq \rho_3(t) \qquad \forall\, r \in [t-1, t],\\ \int_{r-1}^{r} |u'(\theta;\tau,u^\tau)|^2\, d\theta &\leq \rho_4(t) \qquad \forall\, r \in [t-1, t], \end{aligned}\right. \tag{2.16}$$

where

$$\rho_1(t) = 1 + \frac{e^{-\mu(t-3)}}{2\nu\lambda_1 - \mu}\int_{-\infty}^{t} e^{\mu\theta}|f(\theta)|^2 d\theta, \tag{2.17}$$

$$\begin{aligned} \rho_2(t) &= \nu^{-1}\Big(\rho_1(t) + (\nu^{-1}\lambda_1^{-1} + 2)\int_{t-3}^{t} |f(\theta)|^2 d\theta\Big) \\ &\qquad \times \exp\Big[2\nu^{-1}C^{(\nu)}\rho_1(t)\Big(\rho_1(t) + \nu^{-1}\lambda_1^{-1}\int_{t-3}^{t}|f(\theta)|^2 d\theta\Big)\Big], \end{aligned} \tag{2.18}$$

$$\rho_3(t) = \nu^{-1}\Big(\rho_2(t) + 2\nu^{-1}\int_{t-2}^{t}|f(\theta)|^2 d\theta + 2C^{(\nu)}\rho_1(t)\rho_2^2(t)\Big),$$

$$\rho_4(t) = \nu\rho_2(t) + 2\int_{t-2}^{t}|f(\theta)|^2 d\theta + 2C_1^2\rho_2(t)\rho_3(t), \tag{2.19}$$

and

$$C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}. \tag{2.20}$$

Proof. Let $\tau_1(\widehat{D}, t) < t - 3$ be such that

$$e^{-\mu(t-3)}e^{\mu\tau}|u^\tau|^2 \leq 1 \quad \forall\, \tau \leq \tau_1(\widehat{D}, t),\ u^\tau \in D(\tau).$$

Consider fixed $\tau \leq \tau_1(\widehat{D}, t)$ and $u^\tau \in D(\tau)$.

First estimate in (2.16) follows directly from (2.12), using the increasing character of the exponential.

In order to obtain the rest of the estimates, we will proceed with the Galerkin approximations and then passing to the limit.

For each integer $m \geq 1$, we denote by $u^m(t) = u^m(t;\tau,u^\tau)$ the Galerkin approximation of the solution $u(t;\tau,u^\tau)$ to (2.1), which is given by

$$u^m(t) = \sum_{j=1}^{m} \alpha_{m,j}(t) w_j,$$

where the upper script m will be used instead of (m) for short, since no confusion is possible with powers of u, and where the coefficients $\alpha_{m,j}$ are required to satisfy the system

$$\begin{cases} \dfrac{d}{dt}(u^m(t), w_j) + \nu((u^m(t), w_j)) + b(u^m(t), u^m(t), w_j) = (f(t), w_j), \\ (u^m(\tau), w_j) = (u^\tau, w_j), \qquad j = 1, \dots, m, \end{cases} \tag{2.21}$$

where $\{w_j : j \geq 1\} \subset D(A)$ is a Hilbert basis of H formed by ortho-normalized eigenfunctions of the Stokes operator A. Observe that by the regularity of Ω, all w_j belong to $(H^2(\Omega))^2$.

Multiplying in (2.21) by $\alpha_{m,j}(t)$, and summing from $j = 1$ to m, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 &= (f(t), u^m(t)) \\ &\leq \frac{1}{2\nu\lambda_1}|f(t)|^2 + \frac{\nu}{2}\lambda_1|u^m(t)|^2, \quad \text{a.e. } t > \tau, \end{aligned} \tag{2.22}$$

where we have used Young's inequality.

Integrating, in particular we deduce that

$$\nu \int_{r-1}^{r} \|u^m(\theta)\|^2\, d\theta \leq |u^m(r-1)|^2 + \frac{1}{\nu\lambda_1}\int_{r-1}^{r} |f(\theta)|^2\, d\theta \quad \forall\, \tau \leq r-1. \tag{2.23}$$

Now, multiplying in (2.21) by $\lambda_j\alpha_{m,j}(t)$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing from $j = 1$ to m, we obtain

$$\frac{1}{2}\frac{d}{d\theta}\|u^m(\theta)\|^2 + \nu|Au^m(\theta)|^2 + b(u^m(\theta), u^m(\theta), Au^m(\theta)) = (f(\theta), Au^m(\theta)), \tag{2.24}$$

a.e. $\theta > \tau$.

Observe that

$$|(f(\theta), Au^m(\theta))| \leq \frac{1}{\nu}|f(\theta)|^2 + \frac{\nu}{4}|Au^m(\theta)|^2,$$

and by (2.5) and Young's inequality,

$$\begin{aligned} |b(u^m(\theta), u^m(\theta), Au^m(\theta))| &\leq C_1|u^m(\theta)|^{1/2}\|u^m(\theta)\||Au^m(\theta)|^{3/2} \\ &\leq \frac{\nu}{4}|Au^m(\theta)|^2 + C^{(\nu)}|u^m(\theta)|^2\|u^m(\theta)\|^4, \end{aligned} \tag{2.25}$$

where $C^{(\nu)}$ is given by (2.20).

Thus, from (2.24) we deduce

$$\frac{d}{d\theta}\|u^m(\theta)\|^2 + \nu|Au^m(\theta)|^2 \leq \frac{2}{\nu}|f(\theta)|^2 + 2C^{(\nu)}|u^m(\theta)|^2\|u^m(\theta)\|^4, \quad \text{a.e. } \theta > \tau. \tag{2.26}$$

From this inequality, in particular we obtain

$$\|u^m(r)\|^2 \leq \|u^m(s)\|^2 + \frac{2}{\nu}\int_{r-1}^{r} |f(\theta)|^2\, d\theta + 2C^{(\nu)}\int_{s}^{r} |u^m(\theta)|^2\|u^m(\theta)\|^4\, d\theta$$

for all $\tau \leq r-1 \leq s \leq r$, and therefore, by Gronwall's lemma,

$$\|u^m(r)\|^2 \leq \Big(\|u^m(s)\|^2 + \frac{2}{\nu}\int_{r-1}^{r} |f(\theta)|^2\, d\theta\Big) \exp\Big(2C^{(\nu)}\int_{r-1}^{r} |u^m(\theta)|^2\|u^m(\theta)\|^2\, d\theta\Big)$$

for all $\tau \leq r-1 \leq s \leq r$.

Integrating this last inequality for s between $r-1$ and r, we obtain

$$\begin{aligned}\|u^m(r)\|^2 \ &\leq\ \Big(\int_{r-1}^{r} \|u^m(s)\|^2\, ds + \frac{2}{\nu}\int_{r-1}^{r} |f(\theta)|^2\, d\theta\Big)\\ &\qquad \times \exp\Big(2C^{(\nu)}\int_{r-1}^{r} |u^m(\theta)|^2\|u^m(\theta)\|^2\, d\theta\Big)\end{aligned}$$

for all $\tau \leq r-1$.

From this, jointly with (2.23) and the first estimate in (2.16) for u^m, we deduce that for any $m \geq 1$,

$$\|u^m(r;\tau,u^\tau)\|^2 \leq \rho_2(t) \quad \forall\, r \in [t-2,t]. \tag{2.27}$$

So, taking inferior limit when m goes to infinity in (2.27), and using the well-known facts that u^m converges to $u(\cdot;\tau,u^\tau)$ weakly-star in $L^\infty(t-2,t;V)$ and $u(\cdot;\tau,u^\tau) \in C([t-2,t];V)$, we obtain the second estimate in (2.16).

On other hand, from (2.26) we also have

$$\begin{aligned}\nu\int_{r-1}^{r} |Au^m(\theta)|^2\, d\theta \ &\leq\ \|u^m(r-1)\|^2 + \frac{2}{\nu}\int_{r-1}^{r} |f(\theta)|^2\, d\theta\\ &\qquad +2C^{(\nu)}\int_{r-1}^{r} |u^m(\theta)|^2\|u^m(\theta)\|^4\, d\theta \quad \forall\, \tau \leq r-1.\end{aligned}$$

Therefore,

$$\nu\int_{r-1}^{r} |Au^m(\theta;\tau,u^\tau)|^2\, d\theta \leq \rho_3(t) \quad \forall\, r \in [t-1,t]. \tag{2.28}$$

Thus, taking inferior limit when m goes to infinity in (2.28), and by the well-known fact that u^m converges to $u(\cdot;\tau,u^\tau)$ weakly in $L^2(t-2,t;D(A))$, we obtain the third inequality in (2.16).

Finally, multiplying in (2.21) by $\alpha'_{m,j}(t)$, and summing again from $j=1$ to m, we obtain

$$|(u^m)'(\theta)|^2 + \frac{\nu}{2}\frac{d}{d\theta}\|u^m(\theta)\|^2 + b(u^m(\theta),u^m(\theta),(u^m)'(\theta)) = (f(\theta),(u^m)'(\theta)), \quad \text{a.e. } \theta > \tau.$$

Observing that by Young's inequality and (2.6),

$$\begin{aligned} |(f(\theta),(u^m)'(\theta))| &\leq \frac{1}{4}|(u^m)'(\theta)|^2 + |f(\theta)|^2, \\ |b(u^m(\theta),u^m(\theta),(u^m)'(\theta))| &\leq C_1|Au^m(\theta)|\|u^m(\theta)\||(u^m)'(\theta)| \\ &\leq \frac{1}{4}|(u^m)'(\theta)|^2 + C_1^2|Au^m(\theta)|^2\|u^m(\theta)\|^2, \end{aligned}$$

we obtain that

$$|(u^m)'(\theta)|^2 + \nu\frac{d}{d\theta}\|u^m(\theta)\|^2 \leq 2|f(\theta)|^2 + 2C_1^2|Au^m(\theta)|^2\|u^m(\theta)\|^2, \quad \text{a.e. } \theta > \tau. \tag{2.29}$$

Integrating above, we conclude

$$\begin{aligned} \int_{r-1}^{r} |(u^m)'(\theta)|^2\,d\theta &\leq \nu\|u^m(r-1)\|^2 + 2\int_{r-1}^{r}|f(\theta)|^2\,d\theta \\ &\quad +2C_1^2\int_{r-1}^{r}|Au^m(\theta)|^2\|u^m(\theta)\|^2\,d\theta \quad \forall\,\tau \leq r-1. \end{aligned}$$

From the first estimate in (2.16) for u^m, (2.27) and (2.28), we deduce that

$$\int_{r-1}^{r} |(u^m)'(\theta;\tau,u^\tau)|^2\,d\theta \leq \rho_4(t) \quad \forall\,r \in [t-1,t]. \tag{2.30}$$

Thus, taking inferior limit when m goes to infinity in (2.30), and using the also well-known fact that $(u^m)'$ converges to $u'(\cdot;\tau,u^\tau)$ weakly in $L^2(t-2,t;H)$, we obtain the fourth inequality in (2.16). ■

Now, we introduce the following universe in $\mathcal{P}(V)$.

Definition 2.11. *For any $\mu > 0$, we will denote by $\mathcal{D}_\mu^V(H)$ the class of all families $\widehat{D}_V$ of elements of $\mathcal{P}(V)$ of the form $\widehat{D}_V = \{D(t) \cap V : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu(H)$.*

Again, accordingly to the notation in the previous chapter, we denote by $\mathcal{D}_F(V)$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of nonempty fixed bounded subsets of V.

Observe that for any $\mu > 0$, $\mathcal{D}_F(V) \subset \mathcal{D}_\mu^V(H) \subset \mathcal{D}_\mu(H)$. It must also be pointed out that the universe $\mathcal{D}_\mu^V(H)$ is inclusion-closed.

We establish now some results on absorbing properties of $U : \mathbb{R}_d^2 \times V \to V$. The first one is a consequence of Corollary 2.7.

Corollary 2.12. *Under the assumptions of Lemma 2.10, the family $\widehat{D}_{0,\mu,V} = \{D_{0,\mu,V}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V)$, where*

$$D_{0,\mu,V}(t) = \overline{B}_H(0,R_H(t)) \cap V,$$

belongs to $\mathcal{D}_\mu^V(H)$ *and satisfies that for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\mu(H)$*, there exists a* $\tau(\widehat{D}, t) < t$ *such that*

$$U(t,\tau)D(\tau) \subset D_{0,\mu,V}(t) \quad \forall \tau \le \tau(\widehat{D}, t).$$

In particular, the family $\widehat{D}_{0,\mu,V}$ *is pullback* $\mathcal{D}_\mu^V(H)$*-absorbing for the process* $U : \mathbb{R}_d^2 \times V \to V$*.*

Lemma 2.13. *Suppose that* $f \in L^2_{\rm loc}(\mathbb{R}; H)$ *satisfies the condition (2.15). Then, for any* $\widehat{D} \in \mathcal{D}_\mu(H)$ *and any* $r > 0$*, the family* $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$*, where* $D^{(r)}(\tau) = U(\tau + r, \tau)D(\tau)$*, for any* $\tau \in \mathbb{R}$*, belongs to* $\mathcal{D}_\mu^V(H)$*.*

Proof. From (2.12), we deduce

$$\sup_{w \in D^{(r)}(\tau)} (e^{\mu\tau}|w|^2) \le e^{-\mu r} \sup_{v \in D(\tau)} (e^{\mu\tau}|v|^2) + \frac{e^{-\mu r}}{2\nu\lambda_1 - \mu} \int_\tau^{\tau+r} e^{\mu s}|f(s)|^2\, ds,$$

which jointly with the regularity property (a) in Theorem 2.3 and (2.15), conclude the proof. ■

Now, we apply an energy method with continuous functions (e.g. cf. [46, 68, 71]) in order to obtain the pullback asymptotic compactness in V for the universe $\mathcal{D}_\mu^V(H)$.

Lemma 2.14. *Suppose that* $f \in L^2_{\rm loc}(\mathbb{R}; H)$ *satisfies the condition (2.15). Then, the process* $U : \mathbb{R}_d^2 \times V \to V$ *is pullback* $\mathcal{D}_\mu^V(H)$*-asymptotically compact.*

Proof. Let us fix $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^V(H)$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{u^{\tau_n}\} \subset V$ with $u^{\tau_n} \in D(\tau_n)$ for all n. We must prove that the sequence $\{u(t; \tau_n, u^{\tau_n})\}$ is relatively compact in V. For short, let us denote by $u^n(s) = u(s; \tau_n, u^{\tau_n})$.

From Lemma 2.10 we know that there exists a $\tau_1(\widehat{D}, t) < t - 3$, such that the subsequence $\{u^n : \tau_n \le \tau_1(\widehat{D}, t)\} \subset \{u^n\}$ is bounded in $L^\infty(t-2, t; V) \cap L^2(t-2, t; D(A))$ with $\{(u^n)'\}$ also bounded in $L^2(t-2, t; H)$. Then, using in particular the Aubin–Lions compactness lemma (see [3], [61] or [84]) there exists an element $u \in L^\infty(t-2, t; V) \cap L^2(t-2, t; D(A))$ with $u' \in L^2(t-2, t; H)$, such that for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-2, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2, t; H), \\ u^n \to u & \text{strongly in } L^2(t-2, t; V), \\ u^n(s) \to u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-2, t). \end{cases} \tag{2.31}$$

Observe that $u \in C([t-2, t]; V)$, and due to (2.31), u satisfies the equation (2.7) in the interval $(t-2, t)$.

From (2.31) we also deduce that $\{u^n\}$ is equi-continuous on $[t-2, t]$ with values in H. Thus, taking into account that the sequence $\{u^n\}$ is bounded in $C([t-2, t]; V)$, by the compactness of the injection of V into H, and the Ascoli–Arzelà theorem, we obtain that

$$u^n \to u \quad \text{strongly in } C([t-2, t]; H). \tag{2.32}$$

Again by the boundedness of $\{u^n\}$ in $C([t-2,t];V)$, we have that for all sequence $\{s_n\} \subset [t-2,t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{2.33}$$

where we have used (2.32) to identify the weak limit.

Actually, we claim that

$$u^n \to u \quad \text{strongly in } C([t-1,t];V), \tag{2.34}$$

which in particular will imply the relative compactness.

Indeed, if (2.34) does not hold, there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1,t]$, without loss of generality converging to some $t_* \in [t-1,t]$, and such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall\, n \geq 1. \tag{2.35}$$

From (2.33) we already have

$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{2.36}$$

On the other hand, using the energy equality (2.9) for u and all u^n, and reasoning as for the obtention of (2.26), it holds that for all $t-2 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} & \|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2\, dr \\ \leq \ & \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4\, dr + \frac{2}{\nu}\int_{s_1}^{s_2} |f(r)|^2\, dr, \end{aligned}$$

and

$$\begin{aligned} & \|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2\, dr \\ \leq \ & \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4\, dr + \frac{2}{\nu}\int_{s_1}^{s_2} |f(r)|^2\, dr. \end{aligned}$$

In particular we can define the functions

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u^n(r)|^2 \|u^n(r)\|^4\, dr - \frac{2}{\nu}\int_{t-2}^{s} |f(r)|^2\, dr, \\ J(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u(r)|^2 \|u(r)\|^4\, dr - \frac{2}{\nu}\int_{t-2}^{s} |f(r)|^2\, dr. \end{aligned}$$

It is clear from the regularity of u and all u^n that these functions are continuous on $[t-2,t]$, and from the corresponding inequalities above, both J_n and J are non-increasing.

Observe now that by the last convergence in (2.31), and (2.32), $\|u^n(s)\| \to \|u(s)\|$ and $|u^n(s)|^2\|u^n(s)\|^4 \to |u(s)|^2\|u(s)\|^4$, a.e. $s \in (t-2,t)$. Moreover, as the sequence $\{u^n\}$ is bounded in $L^\infty(t-2,t;V) \subset L^\infty(t-2,t;H)$, it holds that the sequence $\{|u^n(s)|^2\|u^n(s)\|^4\}$

is bounded in $L^\infty(t-2,t)$. Therefore, from the Lebesgue's dominated convergence theorem we deduce that

$$\int_{t-2}^{s} |u^n(r)|^2\|u^n(r)\|^4\,dr \to \int_{t-2}^{s} |u(r)|^2\|u(r)\|^4\,dr \quad \forall\, s\in[t-2,t].$$

Thus,

$$J_n(s)\to J(s) \quad \text{a.e. } s\in(t-2,t).$$

Hence, there exists a sequence $\{\tilde t_k\}\subset(t-2,t_*)$ such that $\tilde t_k\to t_*$ when $k\to\infty$, and

$$\lim_{n\to\infty} J_n(\tilde t_k)=J(\tilde t_k) \quad \forall\, k.$$

Fix an arbitrary value $\delta>0$. From the continuity of J, there exists k_δ such that

$$|J(\tilde t_k)-J(t_*)|<\delta/2 \quad \forall\, k\ge k_\delta.$$

Now consider $n(k_\delta)$ such that for all $n\ge n(k_\delta)$ it holds

$$t_n\ge \tilde t_{k_\delta} \quad \text{and} \quad |J_n(\tilde t_{k_\delta})-J(\tilde t_{k_\delta})|<\delta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n\ge n(k_\delta)$

$$\begin{aligned} J_n(t_n)-J(t_*) &\le J_n(\tilde t_{k_\delta})-J(t_*)\\ &\le |J_n(\tilde t_{k_\delta})-J(t_*)|\\ &\le |J_n(\tilde t_{k_\delta})-J(\tilde t_{k_\delta})|+|J(\tilde t_{k_\delta})-J(t_*)|<\delta. \end{aligned}$$

This yields that

$$\limsup_{n\to\infty} J_n(t_n)\le J(t_*),$$

and therefore, by (2.31),

$$\limsup_{n\to\infty}\|u^n(t_n)\|\le\|u(t_*)\|,$$

which joined to (2.36) and (2.33) implies that $u^n(t_n)\to u(t_*)$ strongly in V, in contradiction with (2.35). Thus, (2.34) holds and the relatively compactness of $\{u(t;\tau_n,u^{\tau_n})\}$ in V is proved. ■

As a consequence of the previous results, we obtain the existence of minimal pullback attractors for the process U on V.

Theorem 2.15. *Suppose that $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$ satisfies the condition (2.15). Then, there exist the minimal pullback $\mathcal{D}_F(V)$-attractor $\mathcal{A}_{\mathcal{D}_F(V)}$ and the minimal pullback $\mathcal{D}^V_\mu(H)$-attractor $\mathcal{A}_{\mathcal{D}^V_\mu(H)}$ for the process $U:\mathbb{R}^2_d\times V\to V$ defined by (2.11), and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V)}(t)\subset\mathcal{A}_{\mathcal{D}_F(H)}(t)\subset\mathcal{A}_{\mathcal{D}_\mu(H)}(t)=\mathcal{A}_{\mathcal{D}^V_\mu(H)}(t) \quad \forall\, t\in\mathbb{R}. \tag{2.37}$$

In particular, for any $\widehat D\in\mathcal{D}_\mu(H)$, the following pullback attraction result in V holds:

$$\lim_{\tau\to-\infty}\mathrm{dist}_V(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\mu(H)}(t))=0 \quad \forall\, t\in\mathbb{R}. \tag{2.38}$$

Finally, if moreover f satisfies

$$\sup_{s\leq 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu\theta}|f(\theta)|^2\,d\theta\right)<\infty, \tag{2.39}$$

then

$$\mathcal{A}_{\mathcal{D}_F(V)}(t)=\mathcal{A}_{\mathcal{D}_F(H)}(t)=\mathcal{A}_{\mathcal{D}_\mu(H)}(t)=\mathcal{A}_{\mathcal{D}_\mu^V(H)}(t)\quad \forall\, t\in\mathbb{R}. \tag{2.40}$$

Proof. Since the process U is continuous on V by Proposition 2.4, there exists a pullback absorbing family $\widehat{D}_{0,\mu,V}\in\mathcal{D}_\mu^V(H)$ by Corollary 2.12, and the process U is pullback $\mathcal{D}_\mu^V(H)$-asymptotically compact by Lemma 2.14, the existence of $\mathcal{A}_{\mathcal{D}_\mu^V(H)}$ and $\mathcal{A}_{\mathcal{D}_F(V)}$ follows from Theorem 1.11 and Corollary 1.13 respectively.

In (2.37), the inclusions follow from Corollary 1.13, Theorem 1.15, and the fact that $\mathcal{D}_F(V)\subset\mathcal{D}_F(H)$. The equality is a consequence of Theorem 1.15 and Remark 1.16, by using Lemma 2.13 with $T=r=1$.

The property (2.38) is a consequence of Lemma 2.13, since for any $\widehat{D}\in\mathcal{D}_\mu(H)$ and any $\tau<t-1$,

$$\begin{aligned}\operatorname{dist}_V(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\mu(H)}(t)) &= \operatorname{dist}_V(U(t,\tau+1)(U(\tau+1,\tau)D(\tau)),\mathcal{A}_{\mathcal{D}_\mu(H)}(t))\\ &= \operatorname{dist}_V(U(t,\tau+1)D^{(1)}(\tau),\mathcal{A}_{\mathcal{D}_\mu^V(H)}(t)).\end{aligned}$$

If moreover f satisfies (2.39), then the equality $\mathcal{A}_{\mathcal{D}_F(H)}(t)=\mathcal{A}_{\mathcal{D}_\mu(H)}(t)$ is a consequence of Remark 1.14, and the equality $\mathcal{A}_{\mathcal{D}_F(V)}(t)=\mathcal{A}_{\mathcal{D}_F(H)}(t)$ holds by applying once more Theorem 1.15, and the second estimate in (2.16), since (2.39) is equivalent to

$$\sup_{s\leq 0}\int_{s-1}^{s}|f(\theta)|^2\,d\theta<\infty. \tag{2.41}$$

■

Remark 2.16. *Observe that if $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$ satisfies the condition (2.15), then it also satisfies*

$$\int_{-\infty}^{0} e^{\sigma s}|f(s)|^2\,ds<\infty\quad \forall\,\sigma\in(\mu,2\nu\lambda_1).$$

Thus, for any $\sigma\in(\mu,2\nu\lambda_1)$ there exists the corresponding minimal pullback $\mathcal{D}_\sigma(H)$-attractor $\mathcal{A}_{\mathcal{D}_\sigma(H)}$.

By Theorem 1.15, since $\mathcal{D}_\mu(H)\subset\mathcal{D}_\sigma(H)$, it is evident that, for any $t\in\mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}_\mu(H)}(t)\subset\mathcal{A}_{\mathcal{D}_\sigma(H)}(t)\quad \forall\,\sigma\in(\mu,2\nu\lambda_1).$$

Moreover, if f also satisfies (2.39), then by (2.40), for any $\sigma\in(\mu,2\nu\lambda_1)$,

$$\mathcal{A}_{\mathcal{D}_F(H)}(t)=\mathcal{A}_{\mathcal{D}_\mu(H)}(t)=\mathcal{A}_{\mathcal{D}_\sigma(H)}(t)\quad \forall\, t\in\mathbb{R}.$$

2.3 H^2-boundedness of the pullback attractors

In this section we prove that, under suitable assumptions, any family of bounded subsets of H which is invariant for the process U, is in fact bounded in $(H^2(\Omega))^2 \cap V$. In particular, we will obtain that any pullback attractor $\mathcal{A}$ for U satisfies that $\mathcal{A}(t)$ is a bounded subset of $(H^2(\Omega))^2 \cap V$, for every $t \in \mathbb{R}$ (for similar results for reaction-diffusion equations see [1], and for related results for Navier–Stokes equations see [37]).

First, we recall a result (cf. [78]) which will be used below.

Lemma 2.17. *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, \infty)$, and $u \in C([t_0, T]; Y)$.*
Then, $u(t) \in X$ and $\|u(t)\|_X \leq \liminf_{n\to\infty} \|u_n\|_{L^\infty(t_0,T;X)}$, for all $t \in [t_0, T]$.

Let us consider again the Galerkin approximations defined by (2.21), already used in Lemma 2.10. For short, denote by $u^m(\cdot) = u^m(\cdot; \tau, u^\tau)$ the Galerkin approximation of the solution $u(\cdot; \tau, u^\tau)$ to problem (2.1).

We first prove the following result.

Proposition 2.18. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > \tau + \varepsilon$, the following three properties are satisfied:*

i) The set $\{u^m(r; \tau, u^\tau) : r \in [\tau + \varepsilon, t],\ u^\tau \in B,\ m \geq 1\}$ is a bounded subset of V.

ii) The set of functions $\{u^m(\cdot; \tau, u^\tau) : u^\tau \in B,\ m \geq 1\}$ is a bounded subset of $L^2(\tau + \varepsilon, t; D(A))$.

iii) The set of time derivative functions $\{(u^m)'(\cdot; \tau, u^\tau) : u^\tau \in B,\ m \geq 1\}$ is a bounded subset of $L^2(\tau + \varepsilon, t; H)$.

Proof. Let us fix a bounded set $B \subset H$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u^\tau \in B$.
Integrating in (2.22) between τ and r, we obtain

$$|u^m(r)|^2 + \nu \int_\tau^r \|u^m(\theta)\|^2\, d\theta \leq |u^\tau|^2 + \frac{1}{\nu\lambda_1} \int_\tau^t |f(\theta)|^2\, d\theta \quad \forall\, r \in [\tau, t],\ m \geq 1. \tag{2.42}$$

On the other, from (2.26) in particular we deduce

$$\|u^m(r)\|^2 \leq \|u^m(s)\|^2 + \frac{2}{\nu} \int_\tau^t |f(\theta)|^2\, d\theta + 2C^{(\nu)} \int_s^r |u^m(\theta)|^2 \|u^m(\theta)\|^4\, d\theta$$

for all $\tau \leq s \leq r \leq t$, and therefore, by Gronwall's lemma,

$$\|u^m(r)\|^2 \leq \left(\|u^m(s)\|^2 + \frac{2}{\nu} \int_\tau^t |f(\theta)|^2\, d\theta \right) \exp\left(2C^{(\nu)} \int_\tau^t |u^m(\theta)|^2 \|u^m(\theta)\|^2\, d\theta \right)$$

for all $\tau \leq s \leq r \leq t$.

Integrating this last inequality for s between τ and r, we obtain

$$\begin{aligned}(r-\tau)\|u^m(r)\|^2 \ \le\ & \left(\int_\tau^t \|u^m(s)\|^2\,ds + \frac{2(t-\tau)}{\nu}\int_\tau^t |f(\theta)|^2\,d\theta\right)\\ & \times \exp\left(2C^{(\nu)}\int_\tau^t |u^m(\theta)|^2\|u^m(\theta)\|^2\,d\theta\right)\end{aligned}$$

for all $\tau \le r \le t$, and in particular,

$$\begin{aligned}\|u^m(r)\|^2 \ \le\ & \frac{1}{\varepsilon}\left(\int_\tau^t \|u^m(s)\|^2\,ds + \frac{2(t-\tau)}{\nu}\int_\tau^t |f(\theta)|^2\,d\theta\right)\\ & \times \exp\left(2C^{(\nu)}\int_\tau^t |u^m(\theta)|^2\|u^m(\theta)\|^2\,d\theta\right)\end{aligned} \tag{2.43}$$

for all $\tau+\varepsilon \le r \le t$, for any $m \ge 1$.

From (2.42) and (2.43), the assertion in i) holds. Moreover, by (2.26),

$$\begin{aligned}\nu\int_{\tau+\varepsilon}^t |Au^m(\theta)|^2\,d\theta \ \le\ & \|u^m(\tau+\varepsilon)\|^2 + \frac{2}{\nu}\int_\tau^t |f(\theta)|^2\,d\theta\\ & + 2C^{(\nu)}\int_{\tau+\varepsilon}^t |u^m(\theta)|^2\|u^m(\theta)\|^4\,d\theta,\end{aligned}$$

and therefore, by i), the assertion in ii) holds.

Finally, integrating in (2.29), we deduce that

$$\begin{aligned}\int_{\tau+\varepsilon}^t |(u^m)'(\theta)|^2\,d\theta \ \le\ & \nu\|u^m(\tau+\varepsilon)\|^2 + 2\int_\tau^t |f(\theta)|^2\,d\theta\\ & + 2C_1^2 \sup_{\theta\in[\tau+\varepsilon,t]} \|u^m(\theta)\|^2 \int_{\tau+\varepsilon}^t |Au^m(\theta)|^2\,d\theta,\end{aligned}$$

and therefore iii) follows from i) and ii). ■

Corollary 2.19. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of V.*

Proof. This is a straightforward consequence of Lemma 2.17, assertion i) in Proposition 2.18, and the well-known fact (e.g. cf. [61, 78, 87, 86]) that for all $u^\tau \in B$ the Galerkin approximations $u^m(\cdot;\tau,u^\tau)$ converge weakly to $u(\cdot;\tau,u^\tau)$ in $L^2(\tau,t;V)$, and $u(\cdot;\tau,u^\tau) \in C([\tau,t];H)$. ■

Assuming additional regularity for the time derivative of f, we can improve the above results.

Proposition 2.20. *Assume that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the following two properties are satisfied:*

iv) The set of time derivatives $\{(u^m)'(r;\tau,u^\tau) : r \in [\tau+\varepsilon,t],\ u^\tau \in B,\ m \geq 1\}$ *is a bounded subset of* H.

v) The set $\{u^m(r;\tau,u^\tau) : r \in [\tau+\varepsilon,t],\ u^\tau \in B,\ m \geq 1\}$ *is a bounded subset of* $D(A)$.

Proof. Let us fix a bounded set $B \subset H$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau+\varepsilon$, and $u^\tau \in B$.

As we are assuming that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R};H)$, we can differentiate with respect to time in (2.21), and then, multiplying by $\alpha'_{m,j}(t)$, and summing from $j=1$ to m, we obtain

$$\frac{1}{2}\frac{d}{d\theta}|(u^m)'(\theta)|^2 + \nu\|(u^m)'(\theta)\|^2 + b((u^m)'(\theta), u^m(\theta), (u^m)'(\theta)) = (f'(\theta), (u^m)'(\theta)),$$

a.e. $\theta > \tau$.

From this inequality, taking into account that

$$|(f'(\theta), (u^m)'(\theta))| \leq \frac{\nu}{2}\|(u^m)'(\theta)\|^2 + \frac{1}{2\nu\lambda_1}|f'(\theta)|^2,$$

and that by (2.4)

$$\begin{aligned} |b((u^m)'(\theta), u^m(\theta), (u^m)'(\theta))| &\leq 2^{-1/2}|(u^m)'(\theta)|\|(u^m)'(\theta)\|\|u^m(\theta)\| \\ &\leq \frac{\nu}{2}\|(u^m)'(\theta)\|^2 + \frac{1}{4\nu}|(u^m)'(\theta)|^2\|u^m(\theta)\|^2, \end{aligned}$$

we deduce

$$\frac{d}{d\theta}|(u^m)'(\theta)|^2 \leq \frac{1}{\nu\lambda_1}|f'(\theta)|^2 + \frac{1}{2\nu}|(u^m)'(\theta)|^2\|u^m(\theta)\|^2, \quad \text{a.e. } \theta > \tau. \tag{2.44}$$

Integrating in the last inequality,

$$|(u^m)'(r)|^2 \leq |(u^m)'(s)|^2 + \frac{1}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta + \frac{1}{2\nu}\int_s^r |(u^m)'(\theta)|^2\|u^m(\theta)\|^2\,d\theta$$

for all $\tau \leq s \leq r \leq t$.

Thus, again by Gronwall's lemma,

$$|(u^m)'(r)|^2 \leq \left(|(u^m)'(s)|^2 + \frac{1}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\right)\exp\left(\frac{1}{2\nu}\int_{\tau+\varepsilon/2}^t \|u^m(\theta)\|^2\,d\theta\right)$$

for all $\tau+\varepsilon/2 \leq s \leq r \leq t$.

Now, integrating this inequality with respect to s between $\tau+\varepsilon/2$ and r, we obtain

$$\begin{aligned} (r-\tau-\varepsilon/2)|(u^m)'(r)|^2 &\leq \left(\int_{\tau+\varepsilon/2}^t |(u^m)'(s)|^2\,ds + \frac{t-\tau}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\right) \\ &\quad \times \exp\left(\frac{1}{2\nu}\int_{\tau+\varepsilon/2}^t \|u^m(\theta)\|^2\,d\theta\right) \end{aligned}$$

for all $\tau + \varepsilon/2 \le r \le t$, and any $m \ge 1$. Thus, in particular,

$$|(u^m)'(r)|^2 \le \frac{2}{\varepsilon}\Big(\int_{\tau+\varepsilon/2}^t |(u^m)'(s)|^2\,ds + \frac{t-\tau}{\nu\lambda_1}\int_\tau^t |f'(\theta)|^2\,d\theta\Big) \times \exp\left(\frac{1}{2\nu}\int_{\tau+\varepsilon/2}^t \|u^m(\theta)\|^2\,d\theta\right)$$

for all $\tau + \varepsilon \le r \le t$, and any $m \ge 1$.

From this inequality and properties i) and iii) in Proposition 2.18, we obtain iv).

On the other hand, multiplying again in (2.21) by $\lambda_j\alpha_{m,j}(t)$, and summing once more from $j = 1$ to m, we obtain

$$((u^m)'(r), Au^m(r)) + \nu|Au^m(r)|^2 + b(u^m(r), u^m(r), Au^m(r)) = (f(r), Au^m(r)) \tag{2.45}$$

for all $r > \tau$. But

$$|((u^m)'(r), Au^m(r))| \le \frac{2}{\nu}|(u^m)'(r)|^2 + \frac{\nu}{8}|Au^m(r)|^2,$$

and

$$|(f(r), Au^m(r))| \le \frac{2}{\nu}|f(r)|^2 + \frac{\nu}{8}|Au^m(r)|^2.$$

Therefore, taking into account (2.25), we deduce from (2.45) that

$$\frac{\nu}{2}|Au^m(r)|^2 \le \frac{2}{\nu}\big(|(u^m)'(r)|^2 + |f(r)|^2\big) + C^{(\nu)}|u^m(r)|^2\|u^m(r)\|^4 \quad \forall\, r \ge \tau. \tag{2.46}$$

Finally, since in particular $f \in C(\mathbb{R}; H)$, from i) in Proposition 2.18, iv) and inequality (2.46), we deduce v). ■

As a direct consequence of the above, we can now establish our main results.

Theorem 2.21. *Assume that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; H)$. Then, for any bounded set $B \subset H$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of $D(A) = (H^2(\Omega))^2 \cap V$.*

Proof. This follows from Lemma 2.17, Proposition 2.20, and the well-known facts that $u^m(\cdot;\tau,u^\tau)$ converges weakly to $u(\cdot;\tau,u^\tau)$ in $L^2(\tau,t;V)$, and $u(\cdot;\tau,u^\tau)$ belongs to $C([\tau+\varepsilon,t];V)$. ■

Theorem 2.22. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, and $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a family of bounded subsets of H, such that $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for any $\tau \le t$. Then:*

i) For any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of V.

ii) If moreover $f' \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of $(H^2(\Omega))^2 \cap V$.

Proof. It is enough to observe that if $\tau < T_1 - 1$ is fixed, then

$$\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t) \subset \bigcup_{t\in[\tau+1,T_2]} U(t,\tau)\mathcal{A}(\tau).$$

Now, apply Corollary 2.19 and Theorem 2.21. ■

2.4 Tempered behaviour of the pullback attractors

The tempered behaviour in H of the pullback attractor $\mathcal{A}_{\mathcal{D}_\mu(H)}$ is given by Theorem 2.9. Indeed, under the assumptions of that result, $\mathcal{A}_{\mathcal{D}_\mu(H)} \in \mathcal{D}_\mu(H)$, i.e. one has

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_\mu(H)}(t)}|v|^2\right)=0.$$

In this section we obtain two results about the tempered behaviour of $\mathcal{A}_{\mathcal{D}_\mu(H)}$, in V and $(H^2(\Omega))^2$, when time goes to $-\infty$. In fact, we will obtain the tempered behaviour for any invariant family belonging to $\mathcal{D}_\mu(H)$.

Proposition 2.23. *Suppose that $f\in L^2_{\rm loc}(\mathbb{R};H)$ satisfies the assumption (2.39) in Theorem 2.15, and let $\widehat{D}\in\mathcal{D}_\mu(H)$ be invariant with respect to the process U defined by (2.11) (i.e. such that $D(t)=U(t,\tau)D(\tau)$ for all $\tau\le t$). Then,*

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)}\|v\|^2\right)=0.$$

Proof. The result is a consequence of the invariance of $\widehat{D}$, the second estimate in (2.16) in Lemma 2.10, and the tempered character of the expression (2.18), since for $f\in L^2_{\rm loc}(\mathbb{R};H)$, the condition (2.39) is equivalent to (2.41). ■

Assuming now that $f'\in L^2_{\rm loc}(\mathbb{R};H)$, we can obtain the tempered behaviour in $(H^2(\Omega))^2$ for any invariant family belonging to $\mathcal{D}_\mu(H)$. We first prove the following result, which completes the estimates obtained in Lemma 2.10.

Proposition 2.24. *If $f\in W^{1,2}_{\rm loc}(\mathbb{R};H)$ and satisfies (2.15), then for each $t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}_\mu(H)$, there exists $\tau_1(\widehat{D},t)<t-3$ such that*

$$|AU(r,\tau)u^\tau|^2\le\rho_6(t)\quad\forall\, r\in[t-1,t],\ \tau\le\tau_1(\widehat{D},t),\ u^\tau\in D(\tau),$$

where

$$\rho_6(t)=4\nu^{-2}\Big(\rho_5(t)+\max_{r\in[t-1,t]}|f(r)|^2\Big)+2\nu^{-1}C^{(\nu)}\rho_1(t)\rho_2^2(t), \tag{2.47}$$

with $\rho_5(t)$ defined by

$$\rho_5(t)=\left(\rho_4(t)+\nu^{-1}\lambda_1^{-1}\int_{t-2}^t|f'(\theta)|^2\,d\theta\right)\exp\left(\frac{1}{2\nu}\rho_2(t)\right), \tag{2.48}$$

and where the $\rho_i(t)$, $i=1,\,2,\,4$, are given in Lemma 2.10.

Proof. We consider again the Galerkin approximations used in the proofs of Lemma 2.10, and Propositions 2.18 and 2.20.

Integrating in (2.44), we obtain

$$|(u^m)'(r)|^2\le|(u^m)'(s)|^2+\frac{1}{\nu\lambda_1}\int_{r-1}^r|f'(\theta)|^2\,d\theta+\frac{1}{2\nu}\int_s^r|(u^m)'(\theta)|^2\|u^m(\theta)\|^2\,d\theta$$

for all $\tau \le r-1 \le s \le r$.

Thus, by Gronwall's inequality,

$$|(u^m)'(r)|^2 \le \left(|(u^m)'(s)|^2 + \frac{1}{\nu\lambda_1}\int_{r-1}^r |f'(\theta)|^2\, d\theta\right) \exp\left(\frac{1}{2\nu}\int_{r-1}^r \|u^m(\theta)\|^2\, d\theta\right)$$

for all $\tau \le r-1 \le s \le r$.

Now, integrating this inequality with respect to s between $r-1$ and r, we obtain

$$\begin{aligned} |(u^m)'(r)|^2 \quad &\le \quad \left(\int_{r-1}^r |(u^m)'(s)|^2\, ds + \frac{1}{\nu\lambda_1}\int_{r-1}^r |f'(\theta)|^2\, d\theta\right) \\ &\qquad \times \exp\left(\frac{1}{2\nu}\int_{r-1}^r \|u^m(\theta)\|^2\, d\theta\right) \end{aligned}$$

for all $\tau \le r-1$ and any $m \ge 1$, and therefore, by (2.27) and (2.30) we deduce that for any $m \ge 1$,

$$|(u^m)'(r;\tau,u^\tau)|^2 \le \rho_5(t) \quad \forall\, r \in [t-1,t],\ \tau \le \tau_1(\widehat{D},t),\ u^\tau \in D(\tau), \tag{2.49}$$

where $\rho_5(t)$ is given by (2.48).

Finally, since in particular $f \in C(\mathbb{R};H)$, from inequality (2.46), the first estimate in (2.16) for u^m, (2.27), and (2.49), we deduce that for any $m \ge 1$,

$$|Au^m(r;\tau,u^\tau)|^2 \le \rho_6(t) \quad \forall\, r \in [t-1,t],\ \tau \le \tau_1(\widehat{D},t),\ u^\tau \in D(\tau), \tag{2.50}$$

where $\rho_6(t)$ is given by (2.47).

The result now is a consequence of Lemma 2.17 and (2.50), taking into account the well-known facts that $u^m(\cdot;\tau,u^\tau)$ converges weakly to $u(\cdot;\tau,u^\tau)$ in $L^2(t-1,t;V)$, and $u(\cdot;\tau,u^\tau) \in C([t-1,t];V)$. ■

Now, we may conclude a result about the tempered behaviour in $(H^2(\Omega))^2$.

Proposition 2.25. *Suppose that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R};H)$ satisfies the assumption (2.39) in Theorem 2.15, and moreover*

$$\lim_{t\to-\infty}\left(e^{\mu t}\int_{t-1}^t |f'(\theta)|^2\, d\theta\right) = 0, \tag{2.51}$$

and

$$\lim_{t\to-\infty}\left(e^{\mu t}|f(t)|^2\right) = 0. \tag{2.52}$$

Then, for every family $\widehat{D} \in \mathcal{D}_\mu(H)$ invariant with respect to the process U defined by (2.11), one has

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)} \|v\|^2_{(H^2(\Omega))^2}\right) = 0.$$

Proof. Observe that

$$|f(r)| \leq |f(t-1)| + \left(\int_{t-1}^{t} |f'(\theta)|^2 \, d\theta \right)^{1/2} \quad \forall \, r \in [t-1, t].$$

Thus, taking into account (2.51) and (2.52), the result follows from the invariance of $\widehat{D}$, Proposition 2.24, (2.17), (2.18), (2.19), and again the fact that, as we observed in the proofs of Theorem 2.15 and Proposition 2.23, the condition (2.39) is equivalent to (2.41).
■

Chapter 3

Pullback Flattening Property for Non-Autonomous 2D Navier–Stokes Equations

Our goal in this chapter is to obtain the flattening property for the non-autonomous 2D Navier–Stokes model stated in the previous chapter, in different norms, namely in H and V, when the forcing has again the minimal regularity for generating weak and strong solutions, respectively. As a consequence of such flattening property, we will also obtain the asymptotic compactness of the associated evolution process, and therefore the existence of minimal pullback attractors in these two different spaces.

While in the case of H a direct proof of asymptotic compactness is no harder than a proof of the flattening property, in V a proof of asymptotic compactness via the flattening property is significantly shorter than the one given in Lemma 2.14, which was based on more involved inequalities. This is due to the fact that there are stronger estimates available for the nonlinear term in V than in H:

$$|b(u,u,q)| \leq c|u|\|u\|\|q\|, \qquad |b(u,u,Aq)| \leq c\|u\||u|^{1/2}\|\nabla u\|^{1/2}\|\nabla q\|,$$

(see properties (2.4) and (2.5) in Chapter 2).

The analysis in the space H for $u^\tau \in H$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ is carried out in Section 3.1. Furthermore, by using the semigroup approach of Fujita and Kato [29] and ideas from the ϵ-regularity theory developed by Arrieta and Carvalho [2], we obtain the existence of a compact pullback absorbing family in H if we strengthen the regularity of f to $f \in L^p_{\mathrm{loc}}(\mathbb{R}; V')$ for some $p > 2$.

We then establish additional regularity results in Section 3.2, under the assumption that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ and the integrability condition (2.10). We obtain the flattening property in the V norm, which implies the asymptotic compactness of the corresponding process in this norm. Again, in this section we are able to show the existence of a compact pullback absorbing family in V if we strengthen the regularity of f to $f \in L^p_{\mathrm{loc}}(\mathbb{R}; H)$ for some $p > 2$.

The main results in this chapter can be found in [35].

3.1 Pullback flattening property in H norm

In this section we prove that the process U on H defined by (2.11) satisfies the pullback flattening property. As already pointed out, the proofs of the flattening property and the asymptotic compactness of this process in this space are in fact very similar (indeed, we will obtain both from Lemma 3.2, the asymptotic compactness following almost immediately).

First, we establish several estimates for the process U in finite intervals of time when the initial time is sufficiently shifted in a pullback sense.

Lemma 3.1. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies (2.13). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $\tau_1(\widehat{D}, t) < t-2$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u^\tau \in D(\tau)$,*

$$|u(r;\tau,u^\tau)|^2 \leq \rho_1^2(t) \quad \forall\, r \in [t-2,t], \tag{3.1}$$

$$\nu \int_{r-1}^{r} \|u(\theta;\tau,u^\tau)\|^2 \, d\theta \leq \rho_2^2(t) \quad \forall\, r \in [t-1,t], \tag{3.2}$$

$$\int_{r-1}^{r} \|u'(\theta;\tau,u^\tau)\|_*^2 \, d\theta \leq \rho_3^2(t) \quad \forall\, r \in [t-1,t], \tag{3.3}$$

where

$$\begin{aligned}
\rho_1^2(t) &= 1 + e^{-\mu(t-2)}(2\nu - \mu\lambda_1^{-1})^{-1} \int_{-\infty}^{t} e^{\mu\theta} \|f(\theta)\|_*^2 \, d\theta,\\
\rho_2^2(t) &= \rho_1^2(t) + \nu^{-1} \int_{t-2}^{t} \|f(\theta)\|_*^2 \, d\theta,\\
\rho_3^2(t) &= 3\nu\rho_2^2(t) + \frac{3}{2}\rho_1^2(t)\frac{\rho_2^2(t)}{\nu} + 3\int_{t-2}^{t} \|f(\theta)\|_*^2 \, d\theta.
\end{aligned}$$

Proof. Let $\tau_1(\widehat{D}, t) < t-2$ be such that

$$e^{-\mu(t-2)} e^{\mu\tau} |u^\tau|^2 \leq 1 \quad \forall\, \tau \leq \tau_1(\widehat{D}, t),\ u^\tau \in D(\tau).$$

Consider fixed $\tau \leq \tau_1(\widehat{D}, t)$ and $u^\tau \in D(\tau)$.

The estimate (3.1) follows directly from (2.12), using the increasing character of the exponential.

Now, observing that

$$\frac{d}{d\theta}|u(\theta)|^2 + 2\nu\|u(\theta)\|^2 \leq \nu^{-1}\|f(\theta)\|_*^2 + \nu\|u(\theta)\|^2, \quad \text{a.e. } \theta > \tau, \tag{3.4}$$

and using (3.1), we obtain (3.2).

Finally, from (2.2), (2.4), (2.7), and the fact that A is an isometric isomorphism, we have

$$\|u'(\theta)\|_* \leq \nu\|u(\theta)\| + 2^{-1/2}|u(\theta)|\|u(\theta)\| + \|f(\theta)\|_*, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\|u'(\theta)\|_*^2 \leq 3\nu^2\|u(\theta)\|^2 + \frac{3}{2}|u(\theta)|^2\|u(\theta)\|^2 + 3\|f(\theta)\|_*^2, \quad \text{a.e. } \theta > \tau,$$

whence, using (3.1) and (3.2), the estimate (3.3) follows. ■

Now, in order to prove the pullback $\widehat{D}_{0,\mu}$-flattening property for the process U on H, where the family $\widehat{D}_{0,\mu}$ was defined in Corollary 2.7, we need the following auxiliary result. The proof is similar to that in Lemma 2.14.

Lemma 3.2. *Under the assumptions of Lemma 3.1, for any $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}_\mu(H)$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ and $\{u^{\tau_n}\} \subset H$ such that $\tau_n \to -\infty$ and $u^{\tau_n} \in D(\tau_n)$ for all n, the sequence $\{u(\cdot;\tau_n, u^{\tau_n})\}$ is relatively compact in $C([t-1,t];H)$.*

Proof. Consider fixed $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu(H)$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ with $\tau_n \to -\infty$, and $\{u^{\tau_n}\}$ with $u^{\tau_n} \in D(\tau_n)$ for all n. For simplicity of notation we write $u^n(\cdot) = u(\cdot;\tau_n, u^{\tau_n})$.

From Lemma 3.1 and compactness arguments, there exist a value $\tau_1(\widehat{D}, t) < t-2$ and a function $u \in C([t-2,t];H) \cap L^2(t-2,t;V)$ with $u' \in L^2(t-2,t;V')$, such that for a subsequence of $\{u^n : \tau_n \leq \tau_1(\widehat{D}, t)\} \subset \{u^n\}$, which we relabel the same, it holds that $u^n \overset{*}{\rightharpoonup} u$ weakly-star in $L^\infty(t-2,t;H)$, $u^n \rightharpoonup u$ weakly in $L^2(t-2,t;V)$, and $(u^n)' \rightharpoonup u'$ weakly in $L^2(t-2,t;V')$. Therefore, again up to a subsequence (relabelled the same), $u^n \to u$ strongly in $L^2(t-2,t;H)$, and $u^n(s) \to u(s)$ strongly in H a.e. $s \in (t-2,t)$.

From these convergences, the function u satisfies (2.7) in the interval $(t-2,t)$.

By the Ascoli–Arzelà Theorem, we deduce that $u^n \to u$ in $C([t-2,t];V')$, and so, for any sequence $\{s_n\} \subset [t-2,t]$ with $s_n \to s_*$, we have

$$u^n(s_n) \rightharpoonup u(s_*) \text{ weakly in } H. \tag{3.5}$$

We claim that

$$u^n \to u \text{ in } C([t-1,t];H), \tag{3.6}$$

which in particular will imply the relative compactness. If this were not true, there would exist a subsequence $\{u^n\}$ (relabelled the same), $\epsilon > 0$, and $\{t_n\} \subset [t-1,t]$ with $t_n \to t_*$ such that

$$|u^n(t_n) - u(t_*)| \geq \epsilon \quad \forall\, n \geq 1. \tag{3.7}$$

Recall that by (3.5) we already have

$$|u(t_*)| \leq \liminf_{n\to\infty} |u^n(t_n)|. \tag{3.8}$$

On the other hand, applying the energy equality to $z = u^n$ and $z = u$, and reasoning as in (3.4), we obtain in particular that

$$|z(s_2)|^2 \leq |z(s_1)|^2 + \nu^{-1}\int_{s_1}^{s_2} \|f(\theta)\|_*^2\, d\theta \quad \forall\, t-2 \leq s_1 \leq s_2 \leq t.$$

We may now define the functions

$$\begin{aligned} J_n(s) &= |u^n(s)|^2 - \nu^{-1}\int_{t-2}^{s} \|f(\theta)\|_*^2\, d\theta, \\ J(s) &= |u(s)|^2 - \nu^{-1}\int_{t-2}^{s} \|f(\theta)\|_*^2\, d\theta. \end{aligned}$$

Observe that J and all J_n are continuous functions on $[t-2,t]$, non-increasing, and $J_n(s) \to J(s)$ a.e. $s \in (t-2,t)$.

Take now $\{\tilde{t}_k\} \subset (t-2, t_*)$ such that $\tilde{t}_k \uparrow t_*$ and $\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k)$ for all $k \geq 1$.

Fix an arbitrary value $\eta > 0$. There exists k_η such that $|J(\tilde{t}_k) - J(t_*)| < \eta/2$ for all $k \geq k_\eta$. Now consider $n(k_\eta)$ such that for any $n \geq n(k_\eta)$ it holds that

$$t_n \geq \tilde{t}_{k_\eta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\eta}) - J(\tilde{t}_{k_\eta})| < \eta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n \geq n(k_\eta)$,

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\eta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\eta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\eta}) - J(\tilde{t}_{k_\eta})| + |J(\tilde{t}_{k_\eta}) - J(t_*)| < \eta. \end{aligned}$$

Thus, we conclude that $\limsup_{n\to\infty} |u^n(t_n)| \leq |u(t_*)|$, with joined to (3.5) and (3.8), proves that (3.7) is absurd, and so claim (3.6) is true. This finishes the proof. ■

Note that the asymptotic compactness of U is an immediate corollary of this result.

However, in order to prove the pullback flattening property directly (it is known that it is equivalent to asymptotic compactness in any uniformly convex Banach space, see [12, 35]) we need to do a little more, beginning with the next corollary of Lemma 3.2.

Corollary 3.3. *Under the assumptions of Lemma 3.1, for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $\delta = \delta(\varepsilon, t, \widehat{D}) \in (0,1)$, such that*

$$\nu^{-1}\big||u(t;\tau,u^\tau)|^2 - |u(t-s;\tau,u^\tau)|^2\big| < \varepsilon/2 \quad \forall\, s \in [0,\delta],\ \tau \leq \tau_1(\widehat{D},t),\ u^\tau \in D(\tau), \tag{3.9}$$

where $\tau_1(\widehat{D},t)$ is given in Lemma 3.1.

In particular,

$$\int_{t-\delta}^{t} \|u(\theta;\tau,u^\tau)\|^2\, d\theta < \varepsilon \quad \forall\, \tau \leq \tau_1(\widehat{D},t),\ u^\tau \in D(\tau). \tag{3.10}$$

Proof. First at all, observe that if we consider $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$, for any $\delta \in (0,1)$ and $\tau \leq t-1$, an integration in (3.4) with any $u^\tau \in D(\tau)$ yields

$$\nu \int_{t-\delta}^{t} \|u(\theta;\tau,u^\tau)\|^2\, d\theta \leq |u(t-\delta;\tau,u^\tau)|^2 - |u(t;\tau,u^\tau)|^2 + \nu^{-1}\int_{t-\delta}^{t} \|f(\theta)\|_*^2\, d\theta.$$

Therefore, since $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, (3.10) is a consequence of (3.9).

We prove now (3.9) by a contradiction argument. If (3.9) were not true, there would exist $\varepsilon > 0$, $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu(H)$, and sequences $\{\tau_n\} \subset (-\infty, t-1]$ with $\tau_n \to -\infty$, $\{s_n\}$ with $0 \leq s_n \leq 1/n$, and $\{u^{\tau_n}\}$ with $u^{\tau_n} \in D(\tau_n)$ for all n, such that

$$\nu^{-1}\big||u(t;\tau_n,u^{\tau_n})|^2 - |u(t-s_n;\tau_n,u^{\tau_n})|^2\big| \geq \varepsilon/2 \quad \forall n \geq 1,$$

which is absurd, since from (3.6) we know that for a subsequence (which we relabel the same) it holds that $u(t;\tau_n,u^{\tau_n})$ and $u(t-s_n;\tau_n,u^{\tau_n})$ converge to $u(t)$. ■

We will also use the following result, whose proof is analogous to that of [49, Lemma 12].

Lemma 3.4. *If $f \in L^2_{\rm loc}(\mathbb{R}; V')$ satisfies the condition (2.13), then, for any $t \in \mathbb{R}$,*

$$\lim_{\rho\to\infty} e^{-\rho t}\int_{-\infty}^t e^{\rho s}\|f(s)\|_*^2\,ds = 0.$$

Now, we are able to prove the pullback $\widehat{D}_{0,\mu}$-flattening property for the process U on H defined by (2.11). Actually, we will prove that U satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu(H)$.

Proposition 3.5. *Under the assumptions of Lemma 3.1, for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $m = m(\varepsilon, t, \widehat{D}) \in \mathbb{N}$, such that the projection $P_m : H \to H_m :=$span$[w_1,\dots,w_m]$ (where $\{w_j\}_{j\geq 1}$ is a Hilbert basis of H formed by ortho-normalized eigenfunctions of the Stokes operator A) satisfies the following properties:*

$$\{P_m U(t,\tau)D(\tau) : \tau \leq \tau_1(\widehat{D},t)\} \text{ is bounded in } H, \tag{3.11}$$

$$|(I-P_m)U(t,\tau)u^\tau| < \varepsilon \quad \text{for any } \tau \leq \tau_1(\widehat{D},t),\ u^\tau \in D(\tau), \tag{3.12}$$

where $\tau_1(\widehat{D},t)$ is given in Lemma 3.1.

In particular, the process U on H satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu(H)$.

Proof. Let $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu(H)$ be fixed.

Since P_m is non-expansive and taking into account (3.1), property (3.11) is automatically satisfied for any $m \in \mathbb{N}$. Therefore, we concentrate on proving (3.12).

Consider fixed $\tau \leq \tau_1(\widehat{D},t)$, $u^\tau \in D(\tau)$, and let us define $u(r) = U(r,\tau)u^\tau$ and $q_m(r) = u(r) - P_m u(r)$.

Then, using the energy equality, for each $m \geq 1$ we have

$$\frac{1}{2}\frac{d}{dr}|q_m(r)|^2 + \nu\|q_m(r)\|^2 + b(u(r),u(r),q_m(r)) = \langle f(r), q_m(r)\rangle, \quad \text{a.e. } r > \tau.$$

Observing that by (2.2), (2.4), and Young's inequality,

$$\begin{aligned} |b(u(r),u(r),q_m(r))| &\leq 2^{-1/2}|u(r)|\|u(r)\|\|q_m(r)\| \\ &\leq \frac{\nu}{4}\|q_m(r)\|^2 + \frac{1}{2\nu}|u(r)|^2\|u(r)\|^2, \end{aligned}$$

we obtain

$$\frac{d}{dr}|q_m(r)|^2 + \nu\|q_m(r)\|^2 \le \nu^{-1}|u(r)|^2\|u(r)\|^2 + 2\nu^{-1}\|f(r)\|_*^2, \quad \text{a.e. } r > \tau.$$

Consequently, as $\|q_m(r)\|^2 \ge \lambda_{m+1}|q_m(r)|^2$, where λ_{m+1} is the eigenvalue associated to the eigenfunction w_{m+1}, we deduce that

$$\frac{d}{dr}|q_m(r)|^2 + \nu\lambda_{m+1}|q_m(r)|^2 \le \nu^{-1}|u(r)|^2\|u(r)\|^2 + 2\nu^{-1}\|f(r)\|_*^2, \quad \text{a.e. } r > \tau.$$

Thus, multiplying this last inequality by $e^{\nu\lambda_{m+1}r}$, integrating in $[t-1,t]$, and again taking into account (3.1), we obtain

$$\begin{aligned} e^{\nu\lambda_{m+1}t}|q_m(t)|^2 \;\le\;& e^{\nu\lambda_{m+1}(t-1)}|q_m(t-1)|^2 + \nu^{-1}\rho_1^2(t)\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ &+2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|f(r)\|_*^2\,dr. \end{aligned}$$

Therefore, from Lemma 3.4, and since $|q_m(t-1)|^2 \le |u(t-1)|^2 \le \rho_1^2(t)$ and $\lambda_m \to \infty$ as $m \to \infty$, in order to have (3.12), it suffices to check that for the previously fixed $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $m = m(\varepsilon, t, \widehat{D}) \in \mathbb{N}$, such that for any $\tau \le \tau_1(\widehat{D}, t)$ and $u^\tau \in D(\tau)$,

$$e^{-\nu\lambda_{m+1}t}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r;\tau,u^\tau)\|^2\,dr < \frac{\varepsilon\nu}{3\rho_1^2(t)}. \tag{3.13}$$

Take $\delta = \delta\big(\frac{\varepsilon\nu}{6\rho_1^2(t)}, t, \widehat{D}\big) \in (0,1)$ as in Corollary 3.3. Then, using (3.2), for each $m \ge 1$ we have

$$\begin{aligned} & e^{-\nu\lambda_{m+1}t}\int_{t-1}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ =\;& e^{-\nu\lambda_{m+1}t}\int_{t-1}^{t-\delta} e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr + e^{-\nu\lambda_{m+1}t}\int_{t-\delta}^t e^{\nu\lambda_{m+1}r}\|u(r)\|^2\,dr \\ \le\;& e^{-\nu\lambda_{m+1}\delta}\int_{t-1}^{t-\delta}\|u(r)\|^2\,dr + \int_{t-\delta}^t\|u(r)\|^2\,dr \\ \le\;& e^{-\nu\lambda_{m+1}\delta}\nu^{-1}\rho_2^2(t) + \int_{t-\delta}^t\|u(r)\|^2\,dr. \end{aligned}$$

By taking now $m = m(\varepsilon, t, \widehat{D})$ such that $e^{-\nu\lambda_{m+1}\delta}\nu^{-1}\rho_2^2(t) < \frac{\varepsilon\nu}{6\rho_1^2(t)}$, jointly with (3.10) in Corollary 3.3, we conclude (3.13). ■

As a consequence of the above result and Proposition 1.18, we obtain the asymptotic compactness of the process U defined by (2.11) in the H norm, and therefore the same statements claimed in Theorem 2.9 about the existence of minimal pullback attractors in H, are also satisfied.

3.1.1 A compact pullback absorbing family using semigroup theory

With only a slightly more stringent requirement on the forcing function f, namely that

$$f \in L^p_{\rm loc}(\mathbb{R}; V') \qquad \text{for some} \quad p > 2$$

we can in fact use the semigroup approach of Fujita and Kato [29] (see also [41]) to show, using Corollary 3.3 again (and hence Lemma 3.2 once more) that in fact there is a compact pullback absorbing family in H. Our analysis is inspired by the paper by Arrieta and Carvalho [2] on the ϵ-regularity method for proving existence and uniqueness of semilinear problems.

In order to state the result precisely we need to define the fractional power spaces $D(A^\alpha)$ ($\alpha > 0$) as the domains of the operators A^α, where

$$A^\alpha u := \sum_{j=1}^\infty \lambda_j^\alpha (u, w_j) w_j,$$

where as usual λ_j and w_j are the eigenvalues and eigenfunctions of the Stokes operator. We recall the key estimate

$$|A^\gamma e^{-At} x| \le c_\gamma t^{-\gamma} |x| \tag{3.14}$$

for any $0 \le \gamma < 1$ (e.g., see Henry [41]). In the proof we write

$$\|u\|_s = |A^s u|,$$

and in particular we have $\|\cdot\|_0 = |\cdot|$, $\|\cdot\|_{1/2} = \|\cdot\|$, and $\|\cdot\|_{-1/2} = \|\cdot\|_*$.

Theorem 3.6. *Suppose that $f \in L^p_{\rm loc}(\mathbb{R}; V')$ for some $p > 2$ and that*

$$\int_{-\infty}^0 e^{\mu s} \|f(s)\|_*^2 \, ds < \infty \quad \text{for some } \mu \in (0, 2\nu\lambda_1).$$

Choose $\epsilon < \frac{1}{2} - \frac{1}{p}$. Then there exists a function $\rho_\epsilon : \mathbb{R} \to \mathbb{R}$ such that for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$,

$$|A^\epsilon u(t; \tau, u^\tau)| \le \rho_\epsilon(t) \qquad \text{for all} \quad u^\tau \in D(\tau),\ \tau \le \tau_1(\widehat{D}, t),$$

where $\tau_1(\widehat{D}, t)$ is the same as in Lemma 3.1; hence there is a compact pullback absorbing family in H. In particular if $f \in L^\infty_{\rm loc}(\mathbb{R}; V')$ then we obtain a bounded absorbing family in $D(A^\epsilon)$ for any $\epsilon < 1/2$; this holds therefore in the autonomous case when $f \in V'$.

Proof. Given the improved regularity assumption on f, any weak solution u to (2.1) satisfies the variation of constants formula

$$u(t) = e^{-A(t-s)} u(s) + \int_s^t e^{-A(t-r)} [B(u(r)) + f(r)] \, dr$$

for all $\tau \le s \le t$ (see [41]). We use this formulation to find estimates on u in $D(A^\epsilon)$. The key observation (after [2]) is that $B : D(A^\epsilon) \to D(A^{-(1-2\epsilon)})$, which can be seen by taking $w \in D(A^{1-2\epsilon})$ and using Hölder's inequality to obtain

$$\begin{aligned}|\langle B(u), w\rangle| &= |\langle B(u, w), u\rangle| \\ &\le \|u\|^2_{L^{2/(1-2\epsilon)}} \|\nabla w\|_{L^{1/(2\epsilon)}} \\ &\le \tilde{c}_\epsilon \|u\|^2_\epsilon \|w\|_{1-2\epsilon},\end{aligned}$$

from which it follows that

$$\|B(u)\|_{-(1-2\epsilon)} \le \tilde{c}_\epsilon \|u\|^2_\epsilon. \tag{3.15}$$

Lemma 3.1 guarantees that for any $\widehat{D} \in \mathcal{D}_\mu(H)$ and any $t \in \mathbb{R}$ there exists a $\tau_1(\widehat{D}, t) < t - 2$ such that for any $r \in [t-2, t]$,

$$|u(r; \tau, u^\tau)| \le \rho_1(t) \qquad \forall\, u^\tau \in D(\tau),\ \tau \le \tau_1(\widehat{D}, t). \tag{3.16}$$

Fix $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}^H_\mu$, $\tau \le \tau_1(\widehat{D}, t)$, and $u^\tau \in D(\tau)$ and write

$$u_\sigma(s) = u(\sigma + s; \tau, u^\tau) \qquad \text{and} \qquad f_\sigma(s) = f(\sigma + s).$$

We can now rewrite the variation of constants formula in the notationally convenient form (for $\sigma \ge \tau$)

$$u_\sigma(s) = e^{-As} u_\sigma(0) + \int_0^s e^{-A(s-r)} [B(u_\sigma(r)) + f_\sigma(r)]\, dr \qquad \forall\, s \in [0, t-\sigma], \tag{3.17}$$

noting from (3.16) that

$$|u_\sigma(s)| \le \rho_1(t) \qquad \forall\, \sigma \in [t-1, t],\ s \in [0, t-\sigma]. \tag{3.18}$$

Pick $\sigma \in [t-1, t]$, let $s \le t - \sigma$, then take the norm of (3.17) in $D(A^\epsilon)$ and multiply by s^ϵ to obtain

$$\begin{aligned}s^\epsilon \|u_\sigma(s)\|_\epsilon &\le s^\epsilon \|e^{-As} u_\sigma(0)\|_\epsilon + s^\epsilon \int_0^s \left\| e^{-A(s-r)} [B(u_\sigma(r)) + f_\sigma(r)] \right\|_\epsilon dr \\ &\le c_\epsilon |u_\sigma(0)| + c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|B(u_\sigma(r))\|_{-(1-2\epsilon)}\, dr \\ &\qquad + c_{1/2+\epsilon} s^\epsilon \int_0^s (s-r)^{-1/2-\epsilon} \|f_\sigma(r)\|_*\, dr \\ &\le c_\epsilon \rho_1(t) + \tilde{c}_\epsilon c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|u_\sigma(r)\|^2_\epsilon\, dr \\ &\qquad + c_{1/2+\epsilon} s^\epsilon \int_0^s (s-r)^{-1/2-\epsilon} \|f_\sigma(r)\|_*\, dr,\end{aligned}$$

using (3.14), (3.15), and (3.18). The second term on the right-hand side can be bounded by

$$\begin{aligned}&\tilde{c}_\epsilon c_{1-\epsilon} s^\epsilon \int_0^s (s-r)^{-(1-\epsilon)} \|u_\sigma(r)\|^{3/2}_\epsilon |u_\sigma(r)|^{1/2-\epsilon} \|u_\sigma(r)\|^\epsilon\, dr \\ &\qquad \le R(s) s^\epsilon \left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} \|u_\sigma(r)\|^{3/(2-\epsilon)}_\epsilon\, dr \right)^{1-\epsilon/2},\end{aligned}$$

where

$$R(s) = \tilde{c}_\epsilon c_{1-\epsilon}\left(\sup_{0\le r\le s}|u_\sigma(r)|\right)^{1/2-\epsilon}\left(\int_0^s \|u_\sigma(r)\|^2\,dr\right)^{\epsilon/2}$$
$$\le \tilde{c}_\epsilon c_{1-\epsilon}\rho_1(t)^{1/2-\epsilon}\left(\int_\sigma^t \|u(r)\|^2\,dr\right)^{\epsilon/2} =: P(\sigma,t).$$

Setting $X(s) = s^\epsilon\|u_\sigma(s)\|_\epsilon$ we obtain an integral inequality for $X(s)$:

$$X(s) \le \delta(s) + P(\sigma,t)s^\epsilon\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)}r^{-3\epsilon/(2-\epsilon)}X(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}, \qquad (3.19)$$

where[1]

$$\delta(s) = c_\epsilon\rho_1(t) + c_{1/2+\epsilon}s^\epsilon\int_0^s (s-r)^{-1/2-\epsilon}\|f_\sigma(r)\|_*\,dr$$
$$\le c_\epsilon\rho_1(t) + c_{1/2+\epsilon}s^\epsilon\left(\int_0^s (s-r)^{-p(1/2+\epsilon)/(p-1)}\,dr\right)^{1-(1/p)}\left(\int_0^s \|f_\sigma(r)\|_*^p\,dr\right)^{1/p}$$
$$\le c_\epsilon\rho_1(t) + C_{\epsilon,p}\left(\int_{t-1}^t \|f(r)\|_*^p\,dr\right)^{1/p} =: \Phi(t),$$

using Hölder's inequality and the choice of ϵ which ensures that $p(1/2+\epsilon)/(p-1) < 1$.

In order to find an upper bound on $X(s)$ it suffices to find a continuous function $Y(t)$ with $X(0) < Y(0)$ that is a supersolution of (3.19), i.e. that satisfies

$$Y(s) \ge \Phi(t) + P(\sigma,t)s^\epsilon\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)}r^{-3\epsilon/(2-\epsilon)}Y(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \qquad (3.20)$$

for all $s \in [0, t-\sigma]$, to conclude that $X(s) \le Y(s)$ for all $s \in [0, t-\sigma]$. Indeed, if it were not true, there would exists $\hat{s} \in (0, t-\sigma]$ with $Y(\hat{s}) < X(\hat{s})$ and so we may define $0 < \tilde{s} = \sup\{s \in [0, t-\sigma] : X(r) - Y(r) < 0 \ \forall r \in [0, s)\}$. Then, we would have $X(s) < Y(s)$ for all $s \in [0, \tilde{s})$ and $X(\tilde{s}) = Y(\tilde{s})$ (by continuity), but therefore

$$Y(\tilde{s}) \ge \Phi(t) + P(\sigma,t)\tilde{s}^\epsilon\left(\int_0^{\tilde{s}} (\tilde{s}-r)^{-(1-\epsilon)/(1-\epsilon/2)}r^{-3\epsilon/(2-\epsilon)}Y(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}$$
$$> \Phi(t) + P(\sigma,t)\tilde{s}^\epsilon\left(\int_0^{\tilde{s}} (\tilde{s}-r)^{-(1-\epsilon)/(1-\epsilon/2)}r^{-3\epsilon/(2-\epsilon)}X(r)^{3/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}$$
$$\ge X(\tilde{s}),$$

i.e. $Y(\tilde{s}) > X(\tilde{s})$, a contradiction.

[1]Note that in fact $\delta(s)$ is simply a bound on $s^\epsilon\tilde{U}(s+\sigma,\sigma)$ in $D(A^\epsilon)$, where $\tilde{U}(s,\sigma)$ is a solution of the linear equation $u_t + Au = f(t)$ with initial data $u(\sigma) = u_\sigma(0)$; this where we require the additional regularity for f, so it has nothing to do with the nature of the nonlinear term. We return briefly to this issue in the Conclusion.

Now, we will prove that $Y(s) = 2\Phi(t)$ satisfies (3.20) for $s \in [0, t-\sigma]$, i.e. we need to ensure that

$$2\Phi(t) \geq \Phi(t) + 2\sqrt{2}P(\sigma,t)\Phi^{3/2}(t)s^{\epsilon}\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2}$$

for all $s \in [0, t-\sigma]$. This holds if

$$2\sqrt{2}P(\sigma,t)\Phi^{1/2}(t)s^{\epsilon}\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \leq 1 \quad \forall\, s \in [0, t-\sigma].$$

By substituting $r = s\theta$ one can see that

$$\begin{aligned} s^{\epsilon}&\left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)}\,dr\right)^{1-\epsilon/2} \\ &= s^{\epsilon}\left(\int_0^1 s^{-(1-\epsilon)/(1-\epsilon/2)}(1-\theta)^{-(1-\epsilon)/(1-\epsilon/2)} s^{-3\epsilon/(2-\epsilon)}\theta^{-3\epsilon/(2-\epsilon)} s\,d\theta\right)^{1-\epsilon/2} \\ &= \left(\int_0^1 (1-\theta)^{-(1-\epsilon)/(1-\epsilon/2)}\theta^{-3\epsilon/(2-\epsilon)}\,d\theta\right)^{1-\epsilon/2} =: C_{\epsilon}. \end{aligned}$$

So in order to guarantee that $X(s) \leq 2\Phi(t)$ for $s \in [0, t-\sigma]$ it suffices to ensure that

$$2\sqrt{2}P(\sigma,t)\Phi^{1/2}(t) \leq C_{\epsilon}^{-1}. \tag{3.21}$$

Now, recall that

$$P(\sigma,t) = \tilde{c}_{\epsilon}c_{1-\epsilon}\rho_1(t)^{1/2-\epsilon}\left(\int_{\sigma}^t \|u(r)\|^2\,dr\right)^{\epsilon/2},$$

and that it follows from Corollary 3.3 that for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_{\mu}(H)$, there exists a $\sigma = \sigma(\varepsilon, t, \widehat{D}) \in (t-1, t)$, such that

$$\int_{\sigma}^t \|u(\theta;\tau,u^{\tau})\|^2\,d\theta < \varepsilon \quad \forall\, \tau \leq \tau_1(\widehat{D},t),\ u^{\tau} \in D(\tau),$$

(this was (3.10)). We can therefore satisfy the condition (3.21), and deduce that

$$|A^{\epsilon}u(t;\tau,u^{\tau})| = (t-\sigma)^{-\epsilon}X(t-\sigma) \leq 2(t-\sigma)^{-\epsilon}\Phi(t) =: \rho_{\epsilon}(t) \qquad \forall\, u^{\tau} \in D(\tau),\ \tau \leq \tau_1(\widehat{D},t).$$

Since $D(A^{\epsilon})$ is compactly embedded in H, the existence of a compact pullback absorbing family in H follows immediately. ■

3.2 Pullback flattening property in V norm

The goal of this section is to show a sharper conclusion than above. More precisely, we will prove the flattening property for the process U defined on V, and as a consequence, we will obtain the pullback asymptotic compactness for this process in V, which was already proved by using an energy method in Lemma 2.14, but here with a shorter proof.

We have the following result, which is similar to Lemma 2.10.

Lemma 3.7. *Suppose that $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies (2.15). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $\tau_1(\widehat{D}, t) < t-2$ (the one given in Lemma 3.1), such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u^\tau \in D(\tau)$,*

$$
\begin{aligned}
|u(r;\tau,u^\tau)|^2 &\leq R_1^2(t) \quad \forall\, r \in [t-2,t], && (3.22)\\
\|u(r;\tau,u^\tau)\|^2 &\leq R_2^2(t) \quad \forall\, r \in [t-1,t], && (3.23)\\
\nu \int_{t-1}^{t} |Au(\theta;\tau,u^\tau)|^2 \, d\theta &\leq R_3^2(t), && (3.24)
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^2(t) &= 1 + e^{-\mu(t-2)}(2\nu\lambda_1 - \mu)^{-1} \int_{-\infty}^{t} e^{\mu\theta} |f(\theta)|^2 \, d\theta,\\
R_2^2(t) &= \nu^{-1} \left(R_1^2(t) + (\nu^{-1}\lambda_1^{-1} + 2) \int_{t-2}^{t} |f(\theta)|^2 \, d\theta \right)\\
&\quad \times \exp\left[2\nu^{-1} C^{(\nu)} R_1^2(t) \left(R_1^2(t) + \nu^{-1}\lambda_1^{-1} \int_{t-2}^{t} |f(\theta)|^2 \, d\theta \right) \right],\\
R_3^2(t) &= R_2^2(t) + 2\nu^{-1} \int_{t-1}^{t} |f(\theta)|^2 \, d\theta + 2C^{(\nu)} R_1^2(t) R_2^4(t),
\end{aligned}
$$

with $C^{(\nu)}$ defined by (2.20).

Proof. The first estimate (3.22) follows immediately from (3.1).

On the other hand, the estimates (3.23) and (3.24) can be obtained analogously as the second and third estimates in (2.16). In fact, thanks to the regularity result (a) in Theorem 2.3, and by applying the energy equality (2.9), we do not need to use the Galerkin approximations. ■

Now, we will prove that the process $U : \mathbb{R}^2_d \times V \to V$ satisfies the pullback $\widehat{D}_{0,\mu,V}$-flattening property. In fact, we will prove that U satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu(H)$.

Analogously to Lemma 3.4, we have the following result.

Lemma 3.8. *If $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies (2.15), then, for any $t \in \mathbb{R}$,*

$$
\lim_{\rho \to \infty} e^{-\rho t} \int_{-\infty}^{t} e^{\rho s} |f(s)|^2 \, ds = 0.
$$

Proposition 3.9. *Under the assumptions of Lemma 3.7, for any $\varepsilon > 0$ and $t \in \mathbb{R}$, there exists $m = m(\varepsilon, t) \in \mathbb{N}$ such that for any $\widehat{D} \in \mathcal{D}_\mu(H)$, the projection $P_m : V \to V_m :=$ $\mathrm{span}[w_1, \ldots, w_m]$ satisfies the following properties:*

$$
\{P_m U(t,\tau) D(\tau) : \tau \leq \tau_1(\widehat{D}, t)\} \text{ is bounded in } V,
$$

and

$$\|(I-P_m)U(t,\tau)u^\tau\| < \varepsilon \quad \text{for any } \tau \le \tau_1(\widehat{D},t),\ u^\tau \in D(\tau),$$

where $\tau_1(\widehat{D},t)$ is given in Lemma 3.1.

In particular, the process U on V satisfies the pullback $\widehat{D}$-flattening property for any $\widehat{D} \in \mathcal{D}_\mu(H)$.

Proof. Let $\varepsilon > 0$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}_\mu(H)$ be fixed.

Since $\{w_j\}_{j\ge1}$ is a special basis, P_m is non-expansive in V. From this and (3.23), we deduce the boundedness in V of the set $\{P_m U(t,\tau)D(\tau) : \tau \le \tau_1(\widehat{D},t)\}$, for all $m \ge 1$.

On the other hand, let us fix $\tau \le \tau_1(\widehat{D},t)$, $u^\tau \in D(\tau)$, and let us define again $u(r) = U(r,\tau)u^\tau$ and $q_m(r) = u(r) - P_m u(r)$.

Then, by (2.5) and Lemma 3.7, for each $m \ge 1$ one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dr}\|q_m(r)\|^2 + \nu|Aq_m(r)|^2 &= -b(u(r),u(r),Aq_m(r)) + (f(r),Aq_m(r)) \\ &\le \frac{\nu}{2}|Aq_m(r)|^2 + \frac{1}{\nu}|f(r)|^2 + \frac{C_1^2}{\nu}R_1(t)R_2^2(t)|Au(r)| \end{aligned}$$

a.e. $t-1 < r < t$.

Consequently, as $|Aq_m(r)|^2 \ge \lambda_{m+1}\|q_m(r)\|^2$, from above we deduce that

$$\frac{d}{dr}\|q_m(r)\|^2 + \nu\lambda_{m+1}\|q_m(r)\|^2 \le 2\nu^{-1}|f(r)|^2 + 2C_1^2\nu^{-1}R_1(t)R_2^2(t)|Au(r)|$$

a.e. $t-1 < r < t$.

Thus, multiplying this last inequality by $e^{\nu\lambda_{m+1}r}$, integrating from $t-1$ to t, and taking into account Lemma 3.7, we obtain

$$\begin{aligned} e^{\nu\lambda_{m+1}t}\|q_m(t)\|^2 &\le e^{\nu\lambda_{m+1}(t-1)}\|q_m(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\ &\quad +2C_1^2\nu^{-1}R_1(t)R_2^2(t)\int_{t-1}^t e^{\nu\lambda_{m+1}r}|Au(r)|\,dr \\ &\le e^{\nu\lambda_{m+1}(t-1)}\|u(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\ &\quad +2C_1^2\nu^{-1}R_1(t)R_2^2(t)\left(\int_{t-1}^t e^{2\nu\lambda_{m+1}r}\,dr\right)^{1/2}\left(\int_{t-1}^t |Au(r)|^2\,dr\right)^{1/2} \\ &\le e^{\nu\lambda_{m+1}(t-1)}R_2^2(t) + 2\nu^{-1}\int_{t-1}^t e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\ &\quad +2C_1^2\nu^{-3/2}R_1(t)R_2^2(t)R_3(t)(2\nu\lambda_{m+1})^{-1/2}e^{\nu\lambda_{m+1}t}. \end{aligned}$$

Therefore, from Lemma 3.8 and since $\lambda_m \to \infty$ as $m \to \infty$, we conclude that there exists $m = m(\varepsilon,t) \in \mathbb{N}$ such that $\|(I-P_m)U(t,\tau)u^\tau\| < \varepsilon$ for all $\tau \le \tau_1(\widehat{D},t)$, $u^\tau \in D(\tau)$. ∎

As a consequence of the above result and Proposition 1.18, we obtain the asymptotic compactness in the V norm. It is worth pointing out that in this way the proof is much shorter than that of Lemma 2.14 in Chapter 2. Moreover, observe that, under the assumptions of Lemma 3.7, all the statements asserted in Theorem 2.15 are also true, and so we obtain again the existence of minimal pullback attractors for the process U on V.

In light of the fact that our analysis in H required a similar amount of work to obtain asymptotic compactness or the flattening property, one might ask if one could 'simplify' the direct proof of asymptotic compactness in V from [31] by using some ideas from the above 'flattening' analysis. It then becomes apparent that the idea of 'direct' proof in this case simply means trying to prove asymptotic compactness with resorting to a splitting of the solution into high and low modes; this serves to emphasize that the 'flattening property' can more rightly be thought of as a technique (splitting) that is always available should we require it.

3.2.1 Compactness of the process in V via semigroups

Finally we show that in V, too, a little more regularity of f yields the existence of a compact pullback absorbing family. To this end we assume that

$$f \in L^p_{\mathrm{loc}}(\mathbb{R}; H) \qquad \text{for some } p > 2.$$

With this assumption we show that there is a bounded absorbing family in $D(A^{1/2+\delta})$ for an appropriately chosen $\delta > 0$.

Theorem 3.10. *Suppose that $f \in L^p_{\mathrm{loc}}(\mathbb{R}; H)$ for some $p > 2$ and that*

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < \infty \quad \textit{for some } \mu \in (0, 2\nu\lambda_1).$$

Fix $\delta < \frac{1}{2} - \frac{1}{p}$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu(H)$, there exists $\tau_1(\widehat{D}, t)$ (the one given in Lemma 3.1), such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u^\tau \in D(\tau)$,

$$|A^{1/2+\delta} u(t; \tau, u^\tau)| \leq R_\delta(t),$$

where $R_\delta(t)$ is given below.

Proof. The analysis in V is significantly simpler than in H. Indeed, for any $\epsilon > 0$ the nonlinear term maps V into $D(A^{-\epsilon})$: taking the inner product of $B(u)$ with $w \in D(A^\epsilon)$ we obtain

$$\begin{aligned} |\langle B(u), w\rangle| &\leq \|u\|_{L^{1/\epsilon}} \|\nabla u\|_{L^2} \|w\|_{L^{2/(1-2\epsilon)}} \\ &\leq \tilde{c}_\epsilon \|u\|^2 \|w\|_{L^{2/(1-2\epsilon)}} \\ &\leq \tilde{c}_\epsilon \|u\|^2 \|w\|_\epsilon, \end{aligned}$$

since $D(A^s) \subset (L^{2/(1-2s)}(\Omega))^2$. Thus

$$\|B(u)\|_{-\epsilon} \leq \tilde{c}_\epsilon \|u\|^2.$$

Given a solution $u(t) = u(t;\tau,u^\tau)$ we write

$$u(t) = e^{-A}u(t-1) + \int_{t-1}^{t} e^{-A(t-s)}(B(u(s)) + f(s))\,ds.$$

Take the norm in $D(A^{1/2+\delta})$, using (3.14) and choosing ϵ so that $\delta + \epsilon < 1/2$, we obtain

$$\begin{aligned}|A^{1/2+\delta}u(t)| &\le c_\delta\|u(t-1)\| + \tilde{c}_\epsilon c_{1/2+\delta+\epsilon}\int_{t-1}^{t}(t-s)^{-(1/2+\delta+\epsilon)}\|u(s)\|^2\,ds \\ &\qquad + c_{1/2+\delta}\int_{t-1}^{t}(t-s)^{-(1/2+\delta)}|f(s)|\,ds \\ &\le c_\delta R_2(t) + \frac{\tilde{c}_\epsilon c_{1/2+\delta+\epsilon}}{1/2-\delta-\epsilon}R_2^2(t) + C_{p,\delta}\left(\int_{t-1}^{t}|f(s)|^p\,ds\right)^{1/p} =: R_\delta(t);\end{aligned}$$

the first term is bounded using Lemma 3.7; the second since $(t-s)^{-(1/2+\delta+\epsilon)}$ is integrable and $\|u(s)\|^2 \le R_2^2(t)$ uniformly for $s \in [t-1,t]$ (Lemma 3.7 again); and for the third term we can argue as in the proof of Theorem 3.6 using Hölder's inequality since $\delta < \frac{1}{2} - \frac{1}{p}$. ■

We note in particular that in the autonomous case this gives a very quick method of proving the existence of a compact absorbing set in V when we assume only $f \in H$. As in the more complicated case in H, the higher regularity of f is the same as would be required to obtain a similar result for the linear problem $u_t + Au = f(t)$.

Conclusion

We have shown the existence of pullback attractors in H and V under minimal regularity assumptions on the forcing f, proving asymptotic compactness of the dynamical process via the Fourier splitting method, i.e. a proof of 'Condition (C)'/'the flattening property'. With a little additional regularity we have been able to use the semigroup approach to prove the existence of a compact pullback absorbing family in both cases.

It is interesting that in order to obtain the compact pullback absorbing family we require the same regularity of f as we would in the purely linear problem. One can see that this is to be expected if we consider solutions given by the variation of constants formula

$$u(t) = e^{-A(t-s)}u(s) + \int_{s}^{t} e^{-A(t-r)}[B(u(r)) + f(r)]\,dr,$$

noting that the expression

$$\tilde{U}(t,s) := e^{-A(t-s)}u(s) + \int_{s}^{t} e^{-A(t-r)}f(r)\,dr$$

is simply the solution of the linear equation

$$v_t + Av = f(t), \qquad v(s) = u(s)$$

at time t. So we could write the following variation on the variation-of-constants formula,

$$u(t) = \tilde{U}(t,s) + \int_s^t e^{-A(t-r)} B(u(r))\, dr.$$

For the analysis in H, the key step was the estimate (3.19), which we can write as

$$X(s) \le s^\epsilon |A^\epsilon \tilde{U}(s+\sigma,\sigma)| + P(\sigma,t) s^\epsilon \left(\int_0^s (s-r)^{-(1-\epsilon)/(1-\epsilon/2)} r^{-3\epsilon/(2-\epsilon)} X(r)^{3/(2-\epsilon)}\, dr \right)^{1-\epsilon/2}.$$

Conclusion of the argument requires a bound on $|A^\epsilon \tilde{U}(s+\sigma,\sigma)|$ uniform for $\sigma \in [t-1,t]$, $s \in [0, t-\sigma]$, i.e. relies on solutions of the linear equation.

Similarly, if we estimate u in $D(A^{1/2+\delta})$ as in Section 3.2.1 then we obtain

$$\begin{aligned} |A^{1/2+\delta} u(t)| &\le |A^{1/2+\delta} \tilde{U}(t,t-1)| + \tilde{c}_\epsilon c_{1/2+\delta+\epsilon} \int_{t-1}^t (t-s)^{-(1/2+\delta+\epsilon)} \|u(s)\|^2\, ds \\ &\le |A^{1/2+\delta} v(t)| + \frac{\tilde{c}_\epsilon c_{1/2+\delta+\epsilon}}{1/2-\delta-\epsilon} R_2^2(t), \end{aligned}$$

and the key point is again an estimate on $\tilde{U}$, i.e. smoothness for the linear equation.

Chapter 4

Pullback Attractors for 2D Navier–Stokes Equations with Finite Delay

In this chapter we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth enough boundary $\partial\Omega$, and the following non-autonomous functional Navier–Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \quad \text{in} \ \ \Omega\times(\tau,\infty), \\ \operatorname{div} u = 0 \quad \text{in} \ \ \Omega\times(\tau,\infty), \\ u = 0 \quad \text{on} \ \ \partial\Omega\times(\tau,\infty), \\ u(x,\tau+s) = \phi(x,s), \quad x\in\Omega,\ s\in[-h,0], \end{cases} \tag{4.1}$$

where f is a non-delayed external force field, g is another external force containing some hereditary characteristics, and $\phi(x, s-\tau)$ is the initial datum in the interval of time $[\tau-h,\tau]$, where $h>0$ is the time of memory effect. For each $t\geq\tau$, we denote by u_t the function defined on $[-h,0]$ by the relation $u_t(s)=u(t+s)$, $s\in[-h,0]$.

As it was pointed out in the Introduction, to our knowledge, in all finite delay frameworks the assumptions for the delay terms used to involve estimates in L^2 spaces. The goal of this chapter is to generalize the conditions on the delay terms in the model by allowing just continuous (in time) spaces, which will require less restrictive conditions on the involved operators. Although this implies to restrict the phase space to continuous functions instead of square integrable in time, the delay functions driving the delayed time within this theory can be taken just measurable, without any additional assumption as continuity nor C^1 with bounded derivative, as usual in the literature. Actually, we will provide a simple example where the delay function leading the delayed time is just measurable (cf. Example 4.1 and Remark 4.2 below, for more details).

In Section 4.1 we obtain a result on the existence, uniqueness, and regularity of the solution to (4.1). Section 4.2 is devoted to prove the existence of pullback attractors in the H norm, with respect to two different universes, via asymptotic compactness, and using an energy method which relies strongly on the energy equality associated to the problem. The main results of the chapter are given in Section 4.3. There, we strengthen the regularity of solutions and a second energy equality for them, in order to obtain additional attraction, namely, in the V norm instead of H. Different families of universes (tempered and non-tempered) are introduced. Now, a second (and more involved) energy method is employed to prove asymptotic compactness in the new metric. We finish analyzing the relationships among all these families. Actually, we are able to prove that, under suitable assumptions, in fact all these objects coincide. The results can be found in [33].

4.1 Existence and uniqueness of solution

In this section we prove existence, uniqueness, and regularity of solution to problem (4.1).

To start, we establish some appropriate assumptions on the term in (4.1) containing the delay.

Let us denote by $C_H = C([-h,0];H)$, the space of continuous functions from $[-h,0]$ into H, with the norm

$$|\varphi|_{C_H} = \max_{s\in[-h,0]} |\varphi(s)|.$$

Let us consider over the delay operator from (4.1) that is well defined as $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable,

(II) $g(t,0) = 0$, for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all ξ, $\eta \in C_H$,

$$|g(t,\xi) - g(t,\eta)| \le L_g|\xi - \eta|_{C_H}.$$

Observe that $(I) - (III)$ imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$.

It is worth pointing out that any condition involving L^2 norms of the memory term in g is assumed (e.g. cf. conditions (IV) and (V) in the following chapter).

Example 4.1. *Consider a globally Lipschitz function $G : H \to (L^2(\Omega))^2$, with Lipschitz constant $L_G > 0$, and such that $G(0) = 0$, and a measurable function $\rho : \mathbb{R} \to [0,h]$.*

Then, it is not difficult to check that the operator $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, defined by

$$\mathbb{R} \times C_H \ni (t,\xi) \mapsto g(t,\xi) := G(\xi(-\rho(t)))$$

satisfies the assumptions (I)–(III) given above.

Remark 4.2. *(a) Observe that the only assumption on ρ is that it is measurable, in contrast with the usual conditions appearing in the previous literature, i.e. C^1, with derivative $\rho'(t) \le \rho_* < 1$ (e.g. cf. [36]).*

(b) *The example above can be generalized in several senses. The most immediate generalization is to take into account more than one delay term in the problem. Namely, consider measurable functions $\rho_i : \mathbb{R} \to [0,h]$ for $i = 1,\dots,m$, a measurable mapping $G : \mathbb{R} \times H^m \to (L^2(\Omega))^2$ such that $G(t,\cdot)$ is globally Lipschitz in H^m uniformly with respect to time, and with $G(t,0) = 0$ for all $t \in \mathbb{R}$. Then, consider $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ given by $g(t,\xi) := G(t,\xi(-\rho_1(t)),\dots,\xi(-\rho_m(t)))$. This operator g also satisfies conditions (I)–(III).*

Assume that $\phi \in C_H$, and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$.

Definition 4.3. *A weak solution to (4.1) is a function $u \in C([\tau-h,\infty); H)$ such that $u \in L^2(\tau,T;V)$ for all $T > \tau$, with $u(t) = \phi(t-\tau)$ for all $t \in [\tau-h,\tau]$, and such that for all $v \in V$,*

$$\frac{d}{dt}(u(t),v) + \nu\langle Au(t),v\rangle + b(u(t),u(t),v) = \langle f(t),v\rangle + (g(t,u_t),v), \tag{4.2}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,\infty)$.

Observe that if u is a weak solution to (4.1), then from (4.2) we deduce that for any $T > \tau$, one has $u' \in L^2(\tau,T;V')$, and the following energy equality holds:

$$\begin{aligned} & |u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2\,dr \\ = \ & |u(s)|^2 + 2\int_s^t \big[\langle f(r),u(r)\rangle + (g(r,u_r),u(r))\big]dr \quad \forall\, \tau \le s \le t. \end{aligned} \tag{4.3}$$

As in Chapter 2, we can also define a notion of more regular solution for problem (4.1).

Definition 4.4. *A strong solution to (4.1) is a weak solution u to (4.1) such that $u \in L^2(\tau,T;D(A)) \cap L^\infty(\tau,T;V)$ for all $T > \tau$.*

Note that if $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$ and u is a strong solution to (4.1), then $u' \in L^2(\tau,T;H)$ for all $T > \tau$, and so $u \in C([\tau,\infty);V)$. In this case the following energy equality holds:

$$\begin{aligned} & \|u(t)\|^2 + 2\nu\int_s^t |Au(r)|^2\,dr + 2\int_s^t b(u(r),u(r),Au(r))\,dr \\ = \ & \|u(s)\|^2 + 2\int_s^t (f(r)+g(r,u_r),Au(r))\,dr \quad \forall\, \tau \le s \le t. \end{aligned} \tag{4.4}$$

Concerning the existence and uniqueness of weak solution for (4.1), we have the following result, which improves, in the case of initial datum $\phi \in C_H$ and dimension two, Theorem 2.1 in [9] (see also [36, Theorem 2.3]). In fact, in the theorem below, we neither assume hypotheses (IV) nor (V) of [9].

Theorem 4.5. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), be given. Then, for each $\tau \in \mathbb{R}$ and $\phi \in C_H$, there exists a unique weak solution $u(\cdot) = u(\cdot; \tau, \phi)$ to (4.1).*

Moreover, if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; (L^2(\Omega))^2)$, then

(a) $u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ for all $T > \tau + \varepsilon > \tau$.

(b) If $\phi(0) \in V$, u is in fact a strong solution to (4.1).

Proof. The uniqueness of solution can be obtained in the following way. Consider two weak solutions for (4.1), u and v, with the same initial data, and denote $w = u - v$. We note that by (2.4),

$$\begin{aligned}
|b(u(s), u(s), w(s)) - b(v(s), v(s), w(s))| &= |b(w(s), u(s), w(s))| \\
&\leq 2^{-1/2} |w(s)| \|w(s)\| \|u(s)\|.
\end{aligned}$$

Then, from the equation satisfied by w and the energy equality, we obtain for all $t \geq \tau$ that

$$\begin{aligned}
&|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 \, ds \\
= \; & -2 \int_\tau^t b(w(s), u(s), w(s)) \, ds + 2 \int_\tau^t (g(s, u_s) - g(s, v_s), w(s)) \, ds \\
\leq \; & 2^{1/2} \int_\tau^t |w(s)| \|w(s)\| \|u(s)\| \, ds + 2L_g \int_\tau^t |w_s|_{C_H} |w(s)| \, ds. \qquad (4.5)
\end{aligned}$$

Observe that $w(\theta) = 0$ if $\tau - h \leq \theta \leq \tau$, and therefore,

$$|w_s|_{C_H} = \max_{r \in [\tau, s]} |w(r)| \quad \text{for } \tau \leq s.$$

So, from (4.5), using Young's inequality, we deduce

$$\begin{aligned}
&|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds \\
\leq \; & 2^{1/2} \int_\tau^t |w(s)| \|w(s)\| \|u(s)\| \, ds + 2L_g \int_\tau^t \max_{r \in [\tau, s]} |w(r)| |w(s)| \, ds \\
\leq \; & \nu \int_\tau^t \|w(s)\|^2 \, ds + \frac{1}{2\nu} \int_\tau^t \|u(s)\|^2 |w(s)|^2 \, ds + 2L_g \int_\tau^t \max_{r \in [\tau, s]} |w(r)|^2 \, ds
\end{aligned}$$

for all $t \geq \tau$, and therefore,

$$\max_{r \in [\tau, t]} |w(r)|^2 \leq \left(\frac{1}{2\nu} + 2L_g \right) \int_\tau^t \left(1 + \|u(s)\|^2\right) \max_{r \in [\tau, s]} |w(r)|^2 \, ds$$

for all $t \geq \tau$. Thus, using Gronwall's lemma, we finish the proof of uniqueness.

For the existence, we split the proof in two steps.

Step 1: Galerkin scheme. A priori estimates. Let us consider again $\{w_j\}_{j\geq 1} \subset D(A)$, a Hilbert basis of H formed by ortho-normalized eigenfunctions of the Stokes operator A. Denote $V_m = \mathrm{span}[w_1, \dots, w_m]$ and consider the projector P_m of H onto V_m given by $P_m v = \sum_{j=1}^m (v, w_j) w_j$, for all $v \in H$. Observe that by the choice of the basis $\{w_j\}_{j\geq 1}$, the restriction $P_{m|_V}$ of P_m to V belongs to $\mathcal{L}(V)$, and $\|P_{m|_V}\|_{\mathcal{L}(V)} \leq 1$ for all $m \geq 1$.

Define also

$$u^m(t) = \sum_{j=1}^m \gamma_{m,j}(t) w_j,$$

where the coefficients $\gamma_{m,j}$ are required to satisfy the system

$$\begin{aligned} & \frac{d}{dt}(u^m(t), w_j) + \nu((u^m(t), w_j)) + b(u^m(t), u^m(t), w_j) \\ = \; & \langle f(t), w_j \rangle + (g(t, u_t^m), w_j), \quad \text{a.e. } t > \tau, \quad 1 \leq j \leq m, \end{aligned} \tag{4.6}$$

and the initial condition

$$u^m(\tau + s) = P_m \phi(s) \quad \forall\, s \in [-h, 0]. \tag{4.7}$$

The above system of ordinary functional differential equations with finite delay fulfills the conditions for existence and uniqueness of local solution (see for example [40]).

Next, we will deduce a priori estimates that in particular assure that the solutions u^m do exist for all time $t \in [\tau - h, \infty)$.

Multiplying in (4.6) by $\gamma_{m,j}(t)$, summing from $j = 1$ to $j = m$, we obtain

$$\begin{aligned} \frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 &= 2\langle f(t), u^m(t)\rangle + 2(g(t, u_t^m), u^m(t)) \\ &\leq \nu\|u^m(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + 2L_g|u_t^m|_{C_H}^2, \quad \text{a.e. } t > \tau. \end{aligned}$$

Hence,

$$\begin{aligned} & |u^m(t)|^2 + \nu \int_\tau^t \|u^m(s)\|^2\, ds \\ \leq \; & |\phi(0)|^2 + \int_\tau^t \left(\nu^{-1}\|f(s)\|_*^2 + 2L_g|u_s^m|_{C_H}^2\right) ds \quad \forall\, t \geq \tau. \end{aligned} \tag{4.8}$$

From this inequality, in particular one deduces that

$$|u_t^m|_{C_H}^2 \leq |\phi|_{C_H}^2 + \int_\tau^t \left(\nu^{-1}\|f(s)\|_*^2 + 2L_g|u_s^m|_{C_H}^2\right) ds \quad \forall\, t \geq \tau,$$

and therefore, by Gronwall's lemma we have

$$|u_t^m|_{C_H}^2 \leq e^{2L_g(t-\tau)}\left(|\phi|_{C_H}^2 + \nu^{-1}\int_\tau^t \|f(s)\|_*^2\, ds\right)$$

for all $t \geq \tau$, and any $m \geq 1$.

Then, by (4.8), we deduce that for each $T > \tau$ and $R > 0$, there exists a positive constant $C(\tau, T, R)$, depending on the constants of the problem ν, L_g and f, and on τ, T and R, such that for all $m \geq 1$

$$|u_t^m|_{C_H}^2 + \|u^m\|_{L^2(\tau,T;V)}^2 \leq C(\tau, T, R) \quad \forall\, t \in [\tau, T],\ |\phi|_{C_H} \leq R. \tag{4.9}$$

In particular, this implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau - h, T; H) \cap L^2(\tau, T; V) \quad \forall\, T > \tau. \tag{4.10}$$

From (2.4), (4.6), and because of the choice of the basis, we obtain

$$\|(u^m)'(t)\|_* \leq \nu\|u^m(t)\| + 2^{-1/2}|u^m(t)|\|u^m(t)\| + \|f(t)\|_* + \lambda_1^{-1/2}|g(t, u_t^m)|, \quad \text{a.e. } t > \tau,$$

which combined with (II), (III), (4.9) and (4.10), implies that

$$\{(u^m)'\} \text{ is bounded in } L^2(\tau, T; V') \quad \forall\, T > \tau. \tag{4.11}$$

Step 2: Energy method and compactness results. Now, we combine some well-known compactness results with an energy method (already used in Lemmas 2.14 and 3.2) to pass to the limit in a subsequence of $\{u^m\}$ to obtain a solution for (4.1).

First we observe that

$$u^m{}_{|_{[\tau-h,\tau]}} = P_m\phi \to \phi \quad \text{in } C_H. \tag{4.12}$$

From the assumptions on the operator g and Step 1 we deduce, using the compactness result [61, Theorem 5.1, p. 58] and [86, Lemma 1.2, p. 260], that there exist a subsequence (which we relabel the same) $\{u^m\}$, a function $u \in C([\tau - h, \infty); H)$, with $u_{|_{[\tau-h,\tau]}} = \phi$, $u \in L^2(\tau, T; V)$ and $u' \in L^2(\tau, T; V')$ for all $T > \tau$, and an element $\xi \in L^\infty(\tau, T; (L^2(\Omega))^2)$ for all $T > \tau$, such that

$$\begin{cases} u^m \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H), \\ u^m \rightharpoonup u & \text{weakly in } L^2(\tau, T; V), \\ (u^m)' \rightharpoonup u' & \text{weakly in } L^2(\tau, T; V'), \\ u^m \to u & \text{strongly in } L^2(\tau, T; H), \\ g(\cdot, u^m_\cdot) \overset{*}{\rightharpoonup} \xi & \text{weakly-star in } L^\infty(\tau, T; (L^2(\Omega))^2), \end{cases} \tag{4.13}$$

for all $T > \tau$.

Using (4.13) we can also assume that

$$u^m(t) \to u(t) \quad \text{strongly in } H \text{ a.e. } t \in (\tau, \infty), \tag{4.14}$$

which nevertheless is not enough to deduce that $\xi(\cdot) = g(\cdot, u_\cdot)$.

However, we can obtain convergence for all $t \geq \tau$ with a little more effort and in a more general sense. Observe that

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)\, dr \quad \text{in } V' \quad \forall\, s, t \in [\tau, \infty),$$

and by (4.11) we have that $\{u^m\}$ is equi-continuous on $[\tau, T]$ with values in V', for all $T > \tau$.

Since the injection of V into H is compact, the injection of H into V' is compact too. So, from (4.10) and the equi-continuity in V', by the Ascoli–Arzelà theorem and (4.13), we have that (again, up to a subsequence)

$$u^m \to u \quad \text{in } C([\tau,T];V') \quad \forall\, T>\tau. \tag{4.15}$$

This, jointly with (4.10), allows us to claim that for any sequence $\{t_m\} \subset [\tau,\infty)$, with $t_m \to t$, one has

$$u^m(t_m) \rightharpoonup u(t) \quad \text{weakly in } H, \tag{4.16}$$

where we have used (4.15) in order to identify which is the weak limit.

Our goal now is to prove that in fact

$$u^m \to u \quad \text{in } C([\tau,T];H) \quad \forall\, T>\tau. \tag{4.17}$$

If it were not so, then taking into account that $u \in C([\tau,\infty);H)$, there would exist $T>\tau$, $\varepsilon_0>0$, a value $t_0 \in [\tau,T]$, and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau,T]$, with $\lim_{m\to\infty} t_m = t_0$, such that

$$|u^m(t_m)-u(t_0)| \geq \varepsilon_0 \quad \forall\, m \geq 1.$$

To prove that this is absurd, we will use an energy method.

Observe that the following energy inequality holds for all u^m:

$$\begin{aligned} &\frac{1}{2}|u^m(t)|^2 + \frac{\nu}{2}\int_s^t \|u^m(r)\|^2\,dr \\ \leq\ &\frac{1}{2}|u^m(s)|^2 + \int_s^t \langle f(r),u^m(r)\rangle\,dr + C(t-s) \quad \forall\, \tau \leq s \leq t \leq T, \end{aligned} \tag{4.18}$$

where $C = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |g(r,u^m_r)|^2\,dr \leq D(t-s) \quad \forall\, \tau \leq s \leq t \leq T,$$

by (II), (III) and (4.9). On the other hand, observe that by (4.13), passing to the limit in (4.6), we have that $u \in C([\tau,T];H)$ is a solution of a similar problem to (4.1), namely,

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) = \langle f(t),v\rangle + (\xi(t),v) \quad \forall\, v \in V,$$

fulfilled with the initial datum $u(\tau)=\phi(0)$. Therefore, it satisfies the energy equality

$$\begin{aligned} &|u(t)|^2 + 2\nu\int_s^t \|u(r)\|^2\,dr \\ =\ &|u(s)|^2 + 2\int_s^t \big(\langle f(r),u(r)\rangle + (\xi(r),u(r))\big)\,dr \quad \forall\, \tau \leq s \leq t \leq T. \end{aligned}$$

On other hand, from the last convergence in (4.13) we deduce that

$$\begin{aligned}\int_s^t |\xi(r)|^2\,dr &\le \liminf_{m\to\infty}\int_s^t |g(r,u_r^m)|^2\,dr\\ &\le D(t-s)\quad \forall\,\tau\le s\le t\le T.\end{aligned}$$

So, u also satisfies inequality (4.18) with the same constant C.

Now, consider the functions J_m, $J:[\tau,T]\to\mathbb{R}$ defined by

$$J_m(t)=\frac{1}{2}|u^m(t)|^2-\int_\tau^t\langle f(r),u^m(r)\rangle\,dr-Ct,$$

$$J(t)=\frac{1}{2}|u(t)|^2-\int_\tau^t\langle f(r),u(r)\rangle\,dr-Ct,$$

with C the constant given in (4.18). From (4.18) and the analogous inequality for u, it is clear that J_m and J are non-increasing (and continuous) functions. Moreover, by (4.13) and (4.14),

$$J_m(t)\to J(t)\quad \text{a.e. } t\in(\tau,T). \tag{4.19}$$

Now we are ready to prove that

$$u^m(t_m)\to u(t_0)\quad \text{strongly in } H. \tag{4.20}$$

Firstly, recall from (4.16) that

$$u^m(t_m)\rightharpoonup u(t_0)\quad \text{weakly in } H. \tag{4.21}$$

So, we have that

$$|u(t_0)|\le\liminf_{m\to\infty}|u^m(t_m)|.$$

Therefore, if we show that

$$\limsup_{m\to\infty}|u^m(t_m)|\le|u(t_0)|, \tag{4.22}$$

we obtain that $\lim_{m\to\infty}|u^m(t_m)|=|u(t_0)|$, which jointly with (4.21) implies (4.20).

Now, observe that the case $t_0=\tau$ follows directly from (4.12) and (4.18) with $s=\tau$. So, we may assume that $t_0>\tau$. This is important, since we will approach this value t_0 from the left by a sequence $\{\tilde t_k\}$, i.e., $\lim_{k\to\infty}\tilde t_k\nearrow t_0$, being $\{\tilde t_k\}$ values where (4.19) holds. Since $J(\cdot)$ is continuous at t_0, for any $\varepsilon>0$ there is k_ε such that

$$|J(\tilde t_k)-J(t_0)|<\varepsilon/2\quad\forall\,k\ge k_\varepsilon.$$

On other hand, taking $m\ge m(k_\varepsilon)$ such that $t_m>\tilde t_{k_\varepsilon}$, as J_m is non-increasing and for all $\tilde t_k$ the convergence (4.19) holds, one has

$$J_m(t_m)-J(t_0)\le|J_m(\tilde t_{k_\varepsilon})-J(\tilde t_{k_\varepsilon})|+|J(\tilde t_{k_\varepsilon})-J(t_0)|,$$

and obviously, taking $m\ge m'(k_\varepsilon)$, it is possible to obtain $|J_m(\tilde t_{k_\varepsilon})-J(\tilde t_{k_\varepsilon})|<\varepsilon/2$. It can also be deduced from (4.13) that

$$\int_\tau^{t_m}\langle f(r),u^m(r)\rangle\,dr\to\int_\tau^{t_0}\langle f(r),u(r)\rangle\,dr,$$

so we conclude that (4.22) holds. Thus, (4.20) and finally (4.17) are also true, as we wanted to check.

This also implies, thanks to (4.12), that

$$u_t^m \to u_t \quad \text{in } C_H \quad \forall\, t \geq \tau.$$

Therefore, we identify the weak limit ξ from (4.13), and indeed, from the above convergence and since g satisfies (III), we have

$$g(\cdot, u_\cdot^m) \to g(\cdot, u_\cdot) \quad \text{in } L^2(\tau, T; (L^2(\Omega))^2) \quad \forall\, T > \tau.$$

Thus, we can pass to the limit finally in (4.6) concluding that u solves (4.1).

Finally, the regularity in (a) and (b) is a consequence of the well-known regularity results stated in Theorem 2.3 and the fact that, if $f \in L^2_{\rm loc}(\mathbb{R}; (L^2(\Omega))^2)$, then the function $\hat{f}$ defined by $\hat{f}(t) = f(t) + g(t, u_t)$, $t > \tau$, belongs to $L^2_{\rm loc}(\tau, \infty; (L^2(\Omega))^2)$. ■

Remark 4.6. *Observe that by the uniqueness of the weak solution to (4.1), the convergences in (4.13) hold for the entire sequence $\{u^m\}$ of the Galerkin approximations defined by (4.6) and (4.7).*

We also have the following result on continuity of solutions with respect to the initial datum ϕ.

Proposition 4.7. *Let $f \in L^2_{\rm loc}(\mathbb{R}; V')$, $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), $\tau \in \mathbb{R}$, and ϕ, $\psi \in C_H$, be given.*

Let us denote $u = u(\cdot; \tau, \phi)$ and $v = v(\cdot; \tau, \psi)$ the corresponding weak solutions to (4.1). Then, the following estimate holds:

$$|u_t - v_t|^2_{C_H} \leq |\phi - \psi|^2_{C_H} \exp\left\{ \int_\tau^t \left((2\nu)^{-1}\|u(s)\|^2 + 2L_g\right) ds \right\}$$

for all $t \geq \tau$.

Proof. Let us denote $w = u - v$. Analogously to the obtention of (4.5) in the proof of uniqueness of weak solution to (4.1), we obtain that

$$\begin{aligned} & |w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + 2^{1/2} \int_\tau^t |w(s)|\|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t |w_s|_{C_H} |w(s)|\, ds \end{aligned}$$

for all $t \geq \tau$.

So,

$$\begin{aligned} & |w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + 2^{1/2} \int_\tau^t |w_s|_{C_H} \|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t |w_s|^2_{C_H}\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + \nu \int_\tau^t \|w(s)\|^2\, ds + \int_\tau^t \left((2\nu)^{-1}\|u(s)\|^2 + 2L_g\right) |w_s|^2_{C_H}\, ds \end{aligned}$$

for all $t \geq \tau$, and in particular

$$|w(t)|^2 \leq |\phi(0) - \psi(0)|^2 + \int_\tau^t \big((2\nu)^{-1}\|u(s)\|^2 + 2L_g\big)|w_s|^2_{C_H}\,ds \tag{4.23}$$

for all $t \geq \tau$.

Taking into account that

$$|w(\tau + s)|^2 \leq |\phi - \psi|^2_{C_H} \quad \forall\, s \in [-h, 0],$$

from (4.23) we deduce

$$|w_t|^2_{C_H} \leq |\phi - \psi|^2_{C_H} + \int_\tau^t \big((2\nu)^{-1}\|u(s)\|^2 + 2L_g\big)|w_s|^2_{C_H}\,ds$$

for all $t \geq \tau$.

From this inequality and Gronwall's lemma, we can conclude the result. ■

4.2 Existence of minimal pullback attractors in H norm

Now, by the previous results, we are able to define correctly a process U on C_H associated to (4.1), and to obtain the existence of minimal pullback attractors.

Proposition 4.8. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), be given. Then, the bi-parametric family of mappings $U(t, \tau) : C_H \to C_H$, with $\tau \leq t$, given by*

$$U(t, \tau)\phi = u_t, \tag{4.24}$$

where $u(\cdot) = u(\cdot; \tau, \phi)$ is the unique weak solution to (4.1), defines a continuous process on C_H.

Proof. It is a consequence of Theorem 4.5 and Proposition 4.7. ■

We establish now several estimates for the solution to problem (4.1).

Lemma 4.9. *Consider that the assumptions of Proposition 4.8 are satisfied and let μ be such that $0 < \mu < 2\nu\lambda_1$. Then, for any $\phi \in C_H$, the following estimates hold for the solution to (4.1) for all $t \geq \tau$:*

$$\begin{aligned} |u_t|^2_{C_H} \ &\leq \ e^{\mu h} e^{-(\mu - 2e^{\mu h}L_g)(t-\tau)}|\phi|^2_{C_H} \\ &\qquad + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1}\int_\tau^t e^{-(\mu - 2e^{\mu h}L_g)(t-s)}\|f(s)\|_*^2\,ds, \end{aligned} \tag{4.25}$$

$$\nu\int_\tau^t \|u(s)\|^2\,ds \ \leq \ |u(\tau)|^2 + \nu^{-1}\int_\tau^t \|f(s)\|_*^2\,ds + 2L_g\int_\tau^t |u_s|^2_{C_H}\,ds. \tag{4.26}$$

Proof. Take a μ such that $0 < \mu < 2\nu\lambda_1$. By the energy equality (4.3), one has

$$\begin{aligned}
&\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\
= \;& 2\langle f(t), u(t)\rangle + 2(g(t,u_t), u(t)) \\
\leq \;& 2\|f(t)\|_*\|u(t)\| + 2L_g|u_t|_{C_H}|u(t)| \\
\leq \;& (2\nu - \mu\lambda_1^{-1})\|u(t)\|^2 + (2\nu - \mu\lambda_1^{-1})^{-1}\|f(t)\|_*^2 + 2L_g|u_t|_{C_H}^2, \quad \text{a.e. } t > \tau.
\end{aligned}$$

Thus,

$$\frac{d}{dt}|u(t)|^2 + \mu|u(t)|^2 \leq (2\nu - \mu\lambda_1^{-1})^{-1}\|f(t)\|_*^2 + 2L_g|u_t|_{C_H}^2, \quad \text{a.e. } t > \tau,$$

and therefore,

$$e^{\mu t}|u(t)|^2 \leq e^{\mu\tau}|u(\tau)|^2 + \int_\tau^t e^{\mu s}\big((2\nu - \mu\lambda_1^{-1})^{-1}\|f(s)\|_*^2 + 2L_g|u_s|_{C_H}^2\big)\,ds \quad \forall\, t \geq \tau.$$

From this inequality, we deduce

$$e^{\mu t}|u_t|_{C_H}^2 \leq e^{\mu h}e^{\mu\tau}|\phi|_{C_H}^2 + e^{\mu h}\int_\tau^t e^{\mu s}\big((2\nu - \mu\lambda_1^{-1})^{-1}\|f(s)\|_*^2 + 2L_g|u_s|_{C_H}^2\big)\,ds \quad \forall\, t \geq \tau.$$

Then, by Gronwall's lemma we obtain that (4.25) holds.

Finally, observing that

$$\begin{aligned}
&\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\
\leq \;& 2\|f(t)\|_*\|u(t)\| + 2L_g|u_t|_{C_H}|u(t)| \\
\leq \;& \nu\|u(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + 2L_g|u_t|_{C_H}^2, \quad \text{a.e. } t > \tau,
\end{aligned}$$

we conclude (4.26). ∎

From now on we will assume that

$$\text{there exists } 0 < \mu < 2\nu\lambda_1 \text{ such that } 2e^{\mu h}L_g < \mu, \tag{4.27}$$

and

$$\int_{-\infty}^0 e^{(\mu - 2e^{\mu h}L_g)s}\|f(s)\|_*^2\,ds < \infty. \tag{4.28}$$

Remark 4.10. *If we assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, assumption (4.28) is equivalent to*

$$\int_{-\infty}^t e^{(\mu - 2e^{\mu h}L_g)s}\|f(s)\|_*^2\,ds < \infty \quad \forall\, t \in \mathbb{R}.$$

Having in mind the estimate (4.25), we define the following universe in $\mathcal{P}(C_H)$.

Definition 4.11. *For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\sigma\tau} \sup_{v \in D(\tau)} |v|^2_{C_H} \right) = 0.$$

Once again, accordingly to the notation introduced in Chapter 1, $\mathcal{D}_F(C_H)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H.

Observe that for any $\sigma > 0$, $\mathcal{D}_F(C_H) \subset \mathcal{D}_\sigma(C_H)$, and that $\mathcal{D}_\sigma(C_H)$ is inclusion-closed.

From now on, for brevity, we will denote

$$\sigma_\mu = \mu - 2e^{\mu h} L_g. \tag{4.29}$$

Now, we obtain the existence of a pullback absorbing family for the process U on C_H.

Corollary 4.12. *Under the assumptions of Proposition 4.8, if moreover conditions (4.27) and (4.28) are satisfied, then the family $\widehat{D}_{1,\mu} = \{D_{1,\mu}(t) : t \in \mathbb{R}\}$, with $D_{1,\mu}(t) = \overline{B}_{C_H}(0, \rho_\mu(t))$, the closed ball in C_H of center zero and radius $\rho_\mu(t)$, where*

$$\rho^2_\mu(t) = 1 + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1} \int_{-\infty}^{t} e^{-\sigma_\mu(t-s)} \|f(s)\|^2_* \, ds,$$

is pullback $\mathcal{D}_{\sigma_\mu}(C_H)$-absorbing for the process U defined by (4.24). Moreover, $\widehat{D}_{1,\mu} \in \mathcal{D}_{\sigma_\mu}(C_H)$.

Proof. It follows immediately from Lemma 4.9. ■

By applying again an energy method, we establish the pullback asymptotic compactness of the process $U : \mathbb{R}^2_d \times C_H \to C_H$.

Proposition 4.13. *Under the assumptions of Corollary 4.12, the process U defined by (4.24) is pullback $\widehat{D}_{1,\mu}$-asymptotically compact.*

Proof. Let us fix $t_0 \in \mathbb{R}$. Let $\{u^n\}$ with $u^n = u^n(\cdot; \tau_n, \phi^n)$ be a sequence of weak solutions to (4.1), defined in their respective intervals $[\tau_n - h, \infty)$, with initial data $\phi^n \in D_{1,\mu}(\tau_n) = \overline{B}_{C_H}(0, \rho_\mu(\tau_n))$, where $\{\tau_n\} \subset (-\infty, t_0)$ satisfies that $\tau_n \to -\infty$ as $n \to \infty$. We will prove that the sequence $\{u^n_{t_0}\}$ is relatively compact in C_H, i.e., we will see that there exist a subsequence, relabelled $\{u^n_{t_0}\}$, and a function $\psi \in C_H$, such that $u^n_{t_0} \to \psi$ in C_H.

Consider an arbitrary value $T > h$.

It follows from (4.25) and (4.28) that there exists $n_0(t_0, T)$ such that $\tau_n < t_0 - T$ for $n \geq n_0(t_0, T)$, and

$$|u^n_t|^2_{C_H} \leq R(t_0, T) \quad \forall\, t \in [t_0 - T, t_0],\ n \geq n_0(t_0, T), \tag{4.30}$$

where

$$R(t_0, T) = 1 + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1} e^{-\sigma_\mu(t_0 - T)} \int_{-\infty}^{t_0} e^{\sigma_\mu s} \|f(s)\|^2_* \, ds,$$

so that, in particular,

$$|u^n(t)|^2 \le R(t_0,T) \quad \forall\, t \in [t_0 - T, t_0],\ n \ge n_0(t_0,T). \tag{4.31}$$

Let us denote

$$y^n(t) = u^n(t + t_0 - T) \quad \forall\, t \in [0,T].$$

In particular, by (4.31), the sequence $\{y^n\}_{n\ge n_0(t_0,T)}$ is bounded in $L^\infty(0,T;H)$.

On the other hand, for each $n \ge n_0(t_0,T)$, the function y^n is a weak solution on $[0,T]$ of a problem similar to (4.1), namely with f and g replaced by

$$\tilde{f}(t) = f(t + t_0 - T) \text{ and } \tilde{g}(t,\cdot) = g(t + t_0 - T,\cdot), \quad t \in (0,T),$$

respectively, and with $y_0^n = u_{t_0-T}^n$ and $y_T^n = u_{t_0}^n$. By (4.30), $|y_0^n|_{C_H}^2 \le R(t_0,T)$ for all $n \ge n_0(t_0,T)$. From (4.26) we have

$$\|y^n\|_{L^2(0,T;V)}^2 \le K(t_0,T) \quad \forall\, n \ge n_0(t_0,T),$$

where

$$K(t_0,T) = \nu^{-1}R(t_0,T) + \nu^{-2}\int_0^T \|\widetilde{f}(s)\|_*^2\, ds + \nu^{-1}2L_g R(t_0,T)T.$$

Hence, the sequence $\{y^n\}_{n\ge n_0(t_0,T)}$ is also bounded in $L^2(0,T;V)$, and the sequence of time derivatives $\{(y^n)'\}_{n\ge n_0(t_0,T)}$ is bounded in $L^2(0,T;V')$. Thus, up to a subsequence (relabelled the same), for some function y we have

$$\begin{cases} y^n \overset{*}{\rightharpoonup} y & \text{weakly-star in } L^\infty(0,T;H),\\ y^n \rightharpoonup y & \text{weakly in } L^2(0,T;V),\\ (y^n)' \rightharpoonup y' & \text{weakly in } L^2(0,T;V'),\\ y^n \to y & \text{strongly in } L^2(0,T;H),\\ y^n(t) \to y(t) & \text{strongly in } H, \text{ a.e. } t \in (0,T). \end{cases}$$

Observe also that $y \in C([0,T];H)$, and that for every sequence $\{t_n\} \subset [0,T]$ with $t_n \to t_*$, one has

$$y^n(t_n) \rightharpoonup y(t_*) \quad \text{weakly in } H, \tag{4.32}$$

which follows from the boundedness of the sequences $\{y^n\}_{n\ge n_0(t_0,T)}$ and $\{(y^n)'\}_{n\ge n_0(t_0,T)}$ in $L^\infty(0,T;H)$ and $L^2(0,T;V')$ respectively, and the compactness of the injection of H into V' (see the proof of Theorem 4.5 for a similar argument).

Also, by (II), (III), and (4.30), we obtain

$$\int_0^t |\tilde{g}(s,y_s^n)|^2\, ds \le Ct,$$

where $C > 0$ does not depend neither on n nor $t \in [0,T]$. Thus, eventually extracting a subsequence, there exists $\xi \in L^2(0,T;(L^2(\Omega))^2)$ such that

$$\tilde{g}(\cdot,y_\cdot^n) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;(L^2(\Omega))^2),$$

and therefore

$$\begin{aligned}
&\int_s^t |\tilde{g}(r, y_r^n)|^2\, dr \le C(t-s), \\
&\int_s^t |\xi(r)|^2\, dr \le \liminf_{n\to\infty} \int_s^t |\tilde{g}(r, y_r^n)|^2\, dr \le C(t-s),
\end{aligned} \tag{4.33}$$

for all $0 \le s \le t \le T$.

Then, in a standard way, one can prove that $y(\cdot)$ is the unique weak solution to the problem

$$\begin{cases}
\dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \tilde{f}(t) + \xi(t) \quad \text{in } \ \Omega \times (0, T), \\
\nabla \cdot u = 0 \quad \text{in } \ \Omega \times (0, T), \\
u = 0 \quad \text{on } \ \partial\Omega \times (0, T), \\
u(x, 0) = y(x, 0), \quad x \in \Omega.
\end{cases}$$

By the energy equality and (4.33), we obtain that

$$\frac{1}{2}|z(t)|^2 \le \frac{1}{2}|z(s)|^2 + \int_s^t \langle \tilde{f}(r), z(r)\rangle\, dr + \widetilde{C}(t-s) \quad \forall\, 0 \le s \le t \le T,$$

where $\widetilde{C} = C(4\nu\lambda_1)^{-1}$, and $z = y^n$ or $z = y$. Then, the maps $\tilde{J}_n, \tilde{J} : [0, T] \to \mathbb{R}$ defined by

$$\begin{aligned}
\tilde{J}_n(t) &= \frac{1}{2}|y^n(t)|^2 - \int_0^t \langle \tilde{f}(r), y^n(r)\rangle\, dr - Ct, \\
\tilde{J}(t) &= \frac{1}{2}|y(t)|^2 - \int_0^t \langle \tilde{f}(r), y(r)\rangle\, dr - Ct,
\end{aligned}$$

are non-increasing and continuous, and satisfy

$$\tilde{J}_n(t) \to \tilde{J}(t) \quad \text{a.e. } t \in (0, T). \tag{4.34}$$

We can use the functionals $\tilde{J}_n$ and $\tilde{J}$ to deduce that $y^n \to y$ in $C([\delta, T]; H)$, for any $0 < \delta < T$. If this is not true, then there exist $0 < \delta_* < T$, $\varepsilon > 0$, and subsequences $\{y^m\} \subset \{y^n\}_{n \ge n_0(t_0, T)}$ and $\{t_m\} \subset [\delta_*, T]$, with $t_m \to t_*$, such that

$$|y^m(t_m) - y(t_*)| \ge \varepsilon \quad \forall\, m \ge 1. \tag{4.35}$$

Let us fix $\epsilon > 0$. Observe that $t_* \in [\delta_*, T]$, and therefore, by (4.34) and the continuity and non-increasing character of $\tilde{J}$, there exists $0 < \hat{t}_\epsilon < t_*$ such that

$$\lim_{m\to\infty} \tilde{J}_m(\hat{t}_\epsilon) = \tilde{J}(\hat{t}_\epsilon), \tag{4.36}$$

and

$$0 \le \tilde{J}(\hat{t}_\epsilon) - \tilde{J}(t_*) \le \epsilon. \tag{4.37}$$

As $t_m \to t_*$, there exists an m_ϵ such that $\hat{t}_\epsilon < t_m$ for all $m \geq m_\epsilon$. Then, by (4.37),

$$\begin{aligned}\tilde{J}_m(t_m) - \tilde{J}(t_*) &\leq \tilde{J}_m(\hat{t}_\epsilon) - \tilde{J}(t_*)\\ &\leq |\tilde{J}_m(\hat{t}_\epsilon) - \tilde{J}(\hat{t}_\epsilon)| + |\tilde{J}(\hat{t}_\epsilon) - \tilde{J}(t_*)|\\ &\leq |\tilde{J}_m(\hat{t}_\epsilon) - \tilde{J}(\hat{t}_\epsilon)| + \epsilon\end{aligned}$$

for all $m \geq m_\epsilon$, and consequently, by (4.36),

$$\limsup_{m\to\infty} \tilde{J}_m(t_m) \leq \tilde{J}(t_*) + \epsilon.$$

Thus, as $\epsilon > 0$ is arbitrary, we deduce that

$$\limsup_{m\to\infty} \tilde{J}_m(t_m) \leq \tilde{J}(t_*). \tag{4.38}$$

Taking into account that $t_m \to t_*$, and

$$\int_0^{t_m} \langle \tilde{f}(r), y^m(r)\rangle\, dr \to \int_0^{t_*} \langle \tilde{f}(r), y(r)\rangle\, dr,$$

from (4.38) we deduce that

$$\limsup_{m\to\infty} |y^m(t_m)| \leq |y(t_*)|.$$

This last inequality and (4.32), imply that

$$y^m(t_m) \to y(t_*) \quad \text{strongly in } H,$$

which is in contradiction with (4.35).

We have thus proved that $y^n \to y$ in $C([\delta, T]; H)$, for any $0 < \delta < T$. As $T > h$, we obtain in particular that $u^n_{t_0} \to \psi$ in C_H, where $\psi(s) = y(s+T)$, for $s \in [-h, 0]$. ■

Joining all the above statements we obtain the existence of minimal pullback attractors for the process U on C_H associated to problem (4.1).

Theorem 4.14. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying the assumptions (I)–(III), (4.27) and (4.28), are given. Then, there exist the minimal pullback $\mathcal{D}_F(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H)}$ and the minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$ for the process U defined by (4.24). The family $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$ belongs to $\mathcal{D}_{\sigma_\mu}(C_H)$, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \subset \overline{B}_{C_H}(0, \rho_\mu(t)) \quad \forall\, t \in \mathbb{R}. \tag{4.39}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$ is a consequence of Theorem 1.11, since U is continuous (cf. Proposition 4.8) and therefore closed, the existence of a pullback absorbing family was given by Corollary 4.12, and in Proposition 4.13 we have proved the pullback $\widehat{D}_{1,\mu}$-asymptotic compactness.

By Corollary 1.13, the case of fixed bounded sets follows immediately since $\mathcal{D}_F(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H)$. Then, we also deduce the first inclusion in (4.39).

Finally, Theorem 1.11 also implies the last inclusion in (4.39) and the fact that $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$ belongs to $\mathcal{D}_{\sigma_\mu}(C_H)$, since the sufficient conditions in Remark 1.12 hold. Namely, $\mathcal{D}_{\sigma_\mu}(C_H)$ is inclusion-closed, by construction $D_{1,\mu}(t)$ is closed in C_H for all $t \in \mathbb{R}$, and $\widehat{D}_{1,\mu}$ belongs to $\mathcal{D}_{\sigma_\mu}(C_H)$ (cf. Corollary 4.12). ■

Remark 4.15. (i) If, additionally, we assume that

$$\sup_{r \leq 0} \int_{-\infty}^{r} e^{-\sigma_\mu (r-s)} \|f(s)\|_*^2 \, ds < \infty,$$

where σ_μ is given by (4.29), then, taking into account Remark 1.14, we deduce that

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \forall\, t \in \mathbb{R}.$$

(ii) Observe that a natural question concerning the existence of more families of pullback attractors is to strengthen the conditions on the parameter μ that satisfies (4.27) and (4.28). More exactly, if $\sigma < \sigma'$, then $\mathcal{D}_\sigma(C_H) \subset \mathcal{D}_{\sigma'}(C_H)$. Therefore, in order to obtain attractors for bigger universes, we would wonder if there exists $\mu' \in (0, 2\nu\lambda_1)$ such that $\sigma_{\mu'} > \sigma_\mu$. In such a case, conditions (4.27) and (4.28) would be satisfied automatically. The key point for having $\sigma_{\mu'} > \sigma_\mu$ is to analyze the growth behaviour of the map $\mu \mapsto \sigma_\mu$. Namely, if the map $\mu \mapsto \sigma_\mu$ is non-decreasing, we look for $\mu < \mu' < 2\nu\lambda_1$ (this may involve a smallness condition on the delay); otherwise, we seek for $0 < \mu' < \mu$. Under any of these conditions, we would obtain new families of pullback attractors and new relations among them (see Remark 2.16 in Chapter 2 or [6, Remark 5] for similar results in a simpler context).

4.3 Attraction in V norm and some regularity results for the pullback attractors

Now, we strengthen the regularity of solutions and a second energy equality for them in order to obtain additional attraction, namely, in the H^1 norm instead of L^2 as in Section 4.2.

For any $\tilde{h} \in [0, h]$, let us denote

$$C_H^{\tilde{h},V} = \{\varphi \in C_H : \varphi|_{[-\tilde{h},0]} \in B([-\tilde{h}, 0]; V)\},$$

where $B([-\tilde{h}, 0]; V)$ is the space of bounded functions from $[-\tilde{h}, 0]$ into V. The space $C_H^{\tilde{h},V}$ is a Banach space with the norm

$$\|\varphi\|_{\tilde{h},V} = |\varphi|_{C_H} + \sup_{\theta \in [-\tilde{h},0]} \|\varphi(\theta)\|.$$

Observe that the space $C_V = C([-h, 0]; V)$ is a Banach subspace of $C_H^{h,V}$.

Proposition 4.16. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; (L^2(\Omega))^2)$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying the assumptions (I)–(III), are given. Then, for any bounded set $B \subset C_H$, one has:*

(i) The set of weak solutions to (4.1) $\{u(\cdot; \tau, \phi) : \phi \in B\}$ is bounded in $L^\infty(\tau + \varepsilon, T; V)$, for any $\varepsilon > 0$ and any $T > \tau + \varepsilon$.

(ii) Moreover, if $\{\phi(0) : \phi \in B\}$ is bounded in V, then $\{u(\cdot;\tau,\phi) : \phi \in B\}$ is bounded in $L^\infty(\tau,T;V)$, for all $T > \tau$.

Proof. By (4.4) and the regularity property (a) in Theorem 4.5, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 + b(u(\theta),u(\theta),Au(\theta)) \\
= \;& (f(\theta) + g(\theta,u_\theta), Au(\theta)) \\
\le \;& \frac{2}{\nu}(|f(\theta)|^2 + |g(\theta,u_\theta)|^2) + \frac{\nu}{4}|Au(\theta)|^2, \quad \text{a.e. } \theta > \tau,
\end{aligned}$$

where we have used Young's inequality.

Since the trilinear term b can be estimated using (2.5) as

$$\begin{aligned}
|b(u(\theta),u(\theta),Au(\theta))| &\le C_1|u(\theta)|^{1/2}\|u(\theta)\||Au(\theta)|^{3/2} \\
&\le \frac{\nu}{4}|Au(\theta)|^2 + C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4,
\end{aligned}$$

where $C^{(\nu)}$ is defined by (2.20), this, combined with the above and the properties of g, gives

$$\begin{aligned}
&\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 \\
\le \;& \frac{4}{\nu}|f(\theta)|^2 + \frac{4L_g^2}{\nu}|u_\theta|_{C_H}^2 + 2C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4, \quad \text{a.e. } \theta > \tau. \qquad (4.40)
\end{aligned}$$

Integrating, in particular we deduce that for all $\tau < s \le r$

$$\begin{aligned}
\|u(r)\|^2 \le \;& \|u(s)\|^2 + \frac{4}{\nu}\int_s^r |f(\theta)|^2\,d\theta + \frac{4L_g^2}{\nu}\int_s^r |u_\theta|_{C_H}^2\,d\theta \\
&+ 2C^{(\nu)}\int_s^r |u(\theta)|^2\|u(\theta)\|^4\,d\theta.
\end{aligned}$$

By Gronwall's lemma we obtain again that for all $\tau < s \le r$

$$\begin{aligned}
\|u(r)\|^2 \le \;& \left(\|u(s)\|^2 + \frac{4}{\nu}\int_s^r |f(\theta)|^2\,d\theta + \frac{4L_g^2}{\nu}\int_s^r |u_\theta|_{C_H}^2\,d\theta\right) \\
&\times \exp\left(2C^{(\nu)}\int_s^r |u(\theta)|^2\|u(\theta)\|^2\,d\theta\right). \qquad (4.41)
\end{aligned}$$

Integrating once more with respect to $s \in (\tau, r)$, it yields

$$\begin{aligned}
&(r-\tau)\|u(r)\|^2 \\
\le \;& \left(\int_\tau^T \|u(s)\|^2\,ds + \frac{4(T-\tau)}{\nu}\int_\tau^T |f(\theta)|^2\,d\theta + \frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T |u_\theta|_{C_H}^2\,d\theta\right) \\
&\times \exp\left(2C^{(\nu)}\int_\tau^T |u(\theta)|^2\|u(\theta)\|^2\,d\theta\right) \quad \forall\, \tau < r \le T.
\end{aligned}$$

In particular, for $\tau+\varepsilon\le r\le T$, it holds

$$\begin{aligned}\|u(r)\|^2 \;&\le\; \frac{1}{\varepsilon}\bigg(\int_\tau^T \|u(s)\|^2\,ds+\frac{4(T-\tau)}{\nu}\int_\tau^T |f(\theta)|^2\,d\theta+\frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T |u_\theta|_{C_H}^2\,d\theta\bigg)\\ &\quad\times\exp\bigg(2C^{(\nu)}\int_\tau^T |u(\theta)|^2\|u(\theta)\|^2\,d\theta\bigg).\end{aligned}$$

Taking into account (4.25) and (4.26), the claim (i) is proved.

The proof of claim (ii) is simpler. If $\phi(0)$ belongs to V, then from (4.40) one deduces that for all $\tau\le r\le T$,

$$\|u(r)\|^2\le\|u(\tau)\|^2+\frac{4}{\nu}\int_\tau^r |f(\theta)|^2\,d\theta+\frac{4L_g^2}{\nu}\int_\tau^r |u_\theta|_{C_H}^2\,d\theta+2C^{(\nu)}\int_\tau^r |u(\theta)|^2\|u(\theta)\|^4\,d\theta.$$

Therefore, one may apply directly Gronwall's lemma and proceed analogously as before to conclude (ii). ■

Corollary 4.17. *Under the assumptions of Proposition 4.16, the process U defined by (4.24) satisfies that $U(t,\tau)$ maps bounded sets of C_H into bounded sets of C_H, for all $t\ge\tau$.*

Moreover, for any $\tilde h\in[0,h]$, the family of mappings $U(t,\tau)|_{C_H^{\tilde h,V}}$, with $t\ge\tau$, is also a well defined process on $C_H^{\tilde h,V}$, and maps bounded sets of $C_H^{\tilde h,V}$ into bounded sets of $C_H^{\tilde h,V}$.

Proposition 4.18. *Assume that $f\in L^2_{\rm loc}(\mathbb{R};(L^2(\Omega))^2)$, and $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$ satisfying the assumptions (I)–(III), are given. Let us denote $u=u(\cdot;\tau,\phi)$ and $v=v(\cdot;\tau,\psi)$ the solutions to (4.1) corresponding to initial data ϕ and $\psi\in C_H^{0,V}$. Then, the following estimate holds:*

$$\begin{aligned}&\|u(s)-v(s)\|^2\\ \le\;&\bigg(\|\phi(0)-\psi(0)\|^2+\frac{L_g^2}{\nu}\int_\tau^t |u_\theta-v_\theta|_{C_H}^2\,d\theta\bigg)\\ &\times\exp\bigg[\int_\tau^t\Big(2C^{(\nu)}\lambda_1^{-1}\|u(\theta)\|^4+\frac{2C_1^2}{\nu}|v(\theta)||Av(\theta)|\Big)d\theta\bigg]\quad\forall\,\tau\le s\le t,\end{aligned}\tag{4.42}$$

where $C^{(\nu)}$ is given in (2.20).

As a consequence, for all $\tilde h\in[0,h]$ and any $\tau\le t$, the mapping $U(t,\tau):C_H^{\tilde h,V}\to C_H^{\tilde h,V}$ given by (4.24), is continuous.

Proof. In order to prove the statement, we only have to check (4.42) and combine it with Proposition 4.7, and claim (ii) in Proposition 4.16.

Let us denote $w=u-v$. If we apply the energy equality to w, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\nu|Aw(t)|^2+b(u(t),u(t),Aw(t))-b(v(t),v(t),Aw(t))\\ =\;&(g(t,u_t)-g(t,v_t),Aw(t))\\ \le\;&\frac{L_g^2}{2\nu}|w_t|_{C_H}^2+\frac{\nu}{2}|Aw(t)|^2,\quad\text{a.e. } t>\tau,\end{aligned}$$

where we have used Young's inequality and the property (III) of g.

The trilinear terms can be estimated, using (2.5), as follows:

$$\begin{aligned}
& |b(u(t),u(t),Aw(t)) - b(v(t),v(t),Aw(t))| \\
= \;& |b(w(t),u(t),Aw(t)) + b(v(t),w(t),Aw(t))| \\
\le \;& C_1|w(t)|^{1/2}\|u(t)\||Aw(t)|^{3/2} + C_1|v(t)|^{1/2}|Av(t)|^{1/2}\|w(t)\||Aw(t)| \\
\le \;& C^{(\nu)}|w(t)|^2\|u(t)\|^4 + \frac{C_1^2}{\nu}|v(t)||Av(t)|\|w(t)\|^2 + \frac{\nu}{2}|Aw(t)|^2.
\end{aligned}$$

Therefore, from above we obtain that

$$\frac{d}{dt}\|w(t)\|^2 \le 2C^{(\nu)}|w(t)|^2\|u(t)\|^4 + \frac{2C_1^2}{\nu}|v(t)||Av(t)|\|w(t)\|^2 + \frac{L_g^2}{\nu}|w_t|_{C_H}^2, \quad \text{a.e. } t > \tau.$$

Integrating, it yields for all $\tau \le s \le t$,

$$\begin{aligned}
\|w(s)\|^2 \;\le\; & \|w(\tau)\|^2 + \frac{L_g^2}{\nu}\int_\tau^s |w_\theta|_{C_H}^2\,d\theta \\
& + \int_\tau^s \|w(\theta)\|^2\Big(2C^{(\nu)}\lambda_1^{-1}\|u(\theta)\|^4 + \frac{2C_1^2}{\nu}|v(\theta)||Av(\theta)|\Big)d\theta.
\end{aligned}$$

From this inequality, using Gronwall's lemma, we deduce (4.42). ■

We introduce the following universes in $\mathcal{P}(C_H^{\tilde h,V})$.

Definition 4.19. *For any $\sigma > 0$ and $\tilde h \in [0,h]$, we will denote by $\mathcal{D}_\sigma^{\tilde h,V}(C_H)$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma(C_H)$ such that for any $t \in \mathbb{R}$ and for any $\varphi \in D(t)$, it holds that $\varphi|_{[-\tilde h,0]} \in B([-\tilde h,0];V)$.*

Analogously, we will denote by $\mathcal{D}_F^{\tilde h,V}(C_H)$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H such that for any $\varphi \in D$, it holds that $\varphi|_{[-\tilde h,0]} \in B([-\tilde h,0];V)$.

Finally, we will denote by $\mathcal{D}_F(C_H^{\tilde h,V})$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_H^{\tilde h,V}$.

Remark 4.20. *The chain of inclusions for the universes in the above definition and the universes introduced in Section 4.2, is the following:*

$$\mathcal{D}_F(C_H^{\tilde h,V}) \subset \mathcal{D}_F^{\tilde h,V}(C_H) \subset \mathcal{D}_\sigma^{\tilde h,V}(C_H) \subset \mathcal{D}_\sigma(C_H),$$

and

$$\mathcal{D}_F(C_H^{\tilde h,V}) \subset \mathcal{D}_F^{\tilde h,V}(C_H) \subset \mathcal{D}_F(C_H) \subset \mathcal{D}_\sigma(C_H),$$

for all $\sigma > 0$ and any $\tilde h \in [0,h]$.

It must also be pointed out that the universe $\mathcal{D}_\sigma^{\tilde h,V}(C_H)$ is inclusion-closed, which will be important (cf. Remark 1.12).

Finally, it is clear that if $0 \le \tilde h_1 < \tilde h_2 \le h$, then

$$\mathcal{D}_F(C_H^{\tilde h_2,V}) \subset \mathcal{D}_F(C_H^{\tilde h_1,V}), \quad \mathcal{D}_F^{\tilde h_2,V}(C_H) \subset \mathcal{D}_F^{\tilde h_1,V}(C_H), \quad \mathcal{D}_\sigma^{\tilde h_2,V}(C_H) \subset \mathcal{D}_\sigma^{\tilde h_1,V}(C_H).$$

We establish now some results on absorbing properties of $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$.

Proposition 4.21. *Let g satisfying assumptions (I)-(III) be given. Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; (L^2(\Omega))^2)$ satisfies that there exists $0 < \mu < 2\nu\lambda_1$ such that $\mu > 2e^{\mu h}L_g$, and*

$$\int_{-\infty}^{0} e^{\sigma_\mu s}|f(s)|^2\,ds < \infty, \tag{4.43}$$

where σ_μ is given by (4.29).

Then, for any $\tilde{h} \in [0, h]$, the family $\widehat{D}_{1,\mu,\tilde{h}} = \{D_{1,\mu,\tilde{h}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$, with

$$D_{1,\mu,\tilde{h}}(t) = D_{1,\mu}(t) \cap C_H^{\tilde{h},V},$$

where $D_{1,\mu}(t)$ is defined in Corollary 4.12, is a family of closed sets of $C_H^{\tilde{h},V}$, which is pullback $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$-absorbing for the process $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ given by (4.24). Moreover, $\widehat{D}_{1,\mu,\tilde{h}}$ belongs to $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$.

Proof. It is a consequence of Corollary 4.12. ■

Lemma 4.22. *Under the assumptions of Proposition 4.21, for any $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$ and any $r > h$, the family $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$, where $D^{(r)}(\tau) = U(\tau + r, \tau)D(\tau)$, for any $\tau \in \mathbb{R}$, belongs to $\mathcal{D}_{\sigma_\mu}^{h,V}(C_H)$.*

Proof. From (4.25), we deduce

$$\begin{aligned}\sup_{\psi\in D^{(r)}(\tau)} \left(e^{\sigma_\mu\tau}|\psi|^2_{C_H}\right) \;\le\;& e^{\mu h-\sigma_\mu r}\sup_{\phi\in D(\tau)}\left(e^{\sigma_\mu\tau}|\phi|^2_{C_H}\right)\\ &+(2\nu\lambda_1-\mu)^{-1}e^{\mu h-\sigma_\mu r}\int_\tau^{\tau+r} e^{\sigma_\mu s}|f(s)|^2\,ds.\end{aligned}$$

From this inequality, property (a) in Theorem 4.5, and assumption (4.43), we deduce the result. ■

Now, we establish several estimates in finite intervals of time when the initial time is sufficiently shifted in a pullback sense (cf. Lemmas 2.10 and 3.7 for similar results in a context without delays).

Lemma 4.23. *Under the assumptions of Proposition 4.21, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$, there exist $\tau_1(\widehat{D}, t, h) < t - 2h - 2$ and functions $\{\rho_i\}_{i=1}^4$ depending on t and h, such that for any $\tau \le \tau_1(\widehat{D}, t, h)$ and any $\phi^\tau \in D(\tau)$, it holds*

$$\begin{cases} |u(r;\tau,\phi^\tau)|^2 \le \rho_1(t) & \forall\, r \in [t-2h-2, t],\\ \|u(r;\tau,\phi^\tau)\|^2 \le \rho_2(t) & \forall\, r \in [t-h-1, t],\\ \nu\displaystyle\int_{r-1}^{r} |Au(\theta;\tau,\phi^\tau)|^2\,d\theta \le \rho_3(t) & \forall\, r \in [t-h, t],\\ \displaystyle\int_{r-1}^{r} |u'(\theta;\tau,\phi^\tau)|^2\,d\theta \le \rho_4(t) & \forall\, r \in [t-h, t],\end{cases} \tag{4.44}$$

where

$$
\begin{aligned}
\rho_1(t) &= 1 + e^{\mu h}(2\nu\lambda_1 - \mu)^{-1} e^{-\sigma_\mu(t-2h-2)} \int_{-\infty}^{t} e^{\sigma_\mu s} |f(s)|^2 \, ds, \\
\rho_2(t) &= \left(\nu^{-1}\left(1 + 2\nu^{-1}\lambda_1^{-1} L_g^2 + 4L_g^2\right)\rho_1(t) + \nu^{-1}\left(4 + 2\nu^{-1}\lambda_1^{-1}\right)\int_{t-h-2}^{t} |f(\theta)|^2 \, d\theta\right) \\
&\quad \times \exp\left\{2\nu^{-1} C^{(\nu)} \rho_1(t) \left[\left(1 + 2\nu^{-1}\lambda_1^{-1} L_g^2\right)\rho_1(t) + 2\nu^{-1}\lambda_1^{-1} \int_{t-h-2}^{t} |f(\theta)|^2 \, d\theta\right]\right\}, \\
\rho_3(t) &= \rho_2(t) + 2C^{(\nu)} \rho_1(t) \rho_2^2(t) + 4L_g^2 \nu^{-1} \rho_1(t) + 4\nu^{-1} \int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta, \\
\rho_4(t) &= \nu \rho_2(t) + 4L_g^2 \rho_1(t) + 2C_1^2 \nu^{-1} \rho_2(t) \rho_3(t) + 4 \int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta,
\end{aligned}
$$

and $C^{(\nu)}$ is given in (2.20).

Proof. Let $\tau_1(\widehat{D}, t, h) < t - 2h - 2$ be such that

$$
e^{\mu h} e^{-\sigma_\mu(t-2h-2)} e^{\sigma_\mu \tau} |\phi^\tau|_{C_H}^2 \le 1 \quad \forall \tau \le \tau_1(\widehat{D}, t, h), \ \phi^\tau \in D(\tau).
$$

Consider fixed $\tau \le \tau_1(\widehat{D}, t, h)$ and $\phi^\tau \in D(\tau)$.

First estimate in (4.44) follows directly from (4.25), using the increasing character of the exponential.

Now, for the rest of the estimates, let us consider again the Galerkin approximations already used in Theorem 4.5, and denote for short $u^m(r) = u^m(r; \tau, \phi^\tau)$.

Multiplying in (4.6) by $\gamma_{m,j}(t)$, and summing from $j = 1$ to m, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu \|u^m(t)\|^2 &= (f(t) + g(t, u_t^m), u^m(t)) \\
&\le \frac{1}{\nu\lambda_1}\left(|f(t)|^2 + |g(t, u_t^m)|^2\right) + \frac{\nu}{2}\lambda_1 |u^m(t)|^2, \quad \text{a.e. } t > \tau,
\end{aligned}
$$

where we have used Young's inequality. Now, by the assumptions (II) and (III) on g, we obtain

$$
\frac{d}{dt}|u^m(t)|^2 + \nu \|u^m(t)\|^2 \le \frac{2}{\nu\lambda_1}\left(|f(t)|^2 + L_g^2 |u_t^m|_{C_H}^2\right), \quad \text{a.e. } t > \tau.
$$

Integrating, in particular we deduce that

$$
\nu \int_{r-1}^{r} \|u^m(\theta)\|^2 \, d\theta \le |u^m(r-1)|^2 + \frac{2}{\nu\lambda_1} \int_{r-1}^{r} \left(|f(\theta)|^2 + L_g^2 |u_\theta^m|_{C_H}^2\right) d\theta \quad \forall \tau \le r - 1. \quad (4.45)
$$

Now, observe that the first estimate in (4.44) and the estimates obtained in the proof of Proposition 4.16 also hold for the u^m.

From (4.41), integrating with respect to $s \in (r-1, r)$, and using the first estimate in (4.44), we obtain

$$
\begin{aligned}
\|u^m(r)\|^2 \le\ & \left(\int_{r-1}^{r} \|u^m(s)\|^2 \, ds + 4\nu^{-1} \int_{r-1}^{r} |f(\theta)|^2 \, d\theta + 4L_g^2 \nu^{-1} \rho_1(t)\right) \\
& \times \exp\left(2C^{(\nu)} \rho_1(t) \int_{r-1}^{r} \|u^m(\theta)\|^2 \, d\theta\right) \quad \forall r \in [t-h-1, t].
\end{aligned}
$$

From this, jointly with (4.45) and the first estimate in (4.44) for u^m, one deduces

$$\|u^m(r;\tau,\phi^\tau)\|^2 \leq \rho_2(t) \quad \forall\, r \in [t-h-1,t]. \tag{4.46}$$

From this inequality and Remark 4.6, we deduce that

$$u^m \overset{*}{\rightharpoonup} u(\cdot;\tau,\phi^\tau) \quad \text{weakly-star in } L^\infty(t-h-1,t;V).$$

So, taking inferior limit when m goes to infinity in (4.46), and using the fact that $u(\cdot;\tau,\phi^\tau) \in C([t-h-1,t];V)$, we obtain the second estimate in (4.44).

On other hand, from (4.40) we also have

$$\begin{aligned}
&\nu \int_{r-1}^{r} |Au^m(\theta)|^2\, d\theta \\
\leq\ & \|u^m(r-1)\|^2 + 4\nu^{-1} \int_{r-1}^{r} |f(\theta)|^2\, d\theta + 2C^{(\nu)} \int_{r-1}^{r} |u^m(\theta)|^2 \|u^m(\theta)\|^4\, d\theta \\
& + 4L_g^2 \nu^{-1} \int_{r-1}^{r} |u_\theta^m|_{C_H}^2\, d\theta \quad \forall\, \tau \leq r-1.
\end{aligned}$$

Therefore,

$$\nu \int_{r-1}^{r} |Au^m(\theta;\tau,\phi^\tau)|^2\, d\theta \leq \rho_3(t) \quad \forall\, r \in [t-h,t]. \tag{4.47}$$

From Remark 4.6 and (4.47), we deduce that

$$u^m \rightharpoonup u(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1,r;D(A)),\ \forall\, r \in [t-h,t].$$

Thus, taking inferior limit when m goes to infinity in (4.47), we obtain the third inequality in (4.44).

Finally, multiplying in (4.6) by $\gamma'_{m,j}(t)$, and summing from $j=1$ to m, we obtain

$$\begin{aligned}
& |(u^m)'(\theta)|^2 + \frac{\nu}{2} \frac{d}{d\theta} \|u^m(\theta)\|^2 + b(u^m(\theta), u^m(\theta), (u^m)'(\theta)) \\
=\ & (f(\theta), (u^m)'(\theta)) + (g(\theta, u_\theta^m), (u^m)'(\theta)), \quad \text{a.e. } \theta > \tau.
\end{aligned}$$

Observing that by Young's inequality and (2.6),

$$\begin{aligned}
|(f(\theta), (u^m)'(\theta))| &\leq \frac{1}{8} |(u^m)'(\theta)|^2 + 2|f(\theta)|^2, \\
|(g(\theta, u_\theta^m), (u^m)'(\theta))| &\leq \frac{1}{8} |(u^m)'(\theta)|^2 + 2|g(\theta, u_\theta^m)|^2, \\
|b(u^m(\theta), u^m(\theta), (u^m)'(\theta))| &\leq C_1 |Au^m(\theta)| \|u^m(\theta)\| |(u^m)'(\theta)| \\
&\leq \frac{1}{4} |(u^m)'(\theta)|^2 + C_1^2 |Au^m(\theta)|^2 \|u^m(\theta)\|^2,
\end{aligned}$$

we obtain that

$$|(u^m)'(\theta)|^2 + \nu \frac{d}{d\theta} \|u^m(\theta)\|^2 \leq 4|f(\theta)|^2 + 4|g(\theta, u_\theta^m)|^2 + 2C_1^2 |Au^m(\theta)|^2 \|u^m(\theta)\|^2, \quad \text{a.e. } \theta > \tau.$$

From the properties of g, and integrating above, we conclude

$$\begin{aligned}
&\int_{r-1}^{r} |(u^m)'(\theta)|^2 \, d\theta \\
\leq\ & \nu\|u^m(r-1)\|^2 + 4\int_{r-1}^{r} |f(\theta)|^2 \, d\theta + 2C_1^2 \int_{r-1}^{r} |Au^m(\theta)|^2 \|u^m(\theta)\|^2 \, d\theta \\
&+4L_g^2 \int_{r-1}^{r} |u_\theta^m|_{C_H}^2 \, d\theta \quad \forall\, \tau \leq r-1.
\end{aligned}$$

From the first estimate in (4.44) for u^m, (4.46) and (4.47), we deduce that

$$\int_{r-1}^{r} |(u^m)'(\theta;\tau,\phi^\tau)|^2 \, d\theta \leq \rho_4(t) \quad \forall\, r \in [t-h,t]. \tag{4.48}$$

From Remark 4.6 and (4.48), we deduce that

$$(u^m)' \rightharpoonup u'(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1,r;H),\ \forall\, r \in [t-h,t].$$

Thus, taking inferior limit when m goes to infinity in (4.48), we obtain the fourth inequality in (4.44). ■

Now, we can prove the $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$-asymptotic compactness of the process U restricted to the space $C_H^{\tilde{h},V}$. The proof relies on an energy method with continuous functions, which is similar to that used in the proof of Proposition 4.13, but starting with the energy equality (4.4), as in Lemma 2.14; we reproduce it here just for the sake of completeness.

Lemma 4.24. *Under the assumptions of Proposition 4.21, and for any $\tilde{h} \in [0,h]$, the process $U : \mathbb{R}_d^2 \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ is pullback $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$-asymptotically compact.*

Proof. Let $\tilde{h} \in [0,h]$ be fixed. Since, taking into account Proposition 4.2, the asymptotic compactness in the norm of C_H was already established in Proposition 4.13, we only must care about the sup norm in $B([-\tilde{h},0];V)$. So, let us fix $t \in \mathbb{R}$, a family $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$, a sequence $\{\tau_n\} \subset (-\infty,t]$ with $\tau_n \to -\infty$, and a sequence $\{\phi^{\tau_n}\} \subset C_H^{\tilde{h},V}$, with $\phi^{\tau_n} \in D(\tau_n)$ for all n.

For short, let us denote $u^n(\cdot) = u(\cdot;\tau_n,\phi^{\tau_n})$. It is enough to prove that the sequence $\{u^n(t+\cdot)\}$ is relatively compact in C_V.

By the asymptotic compactness in the norm of C_H and using a recursive argument in a finite number of steps, we may assume without loss of generality that there exists $\xi \in C([-2h-1,0];H)$ such that

$$u^n(t+\cdot) \to \xi(\cdot) \quad \text{strongly in } C([-2h-1,0];H). \tag{4.49}$$

From Lemma 4.23 we know that there exists a value $\tau_1(\widehat{D},t,h) < t-2h-2$ such that the subsequence $\{u^n : \tau_n \leq \tau_1(\widehat{D},t,h)\}$ is bounded in $L^\infty(t-h-1,t;V) \cap L^2(t-h-1,t;D(A))$ with $\{(u^n)'\}$ bounded in $L^2(t-h-1,t;H)$. Moreover, using the Aubin–Lions compactness lemma (e.g. cf. [61]), and taking into account (4.49), we may ensure that if we denote

$u(t+r) = \xi(r)$ for all $r \in [-2h-1, 0]$, then $u \in L^\infty(t-h-1, t; V) \cap L^2(t-h-1, t; D(A))$ with $u' \in L^2(t-h-1, t; H)$, and for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-h-1, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-h-1, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-h-1, t; H), \\ u^n \to u & \text{strongly in } L^2(t-h-1, t; V), \\ u^n(s) \to u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-h-1, t). \end{cases} \tag{4.50}$$

Indeed, $u \in C([t-h-1, t]; V)$ satisfies, thanks to (4.49) and (4.50), the equation (4.2) in $(t-h-1, t)$.

From the boundedness of $\{u^n\}$ in $C([t-h-1, t]; V)$, we have that for any sequence $\{s_n\} \subset [t-h-1, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{4.51}$$

where we have used (4.49) to identify the weak limit. We will prove that

$$u^n \to u \quad \text{strongly in } C([t-h, t]; V), \tag{4.52}$$

using an energy method for continuous functions analogous to that employed in the proof of Proposition 4.13, but starting with the energy equality (4.4) as in Lemma 2.14.

Indeed, if (4.52) is false, there exist $\varepsilon > 0$, a value $t_* \in [t-h, t]$, and subsequences (which we relabel the same) $\{u^n\}$ and $\{t_n\} \subset [t-h, t]$, with $\lim_{n\to\infty} t_n = t_*$, such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall\, n \geq 1. \tag{4.53}$$

Recall that by (4.51) we have

$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{4.54}$$

On the other hand, using the energy equality (4.4) for u and all u^n, and reasoning as for the obtention of (4.40), we have that for all $t-h-1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} &\|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2\, dr \\ \leq\ &\|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4\, dr + \frac{4}{\nu} \int_{s_1}^{s_2} |f(r)|^2\, dr + \frac{4L_g^2}{\nu} \int_{s_1}^{s_2} |u^n_r|^2_{C_H}\, dr, \end{aligned}$$

and

$$\begin{aligned} &\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2\, dr \\ \leq\ &\|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4\, dr + \frac{4}{\nu} \int_{s_1}^{s_2} |f(r)|^2\, dr + \frac{4L_g^2}{\nu} \int_{s_1}^{s_2} |u_r|^2_{C_H}\, dr. \end{aligned}$$

In particular, we can define the functions

$$
\begin{aligned}
\bar{J}_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u^n(r)|^2 \|u^n(r)\|^4 \, dr - \frac{4}{\nu} \int_{t-h-1}^{s} |f(r)|^2 \, dr \\
&\quad - \frac{4L_g^2}{\nu} \int_{t-h-1}^{s} |u_r^n|_{C_H}^2 \, dr, \\
\bar{J}(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u(r)|^2 \|u(r)\|^4 \, dr - \frac{4}{\nu} \int_{t-h-1}^{s} |f(r)|^2 \, dr \\
&\quad - \frac{4L_g^2}{\nu} \int_{t-h-1}^{s} |u_r|_{C_H}^2 \, dr.
\end{aligned}
$$

These are continuous functions on $[t-h-1, t]$, and from the above inequalities, both $\bar{J}_n$ and $\bar{J}$ are non-increasing. Moreover, by (4.49) and (4.50), we have

$$
\bar{J}_n(s) \to \bar{J}(s) \quad \text{a.e. } s \in (t-h-1, t).
$$

So, there exists a sequence $\{\tilde{t}_k\} \subset (t-h-1, t_*)$ such that $\tilde{t}_k \to t_*$, when $k \to \infty$, and

$$
\lim_{n \to \infty} \bar{J}_n(\tilde{t}_k) = \bar{J}(\tilde{t}_k) \quad \forall\, k.
$$

Fix an arbitrary value $\delta > 0$. From the continuity of $\bar{J}$, there exists k_δ such that

$$
|\bar{J}(\tilde{t}_k) - \bar{J}(t_*)| < \delta/2 \quad \forall\, k \geq k_\delta.
$$

Now, consider $n(k_\delta)$ such that for all $n \geq n(k_\delta)$ it holds

$$
t_n \geq \tilde{t}_{k_\delta} \quad \text{and} \quad |\bar{J}_n(\tilde{t}_{k_\delta}) - \bar{J}(\tilde{t}_{k_\delta})| < \delta/2.
$$

Then, since all $\bar{J}_n$ are non-increasing, we deduce that for all $n \geq n(k_\delta)$

$$
\begin{aligned}
\bar{J}_n(t_n) - \bar{J}(t_*) &\leq \bar{J}_n(\tilde{t}_{k_\delta}) - \bar{J}(t_*) \\
&\leq |\bar{J}_n(\tilde{t}_{k_\delta}) - \bar{J}(t_*)| \\
&\leq |\bar{J}_n(\tilde{t}_{k_\delta}) - \bar{J}(\tilde{t}_{k_\delta})| + |\bar{J}(\tilde{t}_{k_\delta}) - \bar{J}(t_*)| < \delta.
\end{aligned}
$$

This yields that

$$
\limsup_{n \to \infty} \bar{J}_n(t_n) \leq \bar{J}(t_*),
$$

and therefore, by (4.49) and (4.50),

$$
\limsup_{n \to \infty} \|u^n(t_n)\| \leq \|u(t_*)\|,
$$

which joined to (4.54) and (4.51) implies that $u^n(t_n) \to u(t_*)$ strongly in V, in contradiction with (4.53). Thus, (4.52) is proved as desired. ■

Now, we can establish the main result of this section.

Theorem 4.25. *Consider given g satisfying assumptions (I)-(III). Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$ satisfies (4.43) for some $0 < \mu < 2\nu\lambda_1$ such that $\mu > 2e^{\mu h}L_g$. Then, for any $\tilde h \in [0,h]$, the process U on $C_H^{\tilde h,V}$ possesses a minimal pullback $\mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)}$, a minimal pullback $\mathcal{D}_F^{\tilde h,V}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F^{\tilde h,V}(C_H)}$, and a minimal pullback $\mathcal{D}_F(C_H^{\tilde h,V})$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde h,V})}$. Besides, the following relations hold:*

$$\begin{aligned}\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde h,V})}(t) &\subset \mathcal{A}_{\mathcal{D}_F^{\tilde h,V}(C_H)}(t)\\ &\subset \mathcal{A}_{\mathcal{D}_F(C_H)}(t)\\ &\subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)\\ &\subset C_V \quad \forall\, t \in \mathbb{R},\end{aligned} \tag{4.55}$$

and for any family $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$,

$$\lim_{\tau\to-\infty} \mathrm{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)) = 0 \quad \forall\, t \in \mathbb{R}. \tag{4.56}$$

Finally, if moreover f satisfies

$$\sup_{s\le 0}\left(e^{-\sigma_\mu s}\int_{-\infty}^{s} e^{\sigma_\mu\theta}|f(\theta)|^2\,d\theta\right) < \infty, \tag{4.57}$$

then all attractors in (4.55) coincide, and this family is tempered in C_V, in the sense that

$$\lim_{t\to-\infty}\left(e^{\sigma_\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)}\|v\|^2_{C_V}\right) = 0, \tag{4.58}$$

where for $v \in C_V$,

$$\|v\|_{C_V} = \max_{s\in[-h,0]}\|v(s)\|.$$

Proof. Let us fix $\tilde h \in [0,h]$. The existence of $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)}$ is a consequence of Theorem 1.11, since the process U on $C_H^{\tilde h,V}$ is continuous (cf. Proposition 4.18) and therefore closed, the existence of a pullback absorbing family was given by Proposition 4.21, and in Lemma 4.24 we have proved the pullback $\mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)$-asymptotic compactness.

The existence of $\mathcal{A}_{\mathcal{D}_F^{\tilde h,V}(C_H)}$ and $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde h,V})}$ follows from the above facts, and the inclusions $\mathcal{D}_F(C_H^{\tilde h,V}) \subset \mathcal{D}_F^{\tilde h,V}(C_H) \subset \mathcal{D}_{\sigma_\mu}^{\tilde h,V}(C_H)$.

In (4.55), the chain of inclusions follows from Corollary 1.13, Theorem 1.15, and Remark 4.20. The equality is a consequence of Theorem 1.15 and Remark 1.16, by using Lemma 4.22 with $T = r = h+1$. The last inclusion is a consequence of the regularity result (a) in Theorem 4.5.

The property (4.56) is a consequence of Lemma 4.22, and the fact that by the regularity result (a) in Theorem 4.5, for any $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$ and any $\tau < t-h-1$,

$$\begin{aligned}&\mathrm{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t))\\ \le\ &\mathrm{dist}_{C_H^{h,V}}(U(t,\tau+h+1)(U(\tau+h+1,\tau)D(\tau)), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t))\\ =\ &\mathrm{dist}_{C_H^{h,V}}(U(t,\tau+h+1)D^{(h+1)}(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}^{h,V}(C_H)}(t)).\end{aligned}$$

The coincidence of all attractors in (4.55) under the additional assumption (4.57) holds by applying once more Theorem 1.15, and the second estimate in (4.44), since (4.57) is equivalent to (2.41).

The tempered condition (4.58) comes from (4.57) (and therefore (2.41)) and the expression of $\rho_2(t)$ given in Lemma 4.23. ■

Remark 4.26. (i) Observe that, under the assumptions of Theorem 4.25, one has $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)} \equiv \mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_H)}$ for any $\tilde{h} \in [0,h]$, i.e., the pullback attractor $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)}$ is independent of $\tilde{h}$.

Actually, if f also satisfies (4.57), then $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{F}(C_H)} \equiv \mathcal{A}_{\mathcal{D}^{h,V}_{F}(C_H)}$, and $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})} \equiv \mathcal{A}_{\mathcal{D}_F(C_H^{h,V})}$.

(ii) Observe that since $\widehat{D}_{1,\mu,h} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_H)$, and that for each $t \in \mathbb{R}$, $D_{1,\mu,h}(t)$ is closed in $C_H^{h,V}$, from Remark 1.12 and Remark 4.20, we deduce that $\mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_H)} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_H)$.

Remark 4.27. We can also consider, for each $0 \le \tilde{h} \le h$, the class $\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})$ of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$ such that

$$\lim_{\tau \to -\infty} \left(e^{\sigma_\mu \tau} \sup_{v \in D(\tau)} \|v\|^2_{\tilde{h},V} \right) = 0.$$

For this universe we have the chain of inclusions

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V}) \subset \mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H).$$

Under the assumptions of Theorem 4.25, we deduce the existence of the minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})}$. Moreover, this pullback attractor satisfies

$$\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \forall\, t \in \mathbb{R}.$$

In fact, if assumption (4.57) is satisfied, then $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})} \equiv \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$.

Chapter 5

Regularity of Pullback Attractors for 2D Navier–Stokes Equations with Finite Delay

In this chapter we strengthen some results on the existence and properties of pullback attractors for a 2D Navier–Stokes model with finite delay formulated in Caraballo and Real [11]. Contrary to Chapter 4, here we will keep all usual conditions for the delay operator of the problem (see conditions (IV) an (V) below).

The chapter is splitted into three sections. In Section 5.1 we will recall some general definitions and some well-known results on the existence of weak and strong solutions to our problem. Moreover, we will see some regularity properties for them.

Section 5.2 is devoted to establish several possible pullback attractors for two different phase spaces (continuous in time, or just square integrable in time) but taking into account the H norm in space. Moreover, we will be also able to compare both kind of attractors, for both possibilities of phase spaces.

Finally, our main results, established in the higher norm V (in space), will be given in Section 5.3. Thanks to regularity results, the attraction from different phase spaces will also happen in $C([-h, 0]; V)$. In this last section, again an energy method that relies on the continuity of the solutions and some non-increasing functions will be used to prove the asymptotic compactness of the associated processes in the respective universes. Moreover, relationships among all these objects will be analyzed and, at the end of the section, in Theorem 5.23, we will actually establish some relations with the attractors previously obtained in Chapter 4.

The results presented in this chapter can be found in [9, 10, 11, 34, 69, 71].

5.1 Statement of the problem

Using the same notation as in the previous chapters, consider the following functional Navier–Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \quad \text{in } \ \Omega\times(\tau,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \ \Omega\times(\tau,\infty), \\ u = 0 \quad \text{on } \ \partial\Omega\times(\tau,\infty), \\ u(x,\tau) = u^\tau(x), \quad x\in\Omega, \\ u(x,\tau+s) = \phi(x,s), \quad x\in\Omega,\ s\in(-h,0), \end{cases} \tag{5.1}$$

where u^τ and $\phi(x,s-\tau)$ are the initial data in τ and $(\tau-h,\tau)$ respectively. For each $t\geq\tau$, we denote by u_t the function defined a.e. on $(-h,0)$ by the relation $u_t(s)=u(t+s)$, a.e. $s\in(-h,0)$.

Now, let us denote by $L^2_X = L^2(-h,0;X)$ for $X=H,\,V$. On the delay operator from (5.1), we consider again that is well defined as $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$, and it satisfies the following assumptions (recall that in Chapter 4 we only assumed conditions $(I)-(III)$):

(I) for all $\xi\in C_H$, the function $\mathbb{R}\ni t\mapsto g(t,\xi)\in(L^2(\Omega))^2$ is measurable,

(II) $g(t,0)=0$, for all $t\in\mathbb{R}$,

(III) there exists $L_g>0$ such that for all $t\in\mathbb{R}$, and for all $\xi,\,\eta\in C_H$,

$$|g(t,\xi)-g(t,\eta)| \leq L_g|\xi-\eta|_{C_H},$$

(IV) there exists $C_g>0$ such that for all $\tau\leq t$, and for all $u,\,v\in C([\tau-h,t];H)$,

$$\int_\tau^t |g(s,u_s)-g(s,v_s)|^2\,ds \leq C_g^2\int_{\tau-h}^t |u(s)-v(s)|^2\,ds.$$

As we pointed out in the previous chapter, given $T>\tau$ and $u\in C([\tau-h,T];H)$, the function $g_u:[\tau,T]\to(L^2(\Omega))^2$ defined by $g_u(t)=g(t,u_t)$ for all $t\in[\tau,T]$, is measurable and belongs to $L^\infty(\tau,T;(L^2(\Omega))^2)$.

Now, thanks to (IV), the mapping

$$\mathcal{G}: u\in C([\tau-h,T];H)\to g_u\in L^2(\tau,T;(L^2(\Omega))^2)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau-h,T;H)$ into $L^2(\tau,T;(L^2(\Omega))^2)$. From now on, we will denote by $g(t,u_t)=\widetilde{\mathcal{G}}(u)(t)$ for each $u\in L^2(\tau-h,T;H)$, and thus property (IV) will also hold for all $u,\,v\in L^2(\tau-h,T;H)$.

Assume that $u^\tau\in H$, $\phi\in L^2_H$, and $f\in L^2_{\text{loc}}(\mathbb{R};V')$.

Definition 5.1. *A weak solution to (5.1) is a function u that belongs to $L^2(\tau-h,T;H) \cap L^2(\tau,T;V) \cap L^\infty(\tau,T;H)$ for all $T>\tau$, with $u(\tau)=u^\tau$ and $u(t)=\phi(t-\tau)$ a.e. $t\in(\tau-h,\tau)$, and such that for all $v\in V$,*

$$\frac{d}{dt}(u(t),v)+\nu\langle Au(t),v\rangle+b(u(t),u(t),v)=\langle f(t),v\rangle+(g(t,u_t),v), \tag{5.2}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,\infty)$.

Once more, if u is a weak solution to (5.1) and $f\in L^2_{\rm loc}(\mathbb{R};V')$, then we deduce that $u\in C([\tau,\infty);H)$, whence the initial datum $u(\tau)=u^\tau$ has full sense. Furthermore, in this case the energy equality (4.3) also holds.

On the other, we define a strong solution to (5.1) in the same way as in the previous chapters, and again, if u is a strong solution to (5.1) and $f\in L^2_{\rm loc}(\mathbb{R};(L^2(\Omega))^2)$, then $u\in C([\tau,\infty);V)$, and it satisfies the energy equality (4.4).

Concerning the existence and uniqueness of weak and strong solutions for (5.1), we have the following result which can be proved similarly as [9, Theorem 2.1] or [10, Theorem 2.5] (see also [36, Theorem 2.3] for a more general case). Actually, as we have proved in Chapter 4, if we restrict to the phase space of continuous in time functions C_H, it is also possible to obtain existence, uniqueness, and regularity of solutions to our problem without conditions (IV) or (V) on the delay operator g (see Theorem 4.5 in Section 4.1).

Theorem 5.2. *Let us consider $u^\tau\in H$, $\phi\in L^2_H$, $f\in L^2_{\rm loc}(\mathbb{R};V')$, and $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$ satisfying (I)–(IV). Then, for each $\tau\in\mathbb{R}$, there exists a unique weak solution $u(\cdot)=u(\cdot;\tau,u^\tau,\phi)$ to (5.1).*

Moreover, if $f\in L^2_{\rm loc}(\mathbb{R};(L^2(\Omega))^2)$, then

(a) $u\in C([\tau+\varepsilon,T];V)\cap L^2(\tau+\varepsilon,T;D(A))$ for all $T>\tau+\varepsilon>\tau$.

(b) If $u^\tau\in V$, u is in fact a strong solution to (5.1).

Before establishing the original results about the regularity of pullback attractors, we recall the main existence results previously studied in the literature.

5.2 Previous results on processes and pullback attractors in H norm

In this section we recall some known results (cf. [11, 69, 71]) on the existence of minimal pullback attractors in the H norm for suitable processes associated to problem (5.1).

In order to apply the abstract theory of pullback attractors studied in Chapter 1 (e.g. see the above cited references), we may consider the Banach space C_H, and the Hilbert space $M^2_H=H\times L^2_H$ with associated norm

$$\|(u^\tau,\phi)\|^2_{M^2_H}=|u^\tau|^2+\int_{-h}^0|\phi(s)|^2\,ds \quad \text{for } (u^\tau,\phi)\in M^2_H.$$

We can define two processes for problem (5.1).

Proposition 5.3. *Assume that* $f \in L^2_{\rm loc}(\mathbb{R}; V')$*, and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfies (I)–(IV). Then, the bi-parametric families of mappings* $U(t,\tau) : C_H \to C_H$ *and* $S(t,\tau) : M^2_H \to M^2_H$ *given respectively by*

$$U(t,\tau)\phi = u_t(\cdot;\tau,\phi(0),\phi) \quad \text{for } \phi \in C_H,\ \tau \le t, \tag{5.3}$$

and

$$S(t,\tau)(u^\tau,\phi) = (u(t;\tau,u^\tau,\phi), u_t(\cdot;\tau,u^\tau,\phi)) \quad \text{for } (u^\tau,\phi) \in M^2_H,\ \tau \le t, \tag{5.4}$$

where u *is the unique weak solution to (5.1), are well defined continuous processes on* C_H *and* M^2_H *respectively.*

Proof. The result follows from Theorem 5.2 above, and from [11, Theorem 9]. ■

Now, in order to establish asymptotic estimates for the solutions to (5.1), we impose a fifth assumption on g and f.

(V) Assume that $\nu\lambda_1 > C_g$, and that there exists a value $\eta \in (0, 2(\nu\lambda_1 - C_g))$ such that for every $u \in L^2(\tau - h, t; H)$,

$$\begin{aligned} \int_\tau^t e^{\eta s}|g(s,u_s)|^2\,ds &\le C_g^2 \int_{\tau-h}^t e^{\eta s}|u(s)|^2\,ds \quad \text{for any } \tau \le t, \quad \text{and} \\ \int_{-\infty}^0 e^{\eta s}\|f(s)\|_*^2\,ds &< \infty. \end{aligned}$$

Lemma 5.4. *Suppose that* $f \in L^2_{\rm loc}(\mathbb{R}; V')$*, and that* f *and* $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$ *satisfy (I)–(V). Then, for any* $(u^\tau,\phi) \in M^2_H$*, the following estimate holds for the solution* u *to (5.1) for all* $t \ge \tau$*:*

$$|u(t)|^2 \le e^{-\eta(t-\tau)} \max\{1, C_g\}\|(u^\tau,\phi)\|^2_{M^2_H} + \beta^{-1}e^{-\eta t}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\,ds, \tag{5.5}$$

where

$$\beta = 2\nu - (\eta + 2C_g)\lambda_1^{-1}. \tag{5.6}$$

Proof. By the energy equality (4.3) and Young's inequality, we have

$$\begin{aligned} &\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\ \le\ & \beta\|u(t)\|^2 + \beta^{-1}\|f(t)\|_*^2 + C_g|u(t)|^2 + C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{d}{dt}\big(e^{\eta t}|u(t)|^2\big) + e^{\eta t}\big(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\big)\|u(t)\|^2 \\ \le\ & e^{\eta t}\beta^{-1}\|f(t)\|_*^2 + e^{\eta t}C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau, \end{aligned}$$

and therefore, integrating above and using property (V), we obtain

$$\begin{aligned} & e^{\eta t}|u(t)|^2 + \left(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\right) \int_\tau^t e^{\eta s}\|u(s)\|^2\, ds \\ \leq\ & e^{\eta\tau}|u^\tau|^2 + \beta^{-1}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\, ds + C_g \int_{\tau-h}^t e^{\eta s}|u(s)|^2\, ds \\ \leq\ & e^{\eta\tau}\max\{1, C_g\}\|(u^\tau, \phi)\|_{M_H^2}^2 + \beta^{-1}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\, ds + C_g \int_\tau^t e^{\eta s}|u(s)|^2\, ds \end{aligned}$$

for all $t \geq \tau$, and from this last inequality and (5.6), in particular we deduce (5.5). ■

After the above result, it turns out appropriate the introduction of the following tempered universe.

Definition 5.5. *We will denote by $\mathcal{D}_\eta(M_H^2)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H^2)$ such that*

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{(w,\varphi)\in D(\tau)}\|(w,\varphi)\|_{M_H^2}^2\right) = 0.$$

Furthermore, we will also consider the universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_F(C_H)$ already defined in Chapter 4 (see Definition 4.11), and $\mathcal{D}_F(M_H^2)$ will denote the universe of fixed bounded sets in M_H^2.

Remark 5.6. *(i) The choices of the above universes are right and convenient to keep, in the sense that, on the one hand, M_H^2 is more general as phase space for the initial data of problem (5.1). On the other hand, the regularity of the solution to (5.1) (cf. Theorem 5.2) makes that, after an elapsed time h, every solution is continuous with values on H. Indeed, as it was observed in [11], in the case of the universes of fixed bounded sets, pullback attractors in both spaces do exist, and they are intrinsically related through the canonical embedding $j : C_H \to M_H^2$ defined by $j(\varphi) = (\varphi(0), \varphi)$ (see Theorem 5.10 below).*

(ii) The universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_\eta(M_H^2)$, which are inclusion-closed, contain respectively the universes $\mathcal{D}_F(C_H)$ and $\mathcal{D}_F(M_H^2)$.

Now, we obtain pullback absorbing families for $U : \mathbb{R}_d^2 \times C_H \to C_H$ and $S : \mathbb{R}_d^2 \times M_H^2 \to M_H^2$.

Corollary 5.7. *Under the assumptions of Lemma 5.4, the family $\widehat{D}_{1,\eta} = \{D_{1,\eta}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ defined by $D_{1,\eta}(t) = \overline{B}_{C_H}(0, r_\eta(t))$, the closed ball in C_H of center zero and radius $r_\eta(t)$, where*

$$r_\eta^2(t) = 1 + \beta^{-1}e^{-\eta(t-h)}\int_{-\infty}^t e^{\eta s}\|f(s)\|_*^2\, ds,$$

with β given by (5.6), is pullback $\mathcal{D}_\eta(C_H)$-absorbing for the process U on C_H defined by (5.3) (and therefore pullback $\mathcal{D}_F(C_H)$-absorbing too), and $\widehat{D}_{1,\eta}$ belongs to $\mathcal{D}_\eta(C_H)$.

Analogously, the family $\widehat{D}_{2,\eta} = \{D_{2,\eta}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H^2)$ *defined by* $D_{2,\eta}(t) = \overline{B}_{M_H^2}(0, R_\eta(t))$*, the closed ball in* M_H^2 *of center zero and radius* $R_\eta(t)$*, with*

$$R_\eta^2(t) = 1 + \beta^{-1}(1 + he^{\eta h})e^{-\eta t} \int_{-\infty}^{t} e^{\eta s}\|f(s)\|_*^2\, ds,$$

is pullback $\mathcal{D}_\eta(M_H^2)$*-absorbing for the process* S *on* M_H^2 *given by (5.4) (and thus also pullback* $\mathcal{D}_F(M_H^2)$*-absorbing), and* $\widehat{D}_{2,\eta}$ *belongs to* $\mathcal{D}_\eta(M_H^2)$*.*

Since it will be useful in order to compare the pullback attractors defined in the spaces C_H and M_H^2, we consider the bi-parametric family of mappings $\widetilde{U}(t,\tau) : M_H^2 \to L_H^2$ defined as

$$\widetilde{U}(t,\tau)(u^\tau, \phi) = u_t(\cdot; \tau, u^\tau, \phi) \quad \text{for } (u^\tau, \phi) \in M_H^2,\ \tau \le t.$$

Remark 5.8. *Observe that* $\widetilde{U}(t,\tau)$ *maps* M_H^2 *into* C_H *if* $t \ge \tau + h$*, and therefore we can write*

$$S(t,\tau)(u^\tau, \phi) = j(\widetilde{U}(t,\tau)(u^\tau, \phi)) \quad \textit{for } (u^\tau, \phi) \in M_H^2,\ t \ge \tau + h,$$

where $S(\cdot,\cdot)$ *is given by (5.4).*

Moreover, it is clear that

$$U(t,\tau)\phi = \widetilde{U}(t,\tau)j(\phi) \quad \textit{for } \phi \in C_H,\ t \ge \tau,$$

with $U(\cdot,\cdot)$ *defined in (5.3).*

Lemma 5.9. *Under the assumptions of Lemma 5.4, for any* $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ *and any* $r \ge h$*, the family* $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$*, where* $D^{(r)}(\tau) = \widetilde{U}(\tau + r, \tau)D(\tau)$ *for any* $\tau \in \mathbb{R}$*, belongs to* $\mathcal{D}_\eta(C_H)$*.*

Proof. From (5.5), we obtain

$$\begin{aligned}\sup_{\psi \in D^{(r)}(\tau)} \left(e^{\eta\tau}|\psi|_{C_H}^2\right) \le\ & e^{-\eta(r-h)} \max\{1, C_g\} \sup_{(u^\tau,\phi)\in D(\tau)} \left(e^{\eta\tau}\|(u^\tau,\phi)\|_{M_H^2}^2\right) \\ & + \beta^{-1}e^{-\eta(r-h)} \int_{\tau}^{\tau+r} e^{\eta s}\|f(s)\|_*^2\, ds.\end{aligned}$$

From this inequality and assumption (V), we deduce the result. ∎

Now, we are able to establish the main result of this section.

Theorem 5.10. *Assume that* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$*, and that* f *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfy (I)–(V). Then, there exist the minimal pullback attractors* $\mathcal{A}_{\mathcal{D}_F(C_H)}$*,* $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$*,* $\mathcal{A}_{\mathcal{D}_F(M_H^2)}$*, and* $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$*, in* C_H *and* M_H^2 *respectively, for the universes of fixed bounded sets and for those with tempered condition given in Definition 5.5.*

Besides, the following relations hold:

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t), \quad \textit{and} \quad \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}, \tag{5.7}$$

$$j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}, \quad \textit{and} \tag{5.8}$$

$$j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}, \tag{5.9}$$

where the map j is the canonical injection of C_H into M_H^2 defined in Remark 5.6 (i).

Finally, if f also satisfies

$$\sup_{s\le 0}\left(e^{-\eta s}\int_{-\infty}^{s} e^{\eta\theta}\|f(\theta)\|_*^2\,d\theta\right)<\infty, \tag{5.10}$$

then, the inclusions in (5.7) and (5.8) are in fact equalities.

Proof. For the existence of the four minimal pullback attractors see [11, Theorem 17, Remark 18], [69, Theorem 20], and [71, Theorem 4].

The relations in (5.7) follow from Corollary 1.13, and the inclusion in (5.8) can be proved analogously as in [71, Theorem 5].

Now, we analyze the equality in (5.9). On the one hand, the inclusion $j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ can be obtained again in a similar way as in [71, Theorem 5]. On the other hand, from Remark 5.8 and Lemma 5.9, we have that for any $\widehat{D}\in\mathcal{D}_\eta(M_H^2)$ and any $\tau<t-h$,

$$\begin{aligned}
&\operatorname{dist}_{M_H^2}(S(t,\tau)D(\tau),j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)))\\
=\;&\operatorname{dist}_{M_H^2}(S(t,\tau+h)(S(\tau+h,\tau)D(\tau)),j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)))\\
=\;&\operatorname{dist}_{M_H^2}(S(t,\tau+h)(j(\widetilde{U}(\tau+h,\tau)D(\tau))),j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)))\\
=\;&\operatorname{dist}_{M_H^2}(j(U(t,\tau+h)D^{(h)}(\tau)),j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)))\\
\le\;&(1+h)^{1/2}\operatorname{dist}_{C_H}(U(t,\tau+h)D^{(h)}(\tau),\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)),
\end{aligned}$$

where in the last inequality we have used that $j\in\mathcal{L}(C_H,M_H^2)$ with $\|j\|_{\mathcal{L}(C_H,M_H^2)}\le(1+h)^{1/2}$. Therefore, the inclusion $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)\subset j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))$ follows since $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ is the minimal closed set in M_H^2 that attracts any family $\widehat{D}\in\mathcal{D}_\eta(M_H^2)$ in the pullback sense.

Finally, if moreover f satisfies (5.10), the coincidences of the pullback attractors in (5.7) is a consequence of Remark 1.14, and the fact that (5.10) is equivalent to have that $\sup_{t\le T} r_\eta(t)$ and $\sup_{t\le T} R_\eta(t)$ are bounded for any $T\in\mathbb{R}$, with $r_\eta(t)$ and $R_\eta(t)$ defined in Corollary 5.7. Now, from these identities and (5.9), the equality in (5.8) follows. ■

Remark 5.11. *Under the assumptions of Theorem 5.10, as a consequence of Remarks 1.12 and 5.6 (ii), and Corollary 5.7, we have that $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$ and $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$ belong to the universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_\eta(M_H^2)$ respectively.*

Actually, if in addition f satisfies (5.10), one can see that for each $T\in\mathbb{R}$, the sets $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t):t\le T\}$ and $\{\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t):t\le T\}$ are bounded in C_H and M_H^2 respectively.

5.3 Regularity of pullback attractors and attraction in V norm

Now, we will improve in a certain way the main result of the previous section, Theorem 5.10, in the sense that we will establish the existence of minimal pullback attractors in the V norm. Moreover, we will check that under suitable assumptions all these families of attractors are in fact the same (here Theorem 1.15 will play an essential role).

For each $\tilde{h} \in [0,h]$, we consider the Banach space $C_H^{\tilde{h},V}$ already defined in Chapter 4, and the Hilbert space $M_V^2 = V \times L_V^2$ with associated norm

$$\|(u^\tau,\phi)\|_{M_V^2}^2 = \|u^\tau\|^2 + \int_{-h}^0 \|\phi(s)\|^2\, ds \quad \text{for } (u^\tau,\phi) \in M_V^2.$$

We have the following result.

Proposition 5.12. *Suppose that $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$, and $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$ satisfies (I)–(IV). Then, for any $\tilde{h} \in [0,h]$, the bi-parametric families of mappings $U(t,\tau)|_{C_H^{\tilde{h},V}}$ and $S(t,\tau)|_{M_V^2}$, with $\tau \le t$, are well defined continuous processes on $C_H^{\tilde{h},V}$ and M_V^2 respectively.*

Proof. The fact that, for any $\tilde{h} \in [0,h]$ and $\tau \le t$, $U(t,\tau)|_{C_H^{\tilde{h},V}}$ and $S(t,\tau)|_{M_V^2}$ are well defined processes follows from Theorem 5.2. The continuity can be proved similarly as Proposition 4.18, using property (IV) instead of (III). ■

We introduce the following universes in $\mathcal{P}(M_V^2)$.

Definition 5.13. *We will denote by $\mathcal{D}_\eta^V(M_H^2)$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(M_H^2)$ such that for any $t \in \mathbb{R}$ and for any $(w,\varphi) \in D(t)$, it holds that $(w,\varphi) \in M_V^2$.*

Moreover, we will denote by $\mathcal{D}_F(M_V^2)$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of M_V^2.

Remark 5.14. *The relations among the universes introduced above and those in $\mathcal{P}(M_H^2)$ defined in Section 5.2, are the following:*

$$\mathcal{D}_F(M_V^2) \subset \mathcal{D}_\eta^V(M_H^2) \subset \mathcal{D}_\eta(M_H^2),$$

and

$$\mathcal{D}_F(M_V^2) \subset \mathcal{D}_F(M_H^2) \subset \mathcal{D}_\eta(M_H^2).$$

Observe also that $\mathcal{D}_\eta^V(M_H^2)$ is inclusion-closed.

Furthermore, for any $\tilde{h} \in [0,h]$, we also consider the universes $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$, $\mathcal{D}_F^{\tilde{h},V}(C_H)$, and $\mathcal{D}_F(C_H^{\tilde{h},V})$, introduced in Chapter 4 (see Definition 4.19).

Now, we establish the existence of pullback absorbing families for the processes $U : \mathbb{R}_d^2 \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ and $S : \mathbb{R}_d^2 \times M_V^2 \to M_V^2$.

Proposition 5.15. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$, and that f and $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, for any $\tilde{h} \in [0,h]$, the family $\widehat{D}_{1,\eta,\tilde{h}} = \{D_{1,\eta,\tilde{h}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$, with*

$$D_{1,\eta,\tilde{h}}(t) = D_{1,\eta}(t) \cap C_H^{\tilde{h},V},$$

where $D_{1,\eta}(t)$ is defined in Corollary 5.7, is a family of closed sets of $C_H^{\tilde{h},V}$, which is pullback $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$-absorbing for the process U on $C_H^{\tilde{h},V}$ given by (5.3), and $\widehat{D}_{1,\eta,\tilde{h}}$ belongs to $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$.

Besides, the family $\widehat{D}_{2,\eta,V} = \{D_{2,\eta,V}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_V^2)$, where

$$D_{2,\eta,V}(t) = D_{2,\eta}(t) \cap M_V^2,$$

with $D_{2,\eta}(t)$ also given in Corollary 5.7, is a family of closed sets of M_V^2, that is pullback $\mathcal{D}_\eta^V(M_H^2)$-absorbing for the process S on M_V^2 defined by (5.4), and $\widehat{D}_{2,\eta,V}$ belongs to $\mathcal{D}_\eta^V(M_H^2)$.

Proof. It is a direct consequence of Corollary 5.7. ■

The following result can be obtained analogously as Lemma 4.23.

Lemma 5.16. *Under the assumptions of Proposition 5.15, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$, there exist $\tau_1(\widehat{D}, t, h) < t - 3h - 2$ and functions $\{\rho_i\}_{i=1}^4$ depending on t and h, such that for any $\tau \leq \tau_1(\widehat{D}, t, h)$ and any $(u^\tau, \phi) \in D(\tau)$, it holds*

$$\left\{\begin{array}{rl} |u(r;\tau,u^\tau,\phi)|^2 \leq \rho_1(t) & \forall\, r \in [t-3h-2, t], \\ \|u(r;\tau,u^\tau,\phi)\|^2 \leq \rho_2(t) & \forall\, r \in [t-2h-1, t], \\ \nu \displaystyle\int_{r-1}^{r} |Au(\theta;\tau,u^\tau,\phi)|^2\, d\theta \leq \rho_3(t) & \forall\, r \in [t-2h, t], \\ \displaystyle\int_{r-1}^{r} |u'(\theta;\tau,u^\tau,\phi)|^2\, d\theta \leq \rho_4(t) & \forall\, r \in [t-2h, t], \end{array}\right. \tag{5.11}$$

where

$$\begin{aligned} \rho_1(t) &= 1 + \beta^{-1} e^{-\eta(t-3h-2)} \int_{-\infty}^{t} e^{\eta s} \|f(s)\|_*^2\, ds, \\ \rho_2(t) &= \left(\nu^{-1}\left(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2 + 4L_g^2\right)\rho_1(t) + \nu^{-1}\left(4 + 2\nu^{-1}\lambda_1^{-1}\right)\int_{t-2h-2}^{t} |f(\theta)|^2\, d\theta\right) \\ &\quad \times \exp\left\{2\nu^{-1}C^{(\nu)}\rho_1(t)\left[\left(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2\right)\rho_1(t) + 2\nu^{-1}\lambda_1^{-1}\int_{t-2h-2}^{t} |f(\theta)|^2\, d\theta\right]\right\}, \\ \rho_3(t) &= \rho_2(t) + 4\nu^{-1}\int_{t-2h-1}^{t} |f(\theta)|^2\, d\theta + 4L_g^2\nu^{-1}\rho_1(t) + 2C^{(\nu)}\rho_1(t)\rho_2^2(t), \\ \rho_4(t) &= \nu\rho_2(t) + 4\int_{t-2h-1}^{t} |f(\theta)|^2\, d\theta + 4L_g^2\rho_1(t) + 2C_1^2\nu^{-1}\rho_2(t)\rho_3(t), \end{aligned}$$

with β and $C^{(\nu)}$ given by (5.6) and (2.20) respectively.

Now, we proceed in a similar way as in Lemma 4.24, by applying the same energy method, in order to obtain the pullback asymptotic compactness in $C_H^{\tilde{h},V}$ and M_V^2 for the universes $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$ and $\mathcal{D}_\eta^V(M_H^2)$ respectively.

Lemma 5.17. *Under the assumptions of Proposition 5.15, for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{(u^{\tau_n}, \phi^n)\} \subset M_H^2$ such that $\tau_n \to -\infty$ and $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n, the sequence $\{u(\cdot; \tau_n, u^{\tau_n}, \phi^n)\}$ is relatively compact in $C([t-h, t]; V)$.*

Proof. Let us fix $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$, and sequences $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n. Denote for short $u^n(\cdot) = u(\cdot; \tau_n, u^{\tau_n}, \phi^n)$.

From Lemma 5.16, we have that there exists a $\tau_1(\widehat{D}, t, h) < t - 3h - 2$ such that the subsequence $\{u^n : \tau_n \le \tau_1(\widehat{D}, t, h)\} \subset \{u^n\}$ is bounded in $L^\infty(t-2h-1, t; V) \cap L^2(t-2h-1, t; D(A))$ with $\{(u^n)'\}$ also bounded in $L^2(t-2h-1, t; H)$. Therefore, using in particular the Aubin–Lions compactness lemma (e.g., cf. [61]), there exists a function $u \in L^\infty(t-2h-1, t; V) \cap L^2(t-2h-1, t; D(A))$ with $u' \in L^2(t-2h-1, t; H)$ such that, for a subsequence which we relabel the same, the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-2h-1, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2h-1, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2h-1, t; H), \\ u^n \to u & \text{strongly in } L^2(t-2h-1, t; V), \\ u^n(s) \to u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-2h-1, t). \end{cases} \tag{5.12}$$

Observe that $u \in C([t-2h-1, t]; V)$ satisfies, thanks to (5.12), the equation (5.2) in the interval $(t-h-1, t)$.

Moreover, from (5.12) we can also deduce that $\{u^n\}$ is equi-continuous on $[t-2h-1, t]$ with values in H. Thus, since $\{u^n\}$ is bounded in $C([t-2h-1, t]; V)$ and the injection of V into H is compact, by the Ascoli–Arzelà theorem, we obtain that (once more, up to a subsequence)

$$u^n \to u \quad \text{strongly in } C([t-2h-1, t]; H). \tag{5.13}$$

Indeed, again from the boundedness of $\{u^n\}$ in $C([t-2h-1, t]; V)$, one has that for any sequence $\{s_n\} \subset [t-2h-1, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V,$$

where we have used (5.13) to identify the weak limit.

Our goal is to show that

$$u^n \to u \quad \text{strongly in } C([t-h, t]; V),$$

which in particular will imply the relative compactness.

Now, the proof follows the same lines as the proof of Lemma 4.24, by using a contradiction argument and the same continuous and non-increasing functions $\bar{J}_n$ and $\bar{J}$ defined in that lemma. ■

As an immediate consequence of the previous lemma, we have the following result.

Corollary 5.18. *Under the assumptions of Lemma 5.17, it holds:*

(a) *For any* $\tilde{h} \in [0,h]$, *the process* $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ *is pullback* $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$*-asymptotically compact.*

(b) *The process* $S : \mathbb{R}^2_d \times M_V^2 \to M_V^2$ *is pullback* $\mathcal{D}_\eta^V(M_H^2)$*-asymptotically compact.*

We establish now the following result about the existence of minimal pullback attractors for the process U on $C_H^{\tilde{h},V}$, which can be proved in a same way as Theorem 4.25.

Theorem 5.19. *Assume that* $f \in L^2_{\rm loc}(\mathbb{R};(L^2(\Omega))^2)$, *and that* f *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfy (I)–(V). Then, for any* $\tilde{h} \in [0,h]$, *the process* U *on* $C_H^{\tilde{h},V}$ *possesses a minimal pullback* $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$*-attractor* $\mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}$, *a minimal pullback* $\mathcal{D}_F^{\tilde{h},V}(C_H)$*-attractor* $\mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)}$, *and a minimal pullback* $\mathcal{D}_F(C_H^{\tilde{h},V})$*-attractor* $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}$. *Besides, the following relations hold:*

$$\begin{aligned}
\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t) &\subset \mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)}(t)\\
&\subset \mathcal{A}_{\mathcal{D}_F(C_H)}(t)\\
&\subset \mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)\\
&\subset C_V \quad \forall\, t \in \mathbb{R},
\end{aligned} \tag{5.14}$$

and for any family $\widehat{D} \in \mathcal{D}_\eta(C_H)$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) = 0 \quad \forall\, t \in \mathbb{R}.$$

Finally, if moreover f *satisfies*

$$\sup_{s \le 0} \left(e^{-\eta s} \int_{-\infty}^{s} e^{\eta\theta} |f(\theta)|^2 \, d\theta \right) < \infty, \tag{5.15}$$

then all attractors in (5.14) coincide, and this family is tempered in C_V, *in the sense that*

$$\lim_{t \to -\infty} \left(e^{\eta t} \sup_{v \in \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)} \|v\|^2_{C_V} \right) = 0.$$

Remark 5.20. *Under the assumptions of Theorem 5.19, since* $\widehat{D}_{1,\eta,h}$ *belongs to* $\mathcal{D}_\eta^{h,V}(C_H)$, *the set* $D_{1,\eta,h}(t)$ *is closed in* $C_H^{h,V}$ *for all* $t \in \mathbb{R}$, *and the universe* $\mathcal{D}_\eta^{h,V}(C_H)$ *is inclusion-closed, from Remark 1.12 and the equality in (5.14), we deduce that* $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$ *belongs to* $\mathcal{D}_\eta^{h,V}(C_H)$.

In fact, if in addition f *satisfies (5.15), then for each* $T \in \mathbb{R}$, *the set* $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t) : t \le T\}$ *is bounded in* $C_H^{h,V}$.

We are also able to obtain the existence of minimal pullback attractors for the process S on M_V^2.

Theorem 5.21. *Suppose that* $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$*, and that* f *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfy assumptions (I)–(V). Then, there exist the minimal pullback* $\mathcal{D}_F(M_V^2)$*-attractor* $\mathcal{A}_{\mathcal{D}_F(M_V^2)}$*, and the minimal pullback* $\mathcal{D}_\eta^V(M_H^2)$*-attractor* $\mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}$ *for the process* S *on* M_V^2*, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) \subset \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}. \tag{5.16}$$

In particular, for any family $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$*,*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{M_V^2}(S(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) = 0 \quad \forall\, t \in \mathbb{R}. \tag{5.17}$$

Finally, if f *also satisfies (5.15), then*

$$\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) = \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t) \quad \forall\, t \in \mathbb{R},$$

and this family is tempered in M_V^2*, i.e.,*

$$\lim_{t\to-\infty} \left(e^{\eta t} \sup_{(w,\varphi)\in\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)} \|(w,\varphi)\|^2_{M_V^2} \right) = 0. \tag{5.18}$$

Proof. From Proposition 5.12, we know that the process S on M_V^2 is continuous, and therefore closed.

From Proposition 5.15, we have that the family $\widehat{D}_{2,\eta,V} \subset \mathcal{P}(M_V^2)$ is pullback $\mathcal{D}_\eta^V(M_H^2)$-absorbing for the process S on M_V^2.

From Corollary 5.18, we also have that S is pullback $\mathcal{D}_\eta^V(M_H^2)$-asymptotically compact.

Therefore, we may apply Theorem 1.11 and Corollary 1.13 to conclude the existence of $\mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}$ and $\mathcal{A}_{\mathcal{D}_F(M_V^2)}$.

In (5.16), the inclusions follow from Corollary 1.13, Theorem 1.15, and the fact that $\mathcal{D}_F(M_V^2) \subset \mathcal{D}_F(M_H^2)$ (see Remark 5.14). The equality is a consequence of Theorem 1.15 and Remark 1.16, by using Lemma 5.9 with $T = r = h+1$, since, by the regularity result (a) in Theorem 5.2, it is clear that the family $\{j(D^{(h+1)}(\tau)) : \tau \in \mathbb{R}\}$ belongs to $\mathcal{D}_\eta^V(M_H^2)$.

The pullback attraction result (5.17) comes from Remark 5.8, Lemma 5.9, and the fact that by the regularity property (a) in Theorem 5.2, for any $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ and any $\tau < t - h - 1$,

$$\begin{aligned}
& \mathrm{dist}_{M_V^2}(S(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\
= \; & \mathrm{dist}_{M_V^2}(S(t,\tau+h+1)(S(\tau+h+1,\tau)D(\tau)), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\
= \; & \mathrm{dist}_{M_V^2}(S(t,\tau+h+1)(j(\widetilde{U}(\tau+h+1,\tau)D(\tau))), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\
= \; & \mathrm{dist}_{M_V^2}(S(t,\tau+h+1)(j(D^{(h+1)}(\tau))), \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t)).
\end{aligned}$$

If moreover f satisfies (5.15), the equality $\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ follows from Remark 1.14, and the equality $\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) = \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t)$ is again a consequence of Theorem 1.15, by using the second estimate in (5.11), since (5.15) is equivalent to (2.41).

Lastly, the tempered property (5.18) comes from (5.15) (and therefore (2.41)) and the tempered character of $\rho_2(t)$ defined in Lemma 5.16. ■

Remark 5.22. *Under the assumptions of Theorem 5.21, reasoning analogously as in Remark 5.20, one has that $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$ belongs to $\mathcal{D}_\eta^V(M_H^2)$.*

To conclude, we relate the minimal pullback attractors obtained in $C_H^{\tilde{h},V}$ and M_V^2 through the canonical injection j.

Theorem 5.23. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R};(L^2(\Omega))^2)$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, the following relations hold:*

$$\begin{aligned} j(\mathcal{A}_{\mathcal{D}_F(C_H^{h,V})}(t)) &\subset \mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) \quad \forall\, t \in \mathbb{R}, \quad \textit{and} && (5.19)\\ j(\mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}(t)) &= \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t) \quad \forall\, \tilde{h} \in [0,h],\ t \in \mathbb{R}. && (5.20) \end{aligned}$$

Actually, if f also satisfies (5.15), then, for any $\tilde{h} \in [0,h]$,

$$j(\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t)) = j(\mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) \quad \forall\, t \in \mathbb{R}. \tag{5.21}$$

Proof. In order to prove the inclusion in (5.19) we proceed similarly as in [71, Theorem 5], taking into account that the map j is continuous from $C_H^{h,V}$ into M_V^2, and that $j(\mathcal{D}_F(C_H^{h,V})) \subset \mathcal{D}_F(M_V^2)$.

The equality in (5.20) is a consequence of property (5.9) in Theorem 5.10, using the equalities in (5.14) and (5.16).

Finally, the equalities in (5.21) follow from (5.20) and the known facts that, under the additional assumption (5.15), all attractors in (5.14) and (5.16) coincide. ∎

Chapter 6

Pullback Attractors for the Non-Autonomous 3D Navier–Stokes–Voigt Equations

In this chapter we consider a non-autonomous 3D Navier–Stokes–Voigt model, to which a continuous process can be associated. We study the existence and relationships between minimal pullback attractors for this process again in two different frameworks, namely, for the universe of fixed bounded sets, and also for another universe given by a tempered condition.

This model was introduced by Oskolkov in [74] as a model for a viscoelastic incompressible fluid and gives an approximate description of the Kelvin–Voigt fluid. Moreover, recently it was proposed as a regularization of the three-dimensional Navier–Stokes equations for the purpose of direct numerical simulations in [5]. It is also worth pointing out that, since the model does not have a regularizing effect (in contrast to the two-dimensional Navier–Stokes model), to obtaining asymptotic compactness for the process is a more involved task. In this chapter we prove this in a relatively simple way just by using an energy method. Our results simplify – and in some aspects generalize – some of those previously obtained for the autonomous and non-autonomous cases, since for example in Section 6.2, regularity is not required for the boundary of the domain and the force may take values in V'. Under additional suitable assumptions, regularity results for these families of attractors are also obtained, via bootstrapping arguments. Finally, we also conclude some results concerning the attraction in $D(A)$ norm.

The structure of the chapter is the following. In Section 6.1 we state our problem in an abstract setting, and we prove the existence and uniqueness of weak solution for such problem, and a regularity property. Two continuous dependence results with respect to the initial datum, in the weak and strong senses, are also provided. In Section 6.3 we prove that the conditions in order to ensure the existence of minimal pullback attractors in V norm are fulfilled. To be more precise, both – pullback absorbing and pullback asymptotic compactness properties – are obtained from a rather general condition on the V' norm of f, square integrable in $(-\infty, 0)$ with an exponential weight. As a consequence two families of pullback attractors, and relations among them, are obtained. In Section 6.3 regularity results for the obtained attractors will be deduced thanks to splitting the solution in sum of two solutions for two different problems, using carefully a bootstrapping argument that involves fractional powers of the Stokes operator. Finally, in Section 6.4 the problem of the attraction in $D(A)$ norm is studied, and indeed under suitable assumptions, all attractors are proved to coincide. All these results can be found in [32].

6.1 Existence and uniqueness of solution

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with, unless otherwise indicated, smooth enough (e.g. C^2) boundary $\partial\Omega$.

We consider the following problem for a system of non-autonomous Navier–Stokes–Voigt equations,

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \alpha^2 \Delta u) - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(t) \quad \text{in } \ \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \quad \text{in } \ \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \ \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u^\tau(x), \quad x \in \Omega, \end{cases} \tag{6.1}$$

where as usual $u = (u_1, u_2, u_3)$ is the unknown velocity field of the fluid and p is the unknown pressure, and we are given the kinematic viscosity $\nu > 0$, a length scale parameter $\alpha > 0$, characterizing the elasticity of the fluid (in the sense that the ratio α^2/ν describes the reaction time that is required for the fluid to respond to the applied force), an initial velocity field u^τ at the initial time $\tau \in \mathbb{R}$, and an external force term f depending on time.

In this section we analyze existence, uniqueness, and regularity properties of the solutions to problem (6.1). At least, part of these results may be found in [74], but for convenience of the reader, they are developed here. In order to proceed, we need previously to pose the problem in an abstract setting, recalling some definitions of functional spaces, operators, and some of their properties.

Throughout this chapter we will consider again the usual function spaces $\mathcal{V}$, H, and V, the trilinear form b, and the operators A and B, already introduced in Chapter 2, but taking into account that we are now in dimension three.

Recall that we are denoting by $\{w_j\}_{j\geq 1} \subset D(A)$ a Hilbert basis of H formed by orthonormalized eigenfunctions of the Stokes operator A ($Aw_j = \lambda_j w_j$, and $|w_j| = 1$).

For each $\beta \geq 0$ we define

$$D(A^\beta) = \Big\{ u \in H : \sum_{j=1}^{\infty} \lambda_j^{2\beta} (u, w_j)^2 < \infty \Big\},$$

and

$$A^\beta u = \sum_{j=1}^\infty \lambda_j^\beta (u, w_j) w_j \quad \forall\, u \in D(A^\beta).$$

Observe that $w_j \in D(A^\beta)$, with $A^\beta w_j = \lambda_j^\beta w_j$, for all $\beta \geq 0$ and any $j \geq 1$.

Endowed with the inner product

$$(u, v)_{D(A^\beta)} = (A^\beta u, A^\beta v) \quad \forall\, u, v \in D(A^\beta),$$

$D(A^\beta)$ is a Hilbert space. In particular, $D(A^0) = H$, $D(A^{1/2}) = V$ and $D(A^1) = D(A)$.

It is also possible to define negative powers of A. For each $\beta > 0$ and $u \in H$, we define

$$|u|_{D(A^{-\beta})} = \left(\sum_{j=1}^\infty \lambda_j^{-2\beta} (u, w_j)^2 \right)^{1/2},$$

and

$$A^{-\beta} u = \sum_{j=1}^\infty \lambda_j^{-\beta} (u, w_j) w_j.$$

Then, we define $D(A^{-\beta})$ as the completion of H for the norm $|\cdot|_{D(A^{-\beta})}$, and we continue denoting by $A^{-\beta}$ the continuous extension of this linear operator to $D(A^{-\beta})$. The space $D(A^{-\beta})$ is identifiable with the dual of $D(A^\beta)$. In particular, $D(A^{-1/2}) = V'$.

One has (cf. [87]),

$$D(A^{\beta_2}) \subset D(A^{\beta_1}) \subset H \subset D(A^{-\beta_1}) \subset D(A^{-\beta_2}) \quad \forall\, 0 < \beta_1 < \beta_2,$$

with compact injections.

Also observe that we have the following inclusions with continuous injection (cf. [85, Chapter III, Lemmas 2.4.2, 2.4.3])

$$D(A^\beta) \subset (L^{6/(3-4\beta)}(\Omega))^3 \quad \forall\, 0 \leq \beta < 3/4, \tag{6.2}$$

$$D(A^{3/4}) \subset (L^p(\Omega))^3 \quad \forall\, 1 \leq p < \infty, \tag{6.3}$$

and

$$D(A^\beta) \subset (L^\infty(\Omega))^3 \quad \forall\, 3/4 < \beta \leq 1. \tag{6.4}$$

Important properties concerning b that will be used later are that there exists a constant $C_2 > 0$ such that

$$|b(u, v, w)| \leq C_2 \|u\| \|v\| \|w\| \quad \forall\, u, v, w \in V, \tag{6.5}$$

and, using Agmon's inequality (e.g. cf. [21]), we can assure that there exists a constant $C_3 > 0$ such that

$$|b(u, v, w)| \leq C_3 |Au|^{1/2} \|u\|^{1/2} \|v\| |w| \quad \forall\, u \in D(A), v \in V, w \in H. \tag{6.6}$$

Thus, by (6.5),

$$\|B(u)\|_* \leq C_2 \|u\|^2 \quad \forall\, u \in V, \tag{6.7}$$

and in particular, by (6.6) and the identification of H' with H, if $u \in D(A)$, then $B(u) \in H$, with

$$|B(u)| \leq C_3 |Au|^{1/2} \|u\|^{3/2} \quad \forall\, u \in D(A). \tag{6.8}$$

In fact, from (6.4), one also deduces that if $u \in D(A^\beta)$ with $3/4 < \beta \leq 1$, then $B(u) \in H$, and more exactly

$$|B(u)| \leq C_{(\beta)} |A^\beta u| \|u\| \quad \forall\, u \in D(A^\beta), \quad \forall\, 3/4 < \beta \leq 1. \tag{6.9}$$

Analogously, if $0 < \beta < 3/4$, from (6.2) one obtains that if $u \in D(A^\beta) \cap V$, $B(u) \in D(A^{\beta - 3/4})$, and more exactly

$$|A^{\beta-3/4} B(u)| \leq C_{(\beta)} |A^\beta u| \|u\| \quad \forall\, u \in D(A^\beta) \cap V, \quad \forall\, 0 < \beta < 3/4. \tag{6.10}$$

Finally, in the case $\beta = 3/4$, from (6.3) one can see that if $u \in D(A^{3/4})$, then $B(u) \in D(A^{-\delta})$ for all $\delta > 0$, and more exactly

$$|A^{-\delta} B(u)| \leq C_{(3/4,\delta)} |A^{3/4} u| \|u\| \quad \forall\, u \in D(A^{3/4}), \quad \forall\, \delta > 0.$$

Before studying (6.1), we treat the autonomous equation $u + \alpha^2 Au = g$.

From Lax–Milgram lemma, we know that for each $g \in V'$ there exists a unique $u_g \in V$ such that

$$u_g + \alpha^2 A u_g = g. \tag{6.11}$$

The mapping $\mathcal{C} : u \in V \mapsto u + \alpha^2 Au \in V'$ is linear and bijective, with $\mathcal{C}^{-1} g = u_g$.

From (6.11), one has

$$\begin{aligned} |u_g|^2 + \alpha^2 \|u_g\|^2 &= \langle g, u_g \rangle \\ &\leq \|g\|_* \|u_g\|, \end{aligned}$$

and in particular,

$$\|u_g\| \leq \alpha^{-2} \|g\|_*,$$

i.e.,

$$\|\mathcal{C}^{-1} g\| \leq \alpha^{-2} \|g\|_* \quad \forall\, g \in V'. \tag{6.12}$$

Observe that by the definition of $D(A)$, we also have that

$$\mathcal{C}^{-1}(H) = D(A),$$

and reasoning as for the obtention of (6.12), we deduce that

$$\begin{aligned} |Au_g| &= \alpha^{-2} |g - u_g| \\ &\leq 2\alpha^{-2} |g| \quad \forall\, g \in H. \end{aligned} \tag{6.13}$$

Assume that $u^\tau \in V$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$.

Definition 6.1. *It is said that u is a weak solution to (6.1) if u belongs to $L^2(\tau, T; V)$ for all $T > \tau$, and satisfies*

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t), \quad \textit{in } \mathcal{D}'(\tau, \infty; V'), \tag{6.14}$$

and

$$u(\tau) = u^\tau. \tag{6.15}$$

Remark 6.2. *If $u \in L^2(\tau, T; V)$ for all $T > \tau$ and satisfies (6.14), then the function v defined by*

$$v(t) = u(t) + \alpha^2 Au(t) \quad t > \tau, \tag{6.16}$$

belongs to $L^2(\tau, T; V')$ for all $T > \tau$, and by (6.7), $v' = \dfrac{dv}{dt} \in L^1(\tau, T; V')$ for all $T > \tau$.

Consequently, $v \in C([\tau, \infty); V')$, and therefore, by (6.12), $u \in C([\tau, \infty); V)$. In particular, (6.15) makes a sense.

Moreover, again by (6.7) and (6.14), $v' \in L^2(\tau, T; V')$ for all $T > \tau$, and therefore, as $u' = \mathcal{C}^{-1}v'$, we deduce that $u' \in L^2(\tau, T; V)$ for all $T > \tau$.

From these considerations, it is clear that u is a weak solution to (6.1) if and only if it is a solution to the problem

$$\begin{cases} u \in C([\tau, \infty); V), \qquad u' \in L^2(\tau, T; V) \text{ for all } T > \tau, \\[2mm] u(t) + \alpha^2 Au(t) + \displaystyle\int_\tau^t (\nu Au(s) + B(u(s)))\, ds \\[2mm] = u^\tau + \alpha^2 Au^\tau + \displaystyle\int_\tau^t f(s)\, ds \quad (\text{equality in } V'), \text{ for all } t \geq \tau. \end{cases}$$

We have the following energy equality for the solutions to (6.1).

Lemma 6.3. *If u is a weak solution to (6.1), then*

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + \alpha^2 \|u(t)\|^2) + \nu \|u(t)\|^2 = \langle f(t), u(t) \rangle, \quad a.e.\ t > \tau. \tag{6.17}$$

Proof. We know from Remark 6.2 that $u \in W^{1,2}(\tau, T; V)$ and $v \in W^{1,2}(\tau, T; V')$ for all $T > \tau$, where v is given by (6.16). Thus,

$$\frac{d}{dt}\langle v(t), u(t) \rangle = \langle v'(t), u(t) \rangle + \langle v(t), u'(t) \rangle, \quad \text{a.e. } t > \tau. \tag{6.18}$$

But observing that $\mathcal{C}$ is self-adjoint, and using the fact that $v(t) = \mathcal{C}u(t)$ and $v'(t) = \mathcal{C}u'(t)$, we have

$$\begin{aligned} \langle v(t), u'(t) \rangle &= \langle \mathcal{C}u(t), u'(t) \rangle \\ &= \langle \mathcal{C}u'(t), u(t) \rangle \\ &= \langle v'(t), u(t) \rangle. \end{aligned}$$

Thus, by (6.18), we have

$$\frac{d}{dt}\langle v(t), u(t) \rangle = 2\langle v'(t), u(t) \rangle, \quad \text{a.e. } t > \tau.$$

From this identity, taking into account (2.3) and (6.14), we obtain (6.17) . ■

With respect to the existence and uniqueness of solution to (6.1), we have the following result.

Theorem 6.4. *Let $f \in L^2_{\rm loc}(\mathbb{R}; V')$ be given. Then, for each $\tau \in \mathbb{R}$ and $u^\tau \in V$, there exists a unique weak solution $u(\cdot) = u(\cdot; \tau, u^\tau)$ to (6.1).*

Moreover, if $f \in L^2_{\rm loc}(\mathbb{R}; H)$ and $u^\tau \in D(A)$, then the weak solution $u(\cdot) = u(\cdot; \tau, u^\tau)$ to (6.1) satisfies

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau, \tag{6.19}$$

and

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \alpha^2|Au(t)|^2) + \nu|Au(t)|^2 + (B(u(t)), Au(t)) = (f(t), Au(t)), \tag{6.20}$$

a.e. $t > \tau$.

Proof. We divide the proof in four steps, according to the claims of existence, uniqueness, regularity, and the energy equality (6.20).

Uniqueness.

Let $u^{(1)}$ and $u^{(2)}$ be two weak solutions to (6.1), corresponding to the same data f, τ and u^τ. Let us denote $\hat{u} = u^{(1)} - u^{(2)}$, and $\hat{v} = \hat{u} + \alpha^2 A\hat{u}$.

We have that $\hat{v} \in C([\tau, \infty); V')$, with

$$\hat{v}(t) = -\nu \int_\tau^t A\hat{u}(s)\, ds - \int_\tau^t (B(u^{(1)}(s)) - B(u^{(2)}(s)))\, ds \quad \forall\, t \geq \tau. \tag{6.21}$$

Observe that by (6.5),

$$\begin{aligned}
&\|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \\
= &\sup_{w \in V, \|w\|=1} |b(u^{(1)}(s) - u^{(2)}(s), u^{(1)}(s), w) - b(u^{(2)}(s), u^{(2)}(s) - u^{(1)}(s), w)| \\
\leq &\ C_2(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)\|u^{(1)}(s) - u^{(2)}(s)\|.
\end{aligned}$$

Thus, if we fix an arbitrary $T > \tau$, and denote by $R_T = C_2 \max_{s\in[\tau,T]}(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)$, we have

$$\|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \leq R_T\|u^{(1)}(s) - u^{(2)}(s)\| \quad \forall\, s \in [\tau, T]. \tag{6.22}$$

Then, as

$$\|A\hat{u}(s)\|_* = \|\hat{u}(s)\|,$$

from (6.21) and (6.22) we deduce that

$$\|\hat{v}(t)\|_* \leq (\nu + R_T) \int_\tau^t \|\hat{u}(s)\|\, ds \quad \forall\, t \in [\tau, T],$$

and therefore, by (6.12),

$$\|\hat{u}(t)\| \leq \alpha^{-2}(\nu + R_T) \int_\tau^t \|\hat{u}(s)\|\, ds \quad \forall\, t \in [\tau, T].$$

From this inequality, by Gronwall's lemma, we deduce that $\|\hat{u}(t)\| = 0$ for all $t \in [\tau, T]$, and therefore, the uniqueness of weak solution to (6.1) holds.

Existence.

We can prove the existence of weak solution to (6.1) reasoning as in [5, pp. 844–846], but then with this method of proof, we do not know how to obtain the regularity result (6.19). We will proceed by using a Galerkin scheme.

Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, $\tau \in \mathbb{R}$, and $u^\tau \in V$, be given.

For each integer $m \geq 1$, let define

$$u^m(t) = \sum_{j=1}^{m} \alpha_{m,j}(t) w_j,$$

where the coefficients $\alpha_{m,j}$ are required to satisfy the system

$$\begin{aligned} & \frac{d}{dt}(u^m(t) + \alpha^2 A u^m(t), w_j) \\ = \; & -\langle \nu A u^m(t) + B(u^m(t)) - f(t), w_j \rangle, \quad \text{a.e. } t > \tau, \quad 1 \leq j \leq m, \end{aligned} \tag{6.23}$$

and the initial condition

$$u^m(\tau) = P_m u^\tau,$$

where recall that $P_m u^\tau = \sum_{j=1}^m (u^\tau, w_j) w_j$, is the orthogonal (in H and in V) projection of u^τ onto the space V_m =span$[w_1, \ldots, w_m]$.

The above system of ordinary differential equations fulfills the conditions of the Picard's theorem for existence and uniqueness of local solution.

Next, we will deduce a priori estimates that assure that the solutions u^m do exist for all time $t \in [\tau, \infty)$.

Multiplying in (6.23) by $\alpha_{m,j}(t)$, summing from $j = 1$ to m, and taking into account (2.3), we obtain that a.e. $t > \tau$,

$$\begin{aligned} \frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2) + 2\nu \|u^m(t)\|^2 &= 2\langle f(t), u^m(t) \rangle \\ &\leq \nu \|u^m(t)\|^2 + \nu^{-1} \|f(t)\|_*^2, \end{aligned}$$

and in particular,

$$|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 \leq |P_m u^\tau|^2 + \alpha^2 \|P_m u^\tau\|^2 + \nu^{-1} \int_\tau^t \|f(s)\|_*^2 \, ds$$

for all $t \geq \tau$, and any $m \geq 1$.

Observe that as $u^\tau \in V$, one has that $|P_m u^\tau| \leq |u^\tau|$, $\|P_m u^\tau\| \leq \|u^\tau\|$, and $\lim\limits_{m \to \infty} \|u^\tau - P_m u^\tau\| = 0$. Thus, the sequence $\{u^m\}$ is bounded in $C([\tau, T]; V)$ for all $T > \tau$.

Now, observe that by (6.23), $v^m = \mathcal{C} u^m$ satisfies

$$\frac{d}{dt}(v^m(t)) = \tilde{P}_m(-\nu A u^m(t) - B(u^m(t)) + f(t)), \quad \text{a.e. } t > \tau, \tag{6.24}$$

where

$$\langle \tilde{P}_m g, w\rangle = \langle g, P_m w\rangle \quad \forall\, g \in V',\, w \in V.$$

Consequently, as $\|\tilde{P}_m\|_{\mathcal{L}(V')} \le 1$ for all $m \ge 1$, we deduce that the sequence $\{dv^m/dt\}$ is bounded in $L^2(\tau, T; V')$ for all $T > \tau$, and therefore, taking into account that $du^m/dt = \mathcal{C}^{-1}(dv^m/dt)$, we have that the sequence $\{du^m/dt\}$ is bounded in $L^2(\tau, T; V)$ for all $T > \tau$.

Thus, by the compactness of the injection of V into H and the Ascoli–Arzelà theorem, we deduce that there exist a subsequence $\{u^{m'}\} \subset \{u^m\}$ and a function $u \in W^{1,2}(\tau, T; V)$ for all $T > \tau$, such that

$$\begin{cases} u^{m'} \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(\tau, T; V), \\ u^{m'} \to u \text{ strongly in } C([\tau, T]; H), \\ u^{m'} \to u \text{ a.e. in } \Omega \times (\tau, T), \\ \dfrac{du^{m'}}{dt} \rightharpoonup \dfrac{du}{dt} \text{ weakly in } L^2(\tau, T; V), \\ \dfrac{dv^{m'}}{dt} = \mathcal{C}\left(\dfrac{du^{m'}}{dt}\right) \rightharpoonup \mathcal{C}\left(\dfrac{du}{dt}\right) \text{ weakly in } L^2(\tau, T; V'), \end{cases} \tag{6.25}$$

for all $T > \tau$.

As in particular $H_0^1(\Omega) \subset L^4(\Omega)$ with continuous injection, for each $1 \le i, j \le 3$, the product $u_i^{m'} u_j^{m'}$ of the corresponding components of $u^{m'}$ is bounded in $L^\infty(\tau, T; L^2(\Omega))$, for all $T > \tau$, and by (6.25), $u_i^{m'} u_j^{m'} \to u_i u_j$ a.e. in $\Omega \times (\tau, T)$. So, by [61, Chapter 1, Lemma 1.3], we deduce that $u_i^{m'} u_j^{m'} \rightharpoonup u_i u_j$ weakly in $L^2(\Omega \times (\tau, T))$, for all $T > \tau$.

Therefore, taking into account (2.2), if $w \in L^2(\tau, T; V)$,

$$\begin{aligned} \int_\tau^T \langle B(u^{m'}(t)), w(t)\rangle\, dt &= -\int_\tau^T b(u^{m'}(t), w(t), u^{m'}(t))\, dt \\ &= -\sum_{i,j=1}^3 \int_\tau^T \int_\Omega u_i^{m'}(x,t) u_j^{m'}(x,t) \frac{\partial w_j}{\partial x_i}(x,t)\, dx\, dt \\ &\to -\sum_{i,j=1}^3 \int_\tau^T \int_\Omega u_i(x,t) u_j(x,t) \frac{\partial w_j}{\partial x_i}(x,t)\, dx\, dt \\ &= \int_\tau^T \langle B(u(t)), w(t)\rangle\, dt. \end{aligned}$$

Hence, $B(u^{m'}) \rightharpoonup B(u)$ weakly in $L^2(\tau, T; V')$, for all $T > \tau$.

From all the convergences above, and (6.24), we can take limits and we obtain that u satisfies (6.14).

Observe that $u(\tau) = \lim_{m' \to \infty} u^{m'}(\tau) = \lim_{m' \to \infty} P_{m'} u^\tau = u^\tau$. Thus, u is the weak solution to (6.1).

Regularity.

Assume now that $u^\tau \in D(A)$ and $f \in L^2_{\rm loc}(\mathbb{R}; H)$.

Multiplying in (6.23) by $\lambda_j \alpha_{m,j}(t)$, and summing from $j = 1$ to m, we obtain that

$$\begin{aligned} & \frac{d}{dt}(\|u^m(t)\|^2 + \alpha^2 |Au^m(t)|^2) + 2\nu |Au^m(t)|^2 \\ = \; & -2(B(u^m(t)), Au^m(t)) + 2(f(t), Au^m(t)), \quad \text{a.e. } t > \tau. \end{aligned} \tag{6.26}$$

But by (6.8) and Young's inequality,

$$\begin{aligned} 2|(B(u^m(t)), Au^m(t))| & \leq 2C_3 \|u^m(t)\|^{3/2} |Au^m(t)|^{3/2} \\ & \leq C_\nu \|u^m(t)\|^6 + \nu |Au^m(t)|^2, \end{aligned}$$

where $C_\nu = 27 C_3^4 (16\nu^3)^{-1}$.

Also,

$$2|(f(t), Au^m(t))| \leq \nu |Au^m(t)|^2 + \nu^{-1} |f(t)|^2.$$

Thus, observing that $|AP_m u^\tau| \leq |Au^\tau|$ and $\|P_m u^\tau\| \leq \|u^\tau\|$, from (6.26) we deduce in particular that

$$\alpha^2 |Au^m(t)|^2 \leq \|u^\tau\|^2 + \alpha^2 |Au^\tau|^2 + \nu^{-1} \int_\tau^t |f(s)|^2 \, ds + C_\nu (t - \tau) \sup_{s \in [\tau, t]} \|u^m(s)\|^6$$

for all $t \geq \tau$, and any $m \geq 1$.

Consequently, as $\{u^m\}$ is bounded in $C([\tau, T]; V)$, we have that $\{u^m\}$ is bounded in $C([\tau, T]; D(A))$, for all $T > \tau$, and therefore, extracting a subsequence weakly-star convergent in $L^\infty(\tau, T; D(A))$, we see that $u \in L^\infty(\tau, T; D(A))$, for all $T > \tau$.

But then, $v = u + \alpha^2 Au \in L^\infty(\tau, T; H)$, with $v' = -\nu Au - B(u) + f \in L^2(\tau, T; H)$, for all $T > \tau$, and therefore, $v \in C([\tau, \infty); H)$.

Thus, $Au = \alpha^{-2}(v - u) \in C([\tau, \infty); H)$, i.e., $u \in C([\tau, \infty); D(A))$.

Moreover, as $v' \in L^2(\tau, T; H)$, by (6.13), then $u' = \mathcal{C}^{-1} v' \in L^2(\tau, T; D(A))$, for all $T > \tau$.

Identity (6.20).

If $u^\tau \in D(A)$ and $f \in L^2_{\rm loc}(\mathbb{R}; H)$, we have seen that $u \in W^{1,2}(\tau, T; D(A))$ and $v = \mathcal{C}u \in W^{1,2}(\tau, T; H)$, for all $T > \tau$. Then,

$$\frac{d}{dt}|v(t)|^2 = 2(v'(t), v(t)), \quad \text{a.e. } t > \tau,$$

and

$$\begin{aligned} \frac{d}{dt}(u(t), v(t)) & = \frac{d}{dt}(u(t), \mathcal{C}u(t)) \\ & = 2(u(t), \mathcal{C}u'(t)) \\ & = 2(u(t), v'(t)), \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}(Au(t),v(t)) &= \alpha^{-2}\frac{d}{dt}(v(t)-u(t),v(t))\\ &= 2\alpha^{-2}(v'(t),v(t))-2\alpha^{-2}(v'(t),u(t))\\ &= 2(v'(t),Au(t)), \quad \text{a.e. } t>\tau.\end{aligned}$$

From this equality, we have (6.20). ■

Remark 6.5. *Observe that in the above proof, using the uniqueness of solution for the problem, for any $T>\tau$ the whole sequence of the Galerkin approximations satisfies that u^m converges to u in $C([\tau,T];H)$, and actually, all convergences in (6.25), except the third one, hold for the whole sequence. Analogously, one also deduces that for any $t\in[\tau,T]$, $u^m(t)\rightharpoonup u(t)$ weakly in V.*

Moreover, if $u^\tau\in D(A)$ and $f\in L^2_{\rm loc}(\mathbb{R};H)$, then in fact for any $T>\tau$ the sequence u^m converges to u in $C([\tau,T];V)$, and weakly-star in $L^\infty(\tau,T;D(A))$, for any $t\in[\tau,T]$, $u^m(t)\rightharpoonup u(t)$ in $D(A)$, and the sequence du^m/dt converges to du/dt weakly in $L^2(\tau,T;D(A))$.

Now, we establish a result on the sequential weak continuity of the solutions to (6.1) with respect to the initial datum u^τ.

Theorem 6.6. *Let $f\in L^2_{\rm loc}(\mathbb{R};V')$ and $\tau<t$ be given. Consider a sequence $\{u^{\tau,n}\}\subset V$ weakly converging to u^τ in V. Then, the following convergences hold for the sequence of solutions $u(\cdot;\tau,u^{\tau,n})$ towards the solution $u(\cdot;\tau,u^\tau)$:*

$$u(\cdot;\tau,u^{\tau,n})\overset{*}{\rightharpoonup}u(\cdot;\tau,u^\tau) \text{ weakly-star in } L^\infty(\tau,t;V),$$

$$u(\cdot;\tau,u^{\tau,n})\to u(\cdot;\tau,u^\tau) \text{ strongly in } C([\tau,t];H),$$

$$u(t;\tau,u^{\tau,n})\rightharpoonup u(t;\tau,u^\tau) \text{ weakly in } V.$$

Moreover, if $f\in L^2_{\rm loc}(\mathbb{R};H)$ and the sequence $\{u^{\tau,n}\}\subset D(A)$ converges weakly to u^τ in $D(A)$, then, in fact,

$$u(\cdot;\tau,u^{\tau,n})\overset{*}{\rightharpoonup}u(\cdot;\tau,u^\tau) \text{ weakly-star in } L^\infty(\tau,t;D(A)),$$

$$u(\cdot;\tau,u^{\tau,n})\to u(\cdot;\tau,u^\tau) \text{ strongly in } C([\tau,t];V),$$

$$u(t;\tau,u^{\tau,n})\rightharpoonup u(t;\tau,u^\tau) \text{ weakly in } D(A).$$

Proof. The proof can be done analogously to that of Theorem 6.4, since the a priori estimates follow exactly the same. The fact that the whole sequence satisfies the above convergences is a consequence of the uniqueness of solution for the problem (cf. Remark 6.5). ■

Remark 6.7. *Although the above result will be enough for our purposes, let us observe that the solution also depends continuously of the initial datum in the strong topology* *of V. Moreover, when $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, the solution depends continuously of the initial datum in the strong topology of $D(A)$. Indeed, this can be proved similarly to the proof of uniqueness of weak solution to (6.1), considering the difference of two solutions and using Gronwall's lemma.*

Remark 6.8. *Observe that actually in the existence and uniqueness part of Theorem 6.4 and also in the first part of Theorem 6.6 we do not need any regularity assumption on the boundary of the domain. This assumption is only required for the additional regularity results.*

6.2 Existence of minimal pullback attractors in V norm

Now, by the previous results, we are able to define a process U on V associated to (6.1), and under suitable assumptions on f, we can obtain the existence of minimal pullback attractors. As pointed out at the beginning of the chapter, in the results of this section we do not require any regularity assumption on $\partial\Omega$, and the force term may take values in V' instead of in L^2 as appears in [45].

Proposition 6.9. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ is given. Then, the bi-parametric family of mappings $U(t, \tau) : V \to V$, with $\tau \leq t$, given by*

$$U(t, \tau)u^\tau = u(t; \tau, u^\tau), \tag{6.27}$$

where $u = u(\cdot; \tau, u^\tau)$ is the unique weak solution to (6.1), defines a closed process on V.

Proof. It is a consequence of Theorem 6.4 and Theorem 6.6. ■

Remark 6.10. *Observe that, by Remark 6.7, U is in fact a continuous process on V.*

For the obtention of a pullback absorbing family for the process U, we have the following result.

Lemma 6.11. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and $u^\tau \in V$. Then, for any*

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}, \tag{6.28}$$

the solution $u = u(\cdot; \tau, u^\tau)$ to (6.1) satisfies

$$\begin{aligned} &\|u(t)\|^2 + \varepsilon\alpha^{-2} \int_\tau^t e^{\sigma(s-t)} \|u(s)\|^2 \, ds \\ \leq\ & (1 + \alpha^{-2}\lambda_1^{-1}) e^{\sigma(\tau - t)} \|u^\tau\|^2 + \alpha^{-2}\varepsilon^{-1} \int_\tau^t e^{\sigma(s-t)} \|f(s)\|_*^2 \, ds \end{aligned} \tag{6.29}$$

for all $t \geq \tau$, where

$$\varepsilon = \nu - \frac{\sigma}{2}(\lambda_1^{-1} + \alpha^2). \tag{6.30}$$

Proof. By (6.17), for all $\varepsilon > 0$,

$$\begin{aligned}
& \frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) \\
= \ & \sigma e^{\sigma t}|u(t)|^2 + \alpha^2 \sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 + 2e^{\sigma t}\langle f(t), u(t)\rangle \\
\leq \ & \{\sigma(\lambda_1^{-1} + \alpha^2) - 2\nu + \varepsilon\} e^{\sigma t}\|u(t)\|^2 + \varepsilon^{-1} e^{\sigma t}\|f(t)\|_*^2, \quad \text{a.e. } t > \tau.
\end{aligned}$$

Thus, if σ satisfies (6.28), then ε given by (6.30) is positive, and for this ε we have

$$\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) + \varepsilon e^{\sigma t}\|u(t)\|^2 \leq \varepsilon^{-1} e^{\sigma t}\|f(t)\|_*^2, \quad \text{a.e. } t > \tau.$$

From this inequality we obtain (6.29). ∎

Taking into account the estimate (6.29), we define the following universe.

Definition 6.12. *For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(V)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V)$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\sigma \tau} \sup_{v \in D(\tau)} \|v\|^2 \right) = 0.$$

Once more, accordingly to the notation introduced in Chapter 1, $\mathcal{D}_F(V)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of V.

Observe that for any $\sigma > 0$, $\mathcal{D}_F(V) \subset \mathcal{D}_\sigma(V)$, and that the universe $\mathcal{D}_\sigma(V)$ is inclusion-closed.

As an evident consequence of Lemma 6.11, we have the following result.

Corollary 6.13. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies that*

$$\int_{-\infty}^0 e^{\sigma s}\|f(s)\|_*^2\, ds < \infty \quad \textit{for some } 0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}. \tag{6.31}$$

Then, the family $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ defined by

$$D_\sigma(t) = \overline{B}_V(0, R_\sigma(t)), \tag{6.32}$$

the closed ball in V of center zero and radius $R_\sigma(t)$, where

$$R_\sigma^2(t) = 1 + \alpha^{-2}\varepsilon^{-1} e^{-\sigma t} \int_{-\infty}^t e^{\sigma s}\|f(s)\|_*^2\, ds, \tag{6.33}$$

with ε given by (6.30), is pullback $\mathcal{D}_\sigma(V)$-absorbing for the process $U : \mathbb{R}^2_d \times V \to V$ given by (6.27) (and therefore $\mathcal{D}_F(V)$-absorbing too), and $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V)$.

In order to prove that the process U is pullback $\widehat{D}_\sigma$-asymptotically compact, we will apply an energy method used by Rosa (cf. [79], see also [69]), which does not require any additional estimate on the solutions in higher norms in contrast with the *energy continuous method* already used in several results of Chapters 2, 3, 4 and 5 (e.g. cf. Lemmas 2.14, 3.2, 4.24 or 5.17), or the method used in [45] with the fractional powers of the operator A. Our proof here relies on a sharp use of the differential equality that leads to the existence of an absorbing family, the use of weak limits in V in a diagonal argument, and the fact that the process is sequentially weakly continuous.

Lemma 6.14. *Assume that $f \in L^2_{\rm loc}(\mathbb{R}; V')$ satisfies (6.31). Then, the process U defined by (6.27) is pullback $\widehat{D}_\sigma$-asymptotically compact, where $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ is defined in Corollary 6.13.*

Proof. Let $t \in \mathbb{R}$, and $\tau_n \to -\infty$ with $\tau_n \leq t$ and $u^{\tau_n} \in D_\sigma(\tau_n)$ for all n, be given. We must prove that the sequence $\{U(t,\tau_n)u^{\tau_n}\}$ is relatively compact in V. By Corollary 6.13, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}_\sigma}(k) \leq t - k$ such that

$$U(t-k,\tau)D_\sigma(\tau) \subset D_\sigma(t-k) \quad \forall\, \tau \leq \tau_{\widehat{D}_\sigma}(k).$$

Recall that each $D_\sigma(t)$, defined in (6.32), is a bounded set of V. From this and a diagonal argument, we can extract a subsequence $\{u^{\tau_{n'}}\} \subset \{u^{\tau_n}\}$ such that

$$U(t-k,\tau_{n'})u^{\tau_{n'}} \rightharpoonup \tilde{w}_k \quad \text{weakly in } V,\ \forall\, k \geq 0, \tag{6.34}$$

where $\tilde{w}_k \in D_\sigma(t-k)$.

Now, applying Theorem 6.6 on each fixed interval $[t-k,t]$ we obtain that

$$\begin{aligned}
\tilde{w}_0 &= V-\text{weak}\lim_{n'\to\infty} U(t,\tau_{n'})u^{\tau_{n'}} \\
&= V-\text{weak}\lim_{n'\to\infty} U(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}} \\
&= U(t,t-k)\left[V-\text{weak}\lim_{n'\to\infty} U(t-k,\tau_{n'})u^{\tau_{n'}}\right] \\
&= U(t,t-k)\tilde{w}_k.
\end{aligned}$$

In particular, observe that

$$\|\tilde{w}_0\| \leq \liminf_{n'\to\infty} \|U(t,\tau_{n'})u^{\tau_{n'}}\|.$$

We will prove now that it also holds that

$$\limsup_{n'\to\infty} \|U(t,\tau_{n'})u^{\tau_{n'}}\| \leq \|\tilde{w}_0\|, \tag{6.35}$$

which combined with (6.34) for $k = 0$, will imply the convergence in the strong topology of V, and the asymptotic compactness.

Observe that, as we already used in Lemma 6.11, for any pair (τ, u^τ) with $u^\tau \in V$, the solution $u(\cdot;\tau,u^\tau)$, for short denoted by $u(\cdot)$, satisfies the differential equality

$$\begin{aligned}
&\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) \\
={}& \sigma e^{\sigma t}|u(t)|^2 + \alpha^2\sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 + 2e^{\sigma t}\langle f(t), u(t)\rangle, \text{ a.e. } t > \tau.
\end{aligned} \tag{6.36}$$

Since we have chosen σ satisfying (6.28), observe that $[\cdot]$, with

$$[v]^2 = (2\nu - \alpha^2\sigma)\|v\|^2 - \sigma|v|^2,$$

defines an equivalent norm to $\|\cdot\|$ in V.

We integrate the above expression in the interval $[t-k,t]$ for the solutions $U(\cdot,\tau_{n'})u^{\tau_{n'}}$ with $\tau_{n'} \le t-k$, which yields

$$\begin{aligned}
&|U(t,\tau_{n'})u^{\tau_{n'}}|^2 + \alpha^2\|U(t,\tau_{n'})u^{\tau_{n'}}\|^2\\
= \;& |U(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}|^2 + \alpha^2\|U(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}\|^2\\
= \;& e^{-\sigma k}\left(|U(t-k,\tau_{n'})u^{\tau_{n'}}|^2 + \alpha^2\|U(t-k,\tau_{n'})u^{\tau_{n'}}\|^2\right)\\
&+2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}\rangle\, ds\\
&-\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}]^2\, ds. \qquad (6.37)
\end{aligned}$$

On other hand, by (6.34) and Theorem 6.6, we deduce that

$$U(\cdot,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}} \rightharpoonup U(\cdot,t-k)\tilde{w}_k \quad \text{weakly in } L^2(t-k,t;V).$$

From this, as $e^{\sigma(\cdot-t)}f(\cdot) \in L^2(t-k,t;V')$, it yields

$$\begin{aligned}
&\lim_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}\rangle\, ds\\
= \;& \int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)\tilde{w}_k\rangle\, ds. \qquad (6.38)
\end{aligned}$$

Since $\int_{t-k}^{t} e^{\sigma(s-t)}[v(s)]^2 ds$ defines an equivalent norm in $L^2(t-k,t;V)$, we also deduce from above that

$$\begin{aligned}
&\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)\tilde{w}_k]^2\, ds\\
\le \;& \liminf_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}]^2\, ds. \qquad (6.39)
\end{aligned}$$

From (6.37)–(6.39), taking into account (6.34) with $k=0$, the compactness of the injection of V into H, (6.32), we conclude that

$$\begin{aligned}
&|\tilde{w}_0|^2 + \alpha^2\limsup_{n'\to\infty}\|U(t,\tau_{n'})u^{\tau_{n'}}\|^2\\
\le \;& e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R_\sigma^2(t-k) + 2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), U(s,t-k)\tilde{w}_k\rangle\, ds\\
&-\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)\tilde{w}_k]^2\, ds.
\end{aligned}$$

Now, taking into account that $\tilde{w}_0 = U(t, t-k)\tilde{w}_k$, integrating again in (6.36), we obtain

$$\begin{aligned} |\tilde{w}_0|^2 + \alpha^2 \|\tilde{w}_0\|^2 &= e^{-\sigma k}(|\tilde{w}_k|^2 + \alpha^2 \|\tilde{w}_k\|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)} \langle f(s), U(s, t-k)\tilde{w}_k \rangle \, ds \\ &\quad - \int_{t-k}^{t} e^{\sigma(s-t)} [U(s, t-k)\tilde{w}_k]^2 \, ds. \end{aligned}$$

Comparing the above two expressions, we conclude that in particular

$$|\tilde{w}_0|^2 + \alpha^2 \limsup_{n' \to \infty} \|U(t, \tau_{n'})u^{\tau_{n'}}\|^2 \leq e^{-\sigma k}(\lambda_1^{-1} + \alpha^2) R_\sigma^2(t-k) + |\tilde{w}_0|^2 + \alpha^2 \|\tilde{w}_0\|^2.$$

But from (6.33) and (6.31), we have that $\lim_{k \to \infty} e^{-\sigma k} R_\sigma^2(t-k) = 0$, so (6.35) holds, and the proof is finished. ■

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_d^2 \times V \to V$ defined by (6.27).

Theorem 6.15. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ satisfies (6.31). Then, there exist the minimal pullback $\mathcal{D}_F(V)$-attractor $\mathcal{A}_{\mathcal{D}_F(V)}$ and the minimal pullback $\mathcal{D}_\sigma(V)$-attractor $\mathcal{A}_{\mathcal{D}_\sigma(V)}$ for the process U defined by (6.27), $\mathcal{A}_{\mathcal{D}_\sigma(V)}$ belongs to $\mathcal{D}_\sigma(V)$, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V)}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma(V)}(t) \subset \overline{B}_V(0, R_\sigma(t)) \quad \forall\, t \in \mathbb{R}, \tag{6.40}$$

where R_σ is given by (6.33).

Finally, if f satisfies the stronger requirement

$$\sup_{r \leq 0} \left(e^{-\sigma r} \int_{-\infty}^{r} e^{\sigma s} \|f(s)\|_*^2 \, ds \right) < \infty, \tag{6.41}$$

then

$$\mathcal{A}_{\mathcal{D}_F(V)}(t) = \mathcal{A}_{\mathcal{D}_\sigma(V)}(t) \quad \forall\, t \in \mathbb{R}. \tag{6.42}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_\sigma(V)}$ and $\mathcal{A}_{\mathcal{D}_F(V)}$ is a direct consequence of Theorem 1.11, Corollary 1.13, Proposition 6.9, Corollary 6.13, and Lemma 6.14.

The inclusions in (6.40) are a consequence of Theorem 1.11 and Corollary 1.13.

Finally, the equality (6.42) is a consequence of Remark 1.14, and the fact that (6.41) is equivalent to have that $\sup_{t \leq T} R_\sigma(t)$ is bounded for any $T \in \mathbb{R}$. ■

Remark 6.16. *Observe that, as it can be easily proved, in general, if $g \in L^2_{\mathrm{loc}}(\mathbb{R}; X)$, with X a Banach space with norm $\|\cdot\|_X$, the three following conditions are equivalent:*

(1) $\sup_{r \leq 0} \left(e^{-\sigma r} \int_{-\infty}^{r} e^{\sigma s} \|g(s)\|_X^2 \, ds \right) < \infty$, *for some $\sigma > 0$.*

(2) $\sup_{r \leq 0} \int_{r-1}^{r} \|g(s)\|_X^2 \, ds < \infty.$

(3) $\sup_{r \le 0} \left(e^{-\hat{\sigma} r} \int_{-\infty}^{r} e^{\hat{\sigma} s} \|g(s)\|_X^2 \, ds \right) < \infty$, *for all* $\hat{\sigma} > 0$.

In fact, in Chapters 2, 4 and 5, it was already observed the equivalence among (2.39), (4.57), (5.15), and (2.41).

Remark 6.17. *Observe that if* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ *satisfies (6.31), then it also satisfies*

$$\int_{-\infty}^{0} e^{\hat{\sigma} s} \|f(s)\|_*^2 \, ds < \infty \quad \forall \, \hat{\sigma} \in (\sigma, 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}).$$

So, there exists the corresponding minimal pullback $\mathcal{D}_{\hat{\sigma}}(V)$*-attractor* $\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(V)}$.

In fact, since $\mathcal{D}_{\sigma}(V) \subset \mathcal{D}_{\hat{\sigma}}(V)$, *for any* $t \in \mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}_{\sigma}(V)}(t) \subset \mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(V)}(t) \quad \forall \, \hat{\sigma} \in (\sigma, 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}).$$

Moreover, if (6.41) also holds, then we conclude by (6.42) and Remark 6.16 that for any $\hat{\sigma} \in (\sigma, 2\nu(\lambda_1^{-1} + \alpha^2)^{-1})$,

$$\mathcal{A}_{\mathcal{D}_F(V)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma}(V)}(t) = \mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(V)}(t) \quad \forall \, t \in \mathbb{R}.$$

Thus, $\mathcal{A}_{\mathcal{D}_F(V)}$ *is the minimal pullback* $\mathcal{D}_{max}(V)$*-attractor, where*

$$\mathcal{D}_{max}(V) = \bigcup_{0 < \hat{\sigma} < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}} \mathcal{D}_{\hat{\sigma}}(V).$$

6.3 Regularity of the pullback attractors

The main goal of this paragraph is to provide some extra regularity for the attractors obtained in the previous section. This will be obtained by a bootstrapping argument, and making the most of a representation of the solutions to the problem splitting it in two parts, the linear part with an exponential decay, and the nonlinear part with good enough estimates. In order to achieve these results, we will use the fractional powers of the Stokes operator, introduced in Section 6.1.

Observe that for every $\tau \in \mathbb{R}$, any $u^\tau \in V$, and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, by Theorem 6.4, there exists a unique weak solution u to problem (6.1). Moreover, let us point out that the following representation of the solution holds:

$$u(t) = U(t,\tau)u^\tau = Y(t,\tau)u^\tau + Z(t,\tau)u^\tau \quad \forall \, t \ge \tau,$$

where $y = Y(\cdot,\tau)u^\tau$ and $z = Z(\cdot,\tau)u^\tau$, are solutions of

$$\begin{cases} y \in C([\tau,\infty); V), \\ \dfrac{d}{dt}(y(t) + \alpha^2 A y(t)) + \nu A y(t) = 0, \quad \text{in } \mathcal{D}'(\tau,\infty; V'), \\ y(\tau) = u^\tau, \end{cases} \tag{6.43}$$

and

$$\begin{cases} z \in C([\tau, \infty); V), \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = f(t) - B(u(t)), \quad \text{in } \mathcal{D}'(\tau, \infty; V'), \\ z(\tau) = 0, \end{cases} \tag{6.44}$$

respectively.

The existence and uniqueness of weak solution to (6.43) and to (6.44) can be obtained reasoning as in the proof of Theorem 6.4.

For the problem (6.43) we have the following result.

Lemma 6.18. *For any $\tau \in \mathbb{R}$, $u^\tau \in V$, and σ fulfilling the assumption (6.28), the solution $y = Y(\cdot, \tau)u^\tau$ to (6.43) satisfies*

$$\|Y(t, \tau)u^\tau\|^2 \le (1 + \alpha^{-2}\lambda_1^{-1})e^{\sigma(\tau - t)}\|u^\tau\|^2 \quad \forall\, t \ge \tau. \tag{6.45}$$

Proof. It is analogous to the proof of (6.29), and we omit it. ∎

For the study of the problem (6.44), we will make use of the following lemma.

Lemma 6.19. *Assume that $g \in L^2_{\mathrm{loc}}(\mathbb{R}; D(A^{-\beta}))$ with $0 \le \beta \le 1/2$. Then, for each $\tau \in \mathbb{R}$ and σ satisfying the assumption (6.28), the unique solution z to the problem*

$$\begin{cases} z \in C([\tau, \infty); V), \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = g(t), \quad \textit{in } \mathcal{D}'(\tau, \infty; V'), \\ z(\tau) = 0, \end{cases} \tag{6.46}$$

satisfies

$$z \in C([\tau, \infty); D(A^{1-\beta})), \tag{6.47}$$

and

$$|A^{1-\beta}z(t)|^2 \le \alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|A^{-\beta}g(s)|^2\, ds \quad \forall\, t \ge \tau, \tag{6.48}$$

where ε is given by (6.30).

Proof. We give a formal proof, the rigorous one should be made using the Galerkin approximations constructed with the basis $\{w_j\}_{j\ge1}$ of eigenfunctions of the Stokes operator A.

Multiplying in (6.46) by $A^{1-2\beta}z(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|A^{(1-2\beta)/2}z(t)|^2 + \alpha^2|A^{1-\beta}z(t)|^2\right) + \nu|A^{1-\beta}z(t)|^2 = (A^{-\beta}g(t), A^{1-\beta}z(t)),$$

a.e. $t > \tau$.

Thus,

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{\sigma t}\left(|A^{(1-2\beta)/2}z(t)|^2+\alpha^2|A^{1-\beta}z(t)|^2\right)\right\}+2\nu e^{\sigma t}|A^{1-\beta}z(t)|^2\\
=\ &\sigma e^{\sigma t}\left(|A^{(1-2\beta)/2}z(t)|^2+\alpha^2|A^{1-\beta}z(t)|^2\right)+2e^{\sigma t}(A^{-\beta}g(t),A^{1-\beta}z(t)),
\end{aligned}\tag{6.49}$$

a.e. $t>\tau$.

Now, using that

$$2e^{\sigma t}|(A^{-\beta}g(t),A^{1-\beta}z(t))|\le \varepsilon e^{\sigma t}|A^{1-\beta}z(t)|^2+\varepsilon^{-1}e^{\sigma t}|A^{-\beta}g(t)|^2,$$

and

$$\begin{aligned}
|A^{1-\beta}z(t)|^2 &= |A^{1/2}(A^{(1-2\beta)/2}z(t))|^2\\
&\ge \lambda_1|A^{(1-2\beta)/2}z(t)|^2,
\end{aligned}$$

from (6.49) and the fact that $z(\tau)=0$, we obtain (6.48).

Now, from (6.48) we have $v=z+\alpha^2Az$ and its derivative v' belong to $L^2(\tau,T;D(A^{-\beta}))$ for any $T>\tau$. So, it holds that $v\in C([\tau,\infty);D(A^{-\beta}))$, whence using the mapping $\mathcal{C}$, (6.47) follows. ■

Now, we can prove the following regularity result for the pullback attractors in V norm.

Theorem 6.20. *Assume that $f\in L^2_{\mathrm{loc}}(\mathbb{R};D(A^{-\beta}))$ for some $0\le\beta\le 1/2$, and that*

$$\sup_{r\le 0}\int_{r-1}^{r}\|f(s)\|_*^2\,ds<\infty. \tag{6.50}$$

Then:

(1) If f also satisfies

$$\int_{-\infty}^{0}e^{\sigma s}|A^{-\beta}f(s)|^2\,ds<\infty\quad \text{for some } 0<\sigma<2\nu(\lambda_1^{-1}+\alpha^2)^{-1}, \tag{6.51}$$

and

$$\begin{cases}\displaystyle\sup_{r\le 0}\int_{r-1}^{r}|A^{-1/4-\beta}f(s)|^2\,ds<\infty, & \text{if } 0<\beta<1/4,\\ \displaystyle\sup_{r\le 0}\int_{r-1}^{r}|A^{-\delta}f(s)|^2\,ds<\infty & \text{for some } 0<\delta<1/4,\ \text{if } \beta=0,\end{cases} \tag{6.52}$$

then the pullback attractor $\mathcal{A}_{\mathcal{D}_{max}(V)}=\mathcal{A}_{\mathcal{D}_F(V)}$ fulfills that for any $t_1<t_2$,

$$\bigcup_{t_1\le t\le t_2}\mathcal{A}_{\mathcal{D}_{max}(V)}(t)=\bigcup_{t_1\le t\le t_2}\mathcal{A}_{\mathcal{D}_F(V)}(t)\ \text{is a bounded subset of } D(A^{1-\beta}). \tag{6.53}$$

(2) If f also satisfies

$$\sup_{r\le 0}\int_{r-1}^{r}|A^{-\beta}f(s)|^2\,ds<\infty, \tag{6.54}$$

then for any $t_2\in\mathbb{R}$,

$$\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_{max}(V)}(t)=\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_F(V)}(t)\ \textit{is a bounded subset of}\ D(A^{1-\beta}). \tag{6.55}$$

Proof. Let us fix $t\in\mathbb{R}$ and $v\in\mathcal{A}_{\mathcal{D}_\sigma(V)}(t)=\mathcal{A}_{\mathcal{D}_F(V)}(t)$. By (6.40), (6.50) and Remark 6.16, we see that

$$\bigcup_{r\le t}\mathcal{A}_{\mathcal{D}_\sigma(V)}(r)\subset\overline{B}_V(0,\widetilde{R}_\sigma(t)), \tag{6.56}$$

where

$$\widetilde{R}_\sigma^2(t)=1+\alpha^{-2}\varepsilon^{-1}\sup_{r\le t}\left(e^{-\sigma r}\int_{-\infty}^{r}e^{\sigma s}\|f(s)\|_*^2\,ds\right),$$

with ε given by (6.30).

Let $\{\tau_n\}\subset(-\infty,t]$ be a sequence with $\tau_n\to-\infty$ as $n\to\infty$. By the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V)}$, for each $n\ge 1$ there exists $u^{\tau_n}\in\mathcal{A}_{\mathcal{D}_\sigma(V)}(\tau_n)$ such that $v=U(t,\tau_n)u^{\tau_n}$, and therefore,

$$v=Y(t,\tau_n)u^{\tau_n}+Z(t,\tau_n)u^{\tau_n}.$$

From (6.45) and (6.56) we deduce that $\|Y(t,\tau_n)u^{\tau_n}\|\to 0$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\|Z(t,\tau_n)u^{\tau_n}-v\|=0. \tag{6.57}$$

Let us denote

$$u_n(r)=U(r,\tau_n)u^{\tau_n},\quad r\ge\tau_n,\quad n\ge 1.$$

By (6.56) and the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V)}$,

$$u_n(r)\in\mathcal{A}_{\mathcal{D}_\sigma(V)}(r)\subset\overline{B}_V(0,\widetilde{R}_\sigma(t))\quad\forall\,\tau_n\le r\le t,\quad\forall\,n\ge 1. \tag{6.58}$$

Now we distinguish three cases.

Case 1. If $1/4\le\beta\le 1/2$.

In this case, from (6.10), the continuous injection of V in $D(A^{3/4-\beta})$ and (6.58), we deduce that

$$\begin{aligned}|A^{-\beta}B(u_n(r))| &\le C_{(3/4-\beta)}|A^{3/4-\beta}u_n(r)|\|u_n(r)\|\\ &\le \widetilde{C}_{(3/4-\beta)}\|u_n(r)\|^2\\ &\le \widetilde{C}_{(3/4-\beta)}\widetilde{R}_\sigma^2(t)\quad\forall\,\tau_n\le r\le t,\quad\forall\,n\ge 1.\end{aligned}$$

Thus, if we assume (6.51), from Lemma 6.19 we obtain that

$$|A^{1-\beta}Z(t,\tau_n)u^{\tau_n}|^2\le M_{\sigma,\beta}^2(t), \tag{6.59}$$

where
$$M^2_{\sigma,\beta}(t) = 2\alpha^{-2}\varepsilon^{-1}\bigg(\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t)\bigg).$$

From (6.57), (6.59) and the weak lower semi-continuity of the norm, we deduce that

$$v \in \overline{B}_{D(A^{1-\beta})}(0, M_{\sigma,\beta}(t)),$$

and therefore (6.53) holds.

Moreover, if f satisfies (6.54), then (6.55) holds, and more exactly,

$$\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_\sigma(V)}(t) \subset \overline{B}_{D(A^{1-\beta})}(0, \widetilde{M}_{\sigma,\beta}(t_2)) \quad \forall\, t_2 \in \mathbb{R}, \tag{6.60}$$

where

$$\widetilde{M}^2_{\sigma,\beta}(t_2) = 2\alpha^{-2}\varepsilon^{-1}\bigg(\sup_{t\le t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t_2)\bigg).$$

Case 2. If $0 < \beta < 1/4$.

In this case, if f satisfies (6.52), as $1/4 < 1/4 + \beta < 1/2$, from (6.60) we have

$$\bigcup_{r\le t}\mathcal{A}_{\mathcal{D}_\sigma(V)}(r) \subset \overline{B}_{D(A^{3/4-\beta})}(0, \widetilde{M}_{\sigma,1/4+\beta}(t)).$$

Thus, by (6.10) and (6.58), we obtain that

$$\begin{aligned}|A^{-\beta}B(u_n(r))| &\le C_{(3/4-\beta)}|A^{3/4-\beta}u_n(r)|\|u_n(r)\|\\ &\le C_{(3/4-\beta)}\widetilde{M}_{\sigma,1/4+\beta}(t)\widetilde{R}_\sigma(t) \quad \forall\,\tau_n \le r \le t, \quad \forall\, n \ge 1.\end{aligned}$$

Thus, if we assume (6.51), from Lemma 6.19 we deduce that

$$|A^{1-\beta}Z(t,\tau_n)u^{\tau_n}|^2 \le R^2_{\sigma,\beta}(t), \tag{6.61}$$

where

$$R^2_{\sigma,\beta}(t) = 2\alpha^{-2}\varepsilon^{-1}\bigg(\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t)\widetilde{R}^2_\sigma(t)\bigg).$$

Again, from (6.57), (6.61) and the weak lower semi-continuity of the norm, we deduce that

$$v \in \overline{B}_{D(A^{1-\beta})}(0, R_{\sigma,\beta}(t)),$$

and therefore (6.53) holds.

Moreover, if f satisfies (6.54), then (6.55) holds, and more exactly,

$$\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_\sigma(V)}(t) \subset \overline{B}_{D(A^{1-\beta})}(0, \widetilde{R}_{\sigma,\beta}(t_2)) \quad \forall\, t_2 \in \mathbb{R}, \tag{6.62}$$

where

$$\widetilde{R}^2_{\sigma,\beta}(t_2) = 2\alpha^{-2}\varepsilon^{-1}\left(\sup_{t\le t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t_2)\widetilde{R}^2_\sigma(t_2)\right).$$

Case 3. If $\beta = 0$.
In this case, if f satisfies (6.52), as $0 < \delta < 1/4$, from (6.62) we see that

$$\bigcup_{r\le t}\mathcal{A}_{\mathcal{D}_\sigma(V)}(r) \subset \overline{B}_{D(A^{1-\delta})}(0, \widetilde{R}_{\sigma,\delta}(t)).$$

So, by (6.9) and (6.58), we deduce that

$$\begin{aligned}|B(u_n(r))| &\le C_{(1-\delta)}|A^{1-\delta}u_n(r)|\|u_n(r)\|\\ &\le C_{(1-\delta)}\widetilde{R}_{\sigma,\delta}(t)\widetilde{R}_\sigma(t) \quad \forall\,\tau_n \le r \le t, \quad \forall\, n \ge 1.\end{aligned}$$

Thus, if we assume (6.51), from Lemma 6.19 we deduce that

$$|AZ(t,\tau_n)u^{\tau_n}|^2 \le R^2_{\sigma,\delta,0}(t), \tag{6.63}$$

where

$$R^2_{\sigma,\delta,0}(t) = 2\alpha^{-2}\varepsilon^{-1}\left(\int_{-\infty}^{t} e^{\sigma(s-t)}|f(s)|^2\,ds + \sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t)\widetilde{R}^2_\sigma(t)\right).$$

Again, from (6.57), (6.63) and the weak lower semi-continuity of the norm, we deduce that

$$v \in \overline{B}_{D(A)}(0, R_{\sigma,\delta,0}(t)),$$

and therefore (6.53) holds.

Moreover, if f satisfies (6.54), then (6.55) holds, and more exactly,

$$\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_\sigma(V)}(t) \subset \overline{B}_{D(A)}(0, \widetilde{R}_{\sigma,\delta,0}(t_2)) \quad \forall\, t_2 \in \mathbb{R},$$

where

$$\widetilde{R}^2_{\sigma,\delta,0}(t_2) = 2\alpha^{-2}\varepsilon^{-1}\left(\sup_{t\le t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|f(s)|^2\,ds + \sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t_2)\widetilde{R}^2_\sigma(t_2)\right).$$

■

6.4 Attraction in $D(A)$ norm

By the previous results, when $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, the restriction to $D(A)$ of the process U defined by (6.27) is a process on $D(A)$. Now, we will prove that under suitable assumptions on f, we can obtain the existence of minimal pullback attractors for U on $D(A)$.

Proposition 6.21. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ is given. Then, the restriction to $D(A)$ of the bi-parametric family of mappings $U(t,\tau)$, with $\tau \le t$, given by (6.27), is a closed process on $D(A)$.*

Proof. It is a consequence of Theorem 6.4 and Theorem 6.6. ■

Remark 6.22. *Observe that, by Remark 6.7, U restricted to $D(A)$ is in fact a continuous process on $D(A)$.*

In order to obtain a pullback absorbing family for the process U restricted to $D(A)$, we first establish the following result.

Lemma 6.23. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies (6.50). Then, for any $\tau \in \mathbb{R}$, $u^\tau \in D(A)$,*

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}, \quad \textit{and} \quad 0 < \underline{\sigma} < \sigma/3, \tag{6.64}$$

the solution $u = u(\cdot; \tau, u^\tau)$ to (6.1) satisfies

$$\begin{aligned} \|u(t)\|^2 + \alpha^2|Au(t)|^2 \ \le\ & e^{\sigma(\tau-t)}(\|u^\tau\|^2 + \alpha^2|Au^\tau|^2) + 2\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|f(s)|^2\,ds \\ & + 4C_\varepsilon C_{\underline{\sigma}}^3(\sigma - 3\underline{\sigma})^{-1}\Big(e^{-3\underline{\sigma}(t-\tau)}\|u^\tau\|^6 + M_{t,\underline{\sigma}}^3\Big) \end{aligned} \tag{6.65}$$

for all $t \ge \tau$, where $\varepsilon > 0$ is given by (6.30),

$$C_\varepsilon = 27C_3^4(2\varepsilon^3)^{-1}, \tag{6.66}$$

$$C_{\underline{\sigma}} = \alpha^{-2}\max\left\{(\alpha^2 + \lambda_1^{-1}), \Big(\nu - \frac{\underline{\sigma}}{2}(\lambda_1^{-1} + \alpha^2)\Big)^{-1}\right\}, \tag{6.67}$$

and

$$M_{t,\underline{\sigma}} = \sup_{r\le t}\int_{-\infty}^r e^{\underline{\sigma}(s-r)}\|f(s)\|_*^2\,ds. \tag{6.68}$$

Proof. Let $\tau \in \mathbb{R}$, $u^\tau \in D(A)$, σ and $\underline{\sigma}$ satisfying (6.64) be fixed. From Lemma 6.11 we deduce in particular that $u = u(\cdot; \tau, u^\tau)$ satisfies

$$\|u(s)\|^2 \le C_{\underline{\sigma}}\Big(e^{\underline{\sigma}(\tau-s)}\|u^\tau\|^2 + M_{t,\underline{\sigma}}\Big) \quad \forall\, \tau \le s \le t. \tag{6.69}$$

On the other hand, by (6.20),

$$\begin{aligned} & \frac{d}{dt}(e^{\sigma t}\|u(t)\|^2 + \alpha^2 e^{\sigma t}|Au(t)|^2) + 2\nu e^{\sigma t}|Au(t)|^2 + 2e^{\sigma t}(B(u(t)), Au(t)) \\ =\ & \sigma e^{\sigma t}\|u(t)\|^2 + \alpha^2\sigma e^{\sigma t}|Au(t)|^2 + 2e^{\sigma t}(f(t), Au(t)), \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus, taking into account that $\|u(t)\|^2 \le \lambda_1^{-1}|Au(t)|^2$,

$$\begin{aligned} 2|(B(u(t)), Au(t))| \ &\le\ 2C_3\|u(t)\|^{3/2}|Au(t)|^{3/2} \\ &\le\ C_\varepsilon\|u(t)\|^6 + \frac{\varepsilon}{2}|Au(t)|^2, \end{aligned}$$

and

$$2|(f(t), Au(t))| \le \frac{2}{\varepsilon}|f(t)|^2 + \frac{\varepsilon}{2}|Au(t)|^2,$$

we deduce that

$$\begin{aligned}\|u(t)\|^2+\alpha^2|Au(t)|^2 \ \le\ & e^{\sigma(\tau-t)}(\|u^\tau\|^2+\alpha^2|Au^\tau|^2)+2\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|f(s)|^2\,ds\\ &+C_\varepsilon\int_\tau^t e^{\sigma(s-t)}\|u(s)\|^6\,ds \quad \forall\, t\ge\tau.\end{aligned}$$

From this inequality and (6.69), we easily obtain (6.65). ■

We introduce now the following tempered universe in $\mathcal{P}(D(A))$.

Definition 6.24. *For any σ, $\underline{\sigma}>0$, we will consider the universe $\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$ formed by the class of all families of nonempty subsets $\widehat{D}=\{D(t): t\in\mathbb{R}\}\subset\mathcal{P}(D(A))$ such that*

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{v\in D(\tau)}|Av|^2\right)=\lim_{\tau\to-\infty}\left(e^{\underline{\sigma}\tau}\sup_{v\in D(\tau)}\|v\|^2\right)=0.$$

Moreover, $\mathcal{D}_F(D(A))$ will denote the class of families $\widehat{D}=\{D(t)=D: t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of $D(A)$.

Note that for any σ, $\underline{\sigma}>0$, $\mathcal{D}_F(D(A))\subset\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$, and that the universe $\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$ is inclusion-closed.

As a consequence of Lemma 6.23, we have the following result.

Corollary 6.25. *Assume that $f\in L^2_{\rm loc}(\mathbb{R};H)$ satisfies (6.50) and*

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds<\infty \quad \textit{for some } 0<\sigma<2\nu(\lambda_1^{-1}+\alpha^2)^{-1}. \tag{6.70}$$

Then, for any $0<\underline{\sigma}<\sigma/3$, the family $\widehat{D}_{\sigma,\underline{\sigma}}=\{D_{\sigma,\underline{\sigma}}(t): t\in\mathbb{R}\}$ defined by

$$D_{\sigma,\underline{\sigma}}(t)=\overline{B}_{D(A)}(0,R_{\sigma,\underline{\sigma}}(t)), \tag{6.71}$$

the closed ball in $D(A)$ of center zero and radius $R_{\sigma,\underline{\sigma}}(t)$, where

$$R^2_{\sigma,\underline{\sigma}}(t)=\alpha^{-2}\left(1+2\varepsilon^{-1}\int_{-\infty}^t e^{\sigma(s-t)}|f(s)|^2\,ds+4C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}M^3_{t,\underline{\sigma}}\right), \tag{6.72}$$

with ε, C_ε, $C_{\underline{\sigma}}$ and $M_{t,\underline{\sigma}}$, given by (6.30), (6.66), (6.67) and (6.68), respectively, is pullback $\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$-absorbing for the restriction to $D(A)$ of the process U given by (6.27) (and therefore $\mathcal{D}_F(D(A))$-absorbing too).

Now, we prove that the process U is pullback $\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$-asymptotically compact. We will apply, with obvious necessary changes, the same energy method used in the proof of Lemma 6.14.

Lemma 6.26. *Assume that $f\in L^2_{\rm loc}(\mathbb{R};H)$ satisfies (6.50) and (6.70). Then, for any $0<\underline{\sigma}<\sigma/3$, the restriction to $D(A)$ of the process U defined by (6.27) is pullback $\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)$-asymptotically compact.*

Proof. Let us fix $0 < \underline{\sigma} < \sigma/3$. Let $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$, $t \in \mathbb{R}$, $\tau_n \to -\infty$ with $\tau_n \le t$ and $u^{\tau_n} \in D(\tau_n)$ for all n, be given. We must prove that the sequence $\{U(t,\tau_n)u^{\tau_n}\}$ is relatively compact in $D(A)$. By Corollary 6.25, for each integer $k \ge 0$, there exists $\tau_{\widehat{D}}(k) \le t-k$ such that

$$U(t-k,\tau)D(\tau) \subset D_{\sigma,\underline{\sigma}}(t-k) \quad \forall\, \tau \le \tau_{\widehat{D}}(k). \tag{6.73}$$

From this and a diagonal argument, we can extract a subsequence $\{u^{\tau_{n'}}\} \subset \{u^{\tau_n}\}$ such that

$$U(t-k,\tau_{n'})u^{\tau_{n'}} \rightharpoonup \hat{w}_k \quad \text{weakly in } D(A),\ \forall\, k \ge 0, \tag{6.74}$$

where $\hat{w}_k \in D_{\sigma,\underline{\sigma}}(t-k)$.

Now, applying Theorem 6.6 on each fixed interval $[t-k,t]$ we obtain that

$$\begin{aligned} \hat{w}_0 &= D(A) - \text{weak} \lim_{n'\to\infty} U(t,\tau_{n'})u^{\tau_{n'}} \\ &= D(A) - \text{weak} \lim_{n'\to\infty} U(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}} \\ &= U(t,t-k)\left[D(A) - \text{weak} \lim_{n'\to\infty} U(t-k,\tau_{n'})u^{\tau_{n'}}\right] \\ &= U(t,t-k)\hat{w}_k. \end{aligned}$$

In particular, observe that

$$|A\hat{w}_0| \le \liminf_{n'\to\infty} |AU(t,\tau_{n'})u^{\tau_{n'}}|.$$

We will prove now that it also holds that

$$\limsup_{n'\to\infty} |AU(t,\tau_{n'})u^{\tau_{n'}}| \le |A\hat{w}_0|, \tag{6.75}$$

which combined with (6.74) for $k=0$, will imply the convergence in the strong topology of $D(A)$, and the asymptotic compactness.

Observe that, as we already used in Lemma 6.23, for any pair (τ,u^τ) with $u^\tau \in D(A)$, the solution $u(\cdot;\tau,u^\tau)$, for short denoted by $u(\cdot)$, satisfies the differential equality (6.20).

Since $0 < \sigma < 2\nu(\lambda_1^{-1}+\alpha^2)^{-1}$, we observe that $[[\cdot]]$, with

$$[[v]]^2 = (2\nu - \alpha^2\sigma)|Av|^2 - \sigma\|v\|^2,$$

defines an equivalent norm to $|\cdot|_{D(A)}$ in $D(A)$.

We integrate (6.20) in the interval $[t-k,t]$ for the solutions $U(\cdot,\tau_{n'})u^{\tau_{n'}}$, which yields

$$\begin{aligned} &\|U(t,\tau_{n'})u^{\tau_{n'}}\|^2 + \alpha^2|AU(t,\tau_{n'})u^{\tau_{n'}}|^2 \\ = {} & \|U(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}\|^2 + \alpha^2|AU(t,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}|^2 \\ = {} & e^{-\sigma k}\Big(\|U(t-k,\tau_{n'})u^{\tau_{n'}}\|^2 + \alpha^2|AU(t-k,\tau_{n'})u^{\tau_{n'}}|^2\Big) \\ & +2\int_{t-k}^t e^{\sigma(s-t)}(f(s),AU(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}})\,ds \\ & -2\int_{t-k}^t e^{\sigma(s-t)}(B(U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}),AU(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}})\,ds \\ & -\int_{t-k}^t e^{\sigma(s-t)}[[U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}]]^2\,ds. \end{aligned} \tag{6.76}$$

From (6.74) and Theorem 6.6, we have

$$U(\cdot, t-k)U(t-k, \tau_{n'})u^{\tau_{n'}} \to U(\cdot, t-k)\hat{w}_k \quad \text{strongly in } C([t-k, t]; V),$$

and also

$$U(\cdot, t-k)U(t-k, \tau_{n'})u^{\tau_{n'}} \rightharpoonup U(\cdot, t-k)\hat{w}_k \quad \text{weakly in } L^2(t-k, t; D(A)).$$

Then, it is not difficult to see that

$$\begin{aligned} & \lim_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}), AU(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}})\, ds \\ = & \int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)\hat{w}_k), AU(s,t-k)\hat{w}_k)\, ds. \end{aligned} \tag{6.77}$$

Also, as $e^{\sigma(\cdot-t)}f(\cdot) \in L^2(t-k, t; H)$, it yields

$$\begin{aligned} & \lim_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}})\, ds \\ = & \int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)\hat{w}_k)\, ds. \end{aligned} \tag{6.78}$$

Finally, as $\int_{t-k}^{t} e^{\sigma(s-t)}[[v(s)]]^2 ds$ defines an equivalent norm in $L^2(t-k, t; D(A))$, we also deduce from above that

$$\begin{aligned} & \int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)\hat{w}_k]]^2\, ds \\ \leq & \liminf_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)U(t-k,\tau_{n'})u^{\tau_{n'}}]]^2\, ds. \end{aligned} \tag{6.79}$$

From (6.73), (6.74) with $k = 0$, the compactness of the injection of $D(A)$ into V, and (6.76), (6.77)–(6.79), we conclude that

$$\begin{aligned} & \|\hat{w}_0\|^2 + \alpha^2 \limsup_{n'\to\infty} |AU(t,\tau_{n'})u^{\tau_{n'}}|^2 \\ \leq & e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)\hat{w}_k)\, ds \\ & -2\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)\hat{w}_k), AU(s,t-k)\hat{w}_k)\, ds \\ & -\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)\hat{w}_k]]^2\, ds. \end{aligned}$$

Now, taking into account that $\hat{w}_0 = U(t, t-k)\hat{w}_k$, integrating again in (6.20), we obtain

$$\begin{aligned} \|\hat{w}_0\|^2 + \alpha^2|A\hat{w}_0|^2 = & e^{-\sigma k}(\|\hat{w}_k\|^2 + \alpha^2|A\hat{w}_k|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)\hat{w}_k)\, ds \\ & -2\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)\hat{w}_k), AU(s,t-k)\hat{w}_k)\, ds \\ & -\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)\hat{w}_k]]^2\, ds. \end{aligned}$$

Comparing the above two expressions, we conclude that

$$\begin{aligned}&\|\hat{w}_0\|^2 + \alpha^2 \limsup_{n'\to\infty} |AU(t,\tau_{n'})u^{\tau_{n'}}|^2 \\ \le\ & e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k) + \|\hat{w}_0\|^2 + \alpha^2|A\hat{w}_0|^2 - e^{-\sigma k}(\|\hat{w}_k\|^2 + \alpha^2|A\hat{w}_k|^2).\end{aligned}$$

But from (6.72), we have that $\lim_{k\to\infty} e^{-\sigma k}R^2_{\sigma,\underline{\sigma}}(t-k) = 0$, so (6.75) holds. ■

In general, the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (6.71) does not belong to $\mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$, and we do not know if U is pullback $\widehat{D}_{\sigma,\underline{\sigma}}$-asymptotically compact. Thus, we cannot apply Theorem 1.11 to the family $\widehat{D}_{\sigma,\underline{\sigma}}$. Nevertheless we can prove the following result.

Theorem 6.27. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies (6.50) and (6.70). Then, for any $0 < \underline{\sigma} < \sigma/3$, the family of sets*

$$X_{\sigma,\underline{\sigma}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_\sigma(D(A))\cap\mathcal{D}_{\underline{\sigma}}(V)} \Lambda_{D(A)}(\widehat{D},t)}^{D(A)} \qquad t \in \mathbb{R}, \tag{6.80}$$

has the following properties:

(a) $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), X_{\sigma,\underline{\sigma}}(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$ (pullback attraction).

(b) It is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A))$ is a family of closed subsets of $D(A)$ such that $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), C(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$, then $X_{\sigma,\underline{\sigma}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

(c) $U(t,\tau)X_{\sigma,\underline{\sigma}}(\tau) = X_{\sigma,\underline{\sigma}}(t)$ for all $\tau \le t$ (invariance).

Proof. The assertion (a) is an easy consequence of Proposition 1.4 and Lemma 6.26.

For the proof of (b), assume that $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A))$ is a family of closed subsets of $D(A)$ such that $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), C(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$. Now, let us fix $t \in \mathbb{R}$. In this case, it is easy to see that, for any $x \in \Lambda_{D(A)}(\widehat{D},t)$, with $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$, one has that $\mathrm{dist}_{D(A)}(x, C(t)) = 0$. Thus, as $C(t)$ is closed in $D(A)$, we deduce that $\Lambda_{D(A)}(\widehat{D},t) \subset C(t)$, and therefore, $X_{\sigma,\underline{\sigma}}(t) \subset C(t)$.

Finally, let $\tau \le t$ be fixed. In order to prove (c) we observe that by Proposition 1.5, we also have that

$$U(t,\tau)\Lambda_{D(A)}(\widehat{D},\tau) = \Lambda_{D(A)}(\widehat{D},t) \quad \text{for any } \widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V). \tag{6.81}$$

If $y \in X_{\sigma,\underline{\sigma}}(t)$, there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$ and $\{y_n\} \subset D(A)$, such that $y_n \in \Lambda_{D(A)}(\widehat{D}_n,t)$ and $y = D(A) - \lim_{n\to\infty} y_n$. But by (6.81), $y_n = U(t,\tau)x_n$, with $x_n \in \Lambda_{D(A)}(\widehat{D}_n,\tau) \subset X_{\sigma,\underline{\sigma}}(\tau)$. By Corollary 6.25, we can also deduce that $X_{\sigma,\underline{\sigma}}(\tau) \subset$

$\overline{B}_{D(A)}(0, R_{\sigma,\underline{\sigma}}(\tau))$, and therefore, by the compactness of the injection of $D(A)$ into V, $X_{\sigma,\underline{\sigma}}(\tau)$ is a compact subset of V. Thus, there exists a subsequence $\{x_{n'}\} \subset \{x_n\}$ such that $x_{n'} \to x \in X_{\sigma,\underline{\sigma}}(\tau)$ in V. But then, as U is a closed process on V, $y = U(t,\tau)x$, and this proves that $X_{\sigma,\underline{\sigma}}(t) \subset U(t,\tau)X_{\sigma,\underline{\sigma}}(\tau)$. The reverse inclusion can be proved analogously. ■

Under the additional assumption

$$\sup_{r\le 0} \int_{r-1}^{r} |f(s)|^2 \, ds < \infty, \tag{6.82}$$

the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (6.71) does belong to $\mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$, whence we can apply Theorem 1.11, and actually we have the following result.

Theorem 6.28. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies (6.82). Then, for any $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$ and $0 < \underline{\sigma} < \sigma/3$, we have that:*

- *(a) The family of sets $X_{\sigma,\underline{\sigma}}(t)$ defined by (6.80) is the minimal pullback $\mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$-attractor, and in fact is a family of compact subsets of $D(A)$.*
- *(b) $X_{\sigma,\underline{\sigma}}(t) = \mathcal{A}_{\mathcal{D}_F(V)}(t)$ for all $t \in \mathbb{R}$.*
- *(c) Indeed, $\mathcal{A}_{\mathcal{D}_F(V)}$ is the unique family of closed subsets for the norm of $D(A)$ in any universe of the form $\mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$ that is invariant for U and attracts any $\widehat{D} \in \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$ in the pullback sense.*

Proof. Let us fix $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$ and $0 < \underline{\sigma} < \sigma/3$.

Observe that under the above assumptions on f, the family $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$ defined by (6.71)–(6.72) belongs to $\mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V)$.

Therefore, the assertion (a) is a direct consequence of Theorem 1.11, Proposition 6.21, Corollary 6.25, and Lemma 6.26.

Now, let us fix $t \in \mathbb{R}$. It is evident that by (6.82),

$$\begin{aligned} X_{\sigma,\underline{\sigma}}(t) &\subset \overline{\bigcup_{\widehat{D} \in \mathcal{D}_{\underline{\sigma}}(V)} \Lambda_{D(A)}(\widehat{D}, t)}^{D(A)} \\ &\subset \overline{\bigcup_{\widehat{D} \in \mathcal{D}_{\underline{\sigma}}(V)} \Lambda_{D(A)}(\widehat{D}, t)}^{V} \\ &= \mathcal{A}_{\mathcal{D}_{\underline{\sigma}}(V)}(t) \\ &= \mathcal{A}_{\mathcal{D}_F(V)}(t). \end{aligned}$$

On the other hand, again by (6.82), from Theorem 6.20 we have that $\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}_F(V)}(r)$ is a bounded subset of $D(A)$, and therefore,

$$\begin{aligned} &\mathrm{dist}_{D(A)}\Big(U(t,\tau)\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}_F(V)}(r), X_{\sigma,\underline{\sigma}}(t)\Big) \\ \le\ &\mathrm{dist}_{D(A)}\Big(U(t,\tau)\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}_F(V)}(r), \Lambda_{D(A)}\Big(\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}_F(V)}(r), t\Big)\Big). \end{aligned}$$

From this inequality, Proposition 1.4, Lemma 6.26, and the invariance of $\mathcal{A}_{\mathcal{D}_F(V)}$, we deduce that

$$\mathrm{dist}_{D(A)}(\mathcal{A}_{\mathcal{D}_F(V)}(t), X_{\sigma,\underline{\sigma}}(t)) = 0,$$

and therefore

$$\mathcal{A}_{\mathcal{D}_F(V)}(t) \subset X_{\sigma,\underline{\sigma}}(t).$$

Thus, (b) is proved.

Finally, (c) is a direct consequence of Remark 1.12. ■

Remark 6.29. *Observe that in particular, if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies (6.82), by Remark 1.14 the minimal attractor $\mathcal{A}_{\mathcal{D}_F(D(A))}$ does exist, and it also coincides with the family $\mathcal{A}_{\mathcal{D}_F(V)}$. Moreover, this last family attracts in the pullback sense in the norm of $D(A)$ to all the families of the universe*

$$\mathcal{D}_{max}(D(A), V) = \bigcup_{\substack{0<\sigma<2\nu(\lambda_1^{-1}+\alpha^2)^{-1} \\ 0<\underline{\sigma}<\sigma/3}} \mathcal{D}_\sigma(D(A)) \cap \mathcal{D}_{\underline{\sigma}}(V).$$

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