Weak pullback attractors of setvalued processes

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Abstract

Weak pullback attractors are defined for nonautonomous setvalued processes and their existence and upper semi continuous convergence under perturbation is established. Unlike strong pullback attractors, invariance and pullback attraction here are required only for at least one trajectory rather than all trajectories at each starting point. The concept is useful in, for example, continuous time control systems and is related to that of viability.

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1 Introduction

Typical and important examples of setvalued processes are dynamical systems without uniqueness generated by ordinary differential equations without uniqueness and ordinary differential control systems (i.e., $\dot{x} = f(t, x, u)$ where $u \in U$) or, more generally, inclusion equations (i.e., $\dot{x} \in F(x)$). Obviously, control systems have more significant applications and thus provide a powerful motivation for studying dynamical systems without uniqueness, although historically the original motivation came from ordinary differential equations without uniqueness. Many interesting systems are in fact nonautonomous,

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although most concepts have been developed only in the more convenient setting of autonomous systems. It is of practical importance as well as intellectual interest to see how such concepts generalize to nonautonomous systems.

In many applications, physical, economical and industrial, such problems are typically stated on finite time intervals, e.g. in Viability Theory (see Aubin [3,4] and Aubin and Frankowska [7] and the references there), capture basins represent the points from which at least one trajectory reaches the target in finite time, thus in a *weak sense*. This arises in *minimal time control problems* (cf. Saint-Pierre [19]), or in *tracking control*, cf. Chen et al. [10], where the target is the graph of a single or multi-valued map (obtained by condition (2.8), see their proof of Theorem 2.2); and in controllability theory (e.g. Johnson and Nerurkar [11]), in which the dramatical influence of parameters on the controllability of the system is particularly worth noting. At the same time, the asymptotic behavior of such weakly invariant systems has also been intensively investigated with many meaningful interpretations in biology such as persistence and extinction, and applications in population genetics (cf. Vuillermot [22]), in minimization problems (cf. Attouch and Cominetti [1]) and stabilization in Mechanics (cf. Attouch and Czarnecki [2]). Attractors provide an important means of characterizing the long time behaviour of dynamical systems. They have been extensively investigated, in particular, global attractors in the autonomous case and its pullback version for general nonautonomous situations [9], which are known as strong attractors for setvalued systems. In the autonomous context, Szegö and Treccani [21] introduced the concept of a weak attractor for the continuous time setvalued semigroup generated by differential inclusions. The key difference here is that only at least one trajectory for each starting point must be attracted to or remain in the weak attractor rather than all trajectories as in the case of the usual (strong) attractor. The concept of a weak attractor has been found to be very useful for autonomous control systems as well as for some optimization systems, so the corresponding concept should thus also be of practical usefulness in the nonautonomous case.

In this paper we will introduce the concept of weak pullback attractor for the setvalued processes generated by different types of nonautonomous dynamical systems such as ordinary differential equations without uniqueness, nonautonomous contingent or inclusion equations, nonautonomous ordinary differential control systems, etc. Here the attractor consists of a family of sets invariant, i.e. carried into each other under the dynamics. Thus forward convergence is to a moving target, whereas pullback convergence is to a fixed target, a particular member of the family. Although similar concepts were introduced recently in [15] for nonautonomous difference inclusions, the techniques needed here to prove the existence of the weak pullback attractor in the continuous time case are somewhat different and more complicated, in particular, requiring Barbashin's results on the compactness of set of trajectories and its generalization for a single setvalued process to a setvalued convergent sequence of processes. Moreover, we establish our results for a more general Banach state space, thus removing a long standing restriction to locally compact state spaces in earlier publications.

In sections 2 and 3 we introduce the usual notation for the setvalued framework, and the analogous tool of semiflows and semigroups through what are known as the setvalued process or general(ized) dynamical systems. In section 4 we establish our main results and highlight some of their features with several examples in Section 5. Proofs are given at the end of the paper.

2 Terminology

Let be given a general Banach space $(X, \|\cdot\|)$. Recall that

$$\operatorname{dist}(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in X$ from a nonempty compact set A and that the Hausdorff separation $H^*(A, B)$ of nonempty compact subsets A, B of X is defined as

$$H^*(A, B) := \max_{a \in A} \operatorname{dist}(a, B) = \max_{a \in A} \min_{b \in B} \|a - b\|,$$

while $H(A, B) = \max\{H^*(A, B), H^*(B, A)\}$ is the Hausdorff metric on the space (X) of nonempty compact subsets of X.

Define an open ϵ–neighbourhood of $A \in (X)$ by $N_\epsilon(A) = \{x \in X \ : \ \operatorname{dist}(x, A) < \epsilon\}$ and closed ϵ–neighbourhood of A by $N_\epsilon[A] = \{x \in X \ : \ \operatorname{dist}(x, A) \leq \epsilon\}$.

A mapping $F : X \mapsto (X)$ is upper semi continuous at x_0 if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that $F(x) \subset N_\varepsilon(F(x_0))$ for all $x \in N_\delta(\{x_0\})$ or alternatively if

$$\lim_{x_n \to x_0} H^*(F(x_n), F(x_0)) = 0$$

for all sequences $x_n \to x_0$.

For any $A \in (X)$ define $F(A) := \cup_{a \in A} F(a)$ and define the set composition of two mappings F, $G : X \mapsto (X)$ as $F \circ G(x) := F(G(x))$ for all $x \in X$. Note that $F \circ G$ is upper semi continuous and compact valued if F and G are (see [6]).

3 Setvalued processes

Barbashin [8] investigated setvalued generalized or general dynamical systems generated by ordinary differential equations without uniqueness. Roxin [18] showed that nonautonomous contingent or inclusion equations generated nonautonomous general dynamical systems, as did nonautonomous ordinary differential control systems in which case he called the generated system a general control system [17]. See also [12,13]. We will use the name setvalued process for all such nonautonomous setvalued systems without assumed backwards extendability in time.

Definition 1 *A <u>setvalued process</u> on a state space X is defined in terms of an attainability set mapping $(t, t_0, x_0) \mapsto \Phi(t, t_0, x_0)$ for all $t \geq t_0$ in $\mathbb{R}$ and $x_0 \in X$ which satisfies the following properties:*
1. <u>Compactness</u> $\Phi(t, t_0, x_0)$ is a nonempty compact subset of X for all $t \geq t_0$ in $\mathbb{R}$ and all $x_0 \in X$;
2. <u>Initial condition</u>

$$\Phi(t_0, t_0, x_0) = \{x_0\}$$

for all $t_0 \in \mathbb{R}$ and $x_0 \in X$;
3. <u>Time evolution</u>

$$\Phi(t_2, t_0, x_0) = \Phi\left(t_2, t_1, \Phi(t_1, t_0, x_0)\right)$$

for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}$ and all $x_0 \in X$;
4. <u>Continuity in time</u>

$$\lim_{s \to t} H\left(\Phi(s, t_0, x_0), \Phi(t, t_0, x_0)\right) = 0$$

for all s, $t \geq t_0$ and all $t_0 \in \mathbb{R}$ and $x_0 \in X$;
5. <u>Upper semi continuity in initial conditions</u>

$$\lim_{t_0^{(n)} \to t_0, x_0^{(n)} \to x_0} H^*\left(\Phi(t, t_0^{(n)}, x_0^{(n)}), \Phi(t, t_0, x_0)\right) = 0$$

uniformly in $t \in [T_0, T_1]$ for any $T_0 < T_1 < \infty$ with $T_0 \geq t_0^{(n)}$, t_0 and for all $t_0 \in \mathbb{R}$ and $x_0 \in X$.

Simple examples of differential equations without uniqueness (e.g., Example 1 on page 122 of [13]) show that Condition 5 cannot in general be strengthened to continuity in the initial variables, i.e., with the Hausdorff metric H instead of the semi–metric H^*.

Definition 2 *A trajectory of a setvalued process* Φ *is a single valued mapping* $\phi : [T_0, T_1] \to X$ *which satisfies*

$$\phi(t) \in \Phi(t, s, \phi(s)) \quad \textit{for all} \quad T_0 \le s \le t \le T_1$$

for some $T_0 < T_1$ *in* $\mathbb{R}$*. A trajectory* ϕ *is called an entire trajectory if it is defined on all of* $\mathbb{R}$*.*

Trajectories are in fact continuous functions. See Lemma 6.1 in [17], or Theorem 4.2 in [12] for the systems without assumed backwards extendability under consideration here.

Barbashin [8] proved [1] existence of at least one trajectory $\phi : [t_0, t_1] \to X$ with $\phi(t_0) = x_0$ and $\phi(t_1) = x_1$ for any x_0 and x_1 with $x_1 \in \Phi(t_1, t_0, x_0)$. Barbashin [8] also proved a result on the compactness of trajectories of a setvalued process Φ. The following generalization is due to Roxin [17]; see also [12]).

Theorem 3 *(Barbashin) Let* B *be a nonempty compact subset of* X *and let* $\phi_n : [t_0, t_1] \to X$ *be a sequence of trajectories of a setvalued process* Φ *with* $\phi_n(t_0) \in B$ *for given* $t_0 < t_1 \in \mathbb{R}$*. Then there exists a subsequence* ϕ_{n_j} *and a trajectory* $\bar{\phi} : [t_0, t_1] \to X$ *of* Φ *with* $\bar{\phi}(t_0) \in B$ *such that* $\phi_{n_j}(t) \to \bar{\phi}(t)$ *as* $n_j \to \infty$ *uniformly in* $t \in [t_0, t_1]$*.*

We will also state, prove and use a further generalization of this theorem for sequences of trajectories belonging to a sequence of upper semi continuously convergent setvalued processes; see Theorem 16 in Section 6.

4 Weak attractors of setvalued processes

For setvalued systems arising from control systems, one is often interested in situations where just one or a few rather than all trajectories emanating from each starting point satisfy a given property. Szegö and Treccani [21] introduced concepts of weak invariance and weak attractors for such situations in the autonomous case.

Our aim in this paper is to introduce and investigate pullback versions of these weak concepts for setvalued processes. As with the strong concepts of

[1] Although he worked in finite dimensional spaces, the extension to a general Banach space X is straightforward due to the fact that the constructed objects are contained in the compact integral funnel.

invariance and attraction, it is also less restrictive here to consider families of sets rather than individual sets.

Definition 4 *A family $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ of nonempty compact subsets of X is said to be weakly positively invariant for a setvalued process Φ on X if for every $t_0 \in \mathbb{R}$ and every $x_0 \in A_{t_0}$ there exists a trajectory $\phi : [t_0, \infty) \to X$ of Φ with $\phi(t_0) = x_0$ such that $\phi(t) \in A_t$ for all $t \geq t_0$.*

It is called weakly invariant if, for every $t_0 \in \mathbb{R}$ and every $x_0 \in A_{t_0}$, there is an entire trajectory ϕ with $\phi(t_0) = x_0$ and $\phi(t) \in A_t$ for all $t \in \mathbb{R}$.

Definition 5 *A weakly invariant family $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ of nonempty compact subsets of X is called a weak pullback attractor of a setvalued process Φ on X if it is weakly pullback attracting, i.e., for any $t_0 \in \mathbb{R}$, any nonempty bounded subset D of X and any sequence $d_n \in D$ there exist sequences of positive numbers $\tau_n \to \infty$ as $n \to \infty$ and trajectories $\phi_n : [t_0 - \tau_n, t_0] \to X$ of Φ with $\phi_n(t_0 - \tau_n) = d_n$ such that*

$$\lim_{n\to\infty} \operatorname{dist}\left(\phi_n(t_0), A_{t_0}\right) = 0. \tag{1}$$

Note that a strong pullback attractor, when it exists, is also a weak pullback attractor. Now, one of our main results will be to show that the existence of a weak pullback attractor follows from that of a more easily determined weak pullback absorbing family of sets.

Definition 6 *A weakly positively invariant family $\mathcal{B} = \{B_t, t \in \mathbb{R}\}$ of nonempty compact subsets of X is called a weak pullback absorbing family of a setvalued process Φ on X if for $t_0 \in \mathbb{R}$ and any bounded subset D of X there exists a $T_{t_0,D} \in \mathbb{R}^+$ such that for each $\tau_n \geq T_{t_0,D}$ and $d_n \in D$ there exists a trajectory $\phi_n : [t_0 - \tau_n, t_0] \to X$ of Φ with*

$$\phi_n(t_0 - \tau_n) = d_n \quad \textit{and} \quad \phi_n(t_0) \in B_{t_0}.$$

Note that by the weak positive invariance of $\mathcal{B}$ the trajectories ϕ_n can be extended, using the concatenation property given by the time evolution property 3, to remain in $\mathcal{B}$ for $t \geq t_0$, i.e. $\phi_n(t) \in B_t$ for each $t \geq t_0$.

Theorem 7 *Let Φ be a setvalued process with a weak pullback absorbing family $\mathcal{B}$. Then Φ has a maximal weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ relative to $\mathcal{B}$, which is uniquely determined by*

$$\begin{aligned} A_{t_0} = \{a_0 \in X \ ; \ & \textit{there exist} \ \ \tau_n \to \infty \ \ \textit{as} \ \ n \to \infty, \\ & \qquad \textit{and trajectories} \ \ \phi_n : [t_0 - \tau_n, t_0] \to X \\ & \qquad\quad \textit{with} \ \ \phi_n(t) \in B_t \ \ \textit{for} \ \ t \in [t_0 - \tau_n, t_0] \end{aligned} \tag{2}$$

$$\text{and} \quad \lim_{n\to\infty} \phi_n(t_0) = a_0 \Big\}$$

for each $t_0 \in \mathbb{R}$.

Remark 8 *A weak pullback attractor consists of trajectories that exist and remain in* $\mathcal{B}$ *for the entire time set* $\mathbb{R}$, *but it does not necessarily contain all such trajectories. See Lemma 13 in the Section 6.*

Remark 9 *As well as being weakly invariant, a weak pullback attractor is also negatively strongly invariant, i.e., satisfies* $A_t \subset \Phi(t, t_0, A_{t_0})$ *for all* $t \geq t_0$ *and* $t_0 \in \mathbb{R}$.

Remark 10 *The uniqueness and maximality of a weak attractor cannot be understood in the usual sense, but rather with respect to an absorbing family* $\mathcal{B}$ *of sets under discussion. This is an intrinsic property of weak pullback attractors and is not contradicted by the existence of other weak pullback attractors, with or without intersecting component set, with respect to different families* $\mathcal{B}$. *This is transparent in the examples of weak pullback attractors for nonautonomous difference equations in [15]. A similar example for setvalued processes will be given in Section 5.*

The proof of the following basic continuity property of a weak pullback attractor is not as immediate a consequence of definitions as in the strong case. It is given in Section 8.

Proposition 11 *Let* $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ *be a weak pullback attractor. Then the setvalued mapping* $t \mapsto A_t$ *is continuous.*

Our second objective is to prove some results on the structure of weak pullback attractors for setvalued processes. In fact, we are interested in some kind of upper semi continuous behaviour produced by some perturbations appeared in the model.

Theorem 12 *Suppose that the setvalued process* Φ *has a weak pullback absorbing family* $\mathcal{B} = \{B_t, t \in \mathbb{R}\}$ *and suppose that each perturbed setvalued process* Φ^ϵ *has a weak pullback absorbing family* $\mathcal{B}^\epsilon = \{B_t^\epsilon, t \in \mathbb{R}\}$ *for* $\epsilon > 0$ *such that*

$$\max_{0\leq\delta\leq1} H^*\left(\Phi^\epsilon(t+\delta, t, x), \Phi(t+\delta, t, x)\right) \leq \epsilon \quad \text{for all} \quad t \in \mathbb{R},\ x \in X \tag{3}$$

and

$$H^*\left(B_{t_0}^\epsilon, B_{t_0}\right) \leq \epsilon \quad \text{for all} \quad t_0 \in \mathbb{R}. \tag{4}$$

Then the maximal weak pullback attractors $\mathcal{A}^\epsilon = \{A_t^\epsilon, t \in \mathbb{R}\}$ *w.r.t.* $\mathcal{B}^\epsilon$ *of the* *perturbed processes* Φ^ϵ *converge upper semi continuously to the maximal weak pullback attractor* $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ *w.r.t.* $\mathcal{B}$ *of* Φ *in the sense that*

$$\lim_{\epsilon \to 0} H^* \left(A^\epsilon_{t_0}, A_{t_0} \right) = 0. \tag{5}$$

for each $t_0 \in \mathbb{R}$*.*

The following structural condition on the unperturbed setvalued process Φ provides a simple (if rather strong) condition for $X = \mathbb{R}^d$ ensuring the existence of a nearby uniform weak pullback absorbing family: assume that $\mathcal{B} = \{B_t, t \in \mathbb{R}\}$ is a family of nonempty compact sets of $\mathbb{R}^d$ and that there exists a $\gamma : \mathbb{R}^+ \to [0, 1]$ such that

$$\min_{y \in \Phi(t,t_0,x)} \operatorname{dist}(y, B_t) \leq \gamma(t - t_0) \operatorname{dist}(x, B_{t_0})$$

for all $x \in \mathbb{R}^d$ and $t \geq t_0$ in $\mathbb{R}$ and that for all bounded D and all fixed time t:

$$\lim_{t_0 \to -\infty} \gamma(t - t_0) \sup_{x \in D} \operatorname{dist}(x, B_{t_0}) = 0.$$

We can take $N_\epsilon[B_t] := \{x \in \mathbb{R}^d \ : \ \operatorname{dist}(x, B_t) \leq \epsilon\}$ for $\epsilon > 0$ small enough. Then the family $N_\epsilon[B_t]$ is weakly positively invariant and weakly pullback absorbing.

5 Examples

Our first example involves nonautonomous setvalued process generated by the nonautonomous differential inclusion

$$x' \in F(t, x) := \begin{cases} \{-x\} & \text{if } t < 0 \\ \{-x, 0\} & \text{if } t \geq 0 \end{cases}, \qquad x \in \mathbb{R}.$$

The setvalued process here is given by

$$\Phi(t,t_0,x_0) := \begin{cases} \left\{x_0 e^{-(t-t_0)}\right\} & \text{if } t_0 \le t \le 0,\ x_0 \in \mathbb{R} \\ \left[x_0 e^{-(t-t_0)}, x_0\right] & \text{if } 0 \le t_0 \le t,\ x_0 \ge 0 \\ \left[x_0, x_0 e^{-(t-t_0)}\right] & \text{if } 0 \le t_0 \le t,\ x_0 \le 0 \\ & \end{cases}$$

and the composition of these cases. The family $\mathcal{A} = \{A_t,\ t \in \mathbb{R}\}$ with $A_t \equiv \{0\}$ for all $t \in \mathbb{R}$ is strongly invariant and hence weakly invariant. It is a weak pullback (and forward) attractor with respect to any absorbing family set $\mathcal{B} = \{B_t,\ t \in \mathbb{R}\}$ with component sets $B_t \equiv [-R, R]$ for all $t \in \mathbb{R}$ and any $R \ge 0$.

As a second example we consider the nonautonomous setvalued process generated by the nonautonomous differential inclusion

$$x' \in F(t,x) := \begin{cases} \{-x\} & \text{if } t < 0, \\ \{-x, 1\} & \text{if } t \ge 0. \end{cases}$$

The setvalued process here is given by

$$\Phi(t,t_0,x_0) = \begin{cases} \left\{x_0 e^{-(t-t_0)}\right\} & \text{if } t_0 \le t \le 0, \\ \{x_0\} & \text{if } 0 \le t_0 = t, \\ \left[x_0 e^{-(t-t_0)}, x_0 + t - t_0\right] & \text{if } 0 \le t_0 < t,\ x_0 \ge \dfrac{t-t_0}{e^{t_0-t}-1}, \\ \left[x_0 + t - t_0, x_0 e^{-(t-t_0)}\right] & \text{if } 0 \le t_0 < t,\ x_0 \le \dfrac{t-t_0}{e^{t_0-t}-1}. \end{cases}$$

and the composition of these cases. The family $\mathcal{A} = \{A_t,\ t \in \mathbb{R}\}$ with $A_t \equiv \{0\}$ for all $t \in \mathbb{R}$ is weakly invariant, but not strongly invariant. It is a weak pullback (and forward) attractor with respect to any absorbing family set $\mathcal{B} = \{B_t,\ t \in \mathbb{R}\}$ with component sets $B_t \equiv [-R, R]$ for all $t \in \mathbb{R}$ and any $0 \le R \le 1$.

Our third example illustrates the ambiguity concerning the existence and uniqueness of weak pullback attractors alluded to in Remark 10. It is based on the autonomous differential inclusion

$$x' \in F(t,x) := [-1,1] \qquad x \in \mathbb{R}.$$

and the associated setvalued process

$$\Phi(t,t_0,x_0) = [x_0 - t + t_0, x_0 + t - t_0] \quad t_0 \leq t,\ x_0 \in \mathbb{R}$$

Here every family $\mathcal{A} = \{A_t,\ t \in \mathbb{R}\}$ with $A_t \equiv [R_1, R_2]$ for all $t \in \mathbb{R}$ is weakly but not strongly invariant for any $R_1 \leq R_2$ in $\mathbb{R}$. It is a weak pullback (and forward) attractor with respect to an absorbing family set equal to itself, i.e., $\mathcal{B} = \{B_t,\ t \in \mathbb{R}\}$ with component sets $B_t \equiv A_t$ for all $t \in \mathbb{R}$.

A fourth example shows an attracting time-depending family, which attracts weakly pullback in time, and forward as well. Consider the nonautonomous differential inclusion

$$x' \in F(t,x) = \begin{cases} \{-x+t\} & \text{if } t < 0, \\ \{-x+t, -x\} & \text{if } t \geq 0. \end{cases}$$

Then it holds

$$\Phi(t,t_0,x_0) = \begin{cases} \left\{(x_0+1-t_0)e^{-(t-t_0)} + t - 1\right\} & \text{if } t_0 \leq t \leq 0, \\ \left[x_0 e^{-(t-t_0)}, (x_0+1-t_0)e^{-(t-t_0)} + t - 1\right] & \text{if } 0 \leq t_0 \leq t, \\ [((x_0+1-t_0)e^{t_0} - 1)e^{-t}, (x_0+1-t_0)e^{t_0}e^{-t} + t - 1] & \text{if } t_0 \leq 0 \leq t. \end{cases}$$

Observe that the family $\mathcal{A} = \{A_t,\ t \in \mathbb{R}\}$ with $A_t \equiv \{t-1\}$ for all $t \in \mathbb{R}$ is weakly (but not strongly) invariant and indeed, a weak pullback (and forward) attractor.

Our fifth example has a weak pullback attractor which is not a weak forward attractor. It is based on the nonautonomous differential inclusion

$$x' \in F(t,x) := \begin{cases} \{2tx\} & \text{if } t \leq 0, \\ \{2tx, 4tx\} & \text{if } 0 \leq t, \end{cases} \qquad x \in \mathbb{R}.$$

and the associated setvalued process

$$\Phi(t,t_0,x_0)=\begin{cases}\left\{x_0e^{(t^2-t_0^2)}\right\} & \text{if } t_0\le t\le 0,\\[2ex] \left[x_0e^{2(t^2-t_0^2)},x_0e^{(t^2-t_0^2)}\right] & \text{if } 0\le t_0\le t,\ x_0\le 0,\\[2ex] \left[x_0e^{(t^2-t_0^2)},x_0e^{2(t^2-t_0^2)}\right] & \text{if } 0\le t_0\le t,\ x_0\ge 0,\end{cases}$$

and the composition of these cases. Here every family $\mathcal{A}=\{A_t,\ t\in\mathbb{R}\}$ with $A_t\equiv\{0\}$ for all $t\in\mathbb{R}$ is strongly and hence weakly invariant. It is a global weak pullback attractor but is not a weak forward attractor.

We finish with an example involving almost periodic oscillations, as investigated by Krasnosel'skii et al. [16] (cf. Ch. 10.5 and 11.7). In particular, this example illustrates, in a nonautonomous context, how one can find systems in which the asymptotic behaviour is not only determined by a compact invariant set which attracts all the bounded subsets of the phase space, and which is independent of time (the usual global attractor). Instead, this limit behaviour needs to be determined by a time-dependent family of sets (namely, a weak attractor as introduced in our theory) which are constructed by using a pullback technique.

Let K be a given fixed cone of $\mathbb{R}^d$ and consider a family of problems

$$\frac{dx}{dt}+A(t,\mu)x=g(t,x),\quad \mu\in\mathbb{R}, \tag{6}$$

where we assume that $A(\cdot,\cdot)$ is jointly continuous in both variables, and for each fixed parameter μ, $A(\cdot,\mu)$ is almost periodic and nonnegative w.r.t. the cone K, and g is uniformly concave on K with $g(t,0)=0$. If for certain values of μ, the Green function associated to the operator $\frac{d}{dt}+A(\cdot,\mu)$ is strongly positive or strongly negative w.r.t. K, and there exist nonzero bounded almost periodic sub and super solutions, then there exists (cf. [16, Th. 10.6]) a unique almost periodic solution x_μ^* between these sub and super solutions. Moreover, by [16, Th. 11.7], this solution is asymptotically stable in the cone, i.e. the solution $x_\mu(t,t_0,x_0)$ of (6) starting at $x_0\in\operatorname{int}K$ at time t_0 is attracted by x_μ^* in the forward sense,

$$\lim_{t\to\infty}|x_\mu(t,t_0,x_0)-x_\mu^*(t)|=0.$$

In fact the solution x_μ^* is constructed in terms of pullback attraction,

$$\lim_{t_0\to-\infty}|x_\mu(t,t_0,x_0)-x_\mu^*(t)|=0,\quad \text{for each fixed } t.$$

Now suppose I^* is a maximal compact set of parameters such that for $\mu \in I^*$ the equation (6) possesses an almost periodic solution x^*_μ as we indicated previously, and let I be a larger compact set such that for $\mu \in I \setminus I^*$ there is no such almost periodic solution.

We now consider the following differential inclusion which arises in problems of parametric uncertainty

$$\frac{dx}{dt} \in F(t,x), \tag{7}$$

where $F(t,x) = \cup_{\mu \in I}\{A(\mu,t)x + g(t,x)\}$, and consider the attainability mapping defined, as usual, by

$$\Phi(t,t_0,x_0) = \{x(t) \,|\, x(\cdot) \quad \text{is a solution of (7)} \quad \text{such that} \quad x(t_0) = x_0\}.$$

The family of nonempty compact sets $A_t = \cup_{\mu \in I^*}\{x^*_\mu(t)\}$, $t \in \mathbb{R}$, is weakly invariant and weakly pullback attracting for the setvalued process Φ. However, it is worth mentioning that this family cannot give a description of the whole dynamics, but only of those "good" parameters which we are trying to identify.

6 Preliminary Results

We will need the following lemmata and theorem in the proof of Theorem 12.

Lemma 13 *Suppose that a setvalued process Φ has a weak pullback absorbing family $\mathcal{B} = \{B_t, t \in \mathbb{R}\}$ and a weak pullback attractor $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ related to $\mathcal{B}$ by theorem 7. Then an entire trajectory ϕ of Φ satisfies $\phi(t) \in B_t$ for all $t \in \mathbb{R}$ if and only if $\phi(t) \in A_t$ for all $t \in \mathbb{R}$.*

Proof. Suppose that ϕ is an entire trajectory with $\phi(t) \in B_t$ for each $t \in \mathbb{R}$. Fix $t_0 \in \mathbb{R}$. Then there is a sequence of trajectories $\phi_n : [t_0 - n, t_0] \to X$, namely $\phi_n \equiv \phi$, with $\phi_n(t) = \phi(t) \in B_t$ for each $t \in [t_0 - n, t_0]$. In particular, $\phi_n(t_0) \equiv \phi(t_0) \to \phi(t_0)$ as $n \to \infty$. By the definition, $\phi(t_0) \in A_{t_0}$. Since t_0 was otherwise arbitrary, we thus have $\phi(t) \in A_t$ for all $t \in \mathbb{R}$. The converse follows from the fact that $A_t \subset B_t$ for all $t \in \mathbb{R}$.

Proofs of the following two lemmata can be found in [15].

Lemma 14 *Suppose that $H^*(B_n, B) \to 0$ as $n \to \infty$ for nonempty compact subsets B, B_1, B_2, ... Then for any sequence $b_n \in B_n$, $n \in \mathbb{Z}^+$, there exists a convergent subsequence $b_{n_j} \to b^* \in B$ as $n_j \to \infty$.*

Lemma 15 *Suppose that F, $F^\epsilon : X \to (X)$ with $\epsilon > 0$ are upper semi continuous and satisfy $F^\epsilon(x) \subset N_\epsilon(F(x))$ for all $x \in X$. Then*

$$H^*\left(F^{\epsilon_n}(x_n), F(x^*)\right) \longrightarrow 0 \qquad as \quad n \to \infty$$

for any convergent sequences $x_n \to x^*$ in X and $\epsilon_n \to 0$ as $n \to \infty$.

We also require the following generalization of Theorem 3, which we will prove in Section 10.

Theorem 16 *(Generalized Barbashin Theorem) Suppose that a sequence of setvalued processes Φ^ϵ converges to a setvalued process Φ upper semi continuously in the sense of (3) and let ϕ^{ϵ_j} be a trajectory of Φ^{ϵ_j} on $[t_0, t_1]$ such that $\phi^{\epsilon_j}(t_0) = x_{0,j} \to x_0$ as $\epsilon_j \to 0$. Then there exists a trajectory ϕ of Φ on $[t_0, t_1]$ with $\phi(t_0) = x_0$ and a convergent subsequence $\phi^{\epsilon'_j}(t) \to \phi(t)$ as $\epsilon'_j \to 0$ uniformly in $t \in [t_0, t_1]$.*

7 Proof of Theorem 7

We divide the proof into three parts.

7.1 Existence and compactness

Fix $t_0 \in \mathbb{R}$ and take a sequence $\tau_n \to +\infty$ as $n \to +\infty$. By the weak positive invariance of $\mathcal{B} = \{B_t, t \in \mathbb{R}\}$, given $b_n \in B_{\tau_n}$, there exist trajectories $\phi_n : [t_0 - \tau_n, t_0] \to X$ with $\phi_n(\tau_n) = b_n$ and $\phi_n(t) \in B_t$ for each $t \in [t_0 - \tau_n, t_0]$ and all $n \in \mathbb{Z}^+$. In particular, $\phi_n(t_0) \in B_{t_0}$ for each $n \in \mathbb{Z}^+$. Since B_{t_0} is compact, there exists a convergent subsequence $\phi_{n_j}(t_0) \to a_0 \in B_{t_0}$. Taking this subsequence to be the original sequence in the definition (2) of A_{t_0}, we have $a_0 \in A_{t_0}$, which proves that A_{t_0} is nonempty.

To show that A_{t_0} is compact, we need only to show that it is closed because A_{t_0} is a subset of the compact set B_{t_0}. Suppose that $a_k \in A_{t_0}$ and $a_k \to a^*$ as $k \to \infty$. Then for each $k \in \mathbb{Z}^+$ there exist subsequences $t_{k,n} \to \infty$ as $n \to \infty$ and trajectories $\phi_{k,n} : [t_0 - t_{k,n}, t_0] \to X$ with $\phi_{k,n}(t) \in B_t$ for each $t \in [t_0 - t_{k,n}, t_0]$ and $n \in \mathbb{Z}^+$ for which $\lim_{k\to\infty} \phi_{k,n}(t_0) = a_k$. Pick n_k so that

$$\|\phi_{k,n_k}(t_0) - a_k\| \leq \frac{1}{k} \qquad \text{and} \qquad t_{k+1,n_{k+1}} \geq t_{k,n_k} + 1$$

for each $k \in \mathbb{Z}^+$. Then

$$\|\phi_{k,n_k}(t_0) - a^*\| \leq \|\phi_{k,n_k}(t_0) - a_k\| + \|a_k - a^*\| \leq \frac{1}{k} + \|a_k - a^*\| \to 0$$

as $k \to \infty$. Write $\bar{\phi}_k \equiv \phi_{k,n_k}$ and $\bar{t}_k \equiv t_{k,n_k}$. Then $\bar{\phi}_k : [t_0 - \bar{t}_k, t_0] \to X$ with $\bar{\phi}_k(t) \in B_t$ for each $t \in [t_0 - \bar{t}_k, t_0]$ and $k \in \mathbb{Z}^+$. Moreover, $\bar{t}_k \to \infty$ as $k \to \infty$ with $\bar{\phi}_k(t_0) \to a^*$ as $k \to \infty$. Thus $a^* \in A_{t_0}$, so A_{t_0} is closed and hence compact.

7.2 Weak positive invariance

Fix $t_0 \in \mathbb{R}$ and take $a_0 \in A_{t_0}$. Then, there exists $\tau_n \to +\infty$ and trajectories $\phi_n : [t_0 - \tau_n, t_0] \to X$ with $\phi_n(t) \in B_t$ for each $t \in [t_0 - \tau_n, t_0]$ and such that $\lim_{n\to\infty} \phi_n(t_0) = a_0$. Since $\mathcal{B}$ is weakly positively invariant, each trajectory ϕ_n can be extended to $[t_0 - \tau_n, \infty)$ so that $\phi_n(t) \in B_t$ for all $t \geq t_0$. By (Barbashin's) Theorem 3 applied successively on intervals of the form $[t_0 + N, t_0 + N + 1]$ (because we can extract subsequence converging in both extremes of each interval) we can find a (diagonal) subsequence $n'_k \to \infty$ as $k \to \infty$ and (by concatenation) a trajectory $\bar{\phi}$ of Φ such that $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for each $t \geq t_0$. Obviously $\bar{\phi}(t_0) = a_0 \in A_{t_0}$ since the original subsequence $\phi_{n_k}(t_0) \to a_0$. By the construction, $\bar{\phi}(t) \in A_t$ for all $t \geq t_0$. Now $t_0 \in \mathbb{R}$ was arbitrary, so $\{A_t, t \in \mathbb{R}\}$ is weakly positively invariant.

7.3 Weak negative invariance

To prove the negative invariance property, a similar argument holds with a little more care for all $t \leq t_0$. Fix an $N \in \mathbb{Z}^+$ and take k large enough so that $\tau_{n_k} \geq N$ in the above subsequence of trajectories ϕ_{n_k} in $\mathcal{B}$ with $\phi_{n_k}(t_0) \to a_0$ which we now restrict to the common definition interval $[t_0 - N, t_0] \subset [t_0 - \tau_{n_k}, t_0]$. Because $B_{t_0 - N}$ is compact, by Barbashin's Theorem (Theorem 3) there is a convergent subsequence with $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for each $t \in [t_0 - N, t_0]$, where $\bar{\phi}$ is a trajectory. Obviously $\bar{\phi}(t_0) = a_0$. By a diagonal subsequence argument we have a (diagonal) subsequence such that $\phi_{n'_k}(t) \to \bar{\phi}(t) \in B_t$ for all $t \leq t_0$. It then follows as above that $\bar{\phi}(t) \in A_t$ for all $t \leq t_0$. Concatenating the two parts of $\bar{\phi}$ to all of $\mathbb{R}$ gives us an entire trajectory $\bar{\phi}$ of the setvalued process Φ with $\bar{\phi}(t) \in A_t$ for all $t \in \mathbb{R}$. Thus $\{A_t, t \in \mathbb{R}\}$ is weakly invariant.

Fix $t_0 \in \mathbb{R}$ and a bounded subset D of X. Since $\mathcal{B}$ is weakly pullback absorbing for the setvalued process Φ on X, for every $n \in \mathbb{Z}^+$ there is a $T_{t_0-n,D} \in \mathbb{R}^+$ such that for each $k \geq T_{t_0-n,D}$ and $d_n \in D$ there exists a trajectory $\phi_{k,n}$ of Φ on $[t_0-k-n, t_0]$ with $\phi_{k,n}(t_0-k-n) = d_n$ and $b_{k,n} = \phi_{k,n}(t_0-n) \in B_{t_0-n}$ for all $k \geq T_{t_0-n,D}$ and $n \in \mathbb{Z}^+$. Since $\mathcal{B}$ is weakly positively invariant, each $\phi_{k,n}$ can be extended indefinitely so that $\phi_{k,n}(t) \in B_t$ for all $t \geq t_0 - n$. In particular, $\phi_{k,n}(t_0) \in B_{t_0}$, which is compact, so there is a subsequence $k_n < k_{n+1} \to \infty$ as $n \to \infty$ with $k_n \geq T_{t_0-n,D}$ and $k_{n+1} \geq T_{t_0-n-1,D}$ such that $\phi_{k_n,n}(t_0) \to a^* \in B_{t_0}$ as $n \to \infty$.

Write $\bar{\phi}_n \equiv \phi_{k_n,n}$ and $\tau_n \equiv k_n + n$. Then $\bar{\phi}_n$ is defined on $[t_0 - \tau_n, \infty)$ with $\bar{\phi}_n(t_0-\tau_n) = d_n \in D$ and $\bar{\phi}_n(t_0) \to a^*$ as $n \to \infty$. By the construction, $a^* \in A_{t_0}$, so $\lim_{n\to\infty} \text{dist}\left(\bar{\phi}_n(t_0), A_{t_0}\right) = 0$, which is property (1). Thus $\{A_t, t \in \mathbb{R}\}$ is weakly pullback attracting with the weak pullback absorbing family $\mathcal{B}$.

8 Proof of Proposition 11

Let $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ be a weak pullback attractor.

Firstly, consider the limit $H^*(A_t, A_s) \to 0$ as $s \to t$. If this does not hold there would exist an $\epsilon_0 > 0$ and a sequence $s_n \to t$ such that

$$\epsilon_0 \leq H^*(A_t, A_{s_n}), \qquad n \in \mathbb{N}.$$

We will show that this leads to a contradiction.

Since A_t is compact, there exists an $a_n \in A_t$ such that

$$H^*(A_t, A_{s_n}) = \text{dist}(a_n, A_{s_n}) \leq \text{dist}(a_n, a_{s_n})$$

for all $a_{s_n} \in A_{s_n}$. By the weak invariance of $\mathcal{A}$ there exists a trajectory ϕ_n with $\phi_n(t) = a_n$ and $\phi_n(s) \in A_s$ for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus

$$H^*(A_t, A_{s_n}) \leq \text{dist}(\phi_n(t), \phi_n(s_n)), \qquad n \in \mathbb{N}.$$

By the compactness of A_t again and Barbashin's Theorem, there exists a subsequence of trajectories ϕ_{n_j} which converges to a trajectory $\bar{\phi}$ uniformly on the interval $[t-1, t+1]$. Thus

$$
\begin{aligned}
\epsilon_0 &\le H^*\left(A_t, A_{s_{n_j}}\right) \\
&\le \operatorname{dist}\left(\phi_{n_j}(t), \phi_{n_j}(s_{n_j})\right) \\
&\le \operatorname{dist}\left(\phi_{n_j}(t), \bar{\phi}(t)\right) + \operatorname{dist}\left(\bar{\phi}(t), \bar{\phi}(s_{n_j})\right) + \operatorname{dist}\left(\bar{\phi}(s_{n_j}), \phi_{n_j}(s_{n_j})\right) \\
&\le \operatorname{dist}\left(\phi_{n_j}(t), \bar{\phi}(t)\right) + \operatorname{dist}\left(\bar{\phi}(t), \bar{\phi}(s_{n_j})\right) + \sup_{t-1\le s\le t+1} \operatorname{dist}\left(\bar{\phi}(s), \phi_{n_j}(s)\right) \\
&\to 0
\end{aligned}
$$

as $j \to \infty$ by the uniform convergence of the subsequence in the first and third terms and the continuity of the trajectory $\bar{\phi}$ in the second term. But this is a contradiction, so we must have $H^*(A_t, A_s) \to 0$ as $s \to t$.

Secondly, consider the limit $H^*(A_s, A_t) \to 0$ as $s \to t$. If this does not hold there would exist an $\epsilon_0 > 0$ and a sequence $s_n \to t$ such that

$$
\epsilon_0 \le H^*(A_{s_n}, A_t), \qquad n \in \mathbb{N}.
$$

We will show that this leads to a contradiction.

Since A_{s_n} is compact, there exists an $a_n \in A_{s_n}$ such that

$$
H^*(A_{s_n}, A_t) = \operatorname{dist}(a_n, A_t) \le \operatorname{dist}(a_n, a)
$$

for all $a \in A_t$. By the weak invariance of $\mathcal{A}$ there exists a trajectory ϕ_n with $\phi_n(s_n) = a_n$ and $\phi_n(s) \in A_s$ for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus

$$
H^*(A_{s_n}, A_t) \le \operatorname{dist}(\phi_n(s_n), \phi_n(t)), \qquad n \in \mathbb{N}.
$$

Now $a_n \in A_{s_n} \subset \Phi([t-1, t+1], t-1, A_{t-1})$, which is compact. Here we have used the negative strong invariance of the weak pullback attractor, see Remark 9. Thus we can apply Barbashin's Theorem to obtain the existence of a subsequence of trajectories ϕ_{n_j} which converges to a trajectory $\bar{\phi}$ uniformly on the interval $[t-1, t+1]$. Thus

$$
\begin{aligned}
\epsilon_0 &\le H^*\left(A_{s_{n_j}}, A_t\right) \\
&\le \operatorname{dist}\left(\phi_{n_j}(s_{n_j}), \phi_{n_j}(t)\right) \\
&\le \operatorname{dist}\left(\phi_{n_j}(s_{n_j}), \bar{\phi}(s_{n_j})\right) + \operatorname{dist}\left(\bar{\phi}(s_{n_j}), \bar{\phi}(t)\right) \\
&\le \sup_{t-1\le s\le t+1} \operatorname{dist}\left(\phi_{n_j}(s), \bar{\phi}(s)\right) + \operatorname{dist}\left(\bar{\phi}(s_{n_j}), \bar{\phi}(t)\right) \\
&\to 0
\end{aligned}
$$

as $j \to \infty$ by the uniform convergence of the subsequence in the first term and the continuity of the trajectory $\bar{\phi}$ in the second term. But this is a contradiction, so we must have $H^*(A_s, A_t) \to 0$ as $s \to t$.

Combining the two cases gives the desired result, i.e., $H(A_s, A_t) \to 0$ as $s \to t$.

9 Proof of Theorem 12

Let $\mathcal{A} = \{A_t, t \in \mathbb{R}\}$ be the weak pullback attractor in $\mathcal{B}$ given by (2) for the unperturbed setvalued process Φ and let $\mathcal{A}^\epsilon = \{A_t^\epsilon, t \in \mathbb{R}\}$ be the weak pullback attractor in $\mathcal{B}^\epsilon$ for the perturbed setvalued process Φ^ϵ. Suppose for some $t_0 \in \mathbb{R}$ that

$$\lim_{\epsilon \to 0} H^*\left(A_{t_0}^\epsilon, A_{t_0}\right) \neq 0.$$

Then there exists an $\eta_0 > 0$ and a subsequence $\epsilon_j \to 0$ as $j \to \infty$ such that

$$H^*\left(A_{t_0}^{\epsilon_j}, A_{t_0}\right) \geq \eta_0 \tag{8}$$

for all $j \in \mathbb{Z}^+$. We will show that this leads to a contradiction.

Let $a^{\epsilon_j} \in A_{t_0}^{\epsilon_j}$ be such that $\text{dist}\,(a^{\epsilon_j}, A_{t_0}) = H^*\left(A_{t_0}^{\epsilon_j}, A_{t_0}\right)$, so $\text{dist}\,(a^{\epsilon_j}, A_{t_0}) \geq \eta_0$ for $j \in \mathbb{Z}^+$. This is possible since $A_{t_0}^{\epsilon_j}$ is compact. By Lemma 13 there is an entire trajectory ϕ^{ϵ_j} of the perturbed setvalued process Φ^{ϵ_j} such that $\phi^{\epsilon_j}(t) \in A_t^{\epsilon_j} \subset B_t^{\epsilon_j}$ for each $t \in \mathbb{R}$ with $\phi^{\epsilon_j}(t_0) = a^{\epsilon_j}$.

Since the $B_{t_0}^{\epsilon_j}$ and B_{t_0} are compact with $H^*\left(B_{t_0}^{\epsilon_j}, B_{t_0}\right) \to 0$ as $\epsilon_j \to 0$, by Lemma 14 there exists a convergent subsequence $a^{\epsilon_j'} = \phi^{\epsilon_j'}(t_0) \to \bar{a}_0 \in B_{t_0}$ as $\epsilon_j' \to 0$.

¿From (8) we have

$$\text{dist}\,(\bar{a}_0, A_{t_0}) \geq \eta_0/2. \tag{9}$$

By Theorem 16 (Generalized Barbashin Theorem) applied to the interval $[t_0, t_0 + 1]$, there exists a trajectory $\bar{\phi}$ of Φ on $[t_0, t_0 + 1]$ with $\bar{\phi}(t_0) = \bar{a}_0$ and a subsubsequence $\phi^{\epsilon_j''}$ with $\phi^{\epsilon_j''}(t) \to \bar{\phi}(t)$ as $\epsilon_j'' \to 0$ uniformly in $t \in [t_0, t_0 + 1]$. Moreover, by Lemma 14 we have $\bar{\phi}(t) \in B_t$ for each $t \in [t_0, t_0 + 1]$. We repeat this construction on successive subintervals $[t_0 + n, t_0 + n + 1]$ for n = 1, 2, ... to obtain a trajectory $\bar{\phi}$ of Φ on $[t_0, \infty)$ and a diagonal subsequence (denoted the same as before) $\phi^{\epsilon''_j}$ with $\phi^{\epsilon''_j}(t) \to \bar{\phi}(t) \in B_t$ as $\epsilon''_j \to 0$ for all $t \in [t_0, \infty)$. We can also work backwards in time on successive subintervals $[t_0-n-1, t_0-n]$ for $n = 1, 2, \ldots$ to obtain a trajectory $\bar{\phi}$ of Φ on $(-\infty, t_0]$ with $\bar{\phi}(t_0) = a_0$ and a further diagonal subsequence (denoted the same as before) $\phi^{\epsilon''_j}$ with $\phi^{\epsilon''_j}(t) \to \bar{\phi}(t) \in B_t$ as $\epsilon''_j \to 0$ for all $t \in (-\infty, t_0]$.

Thus $\bar{\phi}$ is an entire trajectory of the unperturbed setvalued process Φ with $\bar{\phi}(t) \in B_t$ for each $t \in \mathbb{R}$. By Lemma 13 it follows that $\bar{\phi}(t) \in A_t$ for each $t \in \mathbb{R}$. In particular, $\bar{\phi}(t_0) \in A_{t_0}$. However, this contradicts (9) and hence (8). This contradiction means that the A_t^ϵ converge upper semi continuously to A_t for each $t \in \mathbb{R}$.

10 Proof of Theorem 16

For convenience, we consider without loss of generality the interval $[0, 1]$ instead of $[t_0, t_1]$. By assumption, there is a sequence of trajectories ϕ^{ϵ_j} of Φ^{ϵ_j} on $[t_0, t_1]$ with $\phi^{\epsilon_j}(0) = x_{0,j} \to x_0$ as $\epsilon_j \to 0$. Write $\phi(0) = x_0$.

By the upper semi continuous convergence (3) and the upper semi continuity of $\Phi(t, 0, \cdot)$ uniformly in $t \in [0, 1]$, we have

$$\begin{aligned} H^*\left(\Phi^{\epsilon_j}(t,0,x_{0,j}), \Phi(t,0,x_0)\right) &\leq H^*\left(\Phi^{\epsilon_j}(t,0,x_{0,j}), \Phi(t,0,x_{0,j})\right) \\ &\quad + H^*\left(\Phi(t,0,x_{0,j}), \Phi(t,0,x_0)\right) \\ &\leq \epsilon_j + H^*\left(\Phi(t,0,x_{0,j}), \Phi(t,0,x_0)\right) \\ &\longrightarrow 0 \quad \text{as} \quad \epsilon_j \to 0 \end{aligned}$$

uniformly in $t \in [0, 1]$. Hence for every $\epsilon > 0$ and taking ϵ_j sufficiently small,

$$\Phi^{\epsilon_j}(t,0,x_{0,j}) \subset N_\epsilon\left[\Phi([0,1],0,x_0)\right],$$

for all $t \in [0, 1]$. The set $\Phi([0, 1], 0, x_0)$ is compact by the continuity of $\Phi(\cdot, 0, x)$ because of the Properties 1 and 4 of a setvalued process, so from $\phi^{\epsilon_j}(1) \in B^*$, there exists a convergent subsequence $\phi^{\epsilon'_j}(1) = x_{1,j} \to x_1 \in \Phi([0, 1], 0, x_0)$ as $\epsilon'_j \to 0$. Write $\phi(1) = x_1$. Moreover, $\phi(1) \in \Phi(1, 0, x_0)$. This follows from the fact that

$$\begin{aligned} \operatorname{dist}\left(\phi(1), \Phi(1,0,\phi(0))\right) &\leq \left\|\phi(1) - \phi^{\epsilon'_j}(1)\right\| + \operatorname{dist}\left(\phi^{\epsilon'_j}(1), \Phi^{\epsilon'_j}(1,0,\phi^{\epsilon'_j}(0))\right) \\ &\quad + H^*\left(\Phi^{\epsilon'_j}(1,0,\phi^{\epsilon'_j}(0)), \Phi(1,0,\phi(0))\right) \\ &= \left\|\phi(1) - \phi^{\epsilon'_j}(1)\right\| + H^*\left(\Phi^{\epsilon'_j}(1,0,\phi^{\epsilon'_j}(0)), \Phi(1,0,\phi(0))\right), \end{aligned}$$

since $\phi^{\epsilon'_j}(1) \in \Phi^{\epsilon'_j}(1,0,\phi^{\epsilon'_j}(0))$ for the trajectories $\phi^{\epsilon'_j}$ of $\Phi^{\epsilon'_j}$. Thus

$$\phi^{\epsilon'_j}(1) \to \phi(1), \qquad \phi^{\epsilon'_j}(0) \to \phi(0) \quad \text{as} \quad \epsilon'_j \to 0.$$

Since the setvalued mappings $\Phi^{\epsilon'_j}(1,0,\cdot)$ and $\Phi(1,0,\cdot)$ are upper semi continuous and the $\Phi^{\epsilon'_j}(1,0,\cdot)$ converge upper semi continuously to $\Phi(1,0,\cdot)$ due to (3), it follows by Lemma 15 that

$$H^*\left(\Phi^{\epsilon'_j}(1,0,\phi^{\epsilon'_j}(0)), \Phi(1,0,\phi(0))\right) \longrightarrow 0 \quad \text{as} \quad \epsilon'_j \to 0.$$

Thus $\operatorname{dist}(\phi(1), \Phi(1,0,\phi(0))) = 0$, i.e., $\phi(1) \in \Phi\,(1,0,\phi(0))$.

Consider the time instant $t = \frac{1}{2}$. We repeat the above argument on the interval $[0, \frac{1}{2}]$, to construct $\phi\left(\frac{1}{2}\right) \in \Phi\left(\frac{1}{2}, 0, \phi(0)\right)$ using a subsequence of the above one that converges at $t = \frac{1}{2}$ as well as at $t = 0$ and 1. Using this same sequence on the interval $[\frac{1}{2}, 1]$ we also obtain $\phi(1) \in \Phi\left(1, \frac{1}{2}, \phi\left(\frac{1}{2}\right)\right)$.

The construction for $\phi(t)$ for dyadic $t \in \bigcup_{q=0,1,2,\dots}\left\{\frac{p}{2^q} \ : \ p = 0,1,2,\dots,q\right\}$ follows recursively, taking subsequences of the previous ones that also converge at the new points under consideration. Suppose that for a given q we have constructed all of the $\phi\left(\frac{p}{2^q}\right)$ such that

$$\phi\left(\frac{p+1}{2^q}\right) \in \Phi\left(\frac{p+1}{2^q}, \frac{p}{2^q}, \phi\left(\frac{p}{2^q}\right)\right), \qquad p = 0,1,2,\dots,q-1. \tag{10}$$

Consider the time instant $\frac{2p+1}{2^{q+1}}$, which is the midpoint of the interval $[\frac{p}{2^q}, \frac{p+1}{2^q}]$. The construction of $\phi\left(\frac{2p+1}{2^{q+1}}\right)$ with

$$\phi\left(\frac{2p+1}{2^{q+1}}\right) \in \Phi\left(\frac{2p+1}{2^{q+1}}, \frac{p}{2^q}, \phi\left(\frac{p}{2^q}\right)\right)$$

and

$$\phi\left(\frac{p+1}{2^q}\right) \in \Phi\left(\frac{p+1}{2^q}, \frac{2p+1}{2^{q+1}}, \phi\left(\frac{2p+1}{2^{q+1}}\right)\right)$$

follows exactly the same as in the case of $p = 0$ and $q = 1$, i.e., for $\phi\left(\frac{1}{2}\right)$ from $\phi(0)$ and $\phi(1)$.

It follows from the 2–parameter semigroup property of Φ, i.e., the time evolution property 3, and the inclusions (10), we have $\phi(t) \in \Phi\,(t,s,\phi(s))$ for all dyadic s, $t \in [0, 1]$ with $s \leq t$. As in the proof of the original Barbashin Theorem, the $\phi(t)$ for nondyadic t are defined by a limiting argument and the fact that $\phi(t) \in \Phi(t, s, \phi(s))$ for all s, $t \in [0, 1]$ with $s \leq t$ follows from the continuity and upper semi continuity properties of Φ. (See [12] for additional details). Thus the function ϕ is a trajectory of Φ with the stated properties. In particular, the function $t \mapsto \phi(t)$ is continuous since ϕ is a trajectory.

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