THE EXPONENTIAL BEHAVIOUR OF NONLINEAR STOCHASTIC FUNCTIONAL EQUATIONS OF SECOND ORDER IN TIME

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Abstract

Sufficient conditions for exponential mean square stability of solutions to delayed stochastic partial differential equations of second order in time are established. As a consequence of these results, some ones on the pathwise exponential stability of the system are proved. The stability results derived are applied also to partial differential equations without hereditary characteristics. The results are illustrated with several examples.

1 Introduction

Stochastic differential delay equations and their asymptotic behaviour have been receiving much attention in the last years (see [1], [2], [4], [8], [6], [7], [9], [11], [12], and the references therein) since these retarded problems often appear in Physics, Biology, Engineering, etc...

The delays can enter in the formulations in very different ways, e.g., as a constant or variable delay, as a distributed one, or even some of them can appear in the model at the same time. Also these delay can be bounded (finite) or unbounded (infinite). However, there is a possibility of considering all of them under a unified formulation by using appropriate differential functional equations. In this sense, we will carry out our analysis in a functional framework which will cover a wide variety of situations containing finite delays (see Section 5).

On the other hand, a very interesting question is to analyse the long-time behaviour of the solution to a stochastic functional equation. We remark that in some problems the history of the phenomenon has a decisive influence on the future behaviour of the system and, in some cases, not only a short period of the past has to be taken into account, but a large one. This fact motivates the present work.

There exists a wide literature concerning pathwise exponential stability of parabolic stochastic evolution equations (with and without delays). We mention here, amongst many others, Caraballo and Liu [3], Liu and Mao [10], Taniguchi [11], Taniguchi *et al.* [12] and the references therein. However, as far as we know, there are no papers on the asymptotic stability of delay stochastic partial differential equations of second order in time, which is the main aim of this paper. In the case without hereditary characteristics this problem has been considered by Curtain [5], where one can find sufficient conditions for the exponential stability of the expected energy of the system, as well as for the exponential decay of the sample paths, when the main operator generates a strongly continuous contraction semi-group. In this paper we shall develop the theory in a variational framework for non-linear operators in general and under a functional formulation which covers several kinds of delay and, in particular, the non-delay case.

In order to motivate our theory let us first study the following example.

The simplest model of continuum mechanics is given by the vibrating string, subjected to a constant tension μ, which executes small longitudinal vibrations about the position of stable equilibrium. Our problem is to determine the lateral displacement $u(x,t)$ of a point on the string from its equilibrium position. For a constant linear density ϱ of the string, if we also consider some friction proportional to velocity, $\vartheta v(t)$, that an external force $f(t,u(t),v(t))$ acts on the string, and that u_0 and v_0 are respectively the initial position and velocity, we then obtain the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} + \vartheta \dfrac{\partial u}{\partial t} + \Phi\left(t, u(t), \dfrac{\partial u}{\partial t}(t)\right) = 0, \text{ in } [0,+\infty) \times [0,1], \\ u(t,0) = u(t,1) = 0, \ t \in [0,+\infty), \\ u(0,x) = u_0, \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \text{ in } [0,1], \end{cases}$$

where $\Phi\left(t, u(t), \frac{\partial u}{\partial t}(t)\right)$ corresponds to $\frac{1}{\varrho} f\left(t, u(t), \frac{\partial u}{\partial t}(t)\right)$, $a = \sqrt{\frac{\mu}{\varrho}}$, and $\vartheta > 0$.

This model can be thought to be more realistic if we suppose that Φ contains some random features, for example we can think of $\Phi\left(t, u(t), \frac{\partial u}{\partial t}(t)\right) = -\sigma \frac{\partial u(t)}{\partial x} \frac{dW(t)}{dt}$, $t \geq 0$, where

$\sigma \in \mathbb{R}$, and $W(t)$ is a one-dimensional Wiener process. Therefore the equation becomes

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + \vartheta \frac{\partial u}{\partial t} = \sigma \frac{\partial u}{\partial x}(t) \frac{dW(t)}{dt}, \ t \geq 0.$$

Curtain proves in [5] that when $\sigma^2 < \frac{4\vartheta\pi^2}{4\pi^2 + \vartheta(\vartheta + \sqrt{\vartheta^2 + 4\pi^2})}$ and $a = 1$, the null solution of this problem is stable in mean square, i.e., if the random term is sufficiently small so that this relation is satisfied. But, is it possible to deduce any exponential stability results for the above system when $\sigma^2 > \frac{4\vartheta\pi^2}{4\pi^2 + \vartheta(\vartheta + \sqrt{\vartheta^2 + 4\pi^2})}$? As a consequence of the theory we will develop in this paper, we will prove that if $\sigma^2 < \frac{2\vartheta\pi^2}{\vartheta^2 + 2\pi^2}$ the system is exponentially stable both in mean square and pathwise, so these results improve the ones in [5].

On the other hand, if we are interested in some problems concerning the stabilization or controllability of systems, it seems natural to study equations in which some hereditary characteristics can appear. For example, in the above system we can consider the load Φ does not depend just on the present, but on the history of the process, i.e., it is modelled for example by the expression $\Phi\left(t, u_t, \frac{\partial u_t}{\partial t}\right) = -\sigma \frac{\partial u(t-\tau(t))}{\partial x} \dot{W}(t),\ t \geq 0$, where $\tau(\cdot)$ is an appropriate delay function (see Example 8, Section 5).

The content of the paper is as follows. In Section 2 we present the framework in which our analysis is carried out, and introduce some basic notations and assumptions. Section 3 is devoted to the main result of this work, that is, we establish a sufficient condition ensuring mean square stability for delay stochastic partial differential equations of second order in time in a very general situation. We also indicate how this result can be applied in some particular cases. As a consequence of the mean square stability, in Section 4 we obtain pathwise exponential stability. Moreover, in this section we will be concerned with a more general situation in which our theory can be established. In Section 5 we include some examples to illustrate these results. Finally, some conclusions are included in the last section.

2 Statement of the problem

Let V and H be two real separable Hilbert spaces such that $V \subset H \equiv H^* \subset V^*$ where the injections are continuous and dense.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V^* respectively; by $(\cdot, \cdot)$ the inner product in H, and by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V. Let us denote by $c > 0$ a constant such that $|x| \leq c \|x\|$, $\forall x \in V$.

Assume $\{\Omega, \mathcal{F}, P\}$ is a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, i.e., $\mathcal{F}_0$ contains the null sets in $\mathcal{F}$, and $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$, for all $t \geq 0$. Denote $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$.

Let us consider a real valued $\{\mathcal{F}_t\}-$Wiener process $\{W(t)\}_{t\geq 0}$.

Given real numbers $a<b$ and a separable Hilbert space $\mathcal{H}$, we will denote by $I^2(a,b;\mathcal{H})$ the closed subspace of $L^2(\Omega\times(a,b),\mathcal{F}\otimes\mathcal{B}([a,b]),\mathrm{d}P\otimes\mathrm{d}t;\mathcal{H})$ of all stochastic processes which are $\mathcal{F}_t$-adapted for almost every t in (a,b) (in what follows, a.e. t), where $\mathcal{B}([a,b])$ denotes the Borel σ-algebra of subsets in $[a,b]$. If $\varphi\in I^2(a,b;\mathcal{H})$ we will write $|\varphi|_{I^2_{\mathcal{H}}}$ to denote the norm $|\varphi|_{I^2(a,b;\mathcal{H})}$.

We denote by $L^2(\Omega;C(a,b;\mathcal{H}))$ the space of processes $X\in L^2(\Omega,\mathcal{F},dP;C(a,b;\mathcal{H}))$ such that $X(t)$ is $\mathcal{F}_t$-measurable for each t in $[a,b]$, where $C(a,b;\mathcal{H})$ denotes the space of all continuous functions from $[a,b]$ into $\mathcal{H}$ equipped with supremum norm.

Let us fix $h>0$ and consider $T>0$. If we have a function $x\in C(-h,T;\mathcal{H})$, for each $t\in[0,T]$ we denote by $x_t\in C(-h,0;\mathcal{H})$ the function defined by $x_t(s)=x(t+s)$, $-h\leq s\leq 0$. Moreover, if $y\in L^2(-h,T;\mathcal{H})$ we also denote by $y_t\in L^2(-h,0;\mathcal{H})$, a.e. $t\in(0,T)$, the function defined by $y_t(s)=y(t+s)$, a.e. $s\in(-h,0)$.

Let $A(t):V\to V^*$, $t\geq 0$, be a family of operators satisfying:

$(A.1)$ $A(t)$ is self-adjoint for each $t\geq 0$.

$(A.2)$ $A(t)\in\mathcal{L}(V,V^*)$ $\forall\ t\geq 0$, and there exists $c_A>0$ such that $\|A(t)u\|_*\leq c_A\|u\|$, $\forall t\geq 0,\forall u\in V$.

$(A.3)$ $\exists\alpha>0$ such that $\langle A(t)u,u\rangle\geq\alpha\|u\|^2$, $\forall t\geq 0,\forall u\in V$.

$(A.4)$ $\langle A(\cdot)u,\widetilde{u}\rangle\in C^1(0,+\infty)$, $\forall u,\widetilde{u}\in V$, and $\langle A'(t)u,u\rangle\leq 0,\forall t\geq 0,\forall u\in V$, where $\langle A'(t)u,\widetilde{u}\rangle$ denotes $\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)u,\widetilde{u}\rangle$.

$(A.5)$ there exists a Banach space X such that $X\subset\{u\in V\ ;\ A(t)u\in H,\forall t\geq 0\}$, the injection of X in V is continuous, and X is dense in H.

Let $B(t,\cdot):H\to H$ be a family of nonlinear operators defined a.e. $t\geq 0$ and satisfying:

$(B.1)$ $\forall v\in H$, the map $t\in(0,+\infty)\to B(t,v)\in H$ is Lebesgue measurable.

$(B.2)$ the map $\theta\in\mathbb{R}\to(B(t,v+\theta w),z)\in\mathbb{R}$ is continuous $\forall v,w,z\in H$, a.e. $t\geq 0$.

$(B.3)$ there exists $c_B>0$ such that $|B(t,v)|\leq c_B|v|$, $\forall v\in H$, a.e. $t\geq 0$.

$(B.4)$ there exists $\beta>0$ such that $(B(t,v)-B(t,\widetilde{v}),v-\widetilde{v})\geq\beta|v-\widetilde{v}|^2$, $\forall v,\widetilde{v}\in H$, a.e. $t\geq 0$.

Let $F : [0,+\infty) \times C(-h,0;V) \times C(-h,0;H) \to H$ and $G : [0,+\infty) \times C(-h,0;V) \times C(-h,0;H) \to H$ be two families of nonlinear operators defined a.e. $t \geq 0$ such that:

$(F.1)$ $\forall(\xi,\eta) \in C(-h,0;V)\times C(-h,0;H)$ the map $t \in (0,+\infty) \to F(t,\xi,\eta) \in H$ is Lebesgue measurable, a.e. $t \geq 0$.

$(F.2)$ $F(t,0,0) = 0$, a.e. $t \geq 0$.

$(F.3)$ there exist $C_{F,H}, C_{F,V} > 0$ such that $\forall \xi, \widetilde{\xi} \in C(-h,0;V), \forall \eta, \widetilde{\eta} \in C(-h,0;H)$ and a.e. $t \geq 0$,

$$|F(t,\xi,\eta) - F(t,\widetilde{\xi},\widetilde{\eta})|^2 \leq C_{F,V} ||\xi - \widetilde{\xi}||^2_{C(-h,0;V)} + C_{F,H} \left|\eta - \widetilde{\eta}\right|^2_{C(-h,0;H)}.$$

$(F.4)$ there exist $m_0 > 0$ and constants $K_{F,H} = K_{F,H}(m_0,h), K_{F,V} = K_{F,V}(m_0,h) \geq 0$ such that for all $m \in [0,m_0]$, $\forall x, \widetilde{x} \in C(-h,T;V), \forall y, \widetilde{y} \in C(-h,T;H)$, and $\forall t \geq 0$

$$\begin{aligned}&\int_0^t \mathrm{e}^{ms} \left|F(s,x_s,y_s) - F(s,\widetilde{x}_s,\widetilde{y}_s)\right|^2 \mathrm{d}s \\ &\leq K_{F,V} \int_{-h}^t \mathrm{e}^{ms} \left\|x(s) - \widetilde{x}(s)\right\|^2 \mathrm{d}s + K_{F,H} \int_{-h}^t \mathrm{e}^{ms} \left|y(s) - \widetilde{y}(s)\right|^2 \mathrm{d}s.\end{aligned}$$

$(G.1)$ $\forall(\xi,\eta) \in C(-h,0;V)\times C(-h,0;H)$ the map $t \in (0,+\infty) \to G(t,\xi,\eta) \in H$ is Lebesgue measurable, a.e. $t \geq 0$.

$(G.2)$ $G(t,0,0) = 0$, a.e. $t \geq 0$.

$(G.3)$ there exist $C_{G,H}, C_{G,V} > 0$ such that $\forall \xi, \widetilde{\xi} \in C(-h,0;V), \forall \eta, \widetilde{\eta} \in C(-h,0;H)$ and a.e. $t \geq 0$,

$$|G(t,\xi,\eta) - G(t,\widetilde{\xi},\widetilde{\eta})|^2 \leq C_{G,V} ||\xi - \widetilde{\xi}||^2_{C(-h,0;V)} + C_{G,H} \left|\eta - \widetilde{\eta}\right|^2_{C(-h,0;H)}.$$

$(G.4)$ there exist $m_0 > 0$ and constants $K_{G,H} = K_{G,H}(m_0,h), K_{G,V} = K_{G,V}(m_0,h) \geq 0$ such that for all $m \in [0,m_0]$, $\forall x, \widetilde{x} \in C(-h,T;V), \forall y, \widetilde{y} \in C(-h,T;H)$, and $\forall t \geq 0$

$$\begin{aligned}&\int_0^t \mathrm{e}^{ms} \left|G(s,x_s,y_s) - G(s,\widetilde{x}_s,\widetilde{y}_s)\right|^2 \mathrm{d}s \\ &\leq K_{G,V} \int_{-h}^t \mathrm{e}^{ms} \left\|x(s) - \widetilde{x}(s)\right\|^2 \mathrm{d}s + K_{G,H} \int_{-h}^t \mathrm{e}^{ms} \left|y(s) - \widetilde{y}(s)\right|^2 \mathrm{d}s.\end{aligned}$$

Remark 1 *Assumptions $(F.2)$, $(G.2)$ are motivated by our interest in analyzing the stability of the zero solution to our problem, but they are not necessary to prove existence of solution.*

We consider the following problem,

$$\begin{cases} u \in I^2(-h,T;V) \cap L^2(\Omega; C(0,T;V)), \text{ for all } T>0, \\ v \in I^2(-h,T;H) \cap L^2(\Omega; C(0,T;H)), \text{ for all } T>0, \\ u'(t) = v(t), \ t \in [0,T], \\ v(t) + \int_0^t A(s)u(s)\mathrm{d}s + \int_0^t B(s,v(s))\mathrm{d}s = v_0 + \int_0^t F(s,u_s,v_s)\mathrm{d}s \\ + \int_0^t G(s,u_s,v_s)\mathrm{d}W(s), \ t \geq 0, \\ u(0) = u_0, \\ u(t) = \varphi_1(t), \ v(t) = \varphi_2(t), \ \text{a.e. } t \in (-h,0), \end{cases} \qquad (P)$$

where $\varphi_1 \in I^2(-h,0;V)$, $\varphi_2 \in I^2(-h,0;H)$, $u_0 \in L^2(\Omega,\mathcal{F}_0,P;V)$ and $v_0 \in L^2(\Omega,\mathcal{F}_0,P;H)$ are given.

A similar analysis to that in Remark 1 in [7] shows that, under our assumptions $(F.1)-(F.4)$ and $(G.1)-(G.4)$, all the integrals appearing in problem (P) are well defined, and therefore, the above problem makes sense.

On the other hand, several results on the existence and uniqueness of solutions for delay stochastic evolution equations of second order in time can be seen in Garrido-Atienza and Real [7]. In particular, the following one is a consequence of Theorem 4 in [7]:

Theorem 2 *Assume that hypotheses* $(A.1)-(A.5),(B.1)-(B.4),(F.1)-(F.4),(G.1)-(G.4)$ *hold. Then, if* $\varphi_1 \in I^2(-h,0;V)$, $\varphi_2 \in I^2(-h,0;H)$, $u_0 \in L^2(\Omega,\mathcal{F}_0,P;V)$ *and* $v_0 \in L^2(\Omega,\mathcal{F}_0,P;H)$, *there exists a unique solution* (u,v) *to problem* (P)*, for all* $T>0$.

3 Exponential stability in mean square

In this section we will establish a result about the mean square stability for the solution to problem (P).

Theorem 3 *Suppose that assumptions* $(A.1)-(A.5),(B.1)-(B.4),(F.1)-(F.4),(G.1)-(G.4)$ *hold. In addition, assume that there exist some constants* $\varepsilon>0$ *and* $\delta>0$ *such that*

$$\begin{cases} 2\alpha^{3/2}C_\delta > cK_\varepsilon, \\ \textit{and} \\ C_\delta\left(2\beta - (2K_{F,H}^{1/2} + \varepsilon + K_{G,H})\right) > (4\alpha^2\delta)^{-1}(4\alpha\delta + c^2(K_{F,H}^{1/2} + c_B)^2)K_\varepsilon, \end{cases} \qquad (1)$$

where $C_\delta = 1 - \delta - cK_{F,V}^{1/2}\alpha^{-1}$ *and* $K_\varepsilon = K_{F,V}\varepsilon^{-1} + K_{G,V}$*. Then, the zero solution of problem (P) is exponentially stable in mean square, i.e., there exist* $m \in (0, m_0]$ *and* $K_1 = K_1(m_0, h) > 0$ *such that, for all* $t \geq 0$,

$$E\left(|v(t)|^2 + \|u(t)\|^2\right) \leq K_1 \left(E\,|v_0|^2 + E\,\|u_0\|^2 + |\varphi_2|^2_{I_H^2} + \|\varphi_1\|^2_{I_V^2}\right) e^{-mt}, \tag{2}$$

for any solution (u, v) *of* (P).

Proof. Let $m \in (0, m_0]$. Applying Itô's formula to the process

$$\mathrm{e}^{mt}\,|v(t)|^2 + \mathrm{e}^{mt}\,\langle A(t)u(t), u(t)\rangle\,,$$

we obtain for each $t \geq 0$ and P-a.s.

$$\begin{aligned}
&\mathrm{e}^{mt}\,|v(t)|^2 + \mathrm{e}^{mt}\,\langle A(t)u(t), u(t)\rangle \\
&= |v_0|^2 + \langle A(0)u_0, u_0\rangle + m\int_0^t \mathrm{e}^{ms}(|v(s)|^2 + \langle A(s)u(s), u(s)\rangle)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{ms}\,\langle A'(s)u(s), u(s)\rangle\,\mathrm{d}s - 2\int_0^t \mathrm{e}^{ms}(B(s, v(s)), v(s))\mathrm{d}s \\
&+ 2\int_0^t \mathrm{e}^{ms}(F(s, u_s, v_s), v(s))\mathrm{d}s + \int_0^t \mathrm{e}^{ms}\,|G(s, u_s, v_s)|^2\,\mathrm{d}s \\
&+ 2\int_0^t \mathrm{e}^{ms}(G(s, u_s, v_s), v(s))\mathrm{d}W(s),
\end{aligned}$$

and thanks to $(A.4)$ and $(B.4)$,

$$\begin{aligned}
&\mathrm{e}^{mt}E\,|v(t)|^2 + \mathrm{e}^{mt}E\,\langle A(t)u(t), u(t)\rangle \\
&\leq E\,|v_0|^2 + E\,\langle A(0)u_0, u_0\rangle + (m - 2\beta)\int_0^t \mathrm{e}^{ms}E\,|v(s)|^2\,\mathrm{d}s \\
&+ m\int_0^t \mathrm{e}^{ms}E\,\langle A(s)u(s), u(s)\rangle\,\mathrm{d}s \\
&+ 2\int_0^t \mathrm{e}^{ms}E(F(s, u_s, v_s), v(s))\mathrm{d}s + \int_0^t \mathrm{e}^{ms}E\,|G(s, u_s, v_s)|^2\,\mathrm{d}s.
\end{aligned} \tag{3}$$

As $\mathrm{e}^{mt} \le 1$, $\forall t \in [-h, 0]$, from $(F.4)$ and $(A.3)$ we obtain

$$2\int_0^t \mathrm{e}^{ms} E(F(s,u_s,v_s),v(s))\mathrm{d}s$$
$$\le 2\left(\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s\right)^{1/2} \left(\int_0^t \mathrm{e}^{ms} E\left|F(s,u_s,v_s)\right|^2 \mathrm{d}s\right)^{1/2}$$
$$\le 2K_{F,H}^{1/2}\int_{-h}^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s + 2\left(K_{F,V}\int_{-h}^t \mathrm{e}^{ms} E\left\|u(s)\right\|^2 \mathrm{d}s\right)^{1/2} \left(\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s\right)^{1/2}$$
$$\le \left(2K_{F,H}^{1/2}+\varepsilon\right)\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s + K_{F,V}(\alpha\varepsilon)^{-1}\int_0^t \mathrm{e}^{ms} E\left\langle A(s)u(s),u(s)\right\rangle \mathrm{d}s$$
$$+ K_{F,V}\varepsilon^{-1}\left\|\varphi_1\right\|_{I_V^2}^2 + 2K_{F,H}^{1/2}\left|\varphi_2\right|_{I_H^2}^2,$$

for $\varepsilon > 0$, and by $(G.4)$ and $(A.3)$,

$$\int_0^t \mathrm{e}^{ms} E\left|G(s,u_s,v_s)\right|^2 \mathrm{d}s \le K_{G,H}\left|\varphi_2\right|_{I_H^2}^2 + K_{G,H}\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s$$
$$+ K_{G,V}\left\|\varphi_1\right\|_{I_V^2}^2 + K_{G,V}\alpha^{-1}\int_0^t \mathrm{e}^{ms} E\left\langle A(s)u(s),u(s)\right\rangle \mathrm{d}s.$$

Thus, if we substitute these inequalities into (3) we have

$$\mathrm{e}^{mt} E\left|v(t)\right|^2 + \mathrm{e}^{mt} E\left\langle A(t)u(t),u(t)\right\rangle \tag{4}$$
$$\le E\left|v_0\right|^2 + E\left\langle A(0)u_0,u_0\right\rangle + K_\varepsilon\left\|\varphi_1\right\|_{I_V^2}^2 + (2K_{F,H}^{1/2}+K_{G,H})\left|\varphi_2\right|_{I_H^2}^2$$
$$+\left(m-2\beta+2K_{F,H}^{1/2}+\varepsilon+K_{G,H}\right)\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s$$
$$+\left(m+K_\varepsilon\alpha^{-1}\right)\int_0^t \mathrm{e}^{ms} E\left\langle A(s)u(s),u(s)\right\rangle \mathrm{d}s.$$

Now, we estimate the last integral on the right hand side of (4). Thanks to

$$\mathrm{d}(\mathrm{e}^{mt}(u(t),v(t))) = m\mathrm{e}^{mt}(u(t),v(t))\mathrm{d}t + \mathrm{e}^{mt}\left|v(t)\right|^2\mathrm{d}t - \mathrm{e}^{mt}\left\langle A(t)u(t),u(t)\right\rangle \mathrm{d}t$$
$$-\mathrm{e}^{mt}(B(t,v(t)),u(t))\mathrm{d}t + \mathrm{e}^{mt}(F(t,u_t,v_t),u(t)) + \mathrm{e}^{mt}(G(t,u_t,v_t),u(t))\mathrm{d}W(t),$$

we deduce

$$\mathrm{e}^{mt}E(u(t),v(t)) = E(u_0,v_0) + m\int_0^t \mathrm{e}^{ms}E(u(s),v(s))\mathrm{d}s$$
$$+\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s - \int_0^t \mathrm{e}^{ms} E\left\langle A(s)u(s),u(s)\right\rangle \mathrm{d}s$$
$$-\int_0^t \mathrm{e}^{ms}E(B(s,v(s)),u(s))\mathrm{d}s + \int_0^t \mathrm{e}^{ms}E(F(s,u_s,v_s),u(s))\mathrm{d}s.$$

Consequently,

$$
\begin{aligned}
&\int_0^t \mathrm{e}^{ms} E \left\langle A(s)u(s), u(s)\right\rangle \mathrm{d}s \\
&\leq E(u_0, v_0) + \int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s \\
&+ c(m + c_B) \left(\int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s\right)^{1/2} \left(\alpha^{-1} \int_0^t \mathrm{e}^{ms} E \left\langle A(s)u(s), u(s)\right\rangle \mathrm{d}s\right)^{1/2} \\
&+ \left(\int_0^t \mathrm{e}^{ms} E \left|F(s, u_s, v_s)\right|^2 \mathrm{d}s\right)^{1/2} \left(c^2 \int_0^t \mathrm{e}^{ms} E \left\|u(s)\right\|^2 \mathrm{d}s\right)^{1/2} \\
&+ \left(\mathrm{e}^{mt} E \left|v(t)\right|^2\right)^{1/2} \left(c^2 \alpha^{-1} \mathrm{e}^{mt} E \left\langle A(t)u(t), u(t)\right\rangle\right)^{1/2} \\
&\leq E(u_0, v_0) + \int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s + cK_{F,V}^{1/2} \left\|\varphi_1\right\|_{I_V^2}^2 + c^2(m + c_B + K_{F,H}^{1/2})^2 (4\alpha\delta)^{-1} \left|\varphi_2\right|_{I_H^2}^2 \\
&+ c^2(m + c_B + K_{F,H}^{1/2})^2 (4\alpha\delta)^{-1} \int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s \\
&+ \left(\delta + cK_{F,V}^{1/2} \alpha^{-1}\right) \int_0^t \mathrm{e}^{ms} E \left\langle A(s)u(s), u(s)\right\rangle \mathrm{d}s \\
&+ c(2\alpha^{1/2})^{-1} \mathrm{e}^{mt} E \left(\left|v(t)\right|^2 + \left\langle A(t)u(t), u(t)\right\rangle\right),
\end{aligned}
$$

and thus

$$
\begin{aligned}
&C_\delta \int_0^t \mathrm{e}^{ms} E \left\langle A(s)u(s), u(s)\right\rangle \mathrm{d}s \\
&\leq E(u_0, v_0) + cK_{F,V}^{1/2} \left\|\varphi_1\right\|_{I_V^2}^2 + c^2(m + c_B + K_{F,H}^{1/2})^2 (4\alpha\delta)^{-1} \left|\varphi_2\right|_{I_H^2}^2 \\
&+ c(2\alpha^{1/2})^{-1} \mathrm{e}^{mt} E \left(\left|v(t)\right|^2 + \left\langle A(t)u(t), u(t)\right\rangle\right) \\
&+ (4\alpha\delta)^{-1} \left(4\alpha\delta + c^2(m + c_B + K_{F,H}^{1/2})^2\right) \int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s.
\end{aligned}
\tag{5}
$$

Due to the first condition in (1), the constant C_δ is positive. Therefore, we have

$$
\begin{aligned}
&\left(m + K_\varepsilon \alpha^{-1}\right) \int_0^t \mathrm{e}^{ms} E \left\langle A(s)u(s), u(s)\right\rangle \mathrm{d}s \\
&\qquad \leq \left(m + K_\varepsilon \alpha^{-1}\right) C_\delta^{-1} \Big[E(u_0, v_0) + c(2\alpha^{1/2})^{-1} \mathrm{e}^{mt} E \left(\left|v(t)\right|^2 + \left\langle A(t)u(t), u(t)\right\rangle\right) \\
&\qquad + cK_{F,V}^{1/2} \left\|\varphi_1\right\|_{I_V^2}^2 + c^2(m + c_B + K_{F,H}^{1/2})^2 (4\alpha\delta)^{-1} \left|\varphi_2\right|_{I_H^2}^2 \\
&\qquad + (4\alpha\delta)^{-1} \left(4\alpha\delta + c^2(m + c_B + K_{F,H}^{1/2})^2\right) \int_0^t \mathrm{e}^{ms} E \left|v(s)\right|^2 \mathrm{d}s\Big].
\end{aligned}
$$

Thus, if we substitute this into (4) it holds

$$\left(1-c\left(m+K_{\varepsilon}\alpha^{-1}\right)(2\alpha^{1/2}C_{\delta})^{-1}\right)\mathrm{e}^{mt}E\left(|v(t)|^{2}+\langle A(t)u(t),u(t)\rangle\right) \tag{6}$$
$$\leq E\left|v_{0}\right|^{2}+E\left\langle A(0)u_{0},u_{0}\right\rangle+\left(m+\alpha^{-1}K_{\varepsilon}\right)C_{\delta}^{-1}E(u_{0},v_{0})$$
$$+\left(K_{\varepsilon}+c\left(m+K_{\varepsilon}\alpha^{-1}\right)K_{F,V}^{1/2}C_{\delta}^{-1}\right)\|\varphi_{1}\|_{I_{V}^{2}}^{2}$$
$$+\left(2K_{F,H}^{1/2}+K_{G,H}+c^{2}(m+c_{B}+K_{F,H}^{1/2})^{2}(m+K_{\varepsilon}\alpha^{-1})(4\alpha\delta C_{\delta})^{-1}\right)|\varphi_{2}|_{I_{H}^{2}}^{2}$$
$$+\left(m-2\beta+2K_{F,H}^{1/2}+\varepsilon+K_{G,H}\right)\int_{0}^{t}\mathrm{e}^{ms}E\left|v(s)\right|^{2}\mathrm{d}s$$
$$+\left(m+K_{\varepsilon}\alpha^{-1}\right)(4\alpha\delta C_{\delta})^{-1}\left(4\alpha\delta+c^{2}(m+c_{B}+K_{F,H}^{1/2})^{2}\right)\int_{0}^{t}\mathrm{e}^{ms}E\left|v(s)\right|^{2}\mathrm{d}s.$$

Finally, taking into account the expressions of C_δ and K_ε, Eq. (1) implies, for $m>0$ small enough, that

$$\begin{aligned}
0 &< 1-c\left(m+K_{\varepsilon}\alpha^{-1}\right)(2\alpha^{1/2}C_{\delta})^{-1},\\
2\beta &> m+2K_{F,H}^{1/2}+\varepsilon+K_{G,H}\\
&\quad+\left(m+K_{\varepsilon}\alpha^{-1}\right)(4\alpha\delta C_{\delta})^{-1}\left(4\alpha\delta+c^{2}(m+c_{B}+K_{F,H}^{1/2})^{2}\right),
\end{aligned}$$

which together with $(A.2)$ and $(A.3)$ finishes the proof. ■

Remark 4 *Observe that condition (1) can be rewritten in an easier way in some particular cases. Indeed, if we assume that $K_{F,V}=0$ and fix $\delta=1/2$ in Theorem 3, then, it is straightforward to check that (1) holds if we suppose the conditions*

$$\begin{cases} K_{G,V}<c^{-1}\alpha^{3/2},\\ K_{G,H}<2\beta-(2\alpha+c^{2}(c_{B}+K_{F,H}^{1/2})^{2})\alpha^{-2}K_{G,V}-2K_{F,H}^{1/2}.\end{cases} \tag{7}$$

Thus, in particular, if $F\equiv0$, then the null solution to the corresponding problem is exponentially stable in mean square if we suppose that

$$\begin{cases} K_{G,V}<\alpha^{3/2}c^{-1},\\ K_{G,H}<2\beta-K_{G,V}\alpha^{-2}(2\alpha+(cc_{B})^{2}).\end{cases} \tag{8}$$

If also in the previous situation we assume that $K_{G,H}=0$, that is, $G(.,u_.,v_.)=G(.,u_.)$, then, (8) *reads*

$$K_{G,V}<\min\left\{c^{-1},2\beta\alpha^{1/2}(2\alpha+(cc_{B})^{2})^{-1}\right\}\alpha^{3/2}. \tag{9}$$

4 Pathwise stability

For our functional problem, we will prove that the sample paths tend to zero exponentially fast as $t \to \infty$ whenever the exponential mean square stability holds, in particular, under the assumptions of Theorem 3. To this end, we will use a technique (based on the Burkholder-Davis-Gundy inequality, the Doob inequality and the Borel-Cantelli lemma) which has proven very fruitful for parabolic equations (see, e.g. [1],[11],[12]).

First, we need the following result:

Lemma 5 *In the conditions of Theorem 3, there exist constants $C_1, C_2 > 0$ such that for any solution (u,v) of problem (P) it follows*

$$\int_0^t e^{ms} E\left|v(s)\right|^2 ds \le C_1 \left(E\left|v_0\right|^2 + E\left\|u_0\right\|^2 + \left|\varphi_2\right|^2_{I_H^2} + \left\|\varphi_1\right\|^2_{I_V^2} \right), \tag{10}$$

$$\int_0^t e^{ms} E\left\|u(s)\right\|^2 ds \le C_2 \left(E\left|v_0\right|^2 + E\left\|u_0\right\|^2 + \left|\varphi_2\right|^2_{I_H^2} + \left\|\varphi_1\right\|^2_{I_V^2} \right), \tag{11}$$

for each $t \ge 0$.

Proof. To prove (10), we first write (6) in the form

$$\begin{aligned}
&\left(1 - c\left(m + K_\varepsilon \alpha^{-1}\right)(2\alpha^{1/2}C_\delta)^{-1}\right) \mathrm{e}^{mt} E\left(\left|v(t)\right|^2 + \langle A(t)u(t), u(t)\rangle\right) \qquad (12)\\
&\le \widetilde{C}_1 (E\left|v_0\right|^2 + E\left\|u_0\right\|^2 + \left|\varphi_2\right|^2_{I_H^2} + \left\|\varphi_1\right\|^2_{I_V^2})\\
&+ \left(m - 2\beta + 2K_{F,H}^{1/2} + \varepsilon + K_{G,H}\right) \int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s\\
&+ \left(m + K_\varepsilon \alpha^{-1}\right)(4\alpha\delta C_\delta)^{-1}\left(4\alpha\delta + c^2(m + c_B + K_{F,H}^{1/2})^2\right) \int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s.
\end{aligned}$$

Thus, thanks to Theorem 3 we can choose $m > 0$ such that the constant $\widetilde{C}_2$ multiplying $\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s$ in (12) is negative, and hence

$$\int_0^t \mathrm{e}^{ms} E\left|v(s)\right|^2 \mathrm{d}s \le -\widetilde{C}_1 \widetilde{C}_2^{-1} (E\left|v_0\right|^2 + E\left\|u_0\right\|^2 + \left|\varphi_2\right|^2_{I_H^2} + \left\|\varphi_1\right\|^2_{I_V^2}),$$

which proves (10) by setting $C_1 = -\widetilde{C}_1 \widetilde{C}_2^{-1}$.

On the other hand, it is straightforward to obtain (11) by considering (5) and taking into account $(A.2)$, $(A.3)$, (2) and (10). ■

Now, using this lemma, we discuss the almost sure stability of solutions to (P).

Theorem 6 *Under assumptions in Theorem 3, there exist $K_2 = K_2(m,h)$, $\gamma > 0$, and a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$, there exists a positive random* *variable $T(\omega)$ such that if (u,v) is a solution of (P), it holds*

$$|v(t)|^2 + \|u(t)\|^2 \leq K_2 \left(E\,|v_0|^2 + E\,\|u_0\|^2 + |\varphi_2|^2_{I_H^2} + \|\varphi_1\|^2_{I_V^2} \right) e^{-\gamma t}, \ \forall t \geq T(\omega),$$

i.e., $(u(t),v(t))$ decays exponentially to zero almost surely.

Proof. We first prove that there exists a positive constant C such that for any $N \in \mathbb{N}$

$$E\left[\sup_{N \leq t \leq N+1} \left(|v(t)|^2 + \|u(t)\|^2 \right) \right] \leq Ce^{-mN} \left(E\,|v_0|^2 + E\,\|u_0\|^2 + |\varphi_2|^2_{I_H^2} + \|\varphi_1\|^2_{I_V^2} \right). \quad (13)$$

Indeed, Ito's formula implies once again

$$\begin{aligned}
&|v(t)|^2 + \langle A(t)u(t), u(t)\rangle \\
&= |v(N)|^2 + \langle A(N)u(N), u(N)\rangle + \int_N^t \langle A'(s)u(s), u(s)\rangle \,\mathrm{d}s \\
&- 2\int_N^t (B(s,v(s)), v(s))\mathrm{d}s + 2\int_N^t (F(s,u_s,v_s), v(s))\mathrm{d}s \\
&+ \int_N^t |G(s,u_s,v_s)|^2 \,\mathrm{d}s + 2\int_N^t (G(s,u_s,v_s), v(s))\mathrm{d}W(s).
\end{aligned}$$

Now, $(A.2), (A.3), (A.4)$ and $(B.4)$, yield that

$$\begin{aligned}
&\sup_{N \leq t \leq N+1} \left(|v(t)|^2 + \alpha\,\|u(t)\|^2 \right) \\
&\leq 2\,|v(N)|^2 + 2c_A\,\|u(N)\|^2 + 2\int_N^{N+1} (F(s,u_s,v_s), v(s))\mathrm{d}s \\
&+ 2\int_N^{N+1} |G(s,u_s,v_s)|^2 \,\mathrm{d}s + 4 \sup_{N \leq t \leq N+1} \left(\left| \int_N^t (G(s,u_s,v_s), v(s))\mathrm{d}W(s) \right| \right).
\end{aligned}$$

¿From $(F.4)$, (10) and (11), it follows

$$\begin{aligned}
&2\int_N^{N+1} E(F(s,u_s,v_s), v(s))\mathrm{d}s \\
&\leq 2\int_N^{N+1} \mathrm{e}^{m(s-N)} E(F(s,u_s,v_s), v(s))\mathrm{d}s \\
&\leq \mathrm{e}^{-mN}(\varepsilon + 2K_{F,H}^{1/2}) \int_0^{N+1} \mathrm{e}^{ms} E\,|v(s)|^2 \,\mathrm{d}s + 2\mathrm{e}^{-mN} K_{F,H}^{1/2}\, |\varphi_2|^2_{I_H^2} \\
&+ \mathrm{e}^{-mN} K_{F,V} \varepsilon^{-1} \int_0^{N+1} \mathrm{e}^{ms} E\,\|u(s)\|^2 \,\mathrm{d}s + \mathrm{e}^{-mN} K_{F,V} \varepsilon^{-1}\, \|\varphi_1\|^2_{I_V^2} \\
&\leq \mathrm{e}^{-mN}(\varepsilon + 2K_{F,H}^{1/2}) C_1 \left(E\,|v_0|^2 + E\,\|u_0\|^2 + |\varphi_2|^2_{I_H^2} + \|\varphi_1\|^2_{I_V^2} \right) \\
&+ \mathrm{e}^{-mN} K_{F,V} \varepsilon^{-1} C_2 \left(E\,|v_0|^2 + E\,\|u_0\|^2 + |\varphi_2|^2_{I_H^2} + \|\varphi_1\|^2_{I_V^2} \right) \\
&+ 2\mathrm{e}^{-mN} K_{F,H}^{1/2}\, |\varphi_2|^2_{I_H^2} + \mathrm{e}^{-mN} K_{F,V} \varepsilon^{-1}\, \|\varphi_1\|^2_{I_V^2}.
\end{aligned}$$

On the other hand, Burkholder-Davis-Gundy's inequality, $(G.4)$, (10), and (11) yield that

$$\begin{aligned}
&4E\left[\sup_{N\leq t\leq N+1}\left(\left|\int_N^t (G(s,u_s,v_s),v(s))\mathrm{d}W(s)\right|\right)\right]\\
&\leq 24E\left[\left(\sup_{N\leq t\leq N+1}|v(t)|^2\right)^{1/2}\left(\int_N^{N+1}|G(s,u_s,v_s)|^2\,\mathrm{d}s\right)^{1/2}\right]\\
&\leq \frac{1}{2}E\sup_{N\leq t\leq N+1}|v(t)|^2+288\int_N^{N+1}E\,|G(s,u_s,v_s)|^2\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\int_N^{N+1}E\,|G(s,u_s,v_s)|^2\,\mathrm{d}s\\
&\leq \int_N^{N+1}\mathrm{e}^{m(s-N)}E\,|G(s,u_s,v_s)|^2\,\mathrm{d}s\\
&\leq \mathrm{e}^{-mN}\left(K_{G,V}(1+C_2)+K_{G,H}(1+C_1)\right)\left(E\,|v_0|^2+E\,\|u_0\|^2+|\varphi_2|^2_{I_H^2}+\|\varphi_1\|^2_{I_V^2}\right).
\end{aligned}$$

Finally, (2) implies

$$E\left(|v(N)|^2+\|u(N)\|^2\right)\leq K_1\left(E\,|v_0|^2+E\,\|u_0\|^2+|\varphi_2|^2_{I_H^2}+\|\varphi_1\|^2_{I_V^2}\right)\mathrm{e}^{-mN},$$

for $t>N$. Then (13) follows from the previous estimates.

Now, given $\varepsilon>0$, the Doob inequality implies

$$\begin{aligned}
&P\left[\sup_{N\leq t\leq N+1}\left(|v(t)|^2+\|u(t)\|^2\right)\geq e^{-(m+\varepsilon)N}\right]\\
&\leq e^{(m-\varepsilon)N}E\left[\sup_{N\leq t\leq N+1}\left(|v(t)|^2+\|u(t)\|^2\right)\right]\\
&\leq Ce^{-\varepsilon N}\left(E\,|v_0|^2+E\,\|u_0\|^2+|\varphi_2|^2_{I_H^2}+\|\varphi_1\|^2_{I_V^2}\right)
\end{aligned}$$

and therefore, the Borel-Cantelli lemma can now be applied to complete the proof. ■

Remark 7 *It is possible to extend our results to cover the case in which a general family of (nonlinear) operators $B(t,\cdot)$ appears. To be precise, assume that $B(t,\cdot):V\to V^*$ is now a family of operators defined a.e. $t\geq 0$ and satisfying:*

$(B.1)$ $\forall v\in V$, *the map* $t\in(0,+\infty)\to B(t,v)\in V^*$ *is Lebesgue measurable,*

$(B.2)$ *the map* $\theta\in\mathbb{R}\to\langle B(t,v+\theta w),z\rangle\in\mathbb{R}$ *is continuous* $\forall v,w,z\in V$, *a.e.* $t\geq 0$,

$(B.3)$ *there exists* $c_B>0$ *such that* $\|B(t,v)\|_*\leq c_B\,\|v\|$, $\forall v\in V$, *a.e.* $t\geq 0$, *(B.4) there exists* $\beta > 0$ *such that* $\langle B(t,v) - B(t,\widetilde{v}), v - \widetilde{v}\rangle \geq \beta \|v - \widetilde{v}\|^2$, $\forall v, \widetilde{v} \in V$, *a.e.* $t \geq 0$.

Then, given $\varphi_1 \in I^2(-h,0;V)$, $\varphi_2 \in I^2(-h,0;V)$, $u_0 \in L^2(\Omega, \mathcal{F}_0, P; V)$ *and* $v_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, *one can prove existence and uniqueness of a solution* $u \in I^2(-h,T;V) \cap L^2(\Omega; C(0,T;V))$, $v \in I^2(-h,T;V) \cap L^2(\Omega; C(0,T;H))$, *for all* $T > 0$, *to the corresponding problem* (P) *(see [7] for the details).*

Moreover, we can prove in the same way as in Theorem 3, that the null solution to corresponding problem (P) *is exponentially stable in mean square if we suppose that there exist some constants* $\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$\begin{cases} 2\alpha^{3/2} C_\delta > cK_\varepsilon, \\ C_\delta \left(2\beta - c^2(2K_{F,H}^{1/2} + \varepsilon + K_{G,H})\right) > (4\alpha^2\delta)^{-1} \left(4c^2\alpha\delta + (c^2 K_{F,H}^{1/2} + c_B)^2\right) K_\varepsilon. \end{cases} \tag{14}$$

If, in addition, we suppose again, as a particular situation, that $K_{F,V} = 0$ *and we fix* $\delta = \frac{1}{2}$, *then as in Remark 4, condition (14) holds if*

$$\begin{cases} K_{G,V} < c^{-1}\alpha^{3/2}, \\ K_{G,H} < 2\beta c^{-2} - 2(\alpha + (cK_{F,H}^{1/2} + c_B c^{-1})^2)\alpha^{-2} K_{G,V} - 2K_{F,H}^{1/2}. \end{cases} \tag{15}$$

5 Examples

In this section we shall illustrate our theory by applying it to several examples.

Example 8

Consider our introductory example with $a = 1$, that is, the lateral displacement of a stretched string subjected to a random loading with delays:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \vartheta \dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial u(t-\tau(t))}{\partial x} \dfrac{dW(t)}{dt}, \text{ in } (0,+\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \ t \in (0,+\infty), \\ u(0,x) = u_0, \quad \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \text{ in } (0,1), \\ u(t) = \varphi_1(t), \quad \dfrac{\partial u(t)}{\partial t} = \varphi_2(t), \ t \in (-h,0), \end{cases}$$

where $\vartheta > 0$, $\sigma \in \mathbb{R}$, $W(t)$ is a one-dimensional Wiener process, and $\tau \in C^1(\mathbb{R}^+)$ is such that $0 \leq \tau(t) \leq h$, $\forall t \geq 0$, being $\tau^* = \sup_{t \geq 0} \tau'(t) < 1$.

This example can be set within our formulation by taking $H = L^2(0,1)$, $V = H_0^1(0,1)$,

$$\begin{aligned} A(t)u(t) &= -\frac{\partial^2 u}{\partial x^2}(t), \\ B(t, v(t)) &= \vartheta v(t), \\ G(t, \xi, \eta) &= \sigma \frac{\partial \xi(-\tau(t))}{\partial x}. \end{aligned}$$

If we denote $\theta(t) = t - \tau(t)$, then there exists $k > 0$ such that $\theta^{-1}(t) \le t + k$, $\forall t \ge -\tau(0)$, and consequently, with the notation above, it is easy to check all the conditions and deduce that

$$\beta = c_B = \vartheta, \ \alpha = 1, \ c = \frac{1}{\pi}, \ K_{G,H}(m_0) = 0, \ K_{G,V}(m_0) = \frac{\sigma^2}{1-\tau^*} e^{m_0 k},$$

with $m_0 > 0$ arbitrarily chosen.

Thus, we can ensure that given $\varphi_1 \in I^2(-h, 0; V)$, $\varphi_2 \in I^2(-h, 0; H)$, $u_0 \in L^2(\Omega, \mathcal{F}_0, P; V)$ and $v_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, there exists a unique solution $u \in I^2(-h, T; V) \cap L^2(\Omega; C(0, T; V))$, $\frac{\partial u}{\partial t} \in I^2(-h, T; H) \cap L^2(\Omega; C(0, T; H))$ to the corresponding problem (P), and using (9) we immediately obtain the exponential stability in mean square and pathwise of the solution to our problem provided

$$\frac{\sigma^2}{1-\tau^*} < \min\left\{\pi, \frac{2\vartheta\pi^2}{2\pi^2 + \vartheta^2}\right\} = \frac{2\vartheta\pi^2}{2\pi^2 + \vartheta^2}.$$

Observe that in the particular case in which $\tau(t) = h$ for all $t \ge 0$, the previous condition reads $\sigma^2 < \dfrac{2\vartheta\pi^2}{2\pi^2 + \vartheta^2}$, a better estimate than the condition $\sigma^2 < \dfrac{4\vartheta\pi^2}{4\pi^2 + \vartheta(\vartheta + \sqrt{\vartheta^2 + 4\pi^2})}$, obtained by Curtain in [5] in the case without delays, i.e., when $h = 0$.

Example 9

Consider now the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - \gamma \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial u}{\partial t}\right) - \dfrac{\partial}{\partial x}\left(k(t, \dfrac{\partial^2 u}{\partial x \partial t})\right) \\ = \int_{-r_1}^0 f(s) \dfrac{\partial u}{\partial t}(t+s)\, ds + g\left(\dfrac{\partial u(t-r_2)}{\partial x}, \dfrac{\partial u(t-r_3)}{\partial t}\right) \dfrac{dW(t)}{dt}, \ \text{in } (0, +\infty) \times (0, \pi), \\ u(t, 0) = u(t, \pi) = 0, \ t \in (0, +\infty), \\ u(0, x) = u_0, \quad \dfrac{\partial u}{\partial t}(0, x) = v_0(x), \ \text{in } (0, \pi), \\ u(t) = \varphi_1(t), \quad \dfrac{\partial u(t)}{\partial t} = \varphi_2(t), \ t \in (-h, 0), \end{cases}$$

where $\gamma > 0$, $W(t)$ is a one-dimensional Wiener process, $k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous map such that there exists $c_k > 0$ such that

$$(k(t, x) - k(t, \widetilde{x}))(x - \widetilde{x}) \ge 0, \ |k(t, x)| \le c_k |x| \quad \forall\, x, \widetilde{x} \in \mathbb{R}, \forall\, t \ge 0,$$

$h > r_1, r_2, r_3 \geq 0$, are given numbers, $f : (-r_1, 0) \to \mathbb{R}$ is a measurable bounded function such that $|f(s)| \leq L_f$, and $g : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function such that $g(0,0) = 0$, with

$$|g(x,y) - g(\widetilde{x},\widetilde{y})|^2 \leq L_g^2(|x-\widetilde{x}|^2 + |y-\widetilde{y}|^2), \quad \forall\,(x,y),(\widetilde{x},\widetilde{y}) \in \mathbb{R}^2.$$

The above problem can be set within our formulation by taking $H = L^2(0,\pi)$, $V = H_0^1(0,\pi)$,

$$\begin{aligned}
A(t)u(t) &= -\frac{\partial^2 u}{\partial x^2}(t),\ \forall u \in V,\ \forall t \geq 0,\\
\langle B(t,v), w\rangle &= \gamma \int_0^\pi \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x}dx + \int_0^\pi \widetilde{k}(t, \frac{\mathrm{d}v}{\mathrm{d}x})\frac{\mathrm{d}w}{\mathrm{d}x}dx,\ \forall v, w \in V,\ \forall t \geq 0,\\
F(t,\xi,\eta) &= \int_{-r_1}^0 f(s)\eta(s)\,ds, \quad \forall (t,\xi,\eta) \in \mathbb{R}^+ \times C(-h,0;V) \times C(-h,0;H),\\
G(t,\xi,\eta) &= g\left(\frac{\partial \xi(-r_2)}{\partial x}, \eta(-r_3)\right), \quad \forall (t,\xi,\eta) \in \mathbb{R}^+ \times C(-h,0;V) \times C(-h,0;H).
\end{aligned}$$

In this situation, operators A, B, F and G satisfy the hypotheses ensuring existence and uniqueness of a solution to the corresponding problem (see Garrido-Atienza [6]). Consequently, for $\varphi_1 \in I^2(-h,0;V)$, $\varphi_2 \in I^2(-h,0;H)$, $u_0 \in L^2(\Omega, \mathcal{F}_0, P; V)$ and $v_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ given, there exists a unique solution $u \in I^2(-h,T;V) \cap L^2(\Omega; C(0,T;V))$, $\dfrac{\partial u}{\partial t} \in I^2(-h,T;H) \cap L^2(\Omega; C(0,T;H))$. The constants are now

$$c_B = c_k,\ \beta = \gamma,\ \alpha = c = 1,\ K_{F,H}(m_0) = r_1^2 L_f^2 e^{m_0 r_1},\ K_{F,V}(m_0) = 0,$$
$$K_{G,V}(m_0) = L_g^2 e^{m_0 r_2},\ K_{G,H}(m_0) = L_g^2 e^{m_0 r_3}.$$

Thanks to (15), the solution is exponentially stable in mean square and almost surely if we impose

$$r_1 L_f < \gamma,\ L_g^2 < \min\left\{1, \frac{2\gamma - 2L_f r_1}{1 + 2(1 + (c_k + L_f r_1)^2)}\right\}.$$

Example 10

Let us take $H = L^2(\mathcal{O})$ and $V = H^1(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Let us consider $A(t) = -\Delta$ for all $t \geq 0$; $B(t,v)$, where $v \in L^2(\mathcal{O})$, the function of $L^2(\mathcal{O})$ defined, a.e. $x \in \mathcal{O}$, by $B(t,v)(x) = k(t, v(x))$, where $k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous map such that there exist c_k, $\beta_k > 0$ such that

$$|k(t,a)| \leq c_k|a|, \quad (k(t,a) - k(t,\widetilde{a}))(a - \widetilde{a}) \geq \beta_k |a - \widetilde{a}|^2 \quad \forall\, a, \widetilde{a} \in \mathbb{R}, \forall\, t \geq 0.$$

Let us consider two measurable functions $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, such that $f(t,0) = g(t,0,0,0) = 0$, $\forall\, t \geq 0$, and we also suppose that there exist $L_f, L_g > 0$

such that

$$|f(t,b)-f(t,\widetilde{b})|\leq L_f|b-\widetilde{b}|,$$
$$|g(t,a,y,b)-g(t,\widetilde{a},\widetilde{y},\widetilde{b})|^2\leq L_g^2(|a-\widetilde{a}|^2+|y-\widetilde{y}|^2+|b-\widetilde{b}|^2),$$

$\forall\, t\geq 0,\ \forall\, a,\widetilde{a},b,\widetilde{b}\in\mathbb{R},\ \forall\, y,\widetilde{y}\in\mathbb{R}^n$. Consider also four functions $\tau_i\in C^1(\mathbb{R}^+)$, $1\leq i\leq 4$, such that $0\leq\tau_i(t)\leq h$, $\forall t\geq 0$, being $\tau_i^*=\sup_{0\leq t}\tau_i'(t)<1$. As in Example 8, if we denote $\theta_i(t)=t-\tau_i(t)$, then there exists $k_i>0$ such that $\theta_i^{-1}(t)\leq t+k_i$, $\forall t\geq\tau_i(0)$, $1\leq i\leq 4$.

For $t\in\mathbb{R}^+,\xi\in C(-h,0;V),\eta\in C(-h,0;H)$, denote by $F(t,\xi,\eta)$ and $G(t,\xi,\eta)$ the families of operators defined, a.e. $x\in\mathcal{O}$, by

$$F(t,\xi,\eta)(x)=f(t,\eta(-\tau_1(t))(x)),$$
$$G(t,\xi,\eta)(x)=g(t,\xi(-\tau_2(t))(x),\nabla\xi(-\tau_3(t))(x),\eta(-\tau_4(t))(x)).$$

Under these hypotheses, we can ensure that given $\varphi_1\in I^2(-h,0;H^1(\mathcal{O}))$, $\varphi_2\in I^2(-h,0;L^2(\mathcal{O}))$, $u_0\in L^2(\Omega,\mathcal{F}_0,P;H^1(\mathcal{O}))$ and $v_0\in L^2(\Omega,\mathcal{F}_0,P;L^2(\mathcal{O}))$, there exists a unique solution $u\in I^2(-h,T;H^1(\mathcal{O}))\cap L^2(\Omega;C(0,T;H^1(\mathcal{O}))$, $v\in I^2(-h,T;L^2(\mathcal{O}))\cap L^2(\Omega;C(0,T;L^2(\mathcal{O}))$, to the correspondent system (P), (see Garrido-Atienza [6]).

This solution can be seen as a solution to the Neumann problem

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-\Delta u+k\left(t,\dfrac{\partial u}{\partial t}\right)=f(t,u(t-\tau_1(t)))\\ +g\left(t,u(t-\tau_2(t)),\nabla u(t-\tau_3(t)),\dfrac{\partial u}{\partial t}(t-\tau_4(t))\right)\dfrac{dW(t)}{dt},\ \text{in }(0,+\infty)\times\mathcal{O},\\ \dfrac{\partial u}{\partial\nu}=0,\ \text{on }(0,+\infty)\times\partial\mathcal{O},\\ u(0,x)=u_0(x),\quad\dfrac{\partial u}{\partial t}(0,x)=v_0(x),\ \text{in }\mathcal{O},\\ u(t)=\varphi_1(t),\quad\dfrac{\partial u(t)}{\partial t}=\varphi_2(t),\ t\in(-h,0),\end{cases}$$

where we denote by ν the outward unit normal to $\partial\mathcal{O}$.

In this situation, it is not hard to check that

$$\beta=\beta_k,\ c_B=c_k,\ \alpha=c=1,$$
$$K_{F,H}(m_0)=\frac{L_f^2}{1-\tau_1^*}e^{m_0k_1},K_{F,V}(m_0)=0,$$
$$K_{G,H}(m_0)=\frac{L_g^2}{1-\tau_4^*}e^{m_0k_4},\ K_{G,V}(m_0)=L_g^2\max\left\{\frac{e^{m_0k_2}}{1-\tau_2^*},\frac{e^{m_0k_3}}{1-\tau_3^*}\right\}.$$

So, using Remark 4, the solution to our problem is exponentially stable in mean square and, therefore, almost sure exponentially stable if we suppose

$$\frac{L_g^2}{1-\tau_i^*} < 1, \quad i = 2, 3,$$
$$\frac{L_g^2}{1-\tau_4^*} < 2\beta_k - \left(2 + \left(c_k + \frac{L_f}{(1-\tau_1^*)^{1/2}}\right)^2\right) \frac{L_g^2}{1-\tau_i^*} - 2\frac{L_f}{(1-\tau_1^*)^{1/2}}, \quad i = 2, 3.$$

6 Conclusions and final remarks

Some results on the exponential stability of functional stochastic partial differential equations of second order in time have been proved, which, in the particular case without delay, also improves a stability criterium in [5].

However, another interesting question is, in our opinion, the analysis of the actual decay rate of solutions when we are in a situation in which the stability may not be exponential (which uses to appear when one deals with nonlinear or non-autonomous problems). Only a few works have been done concerning the non-exponential stability of parabolic stochastic systems. It is worth mentioning the paper by Liu [9] on the polynomial stability for semilinear stochastic evolution equations which also covers the delay situation; on the other hand, Caraballo et al. [2] prove some results on the pathwise stability with a general decay function satisfying suitable conditions in both cases. It is our intention to do an investigation in this direction in a future paper.

Another point is that, although we have only considered the case of a real Wiener process, the results can be extended to a Hilbert valued situation. However, we have preferred to consider this framework for the sake of clarity.

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