# Dimension of attractors of nonautonomous partial differential equations

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**Abstract**

The concept of nonautonomous (or cocycle) attractor has become a proper tool for the study of the asymptotic behaviour of general nonautonomous partial differential equations. This is a time-dependent family of compact sets, invariant for the associated process and attracting "from $-\infty$". In general, the concept is rather different from the classical one of global attractor for autonomous dynamical systems. We prove a general result on the finite fractal dimensionality of each compact set of this family. In this way, we generalize previous results of Chepyzhov and Vishik in [6]. Our results are also applied to differential equations with a nonlinear term having polynomial growth at most.

## Contents



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# 1 Introduction

In this paper, we develop a general theory on the finite dimension of attractors for nonautonomous partial differential equations and we apply it, in particular, to estimate the fractal dimension of the attractor for the following nonautonomous equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(t,u) = h(t), \\ u \mid_{\partial\Omega} = 0, \\ u(\tau) = u_{\tau}, \end{cases}$$

where the function $h(t)$ is allowed to have polynomial growth in time (see condition (9) below). For these kind of nonautonomous systems it is not possible in general to obtain a uniform global attractor in the sense of [5], since the trajectories can be unbounded when time rises to infinity. A different approach was developed in [8], [9], [25] (see also [2], [17], [16], [24]), where the existence of attractors for some stochastic and nonautonomous equations is studied. The main definitions and theorems from the abstract theory of attractors for such systems are given in Section 2.

It is worth pointing out that in such systems the global attractor is not a compact set, but a parameterized family $\mathcal{A}(t)$ of compact sets. We are interested in proving the finite dimensionality of each of the sets $\mathcal{A}(t)$. We note that the union of all the attractors, i.e. $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ can be infinite-dimensional.

In the case of stochastic equations of parabolic and hyperbolic types such results were obtained in [10], [12], [13]. There are some technical tools in the proofs of these papers that do not seem to be applicable to the nonautonomous case. As far as we know, the only result in the nonautonomous case, was proved in [4] under the assumption of being the function $h(t)$ uniformly bounded in the variable $t$. In such case, the union of the whole family of attractors $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is bounded, and the well known technique of Lyapunov exponents, developed in [7], can be adapted with slight modifications. However, when the function $h(t)$ is allowed to have polynomial growth, the supremum of the norm of the global attractor $\mathcal{A}(t)$ can have also polynomial growth, so that we cannot expect that the union of attractors is bounded.

In this paper we extend the general theory on the finite-dimensionality of compact invariant sets in Hilbert spaces (see [1], [15], [20], [23], [27]) to the case of a parameterized family of global attractors with polynomial growth at most. The invariance property for nonautonomous attractors is now stated for a time-dependent family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ and the attraction is defined for trajectories with initial time going to $-\infty$. Thus, the idea is to construct a sequence of coverings of $\mathcal{A}(t)$ by iterating $n$ times an initial covering of $\mathcal{A}(t - nT^*)$, as $n \to \infty$.

Further we apply this abstract theorem to the attractor of the equation given above. We note that we are able to obtain the estimation of dimension in the case where the function $f(t,u)$ is globally Lipschitz on the second variable $u$. In the autonomous case it is possible to change the global Lipschitz condition by a local one by proving that the global attractor is bounded in $L_\infty(\Omega)$ (see [14], [21], [28]). In our case, in order to use a similar idea we would need to obtain an estimation of the norm in $L_\infty(\Omega)$ of the union $\cup_{\tau \leq t} \mathcal{A}(\tau), \forall t$, which is not possible in general as we have already remarked.

# 2 Attractors of nonautonomous equations

In this section, we introduce the general framework in which the theory of attractors for nonautonomous systems is going to be studied (see Crauel et al. [9] and Schmalfuss [26]). In a first step, we define semiprocesses as two-time dependent operators related with the solutions of nonautonomous differential equations. In this way, we are able to treat these equations as dynamical systems. Secondly, we write the general definitions of invariance, absorption and attraction and we finish with a general theorem on the existence of global attractors for these equations.

Let $(H,d)$ be a complete metric space (with the metric $d$) and $\{S(t,s)\}_{t \geq s}$, $t, s \in \mathbb{R}$ be a family of mappings satisfying:

**i)** $S(t,t,\cdot) = Id$,

**ii)** $S(t,s,S(s,\tau,u)) = S(t,\tau,u)$, for all $\tau \leq s \leq t$, $u \in H$,

**iii)** $u \mapsto S(t,\tau,u)$ is continuous in $H$.

This map is called a process (this term was introduce by Dafermos [11]). In general, we have to consider $S(t,\tau,u)$ as the solution of a nonautonomous equation at time $t$ with initial condition $u$ at time $\tau$.

Let $\mathcal{D}$ be a non-empty set of parameterized families of non-empty bounded sets $\widehat{D} = \{D(t)\}_{t \in \mathbb{R}}$. In particular, $\widehat{D} = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}$, where $D(t) \equiv B$ for all $t$, and $B \subset H$ is a bounded set. In what follows, we will consider fixed this set $\mathcal{D}$, so that the concepts of absorption and attraction in our analysis are always referred to it.

For $A, B \subset H$ we define the Hausdorff semidistance as,

$$dist(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b).$$

**Definition 2.1** *Given $t_0 \in \mathbb{R}$, we say that $K(t) \subset H$ is attracting at time $t_0$ if for every $\widehat{D} = \{D(t)\} \in \mathcal{D}$ we have that*

$$\lim_{\tau \to -\infty} dist(S(t_0, \tau, D(\tau)), K(t_0)) = 0.$$

*A family $\widehat{K} = \{K(t)\}_{t \in \mathbb{R}}$ is attracting if $K(t_0)$ is attracting at time $t_0$, for all $t_0 \in \mathbb{R}$.*

The previous concept considers a fixed final time and moves the initial time to $-\infty$. Note that this does not mean that we are going backwards in time, but we consider the state of the system at time $t_0$ starting at $\tau \to -\infty$. This is called *pullback attraction* in the literature (cf. [18], [26]).

**Definition 2.2** *Given $t_0 \in \mathbb{R}$, we say that $B(t_0) \subset H$ is absorbing at time $t_0$ if for every $\widehat{D} = \{D(t)\} \in \mathcal{D}$ there exists $T = T(t, \widehat{D}) \in \mathbb{R}$ such that*

$$S(t_0, \tau, D(\tau)) \subset B(t_0), \text{ for all } \tau \leq T.$$

*A family $\widehat{B} = \{B(t)\}_{t \in \mathbb{R}}$ is absorbing if $B(t_0)$ is absorbing at time $t_0$, for all $t_0 \in \mathbb{R}$.*

Note that every absorbing set at time $t_0$ is attracting.

**Definition 2.3** *Let $\widehat{B} = \{B(t)\}_{t \in \mathbb{R}}$ be a family of subsets of $H$. This family is said to be invariant with respect to the process $S$ if*

$$S(t, \tau, B(\tau)) = B(t), \text{ for all } (\tau, t) \in \mathbb{R}^2, \tau \leq t.$$

Note that this property is a generalization of the classical property of invariance for semigroups. However, in this case we have to define the invariance with respect to a family of sets depending on a parameter.

We define the *omega-limit set* at time $t_0$ of $\widehat{D} \equiv \{D(t)\} \in \mathcal{D}$ as

$$\Lambda(\widehat{D}, t_0) = \bigcap_{s \leq t_0} \overline{\bigcup_{\tau \leq s} S(t_0, \tau, D(\tau))}.$$

From now on, we assume that there exists a family $\widehat{K} = \{K(t)\}_{t \in \mathbb{R}}$ of compact absorbing sets, that is, $K(t) \subset H$ is non-empty, compact and absorbing for each $t \in \mathbb{R}$. Note that, in this case, $\Lambda(\widehat{D}, t_0) \subset K(t_0)$, for all $\widehat{D} = \{D(t)\} \in \mathcal{D}$, $t_0 \in \mathbb{R}$. As in the autonomous case, it is not difficult to prove that under these conditions $\Lambda(\widehat{D}, t_0)$ is non-empty, compact and attracts $\widehat{D} = \{D(t)\} \in \mathcal{D}$ at time $t_0$. The proof is similar to that of [9, Lemma 1.1], where the set $\mathcal{D}$ consists only of bounded sets.

**Definition 2.4** *The family of compact sets* $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be the global attractor associated to the process* $S$ *if it is invariant, attracting every* $\widehat{D} = \{D(t)\} \in \mathcal{D}$ *(for all* $t_0 \in \mathbb{R}$*) and minimal in the sense that if* $\widehat{C} = \{C(t)\}_{t\in\mathbb{R}}$ *is another family of closed attracting sets, then* $\mathcal{A}(t) \subset C(t)$ *for all* $t \in \mathbb{R}$.

**Remark 2.5** *Chepyzhov and Vishik [4] define the concept of kernel sections for nonautonomous dynamical systems which corresponds to our definition of global nonautonomous attractor with* $\widehat{D} = \{D(t) \equiv B\}_{t\in\mathbb{R}}$ *where* $B \subset H$ *is bounded.*

The general result on the existence of nonautonomous attractors is a generalization of the abstract theory for autonomous dynamical systems (Temam [27], Hale [19]):

**Theorem 2.6** *Assume that there exists a family of compact absorbing sets. Then, the family* $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *defined by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}$$

*is the global nonautonomous attractor.*

As the proof of this theorem repeats the same one of [9, Theorem 1.1] with slight modifications, we will omit it.

**Remark 2.7** *All the general theory of nonautonomous attractors can be written in the framework of cocycles (cf., among others, Cheban et al. [2], Crauel and Flandoli [8], Kloeden and Schmalfuss [18], Schmalfuss [26]). We could have also followed this notation here, but we think that, in this case, it is more clear to keep the explicit dependence on time of the attractor, which in addition, allows us to compare more straightforward the results in [4].*

# 3 Dimension of nonautonomous attractors

In [4], Chepyzhov and Vishik prove a general result for the Hausdorff dimension of kernel sections $\mathcal{A}(t)$ associated to a process $\{S(t,\tau)\}$ generated by a nonautonomous differential equation. The main hypothesis is the uniform boundedness of the set $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$. In applications, this is related to the existence of a uniform bound for the nonautonomous terms in the system. In our case, we allow these terms to be unbounded in $t$, so that their results are not suitable for our situation. However, we are able to prove a general result on the finite fractal dimensionality of the nonautonomous attractor. Due to the weaker properties on the nonautonomous terms, it is not expected that a uniform bound for all $t$ is obtained.

Let $H$ be a Hilbert space and $\mathcal{A} \subset H$ be a compact subset of $H$. We firstly recall the definition of the Hausdorff and fractal dimensions of $\mathcal{A}$.

We shall denote by $B(a, r)$ a closed ball of radius $r$ centered at $a$. Let $\mathcal{U}$ be a covering of $\mathcal{A}$ by a finite family of balls $B(x_i, r_i)$ such that $\sup_i(r_i) = \delta(\mathcal{U}) \leq \delta$. Then the $d$-dimensional Hausdorff measure of $\mathcal{A}$ is defined as follows:

$$\mu_H(\mathcal{A}, d) = \lim_{\delta \to 0} \mu_H(\mathcal{A}, d, \delta),$$

where

$$\mu_H(\mathcal{A}, d, \delta) = \inf_{\delta(\mathcal{U}) \leq \delta} \sum_i r_i^d,$$

where the inf is extended to all the possible covering $\mathcal{U}$ of $\mathcal{A}$ such that $\delta(\mathcal{U}) \leq \delta$. It is known that there exists $d = d_H(\mathcal{A}) \in [0, +\infty]$ such that $\mu_H(\mathcal{A}, d) = 0$ for $d > d_H(\mathcal{A})$ and $\mu_H(\mathcal{A}, d) = \infty$ for $d < d_H(\mathcal{A})$. The value $d_H(\mathcal{A})$ is called the Hausdorff dimension of $\mathcal{A}$.

The fractal dimension of $\mathcal{A}$ is given by

$$d_f(\mathcal{A}) = \inf\{d > 0 \mid \mu_f(\mathcal{A}, d) = 0\},$$

where

$$\mu_f(\mathcal{A}, d) = \overline{\lim_{\epsilon \to 0}} \mu_f(\mathcal{A}, \varepsilon, d) = \overline{\lim_{\epsilon \to 0}} \epsilon^d n_\epsilon,$$

and $n_\epsilon$ is the minimum number of balls of radius $r = \epsilon$ which is necessary to cover $\mathcal{A}$. Since $\mu_H(\mathcal{A}, d) \leq \mu_f(\mathcal{A}, d)$ it is clear that $d_H(\mathcal{A}) \leq d_f(\mathcal{A})$, the converse being false in general (Eden et al. [15]).

Before proving our main result in this section, we will recall a technical lemma which will be repeatedly used in the proof (see Lemma 1 in [1]).

**Lemma 3.1** *Let $B(a, \gamma) \subset \mathbb{R}^N$ be a closed ball centered at $a$ of radius $\gamma$. For any $0 < \lambda < \gamma$ the minimum number of balls $n_\lambda$ of radius $\lambda$ which is necessary to cover $B(a, \gamma)$ is less or equal to $\left(3\frac{\gamma}{\lambda}\right)^N$.*

We consider now a process $S(t, \tau, u) : \mathbb{R} \times \mathbb{R} \times H \to H$, $t \geq \tau$, having the family of global attractors $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$.

**Theorem 3.2** *Suppose there exist constants $K_0, K_1, \theta > 0$ such that*

$$\|\mathcal{A}(t)\|^+ \leq K_0 |t|^\theta + K_1, \forall t \in \mathbb{R}, \tag{1}$$

*where* $\|\mathcal{A}(t)\|^{+} = \sup_{y\in\mathcal{A}(t)} \|y\|$.

*Also assume that for any* $t \in \mathbb{R}$ *there exist* $T^* = T^*(t)$, $l = l(t, T^*) \in [1, +\infty)$, $\delta = \delta(t, T^*) \in (0, \frac{1}{\sqrt{2}})$ *and* $N = N(t)$, *such that for any* $u, v \in A(\tau), \tau \leq t - T^*$,

$$\|S(\tau + T^*, \tau, u) - S(\tau + T^*, \tau, v)\| \leq l \|u - v\|, \tag{2}$$

$$\|Q_N (S(\tau + T^*, \tau, u) - S(\tau + T^*, \tau, v))\| \leq \delta \|u - v\|, \tag{3}$$

*where* $Q_N$ *is the projector mapping* $H$ *onto some subspace* $H_N^{\perp}$ *of codimension* $N \in \mathbb{N}$. *Then, for any* $\eta = \eta(t) > 0$ *such that* $\sigma = \sigma(t) = \left(6\sqrt{2}l\right)^N \left(\sqrt{2}\delta\right)^{\eta} < 1$, *the next inequality holds*

$$d_H(\mathcal{A}(t)) \leq d_f(\mathcal{A}(t)) \leq N + \eta. \tag{4}$$

**Proof.** Let us fix $t \in \mathbb{R}$ and choose $\eta > 0$ such that $\sigma < 1$. We also take an arbitrary $\tau \leq t - T^*$, and denote $\varepsilon(\tau) = 2\left(K_0 |\tau|^{\theta} + K_1\right)$. Let $\mathcal{U}_0$ be a covering of $A(\tau)$ by one ball $B(a_1, \varepsilon(\tau))$, $a_1 \in A(\tau)$, of radius $\varepsilon(\tau)$ centered at $a_1$. Hence, $\mathcal{A}(\tau) \subset B(a_1, \varepsilon(\tau))$.

Since $\mathcal{A}(\tau + T^*) = S(\tau + T^*, \tau, \mathcal{A}(\tau))$ and using condition (2) we have

$$\mathcal{A}(\tau + T^*) \subset B(S(\tau + T^*, \tau, a_1), l\varepsilon(\tau)).$$

Let us denote by $\mathcal{P}_N$ the orthoprojector onto the subspace $H_N$ of dimension $N$ which is orthogonal to $H_N^{\perp}$ (and then $\mathcal{P}_N \oplus Q_N = I$, $H_N \oplus H_N^{\perp} = H$). It is clear that $\mathcal{P}_N B(S(\tau + T^*, \tau, a_1), l\varepsilon(\tau)) \subset B^N(\mathcal{P}_N S(\tau + T^*, \tau, a_1), l\varepsilon(\tau))$, where $B^N(a, \beta)$ denotes a closed ball in $H^N$ of radius $\beta$ and centered at $a$.

In view of the preceding lemma we can cover $B^N(\mathcal{P}_N S(\tau+T^*, \tau, a_1), l\varepsilon(\tau))$ by balls $B^N(a_{1j}, \frac{\delta}{2}\varepsilon(\tau))$, $j = 1, ..., m_1$, $a_{1j} \in H_N$ and

$$m_1 = m_1(t) \leq \left(6\frac{l}{\delta}\right)^N.$$

Let us denote $\mathcal{M}_{1j} = (\mathcal{P}_N^{-1} B^N(a_{1j}, \frac{\delta}{2}\epsilon(\tau))) \cap \mathcal{A}(\tau + T^*)$. We take arbitrary $y_{1j} \in \mathcal{M}_{1j}$. We shall show that the set of balls $B(y_{1j}, \gamma\epsilon(\tau))$, $j = 1, ..., m_1$, $\gamma = \sqrt{2}\delta$ (note that we have assumed that $\gamma < 1$), is a new covering of $\mathcal{A}(\tau + T^*)$. Since

$$\mathcal{A}(\tau + T^*) \subset \bigcup_{j=1}^{m_1} \mathcal{M}_{1j},$$

it is sufficient to prove that $\mathcal{M}_{1j} \subset B(y_{1j}, \gamma\epsilon(\tau))$, $\forall j$. Let $y \in \mathcal{M}_{1j}$. There exist $v_1$, $v_2 \in B(a_1, \varepsilon(\tau)) \cap \mathcal{A}(\tau)$ such that $S(\tau+T^*, \tau, v_1) = y$, $S(\tau+T^*, \tau, v_2) =$

$y_{1j}$. Then $\|v_1 - v_2\| \leq \epsilon(\tau)$ and in view of (3), $\|Q_N\, y - Q_N\, y_{1j}\| \leq \delta\epsilon(\tau)$. On the other hand, $\|\mathcal{P}_N\, y - \mathcal{P}_N\, y_{1j}\| \leq \|\mathcal{P}_N\, y - a_{1j}\| + \|\mathcal{P}_N\, y_{1j} - a_{1j}\| \leq \delta\epsilon(\tau)$. Hence, $\|y - y_{1j}\| \leq \sqrt{(\delta\epsilon(\tau))^2 + (\delta\epsilon(\tau))^2} = \gamma\epsilon(\tau)$.

We have obtained a covering $\mathcal{U}_1$ of $\mathcal{A}(\tau + T^*)$ by balls of radius $\gamma\epsilon(\tau)$ such that the number of balls is $m_1$. Therefore,

$$n_{\gamma\epsilon(\tau)} \leq m_1 \leq \left(6\frac{l}{\delta}\right)^N,$$

where $n_{\gamma\varepsilon(\tau)}$ denotes now the minimum number of balls of radius equal to $\gamma\varepsilon(\tau)$ which is necessary to cover $\mathcal{A}(\tau + T^*)$. Then,

$$\mu_f(\mathcal{A}(\tau + T^*), \gamma\epsilon(\tau), d) = n_{\gamma\epsilon(\tau)}(\gamma\epsilon(\tau))^d \leq \left(6\frac{l}{\delta}\right)^N \left(\sqrt{2}\delta\epsilon(\tau)\right)^d.$$

Taking $d = d(t) = N + \eta$, we get

$$\mu_f(\mathcal{A}(\tau + T^*), \gamma\epsilon(\tau), N + \eta) \leq \left(\sqrt{2}\delta\right)^\eta \left(6\sqrt{2}l\right)^N \epsilon(\tau)^{N+\eta} = \sigma\epsilon(\tau)^{N+\eta}.$$

Suppose now that $\tau \leq t - 2T^*$. Take the covering $\mathcal{U}_1 = \{B(y_{1i}, \gamma\varepsilon(\tau))\}_{i=1}^{m_1}$ of $\mathcal{A}(\tau + T^*)$ and define $\mathcal{M}_i = S(\tau + 2T^*, \tau + T^*, A(\tau + T^*) \cap B(y_{1i}, \gamma\varepsilon(\tau))) \cap A(\tau + 2T^*)$, $i = 1, ..., m_1$.

Now, since $\mathcal{A}(\tau + 2T^*) = S(\tau + 2T^*, \tau + T^*, \mathcal{A}(\tau + T^*)))$ and using condition (2), we have

$$\mathcal{A}(\tau + 2T^*) \subset \bigcup_{i=1}^{m_1} \mathcal{M}_i \subset \bigcup_{i=1}^{m_1} B(S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau)).$$

It is clear that $\mathcal{P}_N\, B(S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau)) \subset B^N(\mathcal{P}_N S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau)), \forall i$. In view of the preceding technical lemma, we can cover each $B^N(\mathcal{P}_N S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau))$ by balls $B^N(a_{ij}, \frac{\delta}{2}\gamma\varepsilon(\tau))$, $j = 1, ..., n_i$, $a_{ij} \in H_N$ and

$$n_i = n_i(t) \leq \left(6\frac{l}{\delta}\right)^N, \forall i.$$

Let us denote $\mathcal{M}_{ij} = (\mathcal{P}_N^{-1} B^N(a_{ij}, \frac{\delta}{2}\gamma\epsilon(\tau))) \cap \mathcal{M}_i$. We take arbitrary $y_{ij} \in \mathcal{M}_{ij}$. We shall show that the set of balls $B(y_{ij}, \gamma^2\epsilon)$, $i = 1, ..., m_1$, $j = 1, ..., n_i$, $\gamma = \sqrt{2}\delta$, is a new covering of $\mathcal{A}(\tau + 2T^*)$. Indeed, since

$$\mathcal{A}(\tau + 2T^*) \subset \bigcup_{ij} \mathcal{M}_{ij},$$

it is sufficient to prove that $\mathcal{M}_{ij} \subset B(y_{ij}, \gamma^2\epsilon)$, $\forall i, j$. Let $y \in \mathcal{M}_{ij}$. There exist $v_1$, $v_2 \in B(y_{1i}, \gamma\varepsilon(\tau)) \cap \mathcal{A}(\tau + T^*)$ such that $S(\tau + 2T^*, \tau + T^*, v_1) = y$, $S(\tau + 2T^*, \tau + T^*, v_2) = y_{ij}$. Then $\|v_1 - v_2\| \leq \gamma\epsilon(\tau)$ and in view of (3), $\|Q_N y - Q_N y_{ij}\| \leq \delta\gamma\epsilon(\tau)$. On the other hand, $\|\mathcal{P}_N y - \mathcal{P}_N y_{ij}\| \leq \|\mathcal{P}_N y - a_{ij}\| + \|\mathcal{P}_N y_{ij} - a_{ij}\| \leq \delta\gamma\epsilon(\tau)$. Hence,

$$\|y - y_{ij}\| \leq \sqrt{(\delta\gamma\epsilon)^2 + (\delta\gamma\epsilon)^2} = \gamma^2\epsilon(\tau).$$

We have obtained a covering $\mathcal{U}_2$ of $\mathcal{A}(\tau + 2T^*)$ by balls of radius $\gamma^2\epsilon$ such that the number of balls is $m_2 = m_2(t) = \sum_{i=1}^{m_1} n_i$. Therefore,

$$n_{\gamma^2\epsilon} \leq \sum_{i=1}^{m_1} n_i \leq m_1 \left(6\frac{l}{\delta}\right)^N \leq \left(6\frac{l}{\delta}\right)^{2N}.$$

Let $k \in \mathbb{N}$. If we suppose that $\tau \leq t - kT^*$, we can obtain, in the same way as before, a sequence of coverings $\mathcal{U}_j$, $j = 1, 2, ..., k$ of the sets $\mathcal{A}(\tau + jT^*)$ by balls of radius $\gamma^j\epsilon$ and such that the number of balls is less than or equal to $\left(6\frac{l}{\delta}\right)^{jN}$. Therefore,

$$n_{\gamma^j\epsilon} \leq \left(6\frac{l}{\delta}\right)^{jN},$$

where $n_{\gamma^j\varepsilon(\tau)}$ denotes now the minimum number of balls of radius equal to $\gamma^j\varepsilon(\tau)$ which is necessary to cover $\mathcal{A}(\tau + jT^*)$.

Hence, choosing $\tau = t - kT^*$ we obtain

$$\begin{aligned}\mu_f(\mathcal{A}(t), \gamma^k\epsilon(\tau), N + \eta) &\leq \left(\sqrt{2}\delta\right)^{k\eta} \left(6\sqrt{2}l\right)^{kN} \varepsilon(\tau)^{N+\eta} \\ &\leq \sigma^k \left(K_1 + K_0 \left|(t - kT^*)\right|^\theta\right)^{N+\eta}.\end{aligned}$$

This implies that $\overline{\lim}_{\alpha\to 0} \mu_f(\mathcal{A}(t), \alpha, d) = 0$, for $d = N + \eta$. Indeed, as for $k$ large enough the sequence

$$r(k) = \gamma^k\varepsilon(t - kT^*) = \gamma^k \left(K_1 + K_0 \left|t - kT^*\right|^\theta\right)$$

is decreasing, we have that for any $\alpha > 0$ small enough, one can find some $k \in \mathbb{N}$ such that $r(k) \leq \alpha < r(k-1)$. It is clear that $n_\alpha \leq n_{r(k)} \leq \left(6\frac{l}{\delta}\right)^{kN}$. Then

$$\begin{aligned}\overline{\lim_{\alpha\to 0}} \mu_f(\mathcal{A}(t), \alpha, N + \eta) &= \overline{\lim_{\alpha\to 0}} n_\alpha \alpha^{N+\eta} \leq \lim_{k\to\infty} \left(6\frac{l}{\delta}\right)^{kN} (r(k-1))^{N+\eta} \\ &= \lim_{k\to\infty} \sigma^k \left(\frac{K_1 + K_0 \left|t - (k-1)T^*\right|^\theta}{\gamma}\right)^{N+\eta} = 0.\end{aligned}$$

Hence, $d_H(\mathcal{A}(t)) \leq d_f(\mathcal{A}(t)) \leq N + \eta$. ■

**Corollary 3.3** *Let conditions (1)-(3) hold. Then*

$$d_H(\mathcal{A}(t)) \leq d_f(\mathcal{A}(t)) \leq N(t)\left(1 - \frac{\log 6\sqrt{2}l(t,T^*)}{\log \sqrt{2}\delta(t,T^*)}\right). \tag{5}$$

**Remark 3.4** *Note that if the constants $T^*, N$, $l$ and $\delta$ do not depend on $t$, then the estimate is uniform for all $\mathcal{A}(t)$.*

# 4 Applications to a nonautonomous partial differential equation

Consider now the nonautonomous partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(t,u) = h(t), \\ u\mid_{\partial\Omega} = 0, \\ u(\tau) = u_\tau, \end{cases} \tag{6}$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $h(\cdot) \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and there exist $r \geq 0$, $p \geq 2$, $c_i > 0$, $i = 1,..,7$, such that

$$c_1|u|^p - c_2 \leq f(t,u)u \leq c_3|u|^p + c_4, \tag{7}$$

$$f_u(t,u) \geq -c_5, \tag{8}$$

$$\|h(t)\|_{L^2} \leq c_6|t|^r + c_7, \tag{9}$$

for all $u, t \in \mathbb{R}$.

Denote $H = L^2(\Omega)$ with norm $||\cdot||$, $V = H^1_0(\Omega)$. For a norm in another space $X$ we shall use the notation $||\cdot||_X$.

**Theorem 4.1** *For any $\tau, T \in \mathbb{R}$, $T > \tau$, $u_\tau \in L^2(\Omega)$ there exists a unique solution $u(\cdot) \in C([\tau,T], H) \cap L^2(\tau,T;V) \cap L^p(\tau,T;L^p(\Omega))$. Moreover, for all $u_\tau, v_\tau \in L^2(\Omega)$, $t \in [\tau,T]$ it holds*

$$\|u(t) - v(t)\| \leq \exp(c_5(t-\tau))\|u_\tau - v_\tau\|. \tag{10}$$

**Proof.** The existence of a solution for any $u_\tau \in L^2(\Omega)$ was proved in [6, Theorem 2.1]. The uniqueness property and (10) can be obtained exactly in the same way as in [21, Theorem 1.1] or [6, Theorem 3.1]. ■

Denote $S(t,\tau,u_\tau)=u(t)$, where $u(\tau)=u_\tau$, which is a process. We denote by $SE(H)$ the class of families of bounded sets $\widehat{B}=\{B(\tau)\}_{\tau\in\mathbb{R}}$ $(B(\tau)\subset H)$ such that

$$\lim_{\tau\to\infty}\frac{\max\left\{\log\|B(\tau)\|^{+},0\right\}}{\tau}=0, \tag{11}$$

that is, the class of sets with subexponential growth on the time variable. In this case $\mathcal{D}=SE(H)$ (see the notation in Section 2).

**Lemma 4.2** *For any $t\in\mathbb{R}$, there exists a bounded set $B_0(t)$ in $H$ such that for any family $\widehat{B}\in SE(H)$ and any $t_0<t$, there exists $T=T\left(\widehat{B},t_0\right)<t_0$ such that*

$$S(t_1,\tau,B(\tau))\subset B_0(t),\forall\tau\leq T,\forall t_1\in[t_0,t], \tag{12}$$

**Proof.** Multiplying (6) by $u(s)=S(s,\tau,u_\tau)$, $u_\tau\in B(\tau)$, and using (7), (9) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|u\|^2+\|\nabla u\|^2+c_1\|u\|_{L^p}^p\\
&\leq c_2\mu(\Omega)+\|u\|\,\|h(s)\|\\
&\leq c_2\mu(\Omega)+\frac{\lambda_1}{2}\|u\|^2+\frac{1}{2\lambda_1}\|h(s)\|^2\\
&\leq c_2\mu(\Omega)+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2\lambda_1}\left(c_6|s|^r+c_7\right)^2,
\end{aligned} \tag{13}$$

where $\mu(\Omega)$ is the Lebesgue measure of $\Omega$ in $\mathbb{R}^n$. Therefore,

$$\begin{aligned}
\frac{d}{ds}\|u\|^2+\lambda_1\|u\|^2+2c_1\|u\|_{L^p}^p&\leq\frac{d}{ds}\|u\|^2+\|\nabla u\|^2+2c_1\|u\|_{L^p}^p\\
&\leq 2c_2\mu(\Omega)+\frac{1}{\lambda_1}\left(c_6|s|^r+c_7\right)^2.
\end{aligned}$$

By the Gronwall lemma

$$\begin{aligned}
\|u(t_1)\|^2&\leq\exp\left(-\lambda_1(t_1-\tau)\right)||u_\tau||^2\\
&\quad+\int_\tau^{t_1}\exp\left(-\lambda_1(t_1-s)\right)\left(2c_2\mu(\Omega)+\frac{1}{\lambda_1}\left(c_6|s|^r+c_7\right)^2\right)ds,
\end{aligned}$$

so that the ball

$$B_0(t)=\left\{y\in H:\|y\|\leq\sqrt{K(t)+\alpha}\right\},$$

with $\alpha > 0$, $K(t) = \int_{-\infty}^{t} \exp(-\lambda_1 (t-s)) \left( 2c_2 \mu(\Omega) + \frac{1}{\lambda_1} (c_6 |s|^r + c_7)^2 \right) ds$, satisfies (12). Indeed, in view of condition (11), we can find $T\left(\widehat{B}, t_0\right)$ such that $\exp(-\lambda_1 (t_1 - \tau)) \|u_\tau\|_H^2 \leq \alpha$, $\forall \tau \leq T\left(\widehat{B}, t_0\right), \forall u_\tau \in B(\tau), \forall t_0 \leq t_1 \leq t$. ■

**Corollary 4.3** *For any $u_\tau \in B(\tau)$, $\tau < T\left(\widehat{B}, t_0\right)$,*

$$\int_{t_0}^{t} \left( \|\nabla u\|^2 + 2c_1 \|u\|_{L^p}^p \right) ds \leq R(t_0, t),$$

*where*

$$R(t_0, t) = 2c_2 \mu(\Omega)(t - t_0) + \frac{1}{\lambda_1} \int_{t_0}^{t} (c_6 |s|^r + c_7)^2 ds + K(t) + \alpha.$$

**Proof.** It is a consequence of Lemma 4.2 and (13). ■

**Lemma 4.4** *For any $t \in \mathbb{R}$, $t_0 < t$ there exists a set $B_1(t_0, t)$ bounded in $V$ and compact in $H$ such that for any $\widehat{B} \in SE(H)$ there exists $T = T\left(\widehat{B}, t_0\right) < t_0$ such that $\forall \tau \leq T$,*

$$S(t, \tau, B(\tau)) \subset B_1(t_0, t). \tag{14}$$

**Proof.** Multiplying (6) by $-\Delta u$, where $u(r) = S(r, \tau, u_\tau)$, $u_\tau \in B(\tau)$, and integrating by parts we have

$$\frac{1}{2} \frac{d}{dr} \|\nabla u\|^2 + \|\Delta u\|^2 + (f_u(r, u) \nabla u, \nabla u) = (h, \Delta u) \leq \frac{1}{2} \|h(r)\|^2 + \frac{1}{2} \|\Delta u\|^2.$$

Using (8) we obtain

$$\frac{d}{dr} \|\nabla u\|^2 \leq \frac{d}{dt} \|\nabla u\|^2 + \|\Delta u\|^2 \leq \|h(r)\|^2 + 2c_5 \|\nabla u\|^2. \tag{15}$$

Denote $a_1 = a_1(t) = \int_{t_0}^{t} \|h(r)\|^2 dr$, $a_2 = a_2(t) = \exp(2c_5 (t - t_0))$. Assume that $t_0 \leq s \leq r \leq t$ and multiply (15) by $\exp(-2c_5 (r - t_0))$. Then

$$\frac{d}{dr} \left( \exp(-2c_5 (r - t_0)) \|\nabla u\|^2 \right) \leq \|h(r)\|^2 \exp(-2c_5 (r - t_0)) \leq \|h(r)\|^2.$$

Integrating over $(s,t)$ we obtain

$$\|\nabla u(t)\|^2 \leq \exp(2c_5(t-s))\|\nabla u(s)\|^2 + \exp(2c_5(t-t_0))\int_s^t \|h(r)\|^2 dr$$
$$\leq \left(\|\nabla u(s)\|^2 + a_1\right) a_2.$$

Finally, integrating with respect to $s$ over $(t_0,t)$ and using Corollary 4.3 we have

$$(t-t_0)\|\nabla u(t)\|^2 \leq (R(t_0,t) + a_1(t-t_0))\, a_2, \forall \tau < T\left(\widehat{B}, t_0\right),$$

so that, taking into account the compact embedding $V \subset H$, the closure in $H$ of the set

$$B_1(t_0,t) = \left\{u \in H_0^1(\Omega) : ||\nabla u||^2 \leq \left(\frac{R(t_0,t)}{t-t_0} + a_1\right) a_2\right\}$$

is the desired set. ■

**Theorem 4.5** *The process $S$ has the global attractor $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$. Moreover, there exist $K_0, K_1, \theta > 0$ such that*

$$\|\mathcal{A}(t)\|^+ \leq K_0 |t|^\theta + K_1, \forall t \in \mathbb{R}, \tag{16}$$

*so that $\widehat{\mathcal{A}} \in SE(H)$.*

**Proof.** The existence of the global attractor is a consequence of Lemma 4.4 and Theorem 2.6.

Further, we note that, choosing $t_0$ in Lemma 4.4 such that $t - t_0 = l > 0$, we have

$$\|B_1(t_0,t)\|_H^+ \leq \beta \sup_{y\in B_1(t_0,t)} \|\nabla y\|$$
$$\leq \beta\left(\left(\frac{R(t_0,t)}{l} + a_1\right) a_2\right)^{\frac{1}{2}}$$
$$\leq \beta\left(\left(\nu + \int_{t_0}^t (c_6|s|^r + c_7)^2 ds\right) a_2\right)^{\frac{1}{2}},$$

where

$$\nu = \frac{2c_2\mu(\Omega)\, l + \frac{1}{\lambda_1}\int_{t_0}^t (c_6|s|^r + c_7)^2 ds + K(t) + \alpha}{l}.$$

It follows from the definition of $K(t)$ the existence of $R_1, R_2, \zeta > 0$ such that

$$\|K(t)\|^+ \leq R_1 |t|^\zeta + R_2, \forall t.$$

Hence, since $\mathcal{A}(t) \subset B_1(t_0,t), \forall t$, estimation (16) follows. ■

**Theorem 4.6** *Suppose there exists a positive and nondecreasing function $\xi(t)$ defined for all $t \in \mathbb{R}$ and such that for all $\tau \leq t, u, v \in \mathbb{R}$,*

$$|f(\tau, u) - f(\tau, v)| \leq \xi(t)\,|u - v|\,. \tag{17}$$

*Then, there exist $L_1, L_2 > 0$ depending on $\Omega$ and $n$ such that*

$$d_H(\mathcal{A}(t)) \leq d_f(\mathcal{A}(t)) \leq \max\left\{L_1(\xi(t))^{\frac{n}{2}}, L_2(c_5)^{\frac{n}{2}}\right\}. \tag{18}$$

**Proof.** The proof is similar to that of [1, Theorem 7]. Let us first prove condition (2). We take a fixed $t \in \mathbb{R}$, and arbitrary solutions $u(\cdot), v(\cdot)$. It is easy to obtain in a standard way that

$$\frac{1}{2}\frac{d}{ds}\|u(s) - v(s)\|^2 + \|\nabla(u(s) - v(s))\|^2$$
$$+ (f(s, u(s)) - f(s, v(s)), u(s) - v(s)) = 0. \quad \text{for all } s \in \mathbb{R}.$$

In view of (8)

$$\frac{d}{ds}\|u(s) - v(s)\|^2 \leq 2c_5\|u(s) - v(s)\|^2.$$

Let us now take $T^* > 0$ to be determined later on and depending on $t$. Gronwall's Lemma implies that for any $\tau \in \mathbb{R}$,

$$\|u(\tau + T^*) - v(\tau + T^*)\|^2 \leq \exp(2c_5T^*)\|u(\tau) - v(\tau)\|^2, \tag{19}$$

so that (2) holds with $l(t, T^*) = \exp(c_5T^*)$.

Denote $m(s) = u(s) - v(s)$. Multiplying

$$\frac{dm(s)}{ds} - \Delta m(s) + f(s, u(s)) - f(s, v(s)) = 0$$

by $Q_N\,m(s)$, we get

$$\frac{1}{2}\frac{d}{ds}\|Q_N\,m(s)\|^2 + \|\nabla Q_N\,m(s)\|^2 + (f(s, u(s)) - f(s, v(s)), Q_N\,m(s)) = 0.$$

Let $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N \to \infty$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. Since $\|\nabla Q_N m(s)\| \geq \lambda_{N+1}\|Q_N m(s)\|^2$, and using condition (17) and (19), we obtain for $s \leq \tau + T^* \leq t$,

$$\frac{d}{ds}\|Q_N m(s)\|^2 \leq -2\lambda_{N+1}\|Q_N m(s)\|^2 + 2\xi(t)\|m(s)\|^2$$
$$\leq -2\lambda_{N+1}\|Q_N m(s)\|^2 + 2\xi(t)\exp(2c_5(s - \tau))\|u(\tau) - v(\tau)\|^2.$$

Multiplying both sides by $\exp(2\lambda_{N+1}(s-\tau))$, we have

$$\frac{d}{ds}\left(\|Q_N m(s)\|^2 \exp(2\lambda_{N+1}(s-\tau))\right) \leq 2\xi(t)\exp(2(c_5+\lambda_{N+1})(s-\tau))\|u(\tau)-v(\tau)\|^2 .$$

Integrating over $(\tau, \tau+T^*)$ we get

$$\begin{aligned}&\|Q_N m(\tau+T^*)\|^2 \exp(2\lambda_{N+1}(T^*))\\ &\leq \|u(\tau)-v(\tau)\|^2 \left(1+\frac{\xi(t)}{c_5+\lambda_{N+1}}\left(\exp\left(2(c_5+\lambda_{N+1})T^*\right)-1\right)\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|Q_N m(\tau+T^*)\|^2 &\leq \|u(\tau)-v(\tau)\|^2 \Big(\frac{c_5+\lambda_{N+1}-\xi(t)}{c_5+\lambda_{N+1}}\exp(-2\lambda_{N+1}T^*)\\ &\quad+\frac{\xi(t)}{c_5+\lambda_{N+1}}\exp(2c_5T^*)\Big)\\ &\leq \|u(\tau)-v(\tau)\|^2\left[\exp(-2\lambda_{N+1}T^*)+\frac{\xi(t)}{c_5+\lambda_{N+1}}\exp(2c_5T^*)\right]\\ &= \delta^2(t,T^*,N)\|u(\tau)-v(\tau)\|^2 .\end{aligned}$$

Choosing appropriate $N=N(t)$ and $T^*$, we obtain $\delta(t)=\delta(t,T^*,N)<\frac{1}{\sqrt{2}}$. Then, condition (3) is satisfied. It follows from Theorems 4.5 and 3.2 that

$$d_f(\mathcal{A}(t)) \leq N+\eta,$$

where $\eta$ is given by condition $\left(6\sqrt{2}l(t)\right)^N\left(\sqrt{2}\delta(t)\right)^\eta=\sigma(t)<1$.

We shall further prove (18). It is well known (see [3, p.201], [22, p.136]) that $\lambda_N=O(N^{\frac{2}{n}})$, as $N\to\infty$, so that there exists $D>0$ such that $\frac{\lambda_N}{N^{\frac{2}{n}}}\geq D$, $\forall N\in\mathbb{N}$. If we put $\eta=N$ then

$$\begin{aligned}\sigma^2(t) &= (12\delta(t)\,l(t))^{2N}\\ &= 12^2\left(\exp(-2\lambda_{N+1}T^*+2c_5T^*)+\frac{\xi(t)}{c_5+\lambda_{N+1}}\exp(4c_5T^*)\right).\end{aligned}$$

Denote $\gamma=12$. We have to choose $T^*$ and $\lambda_{N+1}$ in such a way that

$$\exp(-2(\lambda_{N+1}-c_5)T^*)=\frac{1}{2\gamma^2}, \tag{20}$$

$$\frac{\xi(t)}{c_5+\lambda_{N+1}}\exp(4c_5T^*)\leq\frac{1}{2\gamma^2+\alpha}, \tag{21}$$

where $\alpha > 0$.

It follows from (20) that $T^{*} = \frac{\log 2\gamma^{2}}{2(\lambda_{N+1}-c_{5})}$. Hence, (21) will be satisfied if the next inequality holds

$$(\lambda_{N+1} - c_5) \log\left(\frac{c_5 + \lambda_{N+1}}{(2\gamma^2+\alpha)\,\xi(t))}\right) \geq 4c_5 \log \sqrt{2}\gamma.$$

Using the inequality $\lambda_{N+1} \geq D(N+1)^{\frac{2}{n}}$ we get

$$(\lambda_{N+1} - c_5) \log\left(\frac{c_5 + \lambda_{N+1}}{(2\gamma^2+\alpha)\,\xi(t)}\right) \geq \left((N+1)^{\frac{2}{n}} D - c_5\right) \log\left(\frac{c_5 + D(N+1)^{\frac{2}{n}}}{(2\gamma^2+\alpha)\,\xi(t)}\right).$$

Choosing $N = N(t)$ such that $D(N+1)^{\frac{2}{n}} \geq 5c_5$ and $\frac{c_5+D(N+1)^{\frac{2}{n}}}{(2\gamma^2+\alpha)\xi(t)} \geq \sqrt{2}\gamma$ the inequality (21) holds. Hence, it is sufficient to choose $N$ satisfying

$$N \geq \max\left\{ (D_1 \xi(t))^{\frac{n}{2}} - 1, \left(\frac{5c_5}{D}\right)^{\frac{n}{2}} - 1 \right\},$$

where $D_1 = \frac{\sqrt{2}\gamma\left(2\gamma^2+\alpha\right)}{D}$. We take $N = \max\left\{ [(D_1\xi(t))^{\frac{n}{2}}], \left[\left(\frac{5c_5}{D}\right)^{\frac{n}{2}}\right] \right\}$, where $[x]$ denotes the integer part of $x$, and then $N \leq \max\left\{ (D_1\xi(t))^{\frac{n}{2}}, \left(\frac{5c_5}{D}\right)^{\frac{n}{2}} \right\}$.

Finally, Theorem 3.2 implies that

$$\begin{aligned} d_f(\mathcal{A}(t)) &\leq 2N \leq 2\max\left\{ (D_1\xi(t))^{\frac{n}{2}}, \left(\frac{5c_5}{D}\right)^{\frac{n}{2}} \right\} \\ &= \max\left\{ L_1 (\xi(t))^{\frac{n}{2}}, L_2 (c_5)^{\frac{n}{2}} \right\}, \end{aligned}$$

where $L_1 = 2(D_1)^{\frac{n}{2}}, L_2 = 2\left(\frac{5}{D}\right)^{\frac{n}{2}}$. ■

From the previous result we can also obtain a uniform bound in $t$ for the fractal dimension of the attractors:

**Corollary 4.7** *There exists a positive constant $K$ depending on $n$, $\Omega$, $c_5$ and $\xi(\cdot)$ (but not on $t$) such that*

$$d_H(\mathcal{A}(t)) \leq d_f(\mathcal{A}(t)) \leq K,$$

*for all $t \in \mathbb{R}$.*

**Proof.** Fix some $t^{*} \in \mathbb{R}$. Since $\xi(t)$ is non-decreasing, Theorem 4.6 gives

$$\begin{aligned} d_f(\mathcal{A}(t)) &\leq \max\left\{ L_1 (\xi(t))^{\frac{n}{2}}, L_2 (c_5)^{\frac{n}{2}} \right\} \\ &\leq \max\left\{ L_1 (\xi(t^{*}))^{\frac{n}{2}}, L_2 (c_5)^{\frac{n}{2}} \right\}, \text{ for all } t \leq t^{*}. \end{aligned}$$

On the other hand, note that (19) implies that $S(t+T,t)$ is Lipschitz with constant $e^{c_5 T}$, for all $T>0$. Then, by Proposition 13.2 in [23] we get

$$d_f\left(\mathcal{A}\left(t+T\right)\right)=d_f\left(S(t+T,t)\mathcal{A}\left(t\right)\right)\leq d_f\left(\mathcal{A}\left(t\right)\right), \tag{22}$$

so that

$$d_f\left(\mathcal{A}\left(t\right)\right)\leq \max\left\{L_1\left(\xi\left(t^*\right)\right)^{\frac{n}{2}}, L_2\left(c_5\right)^{\frac{n}{2}}\right\}=K,\text{ for all } t\in\mathbb{R}. \tag{23}$$

■

**Remark 4.8** *We note that (23) is satisfied for all* $t^*\in\mathbb{R}$. *Hence, the best estimate is obtained by the limit*

$$K=\lim_{t^*\to-\infty}\max\left\{L_1\left(\xi\left(t^*\right)\right)^{\frac{n}{2}}, L_2\left(c_5\right)^{\frac{n}{2}}\right\},$$

*which exists because the function is non-decreasing and bounded below by* 0.

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