Stochastics and Dynamics

SYNCHRONIZATION OF SYSTEMS WITH MULTIPLICATIVE NOISE

Tomás Caraballo

Departmento de Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla, Apdo. de Correos 1160, 41080 Sevilla, Spain
caraball@us.es

Peter E. Kloeden and Andreas Neuenkirch

Fachbereich Mathematik, Johann Wolfgang Goethe Universität,
D-60054 Frankfurt am Main, Germany
kloeden@math.uni-frankfurt.de, neuenkirch@math.uni-frankfurt.de

The synchronization of Stratonovich stochastic differential equations (SDE) with a one-sided dissipative Lipschitz drift and linear multiplicative noise is investigated by transforming the SDE to random ordinary differential equations (RODE) and synchronizing their dynamics. In terms of the original SDE this gives synchronization only when the driving noises are the same. Otherwise, the synchronization is modulo exponential factors involving Ornstein-Uhlenbeck processes corresponding to the driving noises. Moreover, this occurs no matter how large the intensity coefficients of the noise.

Keywords: Synchronization, linear noise, random attractor, stationary stochastic process, one-sided Lipschitz dissipative condition

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Dedicated to Ludwig Arnold on his 70th birthday.

1. Introduction

The synchronization of coupled systems is a ubiquitous phenomenon in the biological and physical science and is also known to occur in a number of social science contexts. A descriptive account of its diversity of occurrence can be found in the recent book of Strogatz[15], which contains an extensive list of references. In particular, synchronization provides an explanation for the emergence of spontaneous order in the dynamical behavior of coupled systems, which in isolation may exhibit chaotic dynamics.

The synchronization of coupled dissipative systems has been investigated mathematically in the case of autonomous systems by Afraimovich and Rodrigues[1], Carvalho *et al.*[7] and Rodrigues[14], both for asymptotically stable equilibria and general attractors, such as chaotic attractors. Analogous results also hold for nonautonomous systems (Ref.[9]), but require a new concept of a nonautonomous attractor.

Caraballo & Kloeden proved in Ref.[4] that synchronization persists in the presence of environmental (i.e., additive) noise provided that appropriate concepts of random attractors and stochastic stationary solutions are used instead of their deterministic counterparts. Very recently, it was shown in Ref.[3] that the presence of additive noise could lead to a strengthening of the synchronization, i.e., at the level of trajectories rather than attractors, which does not occur in the absence of noise.

In this paper we analyze the synchronization effects produced by linear multiplicative noise. To be more precise, we first consider two Stratonovich stochastic differential equations (SDE) in $\mathbb{R}^N$ with linear noise, i.e. with linear functions as the diffusion coefficients. In addition, we assume that the drift coefficients satisfy one-sided dissipative Lipschitz conditions. These SDE are tranformed to random differential equations using a transformation which involves the corresponding Ornstein-Uhlenbeck processes. The resulting random differential systems are then synchronized by adding suitable coupled synchronizing terms in both systems. In terms of the original stochastic differential equations, synchronization is obtained modulo exponential factors involving the Ornstein-Uhlenbeck processes. These factors reduce to unity when the driving noises are the same in both systems, in which exact synchronization results. This occurs no matter how large the intensity coefficients of the noise.

2. The synchronization problem

Suppose we have two autonomous ordinary differential equations in $\mathbb{R}^d$,

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = g(y), \tag{2.1}$$

which are sufficiently regular to ensure the forwards existence and uniqueness of solutions and satisfy one-sided dissipative Lipschitz conditions

$$\begin{aligned} \langle x_1 - x_2, f(x_1) - f(x_2)\rangle &\le -L|x_1 - x_2|^2, \\ \langle y_1 - y_2, g(y_1) - g(y_2)\rangle &\le -L|y_1 - y_2|^2, \end{aligned} \tag{2.2}$$

on $\mathbb{R}^d$ for some $L > 0$, and thus have unique equilibria $\bar{x}$ and $\bar{y}$, respectively, which are globally asymptotically stable (cf. Ref.[9]). Notice that the continuity of f and g, and the one-sided dissipative Lipschitz conditions (2.2) ensure the forwards existence and uniqueness of solutions to (2.1).

Consider now the dissipatively coupled system

$$\frac{dx}{dt} = f(x) + \nu(y - x), \qquad \frac{dy}{dt} = g(y) + \nu(x - y) \tag{2.3}$$

with $\nu > 0$. It can be shown (see Afraimovich & Rodrigues[1], and Carvalho *et al.*[7]) that this also has a unique equilibrium $(\bar{x}^\nu, \bar{y}^\nu)$, which is globally asymptotically stable. Moreover, $(\bar{x}^\nu, \bar{y}^\nu) \to (\bar{z}, \bar{z})$ as $\nu \to \infty$, where $\bar{z}$ is the unique globally asymptotically stable equilibrium of the "averaged" system

$$\frac{dz}{dt} = \frac{1}{2}\left(f(z) + g(z)\right). \tag{2.4}$$

This phenomena is known as *synchronization*. Analogous results hold for more general autonomous attractors (cf. Afraimovich & Rodrigues[1], Carvalho *et al.*[7]) as well as for nonautonomous systems (Ref.[9]) with appropriately defined nonautonomous attractors.

Caraballo & Kloeden[4] showed that synchronization persists under additive noise provided asymptotically stable stochastic stationary solutions are considered rather than asymptotically stable steady state solutions. Specifically, they considered two Ito stochastic differential equations in $\mathbb{R}^d$,

$$dX_t = f(X_t)\,dt + \alpha\,dW_t^1, \qquad dY_t = g(Y_t)\,dt + \beta\,dW_t^2, \tag{2.5}$$

where α, $\beta \in \mathbb{R}_+^d$ are constant vectors with no components equal to zero, W_t^1, W_t^2 are independent two-sided scalar Wiener processes (alternatively, one could take α, β scalar valued and W_t^1, W_t^2 vector valued), and f, g satisfy the one-sided dissipative Lipschitz conditions (2.2). The synchronized system corresponding to SDEs (2.5)

$$\begin{aligned} dX_t &= \left(f(X_t) + \nu(Y_t - X_t)\right)\,dt + \alpha\,dW_t^1, \\ dY_t &= \left(g(Y_t) + \nu(X_t - Y_t)\right)\,dt + \beta\,dW_t^2 \end{aligned} \tag{2.6}$$

has a unique stationary solution $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$, which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu, \bar{Y}_t^\nu) \to (\bar{Z}_t^\infty, \bar{Z}_t^\infty) \quad \text{as} \quad \nu \to \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$, where $\bar{Z}_t^\infty$ is the unique pathwise globally asymptotically stable stationary solution of the “averaged” SDE

$$dZ_t = \frac{1}{2}\left[f(Z_t) + g(Z_t)\right]\,dt + \frac{1}{2}\alpha\,dW_t^1 + \frac{1}{2}\beta\,dW_t^2. \tag{2.7}$$

For the proof the SDE were transformed to pathwise random ordinary differential equations (RODE), for which pathwise estimates are obtained as for deterministic systems.

3. Synchronization of systems with multiplicative noise

The aim of this paper is to investigate if synchronization can also occur in systems with linear noise, i.e. with linear functions as the diffusion coefficients. We consider two Stratonovich stochastic differential equations in $\mathbb{R}^d$,

$$dX_t = f(X_t)\,dt + \sum_{i=1}^m \alpha_i X_t \circ dW_t^{(i)}, \qquad dY_t = g(Y_t)\,dt + \sum_{i=1}^m \beta_i Y_t \circ dW_t^{(i)}, \tag{3.1}$$

where $W^{(1)}, \ldots, W^{(m)}$ are independent two-sided scalar Wiener processes, $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \ldots, m$ and f, g are as above, in particular, satisfying the one-sided dissipative Lipschitz conditions (2.2).

We will apply the theory of Imkeller & Schmalfuß[8] to transform it to the pathwise random ordinary differential equation (RODE)

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}(\omega)) := e^{-O_t^{(1)}(\omega)} f\left(e^{O_t^{(1)}(\omega)}\, x\right) + O_t^{(1)}(\omega)x, \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}(\omega)) := e^{-O_t^{(2)}(\omega)} g\left(e^{O_t^{(2)}(\omega)}\, y\right) + O_t^{(2)}(\omega)y, \end{aligned} \tag{3.2}$$

using the transformation $x(t,\omega) = e^{-O_t^{(1)}(\omega)} X_t(\omega)$ and $y(t,\omega) = e^{-O_t^{(2)}(\omega)} Y_t(\omega)$, where

$$O_t^{(1)} = \sum_{i=1}^m \alpha_i e^{-t} \int_{-\infty}^t e^{\tau} dW_\tau^{(i)}, \qquad O_t^{(2)} = \sum_{i=1}^m \beta_i e^{-t} \int_{-\infty}^t e^{\tau} dW_\tau^{(i)}, \qquad t \in \mathbb{R},$$

are two stationary Ornstein-Uhlenbeck processes.

We will show in the next section that each of the stochastic systems in (3.1) has a pathwise asymptotically stable random attractor consisting of single stationary stochastic process. For this, the use of the stationary Ornstein-Uhlenbeck process in the transformation will be essential. Then we will study their behaviour after synchronization by linear cross coupling, i.e. we will consider the coupled RODE

$$\begin{aligned} \tfrac{dx}{dt} &= F(x, O_t^{(1)}(\omega)) + \nu(y - x), \\ \tfrac{dy}{dt} &= G(y, O_t^{(2)}(\omega)) + \nu(x - y), \end{aligned} \tag{3.3}$$

which we will show also has a pathwise asymptotically stable random attractor consisting of a single stationary stochastic process $(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega))$. In particular,

$$(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega)) \to (\bar{z}(\omega), \bar{z}(\omega)) \qquad \text{as} \quad \nu \to \infty,$$

where $\bar{z}(\omega)$ is the pathwise asymptotically stable solution of the averaged RODE

$$\frac{dz}{dt} = \frac{1}{2}\left[F(z, O_t^{(1)}) + G(z, O_t^{(2)})\right] \tag{3.4}$$

that is,

$$\frac{dz}{dt} = \frac{1}{2}\left[e^{-O_t^{(1)}(\omega)} f\left(e^{O_t^{(1)}(\omega)}\, z\right) + e^{-O_t^{(2)}(\omega)} g\left(e^{O_t^{(2)}(\omega)}\, z\right) + \left(O_t^{(1)}(\omega) + O_t^{(2)}(\omega)\right) z\right]$$

or the equivalent Stratonovich SDE

$$dZ_t = \frac{1}{2}\left[e^{-\eta_t} f(e^{\eta_t} Z_t) + e^{\eta_t} g(e^{-\eta_t} Z_t)\right] dt + \frac{1}{2}\sum_{i=1}^m (\alpha_i + \beta_i) Z_t \circ dW_t^{(i)}$$

where

$$\eta_t = \frac{1}{2}\left(O_t^{(1)} - O_t^{(2)}\right).$$

3.1. *Direct synchronization of the SDE*

In terms of the original system of SDEs (3.1), the coupled RODE has the form

$$\begin{aligned} dX_t &= \left(f(X_t) + \nu\left(e^{2\eta_t}\, Y_t - X_t\right)\right)\, dt + \textstyle\sum_{i=1}^m \alpha_i\, X_t \circ dW_t^{(i)}, \\ dY_t &= \left(g(Y_t) + \nu\left(e^{-2\eta_t} X_t - Y_t\right)\right)\, dt + \textstyle\sum_{i=1}^m \beta_i Y_t \circ dW_t^{(i)}. \end{aligned} \tag{3.5}$$

Then this system has a unique stationary stochastic solution $(\bar{X}_\nu \circ \theta_t, \bar{Y}_\nu \circ \theta_t)$, which is pathwise globally asymptotically stable with

$$(\bar{X}_\nu(\theta_t\omega), \bar{Y}_\nu(\theta_t\omega)) \to \left(\bar{z}(\theta_t\omega)e^{-O_t^{(1)}(\omega)}, \bar{z}(\theta_t\omega)e^{-O_t^{(2)}(\omega)}\right) \quad \text{as} \quad \nu \to \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$. If $\alpha_i = \beta_i$ for all $i = 1, \ldots, m$, i.e. the driving noises are the same, this yields synchronization of the coupled SDE

$$\begin{aligned} dX_t &= (f(X_t) + \nu\,(Y_t - X_t))\, dt + \textstyle\sum_{i=1}^m \alpha_i\, X_t \circ dW_t^{(i)}, \\ dY_t &= (g(Y_t) + \nu\,(X_t - Y_t))\, dt + \textstyle\sum_{i=1}^m \alpha_i\, Y_t \circ dW_t^{(i)} \end{aligned} \tag{3.6}$$

4. Random dynamical system

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Following Arnold[2] a random dynamical system (RDS) (θ, ϕ) on $\Omega \times \mathbb{R}^d$ consists of a metric dynamical system θ on Ω and a cocycle mapping $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. Essentially (and sufficient for our purposes here), θ represents the driving noise process and ϕ the state space evolution of the system. For example, for a stochastic differential equation on $\mathbb{R}^d$ with a two-sided m-dimensional Wiener process W_t, i.e. defined for all $t \in \mathbb{R}$, θ is defined by $\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$ on a canonical sample space $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ where, by definition, $W_t(\omega) := \omega(t), t \in \mathbb{R}$, and ϕ is defined by $\phi(t, \omega, x_0) = X_t^{x_0}(\omega)$, the solution of the SDE starting at $X_0^{x_0}(\omega) = x_0$. See the expositions in Arnold[2] and Kloeden *et al.*[11] for more details.

A family $\widehat{A} = \{A(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $A(\omega)$ of $\mathbb{R}^d$ is called *ϕ-invariant* if $\phi(t, \omega, A(\omega)) = A(\theta_t\omega)$ for all $t \geq 0$ and is called a *random attractor* if in addition it is pathwise pullback attracting in the sense that

$$H_d^*\left(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)\right) \to 0 \quad \text{as} \quad t \to -\infty$$

for all suitable (i.e. in a given attracting universe as, for instance, in Kloeden *et al.* (1999)) families of $\widehat{D} = \{D(\omega), \omega \in \Omega\}$ of nonempty measurable bounded subsets $D(\omega)$ of $\mathbb{R}^d$. Here H_d^* is the Hausdorff semi-distance on $\mathbb{R}^d$. The following result (cf. Arnold[2], Kloeden *et al.*[11]) ensures the existence of a random attractor.

Theorem 4.1. *Let (θ, ϕ) be an RDS on $\Omega \times \mathbb{R}^d$. If there exists a family $\widehat{B} = \{B(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $B(\omega)$ of $\mathbb{R}^d$ and a $T_{\widehat{D},\omega} \geq 0$ such that*

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \qquad \forall t \geq T_{\widehat{D},\omega}$$

for all families $\widehat{D} = \{D(\omega), \omega \in \Omega\}$ *in the given attracting universe, then the RDS* (θ, ϕ) *has a random attractor* $\widehat{A} = \{A(\omega), \omega \in \Omega\}$ *with the component subsets defined for each* $\omega \in \Omega$ *by*

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}.$$

If the random attractor consists of singleton sets, i.e $A(\omega) = \{X^*(\omega)\}$ for some random variable X^*, then $X_t^*(\omega) := X^*(\theta_t\omega)$ is a stationary stochastic process, if the driving system θ_t is a stationary process.

We will need the following Lemma, which can be found in Caraballo *et al.*[6].

Lemma 4.1. *There exists a* $\{\theta_t\}_{t\in\mathbb{R}}$ *invariant subset* $\overline{\Omega} \in \mathcal{F}$ *of* $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ *of full measure such that*

$$\lim_{t\to\pm\infty} \frac{1}{t} \|\omega(t)\| = 0 \qquad \text{for} \quad \omega \in \overline{\Omega},$$

and there exist random variables $\overline{O}^{(1)}$ *and* $\overline{O}^{(2)}$ *such that*

$$\overline{O}^{(1)}(\theta_t\omega) = O_t^{(1)}(\omega) \quad \text{and} \quad \overline{O}^{(2)}(\theta_t\omega) = O_t^{(2)}(\omega) \qquad \text{for} \quad \omega \in \overline{\Omega}.$$

Moreover, we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \overline{O}^{(1)}(\theta_\tau\omega)\, d\tau = \lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \overline{O}^{(2)}(\theta_\tau\omega)\, d\tau = 0 \qquad \text{for} \quad \omega \in \overline{\Omega}.$$

In what follows, we consider θ defined on $\overline{\Omega}$ instead of Ω. This mapping has the same properties as the original one if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\overline{\Omega}$.

5. The uncoupled system with linear noise

We consider the first of the uncoupled equations in (3.1),

$$dX_t = f(X_t)\, dt + \sum_{i=1}^m \alpha_i\, X_t \circ dW_t^{(i)}. \tag{5.1}$$

Its solution paths are generally not differentiable, so in order to use the one-sided dissipative Lipschitz condition (2.2) we will use the approach of Imkeller & Schmalfuß (2001). We first rewrite (5.1) as

$$dX_t = \left[f(X_t) + O_t^{(1)} X_t\right] dt + X_t \circ dO_t^{(1)}, \tag{5.2}$$

where the $O_t^{(1)}$, $t \in \mathbb{R}$, is the stationary solution of

$$dO_t^{(1)} = -O_t^{(1)}\, dt + \sum_{i=1}^m \alpha_i dW_t^{(i)},$$

that is

$$O_t^{(1)} = \sum_{i=1}^{m} \alpha_i e^{-t} \int_{-\infty}^{t} e^s \, dW_s^{(i)}, \qquad t \in \mathbb{R}.$$

Then we transform (5.2) to the pathwise random ordinary differential equation (RODE)

$$\frac{dx}{dt} = F(x, O_t^{(1)}(\omega)) := e^{-O_t^{(1)}(\omega)} f\left(e^{O_t^{(1)}(\omega)}(\omega) x\right) + O_t^{(1)}(\omega) x, \tag{5.3}$$

using the transformation $x(t,\omega) = e^{-O_t^{(1)}(\omega)} X_t(\omega)$.

The vector-field function

$$\widetilde{F}(x, z) = e^{-z} f\left(e^z x\right)$$

in the RODE (5.3) satisfies a one-sided Lipschitz condition in its first variable uniformly in the second with the same constant as the original drift coefficient f, since we have:

$$\begin{aligned}\left\langle x_1 - x_2, \widetilde{F}(x_1, z) - \widetilde{F}(x_2, z)\right\rangle &= \left\langle x_1 - x_2, e^{-z}\left(f(e^z x_1) - f(e^z x_2)\right)\right\rangle \\ &= e^{-2z}\left\langle e^z x_1 - e^z x_2, f(e^z x_1) - f(e^z x_2)\right\rangle \\ &\leq -e^{-2z} L \left\| e^z x_1 - e^z x_2 \right\|^2 \\ &\leq -L \left\| x_1 - x_2 \right\|^2 .\end{aligned}$$

Thus we obtain that any of two solutions of RODE (5.3) satisfy pathwise the differential inequality

$$\frac{d}{dt} \left\| x_1(t) - x_2(t) \right\|^2 \leq (-2L + 2O_t^{(1)}) \left\| x_1(t) - x_2(t) \right\|^2 , \tag{5.4}$$

and hence we have

$$\left\| x_1(t) - x_2(t) \right\|^2 \leq e^{-2t\left(L - \frac{1}{t}\int_0^t O_\tau^{(1)}\, d\tau\right)} \left\| x_1(0) - x_2(0) \right\|^2 .$$

Thus it follows by Lemma 4.1 that

$$\lim_{t\to\infty} \left\| x_1(t) - x_2(t) \right\|^2 = 0,$$

which means all solutions converge pathwise to each other.

In order to see what they converge to, we first observe that the RODE (5.3) generates a random dynamical system with $\phi(t,\omega,x_0) := x(t,\omega)$, the solution of the RODE (5.3) with (deterministic) initial value x_0 at time $t = 0$.

Then we need to show that the RODE (5.3) is asymptotically dissipative and has a pullback attractor. Omitting ω for brevity, we have pathwise

$$\begin{aligned}
\frac{d}{dt}|x|^2 &= 2\left\langle x, F(x, O_t^{(1)})\right\rangle \qquad (5.5)\\
&= 2\left\langle x, e^{-O_t^{(1)}} f(e^{O_t^{(1)}}x) + O_t^{(1)}x\right\rangle\\
&= 2e^{-O_t^{(1)}}\left\langle e^{O_t^{(1)}}x - 0, f(e^{O_t^{(1)}}x) - f(0)\right\rangle + 2\left\langle x, e^{-O_t^{(1)}}f(0)\right\rangle + 2O_t^{(1)}\|x\|^2\\
&\leq (-2L + 2O_t^{(1)})\,\|x\|^2 + 2\|x\|\|f(0)\|e^{-O_t^{(1)}}\\
&\leq (-L + 2O_t^{(1)})\,\|x\|^2 + \frac{e^{-2O_t^{(1)}}}{L}\|f(0)\|^2.
\end{aligned}$$

Integration yields

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 e^{-L(t-t_0)+2\int_{t_0}^t O_\tau^{(1)}\,d\tau} + \frac{\|f(0)\|^2}{L}\int_{t_0}^t e^{-2O_u^{(1)}} e^{-L(t-u)+2\int_u^t O_\tau^{(1)}\,d\tau}\,du.$$

Moreover, by Lemma 4.1, we have pathwise

$$\lim_{s\to-\infty}\frac{1}{s}\int_s^0 O_\tau^{(1)}\,d\tau = \lim_{t\to\infty}\frac{1}{t}\int_0^t O_\tau^{(1)}\,d\tau = 0.$$

Thus we obtain

$$e^{2\int_s^t O_\tau^{(1)}\,d\tau} \leq e^{\frac{L}{2}(t-s)}$$

for $s \leq 0$, $t \geq 0$ with $|t|, |s| > T_\omega$ and hence

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 e^{-L(t-t_0)/2} + \frac{\|f(0)\|^2}{L}\int_{t_0}^t e^{-2O_\tau^{(1)}} e^{-L(t-\tau)} e^{2\int_\tau^t O_\tau^{(1)}\,du}\,d\tau$$

for $t_0 \leq 0$, $t \geq 0$ with $|t|, |t_0| > T_\omega$. Now we can use pathwise pullback convergence (i.e. with $t_0 \to -\infty$) to show that the closed ball about the origin with random radius

$$R^2(\omega) := 1 + \frac{\|f(0)\|^2}{L}\int_{-\infty}^0 e^{L\tau - 2O_\tau^{(1)}(\omega)} e^{2\int_\tau^0 O_u^{(1)}(\omega)\,du}\,d\tau$$

is a pullback absorbing set for $t > T_\omega$. Note that the integrals on the right hand sides are well defined by Lemma 4.1.

The theory of RDS then gives us a random attractor $\{A(\omega), \omega \in \Omega\}$. The fact that all trajectories converge to each other forwards in time says the sets in this random attractor are singleton sets, i.e. $A(\omega) = \{a(\omega)\}$.
When we transfer back to the SDE we have the pathwise singleton set attractor $a(\theta_t\omega)e^{O_t^{(1)}(\omega)}$, which is a stationary solution of the SDE, since the Ornstein-Uhlenbeck process is stationary.

5.1. *Example*

Consider the scalar Stratonovich SDE

$$dX_t = (-X_t + 1)dt + X_t \circ dW_t,$$

which is equivalent to

$$dX_t = \left((-1 + O_t)X_t + 1\right) dt + X_t \circ dO_t,$$

with

$$O_t = e^{-t} \int_{-\infty}^{t} e^s \, dW_s, \qquad t \in \mathbb{R}.$$

Applying the transformation

$$x(t) = e^{-O_t} X_t \tag{5.6}$$

yields the RODE

$$\frac{dx}{dt} = (-1 + O_t)x + e^{-O_t},$$

which has the explicit solution

$$x(t, \omega) = x(t_0) e^{-(t-t_0) + \int_{t_0}^{t} O_s(\omega)\, ds} + \int_{t_0}^{t} e^{-O_\tau(\omega)} e^{-(t-\tau) + \int_\tau^t O_s(\omega)\, ds} \, d\tau.$$

The pullback limit as $t_0 \to -\infty$ gives a stationary solution of the RODE, which attracts all others pathwise,

$$\bar{x}(t, \omega) = \int_{-\infty}^{t} e^{-O_\tau(\omega)} e^{-(t-\tau) + \int_\tau^t O_s(\omega)\, ds} \, d\tau.$$

Transforming back yields

$$\bar{X}_t = e^{O_t} \int_{-\infty}^{t} e^{-O_\tau} e^{-(t-\tau) + \int_\tau^t O_s \, ds} \, d\tau.$$

Since

$$O_t - O_\tau = -\int_\tau^t O_s \, ds + W_t - W_\tau,$$

we have

$$\bar{X}_t = e^{-t+W_t} \int_{-\infty}^{t} e^{\tau - W_\tau} \, d\tau$$

in terms of the Wiener process, which is a stationary solution of the SDE, which attracts all others pathwise. Note that we would have obtained the same process $\bar{X}_t$, if we had used the Wiener process instead of the Ornstein-Uhlenbeck process in the transformation (5.6). The previous estimates in this section would have been easier, but we would not have been able to conclude (a priori) that $\bar{X}_t$ is stationary.

6. Asymptotic behaviour of the coupled RODE system

Now we consider the coupled RODE system (3.2)

$$\frac{dx}{dt} = F(x, O_t^{(1)}(\omega)) + \nu(y - x),$$

$$\frac{dy}{dt} = G(y, O_t^{(2)}(\omega)) + \nu(x - y),$$

with

$$F(x, O_t^{(1)}(\omega)) = e^{-O_t^{(1)}(\omega)} f(e^{O_t^{(1)}(\omega)} x) + O_t^{(1)}(\omega) x,$$

$$G(y, O_t^{(2)}(\omega)) = e^{-O_t^{(2)}(\omega)} g(e^{O_t^{(2)}(\omega)} y) + O_t^{(2)}(\omega) y.$$

Using the one-sided Lipschitz conditions on f and g we obtain similarly to (5.4) that

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \leq (-2L - v + 2O_t^{(1)})\|x_1(t) - x_2(t)\|^2 + \nu\|y_1(t) - y_2(t)\|^2,$$
$$\frac{d}{dt}\|y_1(t) - y_2(t)\|^2 \leq (-2L - v + 2O_t^{(2)})\|y_1(t) - y_2(t)\|^2 + \nu\|x_1(t) - x_2(t)\|^2.$$

Now define

$$A_\nu(t) = \begin{pmatrix} -2L - \nu + 2O_t^{(1)} & \nu \\ \nu & -2L - \nu + 2O_t^{(2)} \end{pmatrix}, \qquad t \in \mathbb{R},$$

and set

$$\mathbf{x}(t) = \begin{pmatrix} \|x_1(t) - x_2(t)\|^2 \\ \|y_1(t) - y_2(t)\|^2 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Thus we can write the above inequalities as

$$\dot{\mathbf{x}}(t) \leq H_\nu(t, \mathbf{x}(t)),$$

componentwise, where

$$H_\nu(t, \mathbf{x}) = A_\nu(t)\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

Since the the off-diagonal entries of the matrix $A_\nu(t)$ are positive and independent of $t \in \mathbb{R}$, the mapping $H_\nu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is quasi-monotone nondecreasing. Thus we obtain, see e.g. Corollary 1.5.2 in Lakshmikantham & Leela[12], that

$$\mathbf{x}(t) \leq \exp\left(\int_0^t A_\nu(\tau)\, d\tau\right) \mathbf{x}(0), \tag{6.1}$$

componentwise. Now, we need the following simple Lemma:

Lemma 6.1. *We have*

$$\left\| \exp\left(\int_0^t A_\nu(\tau)\, d\tau\right) \mathbf{x} \right\| \leq e^{-Lt}\|\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^2,$$

for $t \geq T_\omega$ *and all* $\nu \geq 1$.

Proof. First note that the matrix $\int_0^t A_\nu(\tau)d\tau$ is symmetric. Thus, there exists a orthonormal basis of eigenvectors $u_{\nu,t}^{(1)}$, $u_{\nu,t}^{(2)}$ with eigenvalues $\lambda_{\nu,t}^{(1)}$, $\lambda_{\nu,t}^{(2)}$, and we have

$$\exp\left(\int_0^t A_\nu(\tau)\,d\tau\right)\mathbf{x} = e^{\lambda_{\nu,t}^{(1)}}\,c_{\mathbf{x},\nu,t}^{(1)}\,u_{\nu,t}^{(1)} + e^{\lambda_{\nu,t}^{(2)}}\,c_{\mathbf{x},\nu,t}^{(2)}\,u_{\nu,t}^{(2)},$$

where

$$c_{\mathbf{x},\nu,t}^{(1)}\,u_{\nu,t}^{(1)} + c_{\mathbf{x},\nu,t}^{(2)}\,u_{\nu,t}^{(2)} = \mathbf{x}.$$

Since $u_{\nu,t}^{(1)}$ and $u_{\nu,t}^{(2)}$ are orthogonal, we obtain

$$\begin{aligned}\left\|\exp\left(\int_0^t A_\nu(\tau)\,d\tau\right)\mathbf{x}\right\|^2 &= e^{2\lambda_{\nu,t}^{(1)}}\|c_{\mathbf{x},\nu,t}^{(1)}u_{\nu,t}^{(1)}\|^2 + e^{2\lambda_{\nu,t}^{(2)}}\|c_{\mathbf{x},\nu,t}^{(2)}u_{\nu,t}^{(2)}\|^2 \\ &\leq e^{2\max\{\lambda_{\nu,t}^{(1)},\lambda_{\nu,t}^{(2)}\}}\|\mathbf{x}\|^2. \qquad (6.2)\end{aligned}$$

The eigenvalues of $\int_0^t A_\nu(\tau)d\tau$ are given by

$$\lambda_{\nu,t}^{(1/2)} = -(2L+\nu)t + \int_0^t \left(O_\tau^{(1)} + O_\tau^{(2)}\right)d\tau \pm \sqrt{\left(\int_0^t O_\tau^{(1)} - O_\tau^{(2)}\,d\tau\right)^2 + \nu^2 t^2},$$

hence it follows by Lemma 4.1 that

$$\lambda_{\nu,t}^{(1/2)} \leq -Lt \qquad (6.3)$$

for $|t| > T_\omega$ and all $\nu \geq 1$. Combining now (6.2) and (6.3) yields the assertion. □

The above Lemma implies now that

$$\lim_{t\to\infty}\|x_1(t) - x_2(t)\|^2 = \lim_{t\to\infty}\|y_1(t) - y_2(t)\|^2 = 0.$$

Hence all solution of the coupled RODE system converge pathwise to each other in the future. We use the theory of random dynamical systems to see what they converge to. Similarly to (5.5) and (6.1) we obtain

$$\begin{aligned}\frac{d}{dt}\|x\|^2 &\leq (-L - v + 2O_t^{(1)})\,\|x\|^2 + v\|y\|^2 + \frac{e^{-2O_t^{(1)}}}{L}\|f(0)\|^2, \\ \frac{d}{dt}\|y\|^2 &\leq (-L - v + 2O_t^{(2)})\,\|y\|^2 + v\|x\|^2 + \frac{e^{-2O_t^{(2)}}}{L}\|g(0)\|^2\end{aligned}$$

and, componentwise,

$$\begin{aligned}\mathbf{x}(t) \leq{}& \exp\left(\int_{t_0}^t \widetilde{A}_\nu(\tau)\,d\tau\right)\mathbf{x}(t_0) \\ &+ \frac{1}{L}\int_{t_0}^t \exp\left(\int_u^t \widetilde{A}_\nu(\tau)\,d\tau\right)\begin{pmatrix} e^{-2O_u^{(1)}}\|f(0)\|^2 \\ e^{-2O_u^{(2)}}\|g(0)\|^2\end{pmatrix} du\end{aligned}$$

with

$$\widetilde{A}_\nu(t) = \begin{pmatrix} -(L+\nu)+2O_t^{(1)} & \nu \\ \nu & -(L+\nu)+2O_t^{(2)} \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} \|x(t)\|^2 \\ \|y(t)\|^2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Analogously to Lemma 6.1 we can show:

Lemma 6.2. *Let $t_0 \leq 0$ and $t \geq 0$. We have*

$$\left\| \exp\left(\int_{t_0}^{t} \widetilde{A}_\nu(\tau) d\tau \right) \mathbf{x} \right\| \leq e^{-\frac{L}{2}(t-t_0)} \|\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^2,$$

for $|t_0|, |t| \geq T_\omega$ and all $\nu \geq 1$.

Now set

$$C_\nu(\omega) := \frac{1}{L} \int_{-\infty}^{0} \exp\left(\int_u^0 \widetilde{A}_\nu(\tau)\, d\tau \right) \begin{pmatrix} e^{-2O_u^{(1)}} \|f(0)\|^2 \\ e^{-2O_u^{(2)}} \|g(0)\|^2 \end{pmatrix} du$$

and define

$$R_\nu^2(\omega) = 1 + \|C_\nu(\omega)\|^2.$$

Then by pullback techniques and Lemma 6.2 we see that the random balls $B_\nu(\omega)$ in $\mathbb{R}^{2d}$ centered on the origin and with radius $R_\nu(\omega)$ are pullback absorbing. Moreover note that

$$\frac{d}{d\nu}\|C_\nu(\omega)\|^2 = 2\left\langle \frac{d}{d\nu} C_\nu(\omega), C_\nu(\omega) \right\rangle = 2\left\langle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} C_\nu(\omega), C_\nu(\omega) \right\rangle \leq 0$$

and consequently $R_\nu(\omega) \leq R_1(\omega)$ for $\nu \geq 1$. The random dynamical system generated by the coupled RODE (3.2) has a random attractor $A_\nu(\omega)$ in $B_\nu(\omega)$ for each ω. But we know that all solutions converge to each other pathwise forwards in time. Thus the $A_\nu(\omega)$ are singleton sets, say $A_\nu(\omega) = \{(\bar{x}_\nu(\omega), \bar{y}_\nu(\omega))\}$.

Let us now estimate the difference of the components of the coupled system. We have pathwise

$$\begin{aligned}\frac{d}{dt}|x-y|^2 &= 2\left\langle x-y, \frac{dx}{dt}-\frac{dy}{dt}\right\rangle \\ &= 2\left\langle x-y, e^{-O_t^{(1)}}f(e^{O_t^{(1)}}x) - e^{-O_t^{(2)}}g(e^{O_t^{(2)}}y)\right\rangle \\ &\qquad +2\left\langle x-y, 2\nu(y-x)+O_t^{(1)}x - O_t^{(2)}y\right\rangle \\ &\leq -4\nu\|x-y\|^2 \\ &\quad +2\|x-y\|\left(e^{-O_t^{(1)}}\|f(e^{O_t^{(1)}}x)\| + e^{-O_t^{(2)}}\|g(e^{O_t^{(2)}}y)\| + \|O_t^{(1)}x - O_t^{(2)}y\|\right) \\ &\leq -\nu\|x-y\|^2 + \frac{e^{-2O_t^{(1)}}}{\nu}\left\|f(e^{O_t^{(1)}}x)\right\|^2 + \frac{|O_t^{(1)}|^2}{\nu}\|x\|^2 \\ &\qquad + \frac{e^{-2O_t^{(2)}}}{\nu}\left\|g(e^{O_t^{(2)}}y)\right\|^2 + \frac{|O_t^{(2)}|^2}{\nu}\|y\|^2.\end{aligned}$$

Hence labelling the solutions with ν to indicate this dependence we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leq -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu}M^\nu_{T_1,T_2,\omega}$$

with

$$\begin{aligned}M^\nu_{T_1,T_2,\omega} &= \sup_{t\in[T_1,T_2]}\left(e^{-2O_t^{(1)}}\left\|f(e^{O_t^{(1)}}x_\nu)\right\|^2 + |O_t^{(1)}|^2\|x_\nu\|^2\right) \\ &\qquad + \sup_{t\in[T_1,T_2]}\left(e^{-2O_t^{(2)}}\left\|g(e^{O_t^{(2)}}y_\nu)\right\|^2 + |O_t^{(2)}|^2\|y_\nu\|^2\right).\end{aligned}$$

We can restrict ourselves without loss of generality to solutions in the compact absorbing balls $B_\nu(\omega)$, which are all contained in the common compact ball $B_1(\omega)$ for $\nu \geq 1$. Hence $M^\nu_{T_1,T_2,\omega}$ is uniformly bounded in ν and we have

$$\frac{d}{dt}\|x_\nu - y_\nu\|^2 \leq -\nu\|x_\nu - y_\nu\|^2 + \frac{1}{\nu}M_{T_1,T_2,\omega}$$

with

$$M_{T_1,T_2,\omega} = \sup_{\nu\geq 1} M^\nu_{T_1,T_2,\omega},$$

from which we conclude that

$$\|x_\nu(t) - y_\nu(t)\|^2 \to 0 \qquad \text{as} \quad \nu\to\infty,$$

uniformly in $t \in [T_1, T_2]$ for any bounded T_1 and T_2.

7. The synchronized solutions as $\nu \to \infty$

Now we can prove the following result:

Theorem 7.1. *$(\bar{x}_{\nu_n}(t,\omega), \bar{y}_{\nu_n}(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ pathwise uniformly on bounded time intervals $[T_1, T_2]$ of $\mathbb{R}$ for any sequence $\nu_n \to \infty$, where $\bar{z}(\omega)$ is the attracting stationary solution of the averaged RODE*

$$\frac{dz}{dt} = \frac{1}{2}\left[e^{-O_t^{(1)}} f\left(e^{O_t^{(1)}} z\right) + e^{-O_t^{(2)}} g\left(e^{O_t^{(2)}} z\right) + (O_t^{(1)} + O_t^{(2)})z\right]. \tag{7.1}$$

The equivalent Stratonovich SDE is given by

$$dZ_t = \frac{1}{2}\left[e^{-\eta_t} f(e^{\eta_t} Z_t) + e^{\eta_t} g(e^{-\eta_t} Z_t)\right] dt + \frac{1}{2}\sum_{i=1}^{m} (\alpha_i + \beta_i) Z_t \circ dW_t^{(i)}, \tag{7.2}$$

with $\eta_t = \frac{1}{2}(O_t^{(1)} - O_t^{(2)})$.

Proof. Define

$$\bar{z}_\nu(\omega) := \frac{1}{2}\left(\bar{x}_\nu(\omega) + \bar{y}_\nu(\omega)\right)$$

and observe that $\bar{z}_\nu(t,\omega) = \bar{z}_\nu(\theta_t\omega)$ satisfies the RODE

$$\frac{d\bar{z}_\nu}{dt} = \frac{1}{2}\left(e^{-O_t^{(1)}} f\left(e^{O_t^{(1)}} \bar{x}_\nu\right) + e^{-O_t^{(2)}} f\left(e^{O_t^{(2)}} \bar{y}_\nu\right) + O_t^{(1)}\bar{x}_\nu + O_t^{(2)}\bar{y}_\nu\right).$$

Thus

$$\sup_{t\in[T_1,T_2]} \left|\frac{d}{dt}\bar{z}_\nu(t,\omega)\right| \le M_{T_1,T_2,\omega} < \infty$$

by continuity and the fact that these solutions belong to the common compact ball $B_1(\omega)$. We can use the Ascoli Theorem to conclude that there is a subsequence $\nu_{n_j} \to \infty$ such that $\bar{z}_{\nu_{n_j}}(t,\omega) \to \bar{z}(t,\omega)$ as $n_j \to \infty$.

Now

$$\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega) = \frac{1}{2}\left(\bar{x}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega)\right) \to 0,$$

$$\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega) = \frac{1}{2}\left(\bar{y}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega)\right) \to 0,$$

as $\nu_{n_j} \to \infty$, see the previous section, so

$$\bar{x}_{\nu_{n_j}}(t,\omega) = 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{y}_{\nu_{n_j}}(t,\omega) \to \bar{z}(t,\omega),$$

$$\bar{y}_{\nu_{n_j}}(t,\omega) = 2\bar{z}_{\nu_{n_j}}(t,\omega) - \bar{x}_{\nu_{n_j}}(t,\omega) \to \bar{z}(t,\omega)$$

as $\nu_{n_j} \to \infty$. Moreover, using the integral equation representation

$$\begin{aligned}\bar{z}_\nu(t,\omega) = \bar{z}_\nu(T_1,\omega) &+ \frac{1}{2}\int_{T_1}^{t} O_s^{(1)}(\omega)\bar{x}_\nu(s,\omega) + O_s^{(2)}(\omega)\bar{y}_\nu(s,\omega)\, ds \\ &+ \frac{1}{2}\int_{T_1}^{t} e^{-O_s^{(1)}(\omega)} f(e^{O_s^{(1)}(\omega)}\bar{x}_\nu(s,\omega))\, ds \\ &+ \frac{1}{2}\int_{T_1}^{t} e^{-O_s^{(2)}(\omega)} g(e^{O_s^{(2)}(\omega)}\bar{y}_\nu(s,\omega))\, ds,\end{aligned}$$

it follows that the ν_{n_j} subsequence converges pathwise to

$$\begin{aligned}\bar{z}(t,\omega) = \bar{z}(T_1,\omega) &+ \frac{1}{2}\int_{T_1}^{t}(O_t^{(1)}(\omega)+O_t^{(2)}(\omega))\bar{z}(s,\omega)\,ds \\ &+\frac{1}{2}\int_{T_1}^{t} e^{-O_s^{(1)}(\omega)} f(e^{O_s^{(1)}(\omega)}\bar{z}(s,\omega))\,ds \\ &+\frac{1}{2}\int_{T_1}^{t} e^{-O_s^{(2)}(\omega)} g(e^{O_s^{(2)}(\omega)}\bar{z}(s,\omega))\,ds\end{aligned}$$

on the interval $[T_1, T_2]$, so $\bar{z}(t,\omega)$ is a solution of the RODE (7.1) for all $t \in \mathbb{R}$. By the same techniques as in the previous sections, it has a random attractor consisting of a singleton set formed by a single stationary stochastic process which thus must be equal to $\bar{z}(t,\omega)$.

Finally, we note that pathwise all possible subsequences here have the same limit, so by Lemma 2.2 in Caraballo & Kloeden[4] every full sequence $\bar{z}_\nu(t,\omega)$ actually converges to $\bar{z}(t,\omega)$ for the whole sequence $\nu_n \to \infty$. To show that averaged Stratonovich SDE (7.2) transforms to the averaged RODE (7.1), we apply the transformation

$$z = e^{\frac{1}{2}\left(O_t^{(1)}+O_t^{(2)}\right)} Z. \qquad \square$$

As a straightforward consequence of the arguments in the previous proof we have

Corollary 7.1. *$(\bar{x}_\nu(t,\omega), \bar{y}_\nu(t,\omega)) \to (\bar{z}(t,\omega), \bar{z}(t,\omega))$ as $\nu \to \infty$ pathwise on any bounded time interval $[T_1, T_2]$ of $\mathbb{R}$.* $\square$

7.1. *Example*

Consider the scalar Stratonovich SDEs

$$dX_t = (-X_t + 1)dt + X_t \circ dW_t^{(1)}, \qquad dY_t = (-Y_t + 2)dt + 2X_t \circ dW_t^{(2)}.$$

The corresponding RODEs are

$$\frac{dx}{dt} = x(-1 + O_t^{(1)}) + e^{-O_t^{(1)}}, \qquad \frac{dy}{dt} = y(-1 + O_t^{(2)}) + 2e^{-O_t^{(2)}}$$

with

$$O_t^{(1)} = \int_{-\infty}^{t} e^{-(t-s)}\,dW_s^{(1)}, \qquad O_t^{(2)} = 2\int_{-\infty}^{t} e^{-(t-s)}\,dW_s^{(2)}.$$

The averaged RODE is

$$\frac{dz}{dt} = z\left(-1 + \frac{1}{2}(O_t^{(1)} + O_t^{(2)})\right) + e^{-O_t^{(1)}} + 2e^{-O_t^{(2)}}$$

with the explicit solution

$$z(t) = e^{-(t-t_0)+\frac{1}{2}\int_{t_0}^{t}(O_\tau^{(1)}+O_\tau^{(2)})\,d\tau} z_0 + \int_{t_0}^{t} e^{-(t-s)+\frac{1}{2}\int_s^t(O_\tau^{(1)}+O_\tau^{(2)})\,d\tau - O_s^{(1)}}\,ds + 2\int_{t_0}^{t} e^{-(t-s)+\frac{1}{2}\int_s^t(O_\tau^{(1)}+O_\tau^{(2)})\,d\tau - O_s^{(2)}}\,ds.$$

The pullback limit as $t_0 \to -\infty$ gives a stationary solution

$$\bar{z}(t) = \int_{-\infty}^{t} e^{-(t-s)+\frac{1}{2}\int_s^t(O_\tau^{(1)}+O_\tau^{(2)})\,d\tau - O_s^{(1)}}\,ds + 2\int_{-\infty}^{t} e^{-(t-s)+\frac{1}{2}\int_s^t(O_\tau^{(1)}+O_\tau^{(2)})\,d\tau - O_s^{(2)}}\,ds,$$

which corresponds to the SDE

$$dZ_t = \left(-Z_t + \frac{1}{2}e^{-\frac{1}{2}(O_t^{(1)}-O_t^{(2)})} + e^{\frac{1}{2}(O_t^{(1)}-O_t^{(2)})}\right)dt + \frac{1}{2}Z_t \circ dW_t^{(1)} + Z_t \circ dW_t^{(2)}$$

and attracts all other solutions pathwise.

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