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SYNCHRONIZATION OF A STOCHASTIC REACTION-DIFFUSION SYSTEM ON A THIN TWO-LAYER DOMAIN*

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Abstract. A system of semilinear parabolic stochastic partial differential equations with additive space-time noise is considered on the union of thin bounded tubular domains $D_{1,\varepsilon} := \Gamma \times (0, \varepsilon)$ and $D_{2,\varepsilon} := \Gamma \times (-\varepsilon, 0)$ joined at the common base $\Gamma \subset \mathbb{R}^d$, where $d \geq 1$. The equations are coupled by an interface condition on Γ which involves a reaction intensity $k(x', \varepsilon)$, where $x = (x', x_{d+1}) \in \mathbb{R}^{d+1}$ with $x' \in \Gamma$ and $|x_{d+1}| < \varepsilon$. Random influences are included through additive space-time Brownian motion, which depend only on the base spatial variable $x' \in \Gamma$ and not on the spatial variable x_{d+1} in the thin direction. Moreover, the noise is the same in both layers $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$. Limiting properties of the global random attractor are established as the thinness parameter of the domain $\varepsilon \to 0$, i.e., as the initial domain becomes thinner, when the intensity function possesses the property $\lim_{\varepsilon \to 0} \varepsilon^{-1} k(x', \varepsilon) = +\infty$. In particular, the limiting dynamics is described by a single stochastic parabolic equation with the averaged diffusion coefficient and a nonlinearity term, which essentially indicates synchronization of the dynamics on both sides of the common base Γ. Moreover, in the case of nondegenerate noise we obtain stronger synchronization phenomena in comparison with analogous results in the deterministic case previously investigated by Chueshov and Rekalo [*EQUADIFF*-2003, F. Dumortier et al., eds., World Scientific, Hackensack, NJ, 2005, pp. 645–650; *Sb. Math.*, 195 (2004), pp. 103–128].

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1. Introduction.

Let $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ be thin bounded domains in $\mathbb{R}^{d+1}$, where $d \geq 1$, of the form

$$D_{1,\varepsilon} = \Gamma \times (0, \varepsilon), \qquad D_{2,\varepsilon} = \Gamma \times (-\varepsilon, 0),$$

where $0 < \varepsilon \leq 1$ and Γ is a bounded C^2-domain in $\mathbb{R}^d$. We write $x \in D_\varepsilon := D_{1,\varepsilon} \cup D_{2,\varepsilon}$ as $x = (x', x_{d+1})$, where $x' \in \Gamma$ and $x_{d+1} \in (0, \varepsilon)$ or $x_{d+1} \in (-\varepsilon, 0)$, and will not distinguish between the sets $\Gamma \times \{0\} \subset \mathbb{R}^{d+1}$ and $\Gamma \subset \mathbb{R}^d$.

We consider the following system of semilinear parabolic equations:

$$\frac{\partial}{\partial t} U^i - \nu_i \Delta U^i + aU^i + f_i(U^i) + h_i(x) = \dot{W}(t, x'), \quad t > 0, \ \ x \in D_{i,\varepsilon}, \ \ i = 1, 2, \tag{1}$$

with the initial data

$$U^i(0, x) = U_0^i(x), \qquad x \in D_{i,\varepsilon}, \ \ i = 1, 2, \tag{2}$$

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where the $\dot{W}(t,x')$ is a Gaussian white noise depending on the spatial variable $x' \in \Gamma$ (but not on the x_{d+1} spatial variable).

We assume that U^1 and U^2 satisfy the Neumann boundary conditions

$$(\nabla U^i, n_i) = 0, \qquad x \in \partial D_{i,\varepsilon} \setminus \Gamma, \quad i = 1, 2, \tag{3}$$

on the external part of the boundary of the compound domain D_ε, where n is the outer normal to ∂D_ε, and a matching condition on Γ of the form

$$\begin{aligned} \left(-\nu_1 \frac{\partial U^1}{\partial x_{d+1}} + k(x',\varepsilon)(U^1 - U^2)\right)\Big|_\Gamma &= 0, \\ \left(\nu_2 \frac{\partial U^2}{\partial x_{d+1}} + k(x',\varepsilon)(U^2 - U^1)\right)\Big|_\Gamma &= 0. \end{aligned} \tag{4}$$

Here the above constants ν_i and a are positive numbers.

We impose the following assumptions:

- for $i = 1, 2$ the function $f_i \in C^1(\mathbb{R})$ possesses the property $f_i'(v) \geq -c$ for all $v \in \mathbb{R}$ and also satisfies the relations

  $$v f_i(v) \geq a_0 |v|^{p+1} - c, \quad |f_i'(v)| \leq a_1 |v|^{p-1} + c, \quad v \in \mathbb{R}, \tag{5}$$

  where a_j and c are positive constants and $1 \leq p < 3$;
- $h_i \in H^1(D_{i,1})$, $i = 1, 2$;
- the interface reaction intensity $k(x',\varepsilon)$ satisfies

  $$k(\cdot,\varepsilon) \in L^\infty(\Gamma), \quad k(x',\varepsilon) > 0 \text{ for } x' \in \Gamma, \ \varepsilon \in (0,1],$$

  and

  $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} k(x',\varepsilon) = +\infty, \ x' \in \Gamma, \text{ in Lebesgue measure (see Remark 1.1);} \tag{6}$$

- $W(t)$, $t \in \mathbb{R}$, is a two-sided $L_2(\Gamma)$-valued Wiener process with covariance operator $K = K^* \geq 0$ such that

  $$\operatorname{tr}\left[K\left(-\Delta_N + 1\right)^{2\beta - 1}\right] < \infty \quad \text{for some} \quad \beta > \max\left\{1, \frac{d}{4}\right\}, \tag{7}$$

  where Δ_N is the Laplace operator in $L_2(\Gamma)$ with the Neumann boundary conditions on $\partial\Gamma$. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the corresponding probability space, and by $\dot{W} \equiv \partial_t W$ the generalized derivative with respect to t.

Remark 1.1. Our main example of the interface reaction intensity is the following function:

$$k(x',\varepsilon) = \varepsilon^\alpha k_0(x') \in L^\infty(\Gamma), \quad k_0(x') > 0 \quad \text{for } x' \in \Gamma, \ \varepsilon \in (0,1],$$

for some $\alpha \in [0,1)$. We also note that the convergence in (Lebesgue) measure to infinity means that

$$\lim_{\varepsilon \to 0} \operatorname{Leb}\left\{x' \in \Gamma \, : \, \varepsilon^{-1} k(x',\varepsilon) \leq N\right\} = 0 \quad \text{for any} \quad N > 0.$$

Problem (1)–(4) is a model for a reaction-diffusion system consisting of two components filling thin contacting layers $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ separated by a penetrable membrane Γ. Reaction of the components is possible on the surface Γ only, and the reaction intensity $k(x', \varepsilon)$ depends on the thickness of the domains filled by the reactants. The deterministic version of the model was considered by Chueshov and Rekalo [12, 13], while Rekalo [27] investigated the special case of identical equations in both layers with $k(\varepsilon, x)$ independent of ε. The stochastic version considered in the present paper allows for irregularities and random effects on the separating membrane.

Hale and Raugel [21, 22] initiated the analysis of asymptotic dynamics of deterministic semilinear reaction-diffusion equations on thin domains. Some extensions of their results can be also found in [16] and [26]. In all these papers, a reaction-diffusion equation is endowed with homogeneous Neumann boundary conditions. To our knowledge stochastic evolution equations have not previously been investigated on thin domains.

In this paper we investigate the pathwise asymptotic behavior of the above stochastic evolution system by converting it into a system of pathwise random partial differential equations (PDEs) to which deterministic methods can be applied in a pathwise manner.

Our main result deals with properties of random (global) pullback attractors for the random dynamical system generated by (1)–(4) in $L_2(D_\varepsilon)$. In particular, we prove that these pullback attractors are closely related to the corresponding object for the problem

$$\frac{\partial}{\partial t}U - \nu\Delta_{x'}U + aU + f(U) + h(x') = \dot{W}(t, x'), \qquad t > 0, \;\; x' \in \Gamma, \tag{8}$$

on the spatial domain Γ with the Neumann boundary conditions on $\partial\Gamma$. Here we denote

$$\nu = \frac{\nu_1 + \nu_2}{2}, \quad f(U) = \frac{f_1(U) + f_2(U)}{2}, \quad h(x') = \frac{h_1(x', 0) + h_2(x', 0)}{2}. \tag{9}$$

This is essentially a statement about the synchronization of the dynamics of the system in the two thin layers at the level of global pullback attractors. Since, in principle, a global attractor can be a rather complicated set, the synchronization at this level does not imply that *any* pair of trajectories becomes asymptotically synchronized. However, in the case of nondegenerate noise ($Kh = 0$ if and only if $h = 0$ and the image of K is dense in $L_2(\Gamma)$) we can prove, in contrast with the deterministic counterpart, that the global pullback attractor for (8) is a singleton. This means that we also have asymptotic synchronization in our system at the level of trajectories. Thus we observe a stronger synchronizing effect of a nondegenerate stochastic noise in the system under consideration.

The synchronization of stochastic stationary solutions (i.e., single-valued random attractors) of finite dimensional stochastic systems was considered in [5]. See also [1, 23] for similar results in deterministic nonautonomous systems and [7, 28] for autonomous infinite dimensional systems.

The synchronization of coupled systems is a ubiquitous phenomenon in the biological and physical science and is also known to occur in a number of social science contexts. A descriptive account of its diversity of occurrence can be found in the recent book of Strogatz [32], which contains an extensive list of references. In particular, synchronization provides an explanation for the emergence of spontaneous order in the dynamical behavior of coupled systems, which in isolation may exhibit chaotic dynamics. It has been shown to persist in the presence of environmental noise provided that appropriate concepts of random attractors and stochastic stationary solutions are used instead of their deterministic counterparts [5]. As mentioned above, in this paper we will see that the presence of additive noise can lead to a strengthening of the synchronization, i.e., at the level of trajectories rather than attractors, which does not occur in the absence of noise.

Since most of our analysis is a pathwise analysis applied to pathwise defined random PDEs, i.e., with the stationary Ornstein–Uhlenbeck process appearing as a space-time dependent coefficient, it is reasonable to expect that similar results will also hold for other kinds of noise, for example, with fractional Brownian motion in the original stochastic partial differential equations (SPDEs). The results will be presented in a forthcoming paper.

The paper is organized as follows. We start with preliminary section 2 containing background material from the theory of random systems which we need to state and discuss our main results in section 3. Further sections are devoted to the proof of our main theorem, Theorem 3.1.

2. Random dynamical systems. In order to formulate our results we need some notation and results from the theory of random dynamical systems (with continuous time) and random attractors.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete separable metric (Polish) space. Arnold [2] defined a random dynamical system (RDS) (θ, ϕ) on $\Omega \times \mathcal{X}$ in terms of a metric dynamical system θ on Ω, which represents the noise driving the system, and a cocycle mapping $\phi : \mathbb{R}_+ \times \Omega \times \mathcal{X} \to \mathcal{X}$, which represents the dynamics in the state space $\mathcal{X}$ and satisfies the following properties:

1. $\phi(0, \omega)\phi_0 = \phi_0$ for all $\phi_0 \in \mathcal{X}$ and $\omega \in \Omega$;
2. $\phi(s + t, \omega)\phi_0 = \phi(s, \theta_t\omega)\phi(t, \omega)\phi_0$ for all $s,\ t \geq 0$, $\phi_0 \in \mathcal{X}$, and $\omega \in \Omega$;
3. $(t, \phi_0) \mapsto \phi(t, \omega)\phi_0$ is continuous for each $\omega \in \Omega$; and
4. $\omega \mapsto \phi(t, \omega)\phi_0$ is $\mathcal{F}$-measurable for all $(t, \phi_0) \in \mathbb{R}_+ \times \mathcal{X}$.

We recall that a metric dynamical system $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is a family of measure-preserving transformations $\{\theta_t : \Omega \mapsto \Omega, t \in \mathbb{R}\}$ such that

(i) $\theta_0 = id, \theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$;
(ii) the map $(t, \omega) \mapsto \theta_t\omega$ is measurable, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

RDSs (with continuous time) are generated by differential equations with random coefficients or stochastic differential equations with a unique and global solution, as well as by infinite dimensional stochastic evolution equations with additive noise. We refer to [2] for more details on the general theory of RDS theory.

To construct an RDS in our case we first need to associate a metric dynamical system θ with the Wiener process W on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $L_2(\Gamma)$. The probability measure $\mathbb{P}$ of this process can be realized on $\mathcal{F} = \mathcal{B}(C_0(\mathbb{R}, L_2(\Gamma)))$, where $C_0(\mathbb{R}, L_2(\Gamma))$ is the Fréchet space of continuous functions on $\mathbb{R}$ with values in $L_2(\Gamma)$ which are zero at time zero. For this realization we introduce the flow $(\theta_t)_{t\in\mathbb{R}}$ given by the *Wiener shift*

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}. \tag{10}$$

Interpreting the above Wiener process in the canonical sense $W(\cdot, \omega) = \omega(\cdot)$, it follows that (10) is the well-known helix property of a Wiener process:

$$W(t + s, \omega) - W(s, \omega) = W(t, \theta_s\omega), \quad s, t \in \mathbb{R}, \quad \omega \in \Omega.$$

We now introduce the Ornstein–Uhlenbeck process as a stationary solution of the linear stochastic evolution equation

$$\frac{\partial}{\partial t}U = \nu\Delta_{x'}U - aU + \dot{W}(t, x'), \qquad t > 0, \;\; x' \in \Gamma,$$

on the spatial domain Γ with Neumann boundary conditions on $\partial\Gamma$. Here, as above, we denote $\nu = (\nu_1 + \nu_2)/2$. This process $\eta(t)$ can be written in the form

$$\eta(t, \omega) := \left(\int_{-\infty}^{t} e^{-(t-\tau)A_0} dW(\tau) \right) (\omega), \tag{11}$$

where $A_0 = -\nu\Delta_N + a$ and Δ_N is the Laplace operator in $L_2(\Gamma)$ with the Neumann boundary conditions on $\partial\Gamma$. The integral in (11) exists as an operator stochastic integral (see, e.g., [24] or [19]) We can also involve a perfection procedure to define $\eta(t, \omega) \equiv \bar{\eta}(\theta_t\omega)$ for *all* $\omega \in \Omega$ (for details see [14]). Moreover, under condition (7), $t \mapsto \bar{\eta}(\theta_t\omega)$ is continuous from $\mathbb{R}$ into $D(A_0^{\beta'}) \subset H^{2\beta'}(\Gamma)$ for each $\omega \in \Omega$, where $\beta' \in [0, \beta)$ is arbitrary, and the *temperedness* condition

$$\sup_{t\in\mathbb{R}} \{ \| A_0^{\beta'} \bar{\eta}(\theta_t\omega) \| \, e^{-\gamma|t|} \} < \infty \quad \forall\, \gamma > 0,\ \omega \in \Omega,$$

is satisfied. We also note that under condition (7), since $H^s(\Gamma) \subset C(\overline{\Gamma})$ for $s > d/2$, we have that $t \mapsto \bar{\eta}(\theta_t\omega)$ is a pathwise continuous tempered process with values in $D(A_0) \cap C(\overline{\Gamma})$. In particular

$$\bar{\eta}(\theta_t\omega) \in C\left(\mathbb{R}; C(\overline{\Gamma}) \cap \left\{ \psi \in H^2(\Gamma) : \psi \text{ satisfies Neumann b.c. on } \partial\Gamma \right\}\right) \tag{12}$$

for every $\omega \in \Omega$. We will use this observation later.

We recall the following definition of a random set (see [2] or [4]).

DEFINITION 2.1 (random set). *Let $\mathcal{X}$ be a Polish space with a metric $d_{\mathcal{X}}$. A multifunction $\omega \mapsto D(\omega) \neq \emptyset$ is said to be a* random set *if the mapping $\omega \mapsto \mathrm{dist}_{\mathcal{X}}(v, D(\omega))$ is measurable for any $v \in \mathcal{X}$, where $\mathrm{dist}_{\mathcal{X}}(v, B)$ is the distance in $\mathcal{X}$ between the element v and the set $B \subset \mathcal{X}$. For ease of notation we denote the random set $\omega \mapsto D(\omega)$ by $\widehat{D}$ or $\{D(\omega)\}$. If $D(\omega)$ is closed for each $\omega \in \Omega$, then $\widehat{D}$ is called a* random closed set*, while if $D(\omega)$ is a compact set for all $\omega \in \Omega$, then $\widehat{D}$ is called a* random compact set*. A random set $\{D(\omega)\}$ is said to be* tempered *if there exists a $v_0 \in \mathcal{X}$ such that $D(\omega) \subset \{v \in \mathcal{X} : d_{\mathcal{X}}(v, v_0) \leq r(\omega)\}$ for all $\omega \in \Omega$, where the random variable $r(\omega) > 0$ is tempered, i.e.,*

$$\sup_{t\in\mathbb{R}} \{ r(\theta_t\omega) e^{-\gamma|t|} \} < \infty \quad \forall\, \gamma > 0,\ \omega \in \Omega.$$

We denote by $\mathcal{D}$ the collection of all tempered random sets in $\mathcal{X}$.

Below we also need the concept of a random attractor for RDSs (see, e.g., [2, 17, 18, 29] and the references therein), which extends the corresponding definition of a global attractor in autonomous systems (cf. [3, 9, 33], for example).

DEFINITION 2.2. *Let (θ, ϕ) be an RDS with the phase space $\mathcal{X}$. A random closed set $\{\mathfrak{A}(\omega)\}$ from $\mathcal{D}$ is said to be a* random pullback attractor *for (θ, ϕ) in $\mathcal{D}$ if* (i) *$\widehat{\mathfrak{A}}$ is an invariant set, i.e., $\phi(t, \omega)\mathfrak{A}(\omega) = \mathfrak{A}(\theta_t\omega)$ for $t \geq 0$ and $\omega \in \Omega$, and* (ii) *$\widehat{\mathfrak{A}}$ is pullback attracting in $\mathcal{D}$, i.e.,*

$$\lim_{t\to+\infty} d_{\mathcal{X}} \{ \varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) \,|\, \mathfrak{A}(\omega) \} = 0, \quad \omega \in \Omega,$$

for all $\widehat{D} \in \mathcal{D}$, where $d_{\mathcal{X}}\{A|B\} = \sup_{a\in A} \mathrm{dist}_{\mathcal{X}}(a, B)$.

Note that a pullback attractor is also a weak *forward* attractor; i.e., we have that

$$\lim_{t\to+\infty} \int_{\Omega} d_{\mathcal{X}} \{ \varphi(t, \omega) D(\omega) \,|\, \mathfrak{A}(\theta_t\omega) \} \mathbb{P}(d\omega) = 0 \quad \forall \widehat{D} \in \mathcal{D}.$$

If the random attractor consists of singleton sets, i.e., $\mathfrak{A}(\omega) = \{X^*(\omega)\}$ for some random variable X^* with $X^*(\omega) \in \mathcal{X}$, then $X^*_t(\omega) := X^*(\theta_t\omega))$ is a stationary stochastic process on $\mathcal{X}$.

The following result [18] ensures the existence of a random attractor for an RDS on a Polish space.

THEOREM 2.3. *Let (θ, ϕ) be a continuous or discrete time RDS on $\Omega \times \mathcal{X}$ such that $\phi(t, \omega, \cdot) : \mathcal{X} \to \mathcal{X}$ is a compact operator for each fixed $t > 0$ and $\omega \in \Omega$. If there exists a tempered random set $\widehat{\mathfrak{B}} = \{\mathfrak{B}(\omega), \omega \in \Omega\}$ and a $T_{\widehat{D},\omega} \geq 0$ such that*

$$\phi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset \mathfrak{B}(\omega) \qquad \forall t \geq T_{\widehat{D},\omega},$$

for every tempered random set $\widehat{D}$, then the RDS (θ, ϕ) has a random pullback attractor $\widehat{\mathfrak{A}} = \{\mathfrak{A}(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by

$$\mathfrak{A}(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega)\mathfrak{B}(\theta_{-t}\omega)}^{d_{\mathcal{X}}}.$$

The family $\{\mathfrak{B}(\omega)\}$ is called a pullback absorbing random set for the RDS.

3. Main results. Now we are in position to state our main results which we formulate in the theorem below. This says that the limiting dynamics of the system (1)–(4) is given by that of the averaged system (8) on Γ, which one can interpret as the synchronization of dynamics of the original system on the two sides of the membrane Γ. In addition, if the system is the same on both sides of the membrane, then the limiting behavior is independent of the thinness parameter ε when it is sufficiently small.

THEOREM 3.1. *Under the conditions above the following assertions hold.*

1. *Problem* (1)–(4) *generates an RDS $(\theta, \bar{\phi}_\varepsilon)$ in the space*

$$\mathcal{H}_\varepsilon = L_2(D_{1,\varepsilon}) \oplus L_2(D_{2,\varepsilon}) \sim L_2(D_\varepsilon)$$

with the metric dynamical system θ generated by the Wiener process W and the cocycle $\bar{\phi}_\varepsilon$ defined by the formula $\bar{\phi}_\varepsilon(t, \omega)U_0 = U(t, \omega)$, where $U(t, \omega) = (U^1(t, \omega); U^2(t, \omega))$ is a strong (*in the sense of stochastic equations* [19]) *solution to problem* (1)–(4) *and $U_0 = (U_0^1; U_0^2)$.*

2. *Similarly, problem* (8) *generates an RDS $(\theta, \bar{\phi}_0)$ in the space $L_2(\Gamma)$.*
3. *The cocycles $\bar{\phi}_\varepsilon$ converge to $\bar{\phi}_0$ in the sense that*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \frac{1}{\varepsilon} \int_{D_\varepsilon} |\bar{\phi}_\varepsilon(t, \omega)v - \bar{\phi}_0(t, \omega)v|^2 dx = 0 \quad \forall \omega \in \Omega,$$

for any $v(x) \in \mathcal{H}_\varepsilon$ independent of the variable x_{d+1}, and for any $T > 0$.

4. *These RDS $(\theta, \bar{\phi}_\varepsilon)$ and $(\theta, \bar{\phi}_0)$ have random compact pullback attractors $\{\bar{\mathfrak{A}}^\varepsilon(\omega)\}$ and $\{\bar{\mathfrak{A}}^0(\omega)\}$ in their corresponding phase spaces. Moreover, if the correlation operator K of the Wiener process W is nondegenerate in the sense that* (i) *$Kh = 0$ if and only if $h = 0$, and* (ii) *the image of K is dense in $L_2(\Gamma)$, then the attractor $\{\bar{\mathfrak{A}}^0(\omega)\}$ is a singleton, i.e., $\bar{\mathfrak{A}}^0(\omega) = \{\bar{v}_0(\omega)\}$, where $\bar{v}_0(\omega)$ is a tempered random variable with values in $L_2(\Gamma)$.*
5. *The attractors $\{\bar{\mathfrak{A}}^\varepsilon(\omega)\}$ are upper semicontinuous as $\varepsilon \to 0$ in the sense that*

$$\lim_{\varepsilon \to 0} \sup_{v \in \bar{\mathfrak{A}}^\varepsilon(\omega)} \left\{ \inf_{v_0 \in \bar{\mathfrak{A}}^0(\omega)} \frac{1}{\varepsilon} \int_{D_\varepsilon} |v(x', x_{d+1}) - v_0(x')|^2 dx \right\} = 0 \quad \forall \omega \in \Omega. \tag{13}$$

6. *In addition, if*

$$\nu_1 = \nu_2 := \nu, \quad f_1(U) = f_2(U) := f(U),$$
$$h_1(x', x_{d+1}) = h(x'), \quad h_2(x', x_{d+1}) = h(x'); \tag{14}$$

$f(U)$ is globally Lipschitz, i.e., there exists a constant $L > 0$ such that

$$|f(U) - f(V)| \le L|U - V|, \quad U, V \in \mathbb{R}, \tag{15}$$

and also that

$$k(x', \varepsilon) > k_\varepsilon \text{ for } x' \in \Gamma, \ \varepsilon \in (0, 1]; \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon^{-1} k_\varepsilon = +\infty, \tag{16}$$

then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the global random pullback attractor $\{\bar{\mathfrak{A}}^\varepsilon(\omega)\}$ for $(\theta, \bar{\phi}_\varepsilon)$ has the form

$$\bar{\mathfrak{A}}^\varepsilon(\omega) \equiv \left\{ v(x', x_{d+1}) \equiv v_0(x') \ : \ v_0 \in \bar{\mathfrak{A}}^0(\omega) \right\},$$

where $\{\bar{\mathfrak{A}}^0(\omega)\}$ is the random pullback attractor for the RDS $(\theta, \bar{\phi}_0)$.

Remark 3.2. In the case when $\bar{\mathfrak{A}}^0(\omega) = \{\bar{v}_0(\omega)\}$ is a singleton, relation (13) turns into the equality

$$\lim_{\varepsilon \to 0} \sup_{v \in \bar{\mathfrak{A}}^\varepsilon(\omega)} \left\{ \frac{1}{\varepsilon} \int_{D_\varepsilon} |v(x', x_{d+1}, \omega) - \bar{v}_0(x', \omega)|^2 dx \right\} = 0 \quad \forall \omega \in \Omega.$$

In particular, this implies that for any $U_0, U_0^* \in \mathcal{H}_\varepsilon$ we have that

$$\lim_{\varepsilon \to 0} \limsup_{t \to +\infty} \left\{ \frac{1}{\varepsilon} \|\bar{\phi}_\varepsilon(t, \theta_{-t}\omega) U_0 - \bar{\phi}_\varepsilon(t, \theta_{-t}\omega) U_0^*\|^2_{L_2(D_\varepsilon)} \right\} = 0 \quad \forall \omega \in \Omega, \tag{17}$$

where we can omit the $\lim_{\varepsilon \to 0}$ under conditions (14)–(16). Thus we obtain the synchronization effect not only at the level of global attractors (see (13)) but also at the level of trajectories in relation (17). We emphasize that this *double* synchronization phenomenon is not true for the deterministic ($K \equiv 0$) counterpart of the problem. In the latter case the global attractor for (8) (without the noise $\dot{W}$) is not a single point when the reaction term $au + f(u)$ has several roots, and thus (17) cannot be true for *all* initial data. In this case we have synchronization at the level of the global attractors only.

Remark 3.3. The statements of Theorem 3.1 deal with the case when the intensity interaction $k(x', \varepsilon)$ between layers is asymptotically strong enough (see condition (6)). However, similarly to [12, 13] we can also consider the case when the limit in (6) is finite by assuming that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} k(x', \varepsilon) = k(x') \quad \text{strongly in } L_2(\Gamma) \tag{18}$$

for some bounded nonnegative function $k(x') \in L_2(\Gamma)$. In this case the limiting problem for (1)–(4) is a system of two parabolic SPDEs on Γ of the form

$$\frac{\partial}{\partial t} U^i - \nu_i \Delta_{x'} U^i + aU^i + f_i(U^i) + k(x')(-1)^{i+1}(U^1 - U^2) + h_i(x', 0) = \dot{W}(t, x'), \tag{19}$$

where $i = 1,2$ and $(t;x') \in \mathbb{R}_+ \times \Gamma$, with the Neumann boundary condition on $\partial\Gamma$. Using the same method as for the case (6) in combination with deterministic arguments given in [13] for a particular case of (18), one can prove upper semicontinuity of $\{\bar{\mathfrak{A}}^\varepsilon(\omega)\}$ in the limit $\varepsilon \to 0$ in the case (18). However we will not present the case because (i) our main point of interest is the phenomenon of synchronization, and (ii) under condition (18) synchronization is possible only in some very special cases.

The proof of Theorem 3.1 is given in the remaining sections of the paper. To begin, in section 4 the problem is reformulated in terms of pathwise random PDEs on a scaled domain and appropriate function spaces are introduced. Then we show that (1)–(4) generates an RDS. In section 5 the existence of a random pullback attractor is proved. Then in section 5.1 the limiting dynamics on finite time intervals as $\epsilon \to 0$ is established, and in section 7 the upper continuous dependence of the attractors as $\epsilon \to 0$ is shown. Finally, in section 8 the synchronization of the systems for fixed $\epsilon > 0$ is considered.

4. Generation of an RDS by the two-layer problem.

4.1. Equivalent random PDEs. We introduce the new dependent variables V^i (which are also stochastic processes):

$$V^i(t,x,\omega) := U^i(t,x',x_{d+1},\omega) - \bar{\eta}(\theta_t\omega,x'), \quad t>0, \ \ x=(x',x_{d+1}) \in D_{i,\varepsilon}, \ \ i=1,2,$$

where $\bar{\eta}(\omega,x')$ is given by (11) after perfection. Let

$$\begin{aligned} h_1(x,\omega) &= -\frac{1}{2}(\nu_1-\nu_2)\Delta\bar{\eta}(\omega) + h_1(x), \\ h_2(x,\omega) &= \frac{1}{2}(\nu_1-\nu_2)\Delta\bar{\eta}(\omega) + h_2(x). \end{aligned} \tag{20}$$

Then equations (1)–(4) can be transformed into the pathwise random semilinear parabolic PDEs

$$\partial_t V^i - \nu_i \Delta V^i + aV^i + f_i\left(V^i + \bar{\eta}(\theta_t\omega)\right) + h_i(x,\theta_t\omega) = 0, \quad t>0, \ x \in D_{i,\varepsilon}, \tag{21}$$

for $i=1,2$, with the random initial data

$$V^i(0,x,\omega) = U_0^i(x) - \bar{\eta}(\omega), \qquad x \in D_{i,\varepsilon}, \ \ i=1,2. \tag{22}$$

Since the Ornstein–Uhlenbeck process $\bar{\eta}(\theta_t\omega;x')$ does not depend on x_{d+1}, due to (12) we obtain the Neumann boundary conditions

$$\left(\nabla V^i(x), n_i(x)\right) = 0, \qquad x \in \partial D_{i,\varepsilon} \setminus \Gamma, \quad i=1,2, \tag{23}$$

on the external part of the boundary of the compound domain D_ε, where n is the outer normal to ∂D_ε. Condition (4) turns into a matching condition on Γ of the form

$$\begin{aligned} \left(-\nu_1 \frac{\partial V^1}{\partial x_{d+1}} + k(x',\varepsilon)(V^1-V^2)\right)\Big|_\Gamma &= 0, \\ \left(\nu_2 \frac{\partial V^2}{\partial x_{d+1}} + k(x',\varepsilon)(V^2-V^1)\right)\Big|_\Gamma &= 0, \end{aligned} \tag{24}$$

which is now pathwise random and homogeneous.

4.2. Scaling and functional spaces. It is convenient to deal with a fixed domain where every equation is defined for $\varepsilon > 0$. Let us introduce the new coordinates $(x, y) \in \mathbb{R}^{d+1}$, as follows:

$$x = x', \quad x \in \Gamma, \quad y = \varepsilon^{-1} x_{d+1}, \quad y \in (-1, 1).$$

In so doing, we transform the domain D_ε into $D = D_1 \cup D_2$, where $D_1 = \Gamma \times (0, 1)$, $D_2 = \Gamma \times (-1, 0)$; the operator $\nabla = (\nabla_{x'}$ and $\partial_{x_{d+1}})$ into $\nabla_\varepsilon = (\nabla_x,\, \varepsilon^{-1}\partial_y)$; and $\Delta = \Delta_{x'} + \partial^2_{x_{d+1}}$ into $\Delta_\varepsilon = \Delta_x + \varepsilon^{-2}\partial_{yy}$. Problem (21)–(24) takes the form

$$\partial_t v^i - \nu_i \Delta_\varepsilon v^i + a v^i + f_i(v^i + \bar{\eta}) + h_i^\varepsilon(x, y, \theta_t \omega) = 0, \quad t > 0,\ (x, y) \in D_i, \tag{25}$$

for $i = 1, 2$, with the initial data

$$v^i(0, x, y) = V_0^i(x, y), \quad (x, y) \in D_i, \quad i = 1, 2, \tag{26}$$

and the boundary conditions

$$\left.\frac{\partial v^i}{\partial n_i}\right|_{\partial D_i \setminus \Gamma} = 0, \quad i = 1, 2, \tag{27}$$

$$\left.\left(\nu_i \frac{\partial v_i}{\partial y} - \varepsilon k(x, \varepsilon)(v_1 - v_2)\right)\right|_{y=0} = 0, \quad i = 1, 2. \tag{28}$$

Here $h_i^\varepsilon(x, y, \omega) = h_i(x, \varepsilon y, \omega)$ and n_i is the outward normal to the boundary ∂D_i. A solution $V(t, x', x_{d+1})$ to problem (21)–(24) is expressed in terms of a solution $v(t, x, y)$ to problem (25)–(28) by the formula $V(t, x', x_{d+1}) = v(t, x', \varepsilon^{-1} x_{d+1})$.

Let us introduce the space

$$\mathcal{H} = L^2(D_1) \oplus L^2(D_2) \simeq L^2(D)$$

endowed with the norm $\|u\|^2 \equiv \|u_1\|^2_{L^2(D_1)} + \|u_2\|^2_{L^2(D_2)}$, where $u = (u_1; u_2)$, $u_i \equiv u|_{D_i}$, and let us define a family of Sobolev spaces

$$\mathcal{H}^1_\varepsilon = H^1(D_1) \oplus H^1(D_2), \quad \varepsilon \in (0, 1],$$

endowed with the norm

$$\|u\|^2_{1,\varepsilon} \equiv \sum_{i=1}^{2} \left(\|u_i\|^2_{H^1(D_i)} + \varepsilon^{-2} \|\partial_y u_i\|^2_{L^2(D_i)} \right).$$

Every element $v \in H^1(\Gamma) \oplus H^1(\Gamma)$ can be extended naturally to an element $u \in \mathcal{H}^1_\varepsilon$ by the formula $u_i(x, y) \equiv v_i(x)$, $(x, y) \in D_i$, $i = 1, 2$; in what follows, this will be done without further comment.

4.3. Abstract representation. Now we represent problem (25)–(27) in the abstract form. To do this we first consider the bilinear form

$$\begin{aligned} a_\varepsilon(u, v) \ &= \sum_{i=1}^{2} \nu_i \left[(\nabla_x u_i, \nabla_x v_i)_{L^2(D_i)} + \frac{1}{\varepsilon^2} (\partial_y u_i, \partial_y v_i)_{L^2(D_i)} \right] + a \cdot (u, v)_{\mathcal{H}} \\ &\quad + \frac{1}{\varepsilon} \int_\Gamma k(x, \varepsilon)(u_1(x, 0) - u_2(x, 0))(v_1(x, 0) - v_2(x, 0))\, dx, \end{aligned}$$

defined on the elements $u = (u_1; u_2)$, $v = (v_1; v_2)$ of the space $\mathcal{H}^1_\varepsilon = H^1(D_1) \oplus H^1(D_2)$. One can show that $a_\varepsilon(u, v)$ is a closed symmetric form in $\mathcal{H}$ possessing the property

$$c_0 \sum_{i=1,2} \|u\|^2_{H^1(D_i)} \le c_1 \|u\|^2_{1,\varepsilon} \le a_\varepsilon(u, u), \quad u \in \mathcal{H}^1_\varepsilon. \tag{29}$$

Here and in what follows we drop the subscript ε in constants which can be chosen independently of $\varepsilon \in (0, 1]$. Therefore, there exists a unique positive self-adjoint operator A_ε such that $\mathcal{D}(A_\varepsilon) \subset \mathcal{H}^1_\varepsilon$ and

$$a_\varepsilon(u, v) = (A_\varepsilon u, v)_{\mathcal{H}}, \quad u \in \mathcal{D}(A_\varepsilon),\ v \in \mathcal{H}^1_\varepsilon.$$

It can be shown that

$$\mathcal{D}(A_\varepsilon) = \left\{ u \in H^2(D_1) \oplus H^2(D_2) \ :\ u \text{ satisfies (27) and (28)} \right\}$$

and also that

$$A_\varepsilon u = (-\nu_1 \Delta_\varepsilon u_1 + a u_1,\ -\nu_2 \Delta_\varepsilon u_2 + a u_2), \quad u = (u_1, u_2) \in \mathcal{D}(A_\varepsilon).$$

Moreover, $\mathcal{D}(A_\varepsilon^{1/2}) = \mathcal{H}^1_\varepsilon$, $a_\varepsilon(u, u) = \|A_\varepsilon^{1/2} u\|^2$. For more details concerning the operator A_ε we refer to [13].

Now we can rewrite the pathwise random PDE in problem (25)–(28) in the abstract form

$$\frac{d}{dt} v + A_\varepsilon v = B(v, \theta_t \omega), \quad v|_{t=0} = v_0, \tag{30}$$

in the space $\mathcal{H}$, where

$$B(v, \omega) = \begin{cases} -f_1(v^1 + \bar{\eta}(\omega)) - h_1(x, \varepsilon y, \omega), & y > 0, \\ -f_2(v^2 + \bar{\eta}(\omega)) - h_2(x, \varepsilon y, \omega), & y < 0. \end{cases}$$

4.4. Generation of an RDS. By the same method as in [25] (see also [30, Chap. 3]) one can prove that there exists a deterministic constant M such that the nonlinear mapping $A_\varepsilon - B(\cdot, \omega) + M$ is a maximal monotone operator on $\mathcal{D}(A_\varepsilon)$. This observation makes it possible (some details can be found in [8, Chap. 15] for the general nonautonomous case) to prove that for each $\omega \in \Omega$ and $v_0 \in \mathcal{H}$ on any time interval $[0, T]$ there exists a unique weak solution $v(t, \omega)$ to (30) from the class

$$L_{p+1}(0, T; L_{p+1}(D)) \cap L_2(0, T; \mathcal{H}^1_\varepsilon) \cap C(0, T; \mathcal{H}).$$

Since this solution can be constructed as a limit of the corresponding Galerkin approximations, the mapping $(t; \omega) \mapsto v(t, \omega)$ is measurable. Moreover, it is easy to derive from the uniqueness property that the mapping $\phi_\varepsilon(t, \omega) : \mathcal{H} \mapsto \mathcal{H}$ defined by the relation $\phi_\varepsilon(t, \omega) v_0 = v(t, \omega)$, where $v(t, \omega)$ solves (30), satisfies the cocycle property. Thus (30) generates an RDS.

Now using inverse transformation we define the cocycle $\bar{\phi}_\varepsilon$ for problem (1)–(4) by the formula

$$\bar{\phi}_\varepsilon(t, \omega) = R_\varepsilon^{-1}(\theta_t \omega) \circ \phi_\varepsilon(t, \omega) \circ R_\varepsilon(\omega),$$

where $R_\varepsilon(\omega) : L_2(D_\varepsilon) \mapsto L_2(D)$ is an affine random mapping of the form

$$[R_\varepsilon(\omega) U](x, y) = U(x, \varepsilon y) - \bar{\eta}(\omega), \quad U \in L_2(D_\varepsilon).$$

This proves the first statement in Theorem 3.1.

It is clear that $R_\varepsilon(\omega)$ maps tempered random sets in $L_2(D_\varepsilon)$ into tempered sets in $L_2(D)$. Therefore all other statements of Theorem 3.1 can be easily reformulated as statements concerning the RDS $(\theta, \phi_\varepsilon)$ generated by the random evolution equation in (30). In our further considerations we deal with this RDS $(\theta, \phi_\varepsilon)$.

5. Random pullback attractors. In this section we prove the existence of a random pullback attractor for problem (25)–(28) for every *fixed* $\varepsilon \in (0, 1]$ and also for the limiting problem (8).

5.1. The case $\varepsilon > 0$. We first want to emphasize that we do not use any information concerning the behavior of the intensity $k(x', \varepsilon)$ as $\varepsilon \to 0$, and hence our results in this subsection cover both of the cases (6) and (18).

Our main result in this section is the following assertion.

PROPOSITION 5.1. *In the space $\mathcal{H}$ the RDS $(\theta, \phi_\varepsilon)$ generated by problem* (25)–(28) *possesses a compact pullback attractor $\widehat{\mathfrak{A}}^\varepsilon$ which belongs to the space $\mathcal{H}^1_\varepsilon$. Moreover, there exists a tempered random variable $R(\omega)$, which does not depend on ε, such that*

$$\mathfrak{A}^\varepsilon(\omega) \subset \left\{ v \in \mathcal{H}^1_\varepsilon \ : \ a_\varepsilon(v, v) + \|v\|^{p+1}_{L_{p+1}(D)} \le R^2(\omega) \right\}, \quad \omega \in \Omega. \tag{31}$$

We split the proof into several lemmata which are also important for the limit transition on finite time intervals.

LEMMA 5.2 (pullback dissipativity). *The RDS $(\theta, \phi_\varepsilon)$ is pullback dissipative in $\mathcal{D}$; i.e., there exists a tempered random variable $R(\omega) > 0$ such that for any random set $\widehat{D}$ from $\mathcal{D}$ we can find $t_0(\omega, \widehat{D}) > 0$ for which*

$$\|\phi_\varepsilon(t, \theta_{-t}\omega)U\|_{\mathcal{H}} \le R(\omega) \quad \forall\ U \in D(\theta_{-t}\omega),\ t \ge t_0(\omega, \widehat{D}).$$

Thus the random ball $B_0(\omega) = \{U \in \mathcal{H} \ : \ \|U\|_{\mathcal{H}} \le R(\omega)\}$ is pullback absorbing. This ball is also forward invariant and absorbing if we take

$$R^2(\omega) = c_1 \int_{-\infty}^0 e^{c_0\tau} \left(1 + \|\bar\eta(\theta_\tau\omega)\|^{p+1}_{L_{p+1}(\Gamma)} + \|\bar\eta(\theta_\tau\omega)\|^2_{H^1(\Gamma)} \right) d\tau,$$

with appropriate $c_0 > 0$ and $c_1 > 0$ independent of $\varepsilon \in (0, 1]$.

Proof. The calculations below are formal, but can be justified by considering Galerkin approximations.

Multiplying (25) by v^i in $L_2(D_i)$ for $i = 1, 2$, after some calculations we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{\mathcal{H}} + a_\varepsilon(v, v) + \sum_{i=1,2} \left[\int_{D_i} f_i(v^i + \bar\eta) v^i dx + (h^\varepsilon, v^i)_{L_2(D_i)} \right] = 0. \tag{32}$$

From (5) we have that

$$\begin{aligned}(f_i(v^i + \bar\eta), v^i) &= \int_{D_i} f(v^i) v^i dx + \int_{D_i} \left[\int_0^1 f_i'(v^i + \lambda\bar\eta) d\lambda \right] \bar\eta v^i dx \\ &\ge a_0 \|v^i\|^{p+1}_{L_{p+1}(D_i)} - c_1 \int_{D_i} \left(1 + |v^i|^{p-1} + |\bar\eta|^{p-1} \right) |\bar\eta||v^i| dx - c_2 \\ &\ge \frac{a_0}{2} \|v^i\|^{p+1}_{L_{p+1}(D_i)} - b_0 \left(1 + \|\bar\eta\|^{p+1}_{L_{p+1}(\Gamma)} \right) \end{aligned} \tag{33}$$

and from (20) and (29) we also have that

$$\sum_{i=1,2}(h^\varepsilon, v^i)_{L_2(D_i)} \le C\left(\|\bar\eta\|_{H^1(\Gamma)} + \sum_{i=1,2}\|h\|_{H^1(D_i)}\right)[a_\varepsilon(v,v)]^{1/2}. \tag{34}$$

Now from (32)–(34) we obtain that

$$\frac{d}{dt}\|v\|^2_{\mathcal{H}} + a_\varepsilon(v,v) + a_0\|v\|^{p+1}_{L_{p+1}(D)} \le R_0^2(\theta_t\omega), \tag{35}$$

where

$$R_0^2(\omega) = c\left(1 + \|\bar\eta(\omega)\|^{p+1}_{L_{p+1}(\Gamma)} + \|\bar\eta(\omega)\|^2_{H^1(\Gamma)}\right). \tag{36}$$

Since $a_\varepsilon(v,v) \ge c_0\|v\|^2_{\mathcal{H}} + \frac{1}{2}a_\varepsilon(v,v)$, by differentiating $e^{\nu_* t}\|v\|^2_{\mathcal{H}}$, taking into account (35), and integrating, we have that

$$\|v(t)\|^2_{\mathcal{H}} + \int_0^t e^{-\nu_*(t-\tau)}V^0_\varepsilon(v(\tau))d\tau \le \|v_0\|^2_{\mathcal{H}}e^{-\nu_* t} + \int_0^t e^{-\nu_*(t-\tau)}R_0^2(\theta_\tau\omega)d\tau, \tag{37}$$

for any $0 < \nu_* \le c_0$, where $R_0(\omega)$ is given by (36) and

$$V^0_\varepsilon(v) = \frac{1}{2}a_\varepsilon(v,v) + a_0\|v\|^{p+1}_{L_{p+1}(D)}. \tag{38}$$

This allows us to complete the proof of Lemma 5.2. ∎

LEMMA 5.3 (compact absorbing set). *For each $\varepsilon \in (0,1]$ there exists a compact, forward invariant tempered absorbing set.*

Proof. Multiplying (25) by $\partial_t v^i$ in $L_2(D_i)$ we find that

$$\begin{aligned}\partial_t\Psi_\varepsilon(v(t)) &+ \|\partial_t v(t)\|^2_{\mathcal{H}} \\ &+ \sum_{i=1,2}\int_{D_i}\left[f(v^i+\bar\eta) - f(v^i)\right]\partial_t v^i dxdy + \int_D h^\varepsilon\partial_t v^i dxdy = 0,\end{aligned} \tag{39}$$

where

$$\Psi_\varepsilon(u) = \frac{1}{2}a_\varepsilon(u,u) + \sum_{i=1}^2\int_{D_i}F_i(u^i)\,dxdy, \quad u = (u^1;u^2) \in \mathcal{H}^1_\varepsilon. \tag{40}$$

Here $F_i(u) = \int_0^u f_i(\xi)d\xi$. It is clear from the assumptions concerning f_i that

$$\begin{aligned}&\left|\sum_{i=1,2}\int_{D_i}\left[f(v^i+\bar\eta) - f(v^i)\right]\partial_t v^i dxdy\right| \\ &\le c\sum_{i=1,2}\int_{D_i}\left|f(v^i+\bar\eta) - f(v^i)\right|^2 dxdy + \frac{1}{4}\|\partial_t v(t)\|^2_{\mathcal{H}} \\ &\le c_1 + c_2\int_D |v|^{p+1}dxdy + c_3\left[|\bar\eta|^{p+1}_{L_{p+1}(\Gamma)} + |\bar\eta|^{p_*}_{L_{p_*}(\Gamma)}\right] + \frac{1}{4}\|\partial_t v(t)\|^2_{\mathcal{H}},\end{aligned}$$

where $p_* = 2(p+1)/(3-p)$. We also have that

$$\left|\int_D h^\varepsilon\partial_t v^i dxdy\right| \le c_1 + c_2\|\bar\eta\|^2_{H^2(\Gamma)} + \frac{1}{4}\|\partial_t v(t)\|^2_{\mathcal{H}}$$

Therefore from (39) we have that

$$\begin{aligned}\partial_t\Psi_\varepsilon(v(t)) &+ \frac{1}{2}\|\partial_t v(t)\|^2_{\mathcal{H}} \\ &\le c_1 + c_2\|v\|^{p+1}_{L_{p+1}(D)} + c_3\left(\|\bar\eta\|^2_{H^2(\Gamma)} + |\bar\eta|^{p+1}_{L_{p+1}(\Gamma)} + |\bar\eta|^{p_*}_{L_{p_*}(\Gamma)}\right).\end{aligned} \tag{41}$$

Consequently, choosing positive constants b_0 and b_1 in an appropriate way one can see that

$$V_\varepsilon(u) := b_0\|u\|^2_{\mathcal{H}} + \Psi_\varepsilon(u) + b_1 \tag{42}$$

with Ψ_ε given by (40) satisfies the relations

$$c_0 V^0_\varepsilon(v) \le V_\varepsilon(v) \le c_1\left[1 + V^0_\varepsilon(v)\right] \tag{43}$$

with $V^0_\varepsilon(v)$ given by (38). Moreover, due to (35) we can choose b_0 and b_1 such that

$$\frac{d}{dt}V_\varepsilon(v) + \gamma V_\varepsilon(v) + \frac{1}{2}\|\partial_t v_\varepsilon(t)\|^2_{\mathcal{H}} \le R_1^2(\theta_t\omega), \tag{44}$$

with positive γ, where

$$R_1^2(\omega) = c\left(1 + \|\bar\eta(\omega)\|^{p+1}_{L_{p+1}(\Gamma)} + |\bar\eta(\omega)|^{p_*}_{L_{p_*}(\Gamma)} + \|\bar\eta(\omega)\|^2_{H^2(\Gamma)}\right), \quad p_* = \frac{2p+2}{3-p}. \tag{45}$$

We note that $R_1(\omega)$ is a tempered random variable because $t \mapsto \bar\eta(\theta_t\omega)$ is a tempered process with values in $H^2(\Gamma)\cap C(\overline{\Gamma})$. From (44) we have that

$$V_\varepsilon(v(t)) \le e^{-\gamma(t-s)}V_\varepsilon(v(s)) + \int_s^t e^{-\gamma(t-\tau)}R_1^2(\theta_\tau\omega)d\tau, \quad t \ge s. \tag{46}$$

By (43) we also have

$$V^0_\varepsilon(v(t)) \le c_1 e^{-\gamma(t-s)}V^0_\varepsilon(v(s)) + c_2\int_s^t e^{-\gamma(t-\tau)}R_1^2(\theta_\tau\omega)d\tau, \quad t \ge s.$$

Therefore using (37) after integration with respect to s over the interval $[0,t]$ we obtain

$$V^0_\varepsilon(v(t)) \le \frac{c_1}{t}\|v_0\|^2_{\mathcal{H}}e^{-\gamma_* t} + c_2\left(1+\frac{1}{t}\right)\int_0^t e^{-\gamma_*(t-\tau)}R_1^2(\theta_\tau\omega)d\tau, \quad t > 0, \tag{47}$$

for some $0 < \gamma_* \le \gamma$. Relations (46) and (47) makes it possible to conclude that there exists a tempered random variable $R_*(\omega)$ such that the set

$$\mathfrak{B}(\omega) = \left\{v : V_\varepsilon(v) \le R_*^2(\omega)\right\} \tag{48}$$

is forward invariant and absorbing. It is clear that $\mathfrak{B}(\omega)$ is compact in $\mathcal{H}$ for each $\omega \in \Omega$. Moreover, $R_*^2(\omega)$ does not depend on ε. ∎

Completion of the proof of Proposition 5.1. The proof follows from Theorem 2.3 and Lemmata 5.2 and 5.3. Relation (31) follows from (47), (48), and properties of the functionals V^0_ε and V_ε given in (38), (42), and (43).

Remark 5.4. It also follows from (44) and (37) that

(49)
$$\int_0^t \tau e^{-\gamma_*(t-\tau)} \|\partial_t v_\varepsilon(\tau)\|^2_{\mathcal{H}} d\tau \le c_1 \|v_0\|^2_{\mathcal{H}} e^{-\gamma_* t} + c_2 \int_0^t (1+\tau) e^{-\gamma_*(t-\tau)} R_1^2(\theta_\tau \omega) d\tau,$$

for all $t \ge 0$, where $\gamma_* > 0$. Below we will also need the next lemma.

LEMMA 5.5. *For any initial data $v, v_* \in \mathcal{H}$ we have the estimate*

$$\|\phi_\varepsilon(t,\omega)v - \phi_\varepsilon(t,\omega)v_*\|_{\mathcal{H}} \le c_1 e^{c_2 t} \|v - v_*\|_{\mathcal{H}}, \quad \omega \in \Omega, \tag{50}$$

where c_1 and c_2 do not depend on $\omega \in \Omega$ and $\varepsilon \in (0,1]$.

Proof. We use the same method as in Lemma 5.2 by considering the difference of two solutions and relying on the property

$$\left(f_i(v^i + \bar{\eta}) - f_i(v^i_* + \bar{\eta})\right)\,(v^i - v^i_*) \ge -c_0 |v^i - v^i_*|^2,$$

where c_0 does not depend on ω and ε. □

5.2. Limiting system. The same change of unknown variable $U = v + \bar{\eta}$ transforms equation (8) into the following random PDE on Γ:

$$\begin{cases} \partial_t v - \nu \Delta v + av + f(v + \bar{\eta}(\theta_t \omega)) + h(x') = 0, \quad t > 0,\ x' \in \Gamma, \\ \frac{\partial v}{\partial n}\big|_{\partial\Gamma} = 0,\ v|_{t=0} = v_0, \end{cases} \tag{51}$$

where ν, $f(v)$, and h are given by (9). The same argument as in section 4 allows us to prove that problem (51) generates an RDS (θ, ϕ_0) in the space $L_2(\Gamma)$ and thus to establish Theorem 3.1(2).

The following assertion states the existence of a pullback attractor for this RDS (θ, ϕ_0).

PROPOSITION 5.6. *In the space $L_2(\Gamma)$, problem* (51) *generates an RDS (θ, ϕ_0) possessing a compact pullback attractor $\{\mathfrak{A}^0(\omega)\}$ which belongs to the space $H^1(\Gamma)$. If the correlation operator K possesses the properties* (i) *$Kh = 0$ if and only if $h = 0$ and* (ii) *the image of K is dense in $L_2(\Gamma)$, then the attractor $\{\mathfrak{A}^0(\omega)\}$ is a singleton; i.e., there exists a tempered random variable $v_0(\omega)$ with values in $H^1(\Gamma)$ such that $\mathfrak{A}^0(\omega) = \{v_0(\omega)\}$ for all $\omega \in \Omega$.*

Proof. To prove the existence of the attractor we argue exactly as in Proposition 5.1 and we do not repeat it again.

As for the second part, we first note that the RDS (θ, ϕ_0) is *monotone*; i.e., the property $v(x) \le v_*(x)$ for almost all $x \in \Gamma$ implies that

$$[\phi_0(t,\omega)v](x) \le [\phi_0(t,\omega)v_*](x) \quad \text{for almost all} \quad x \in \Gamma,$$

for all $t > 0$, and for $\omega \in \Omega$. This monotonicity property can be established by the standard (pathwise) argument (see, e.g., [31]). We also refer to [10] for a general discussion of monotone RDSs. Our next step is to apply a result from [15] which states that, under some conditions, the global pullback attractor of a monotone RDS consists of a single random equilibrium. The main hypothesis in [15] is the weak convergence of distributions of the process $t \mapsto \phi_0(t,\omega)v$ to some limiting probability measure. In our case we can guarantee this property because the noise $\dot{W}$ is nondegenerate in the phase space of the system (θ, ϕ_0). We refer to [15, subsection 4.5] for details. □

Propositions 5.1 and 5.6 imply Theorem 3.1(4).

Remark 5.7. Although it is possible to prove that the RDS $(\theta, \phi_\varepsilon)$ generated by problem (25)–(28) is also monotone, we cannot apply the result from [15] to prove that $\hat{\mathfrak{A}}^\varepsilon$ is a single equilibrium. The point is that the noise $\dot{W}$ is nondegenerate in $L_2(\Gamma)$ (the phase space of the system (θ, ϕ_0)), but it is *degenerate* in $\mathcal{H} = L_2(D)$ (the phase space for $(\theta, \phi_\varepsilon)$), and hence we cannot guarantee the weak convergence of distributions of the process $t \mapsto \phi_\varepsilon(t,\omega)U_0$. Thus the pullback attractor $\hat{\mathfrak{A}}^\varepsilon$ may contain more than one equilibrium. The same conclusion is valid for problem (19). One can prove that (19) generates a monotone RDS with a compact pullback attractor, but to conclude that this attractor is a random equilibrium we need the nondegeneracy of the noise in $L_2(\Gamma) \times L_2(\Gamma)$, which is obviously not true for this case.

Remark 5.8. 1. It is clear from the argument in the proof of Lemma 5.5 that

$$\|\phi_0(t,\omega)v - \phi_0(t,\omega)v_*\|_{L_2(\Gamma)} \le c_1 e^{c_2 t}\|v - v_*\|_{L_2(\Gamma)}, \quad \omega \in \Omega, \tag{52}$$

for some constants c_1 and c_2 independent of ω, where $v, v_* \in L_2(\Gamma)$.

2. Since $L_2(\Gamma)$ can be embedded naturally into $L_2(D) \sim \mathcal{H}$ as the subspace of functions independent of y, we can consider the cocycle ϕ_0 as a mapping from $L_2(\Gamma)$ into $\mathcal{H}$. Therefore we can compare it with ϕ_ε. Below we also consider the image $\widetilde{\mathfrak{A}}^0(\omega)$ of $\mathfrak{A}^0(\omega)$ under this embedding.

6. Limit transition on finite time intervals. Our main result in this section is the following theorem, which implies the third statement in Theorem 3.1.

THEOREM 6.1. *For any time interval we have that*

$$\lim_{\varepsilon\to 0} \sup_{t\in[\delta,T]} \|\phi_\varepsilon(t,\omega)v - \phi_0(t,\omega)v_*\|_{\mathcal{H}} = 0 \quad \forall \delta \in (0,T), \tag{53}$$

where $v_* = \langle v \rangle := \frac{1}{2}\int_{-1}^{1} v(x,y)dy$. *If* v *does not depend on* y, *i.e.*, $v = v_*$, *then*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|\phi_\varepsilon(t,\omega)v - \phi_0(t,\omega)v_*\|_{\mathcal{H}} = 0. \tag{54}$$

Proof. Let $w_\varepsilon(t) = \phi_\varepsilon(t,\omega)v$. It follows from (37), (47), and (49) that

$$\sup_{t\in[0,T]} \sum_{i=1,2} \|w_\varepsilon^i(t)\|^2_{L_2(D_i)} + \int_0^T \|w_\varepsilon^i(t)\|^2_{H^1(D_i)} dt \le C_T(\omega), \tag{55}$$

and, for every $\delta > 0$,

$$\sup_{t\in[\delta,T]} \sum_{i=1,2} \|w_\varepsilon^i(t)\|^2_{H^1(D_i)} + \sum_{i=1,2} \int_\delta^T \|\partial_t w_\varepsilon^i(t)\|^2_{L_2(D_i)} dt \le C_{T,\delta}(\omega), \tag{56}$$

$$\frac{1}{\varepsilon^2}\left[\sup_{t\in[\delta,T]} \sum_{i=1,2} \|\partial_y w_\varepsilon^i(t)\|^2_{L_2(D_i)} + \sum_{i=1,2} \int_0^T \|\partial_y w_\varepsilon^i(t)\|^2_{L_2(D_i)} dt\right] \le C_{T,\delta}(\omega). \tag{57}$$

Moreover, we have that

(58)

$$\sup_{t\in[\delta,T]} \int_\Gamma \frac{k(x',\varepsilon)}{\varepsilon} |w_\varepsilon^1(t) - w_\varepsilon^2(t)|^2 dx' + \int_0^T dt \int_\Gamma \frac{k(x',\varepsilon)}{\varepsilon} |w_\varepsilon^1(t) - w_\varepsilon^2(t)|^2 dx' \le C_{T,\delta}(\omega)$$

for all intervals $[0,T]$ and $\varepsilon \in (0,1]$. Therefore, using relations (55)–(57) and Aubin's compactness theorem we can conclude that there exist a pair of functions

$$u_i \in C(\delta, T; L^2(\Gamma)) \cap L^\infty(\delta, T; H^1(\Gamma)), \quad i = 1, 2, \quad \forall \delta > 0$$

and a sequence $\{\varepsilon_n\}$ such that

$$\lim_{n\to\infty} \sum_{i=1,2} \sup_{t\in[\delta,T]} \|w^i_{\varepsilon_n}(t) - u_i(t)\|_{L_2(D_i)} = 0. \tag{59}$$

Moreover, we also have weak convergence in $L_2(0,T;H^1(D))$. One can also see from (58) and (6) that $u_1(t) = u_2(t) \equiv u(t)$ on the set Γ. Considering a variational form of (25)–(28), one can show that $u(t)$ solves problem (51). The corresponding argument is exactly the same as in [13] for the deterministic case and therefore we do not give details here. Thus (53) follows from (59) and from the uniqueness theorem for (51).

To prove (54) we first consider $v \equiv v_*$ from the space $H^1(\Gamma) \cap L_{p+1}(\Gamma)$. In this case relying on (46) with $s = 0$ and using the fact that $V_\varepsilon(v)$ does not depend on ε for this choice of v, we can easily prove estimates (56) and (57) with $\delta = 0$. Thus the same argument as above gives (54) for $v \equiv v_*$ from $H^1(\Gamma) \cap L_{p+1}(\Gamma)$. To obtain (54) for $v_* \in L_2(\Gamma)$ we use an appropriate approximation procedure and relations (50) and (52). □

Remark 6.2. By a standard argument we can prove that (53) and (54) hold uniformly with respect to v in every compact set.

Remark 6.3. Since the arguments given in Lemmata 5.2 and 5.3 do not depend on the behavior of $k(x,\varepsilon)$ as $\varepsilon \to 0$, the estimates in (55)–(58) hold for *both* cases (6) and (18). Thus, in the latter case, we can also conclude from (55)–(57) that w^1_ε and w^2_ε converge to some functions u^1 and u^2 defined on Γ. However, in that case we cannot prove that u^1 and u^2 are the same because under condition (18) estimate (58) does not lead to the conclusion. In the case (18) the same arguments as in [12, 13] give us the convergence of $\bar{\phi}_\varepsilon(t,\omega)$ generated by (1)–(4) to the cocycle generated by (19).

7. Upper semicontinuity of attractors. In this section we prove the following assertion, which is our first result on synchronization.

THEOREM 7.1. *Let* $\{\mathfrak{A}^\varepsilon(\omega)\}$ *be the global random pullback attractor for the RDS* $(\theta, \phi_\varepsilon)$ *generated by* (25)–(28). *Then*

$$\lim_{\varepsilon\to 0} \sup \left\{ \mathrm{dist}_{\mathcal{H}} \left(u, \widetilde{\mathfrak{A}}^0(\omega) \right) \, : \, u \in \mathfrak{A}^\varepsilon(\omega) \right\} = 0 \quad \forall \omega \in \Omega, \tag{60}$$

where $\widetilde{\mathfrak{A}}^0(\omega) = \left\{ J(v) \, : v \in \mathfrak{A}^0(\omega) \right\} \subset \mathcal{H}$. *Here* $\{\mathfrak{A}^0(\omega)\}$ *is the random pullback attractor for the RDS* (θ, ϕ_0) *and* $J \, : \, L_2(\Gamma) \mapsto L_2(D) = \mathcal{H}$ *is the natural embedding operator.*

Proof. Assume that (60) does not hold for some $\omega \in \Omega$. Then there exist a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and a sequence $u_n \in \mathfrak{A}^{\varepsilon_n}(\omega)$ such that

$$\mathrm{dist}_{\mathcal{H}}(u_n, \widetilde{\mathfrak{A}}^0(\omega)) \ge \delta > 0 \quad \forall \, n = 1, 2, \ldots. \tag{61}$$

By the invariance property of the attractor $\mathfrak{A}^{\varepsilon_n}(\omega)$, for every $t > 0$ there exists $v^t_n \in \mathfrak{A}^{\varepsilon_n}(\theta_{-t}\omega)$ such that $u_n = \phi_{\varepsilon_n}(t, \theta_{-t}\omega)v^t_n$. Since $\mathfrak{A}^{\varepsilon_n}(\omega)$ is compact and estimate (31) holds, we can assume that there exist u_* and v^t_* in $H^1(D_1) \oplus H^1(D_2)$ such that

$$\lim_{n\to\infty} \|u_n - u_*\|_{\mathcal{H}} = 0, \quad \lim_{n\to\infty} \|v^t_n - v^t_*\|_{\mathcal{H}} = 0. \tag{62}$$

As in the proof of Theorem 6.1 one can see that

$$u_* = \tilde{u} \oplus \tilde{u}, \quad v_*^t = \tilde{v}^t \oplus \tilde{v}^t,$$

where $\tilde{u}, \tilde{v}^t \in H^1(\Gamma)$. Therefore, if we show that $\tilde{u} \in \mathfrak{A}^0(\omega)$, then we obtain a contradiction to (61).

It follows from Lemma 5.5 and Theorem 6.1 that

$$\tilde{u} = \phi_0(t, \theta_{-t}\omega)\tilde{v}^t.$$

However, it follows from (31) and (62) that $\tilde{v}^t \in B_0(\theta_{-t}\omega)$, where

$$B_0(\omega) = \left\{ v \in H^1(\Gamma) \ : \ \|v\|_{H^1(\Gamma)} \le \tilde{R}(\omega) \right\},$$

where $\tilde{R}(\omega)$ is a tempered random variable. Thus we have that

$$\tilde{u} \in \phi_0(t, \theta_{-t}\omega) B_0(\theta_{-t}\omega) \quad \text{for every} \quad t > 0.$$

Since $\phi_0(t, \theta_{-t}\omega) B_0(\theta_{-t}\omega) \to \mathfrak{A}^0(\omega)$ as $t \to \infty$, this implies that $\tilde{u} \in \mathfrak{A}^0(\omega)$. □

Theorem 3.1(5) follows from Theorem 7.1.

Remark 7.2. In the case (18), similarly to the deterministic case (see [12, 13]), we can prove the upper convergence of the pullback attractors $\hat{\mathfrak{A}}^\varepsilon$ to the corresponding object for the RDS generated by (19). We also refer to [6] and to the references therein for a general study of upper semicontinuity of random and nonautonomous attractors.

8. Synchronization for fixed $\varepsilon > 0$. Now we consider the case when the equations are the same in both domains; i.e., we assume that relations (14), (15), and (16) hold.

Under conditions (14) the cocycle ϕ_ε has a deterministic forward *invariant* subspace $\mathcal{L}$ in $\mathcal{H}$ consisting of functions which are independent of the variable y, i.e.,

$$\mathcal{L} = \{ u(x, y) \in L_2(D) \ : \ u(x, y) \equiv u(x, 0) \equiv v \in L_2(\Gamma) \}.$$

It is clear that $\phi_\varepsilon(t, \omega)\mathcal{L} \subset \mathcal{L}$ and $\phi_\varepsilon(t, \omega) \equiv \phi_0(t, \omega)$ on $\mathcal{L}$.

THEOREM 8.1. *Under conditions* (14), (15), *and* (16) *there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *the global random pullback attractor* $\mathfrak{A}^\varepsilon(\omega)$ *for* $(\theta, \phi_\varepsilon)$ *has the form*

$$\mathfrak{A}^\varepsilon(\omega) \equiv \widetilde{\mathfrak{A}}^0(\omega) = \left\{ J(v) \ : v \in \mathfrak{A}^0(\omega) \right\} \subset \mathcal{H}, \tag{63}$$

where $J : L_2(\Gamma) \mapsto L_2(D) = \mathcal{H}$ *is the natural embedding operator and* $\mathfrak{A}^0(\omega)$ *is the random pullback attractor for the RDS* (θ, ϕ_0).

Proof. Let P be the orthoprojector in $\mathcal{H}$ onto $\mathcal{L}$. This operator has the form

$$(Pu)(x, y) = \frac{1}{2} \int_{-1}^{1} u(x, \xi) d\xi, \quad u \in \mathcal{H} \sim L_2(D).$$

Let $Q = 1 - P$. Both of the operators P and Q map the domain $\mathcal{D}(A_\varepsilon)$ of the operator A_ε into itself and commute with A_ε. Therefore it follows from (30) that Qv_ε satisfies the equation

$$\frac{d}{dt} Qv_\varepsilon + A_\varepsilon Qv_\varepsilon = QB(v_\varepsilon, \theta_t\omega), \quad Qv|_{t=0} = Qv_0. \tag{64}$$

Multiplying this equation by Qv_ε we obtain

$$\frac{1}{2}\frac{d}{dt}\|Qv_\varepsilon\|^2_{\mathcal{H}} + a_\varepsilon(Qv_\varepsilon, Qv_\varepsilon)_{\mathcal{H}} = (QB(v_\varepsilon, \theta_t\omega), Qv_\varepsilon)_{\mathcal{H}}. \tag{65}$$

From (15) we have that

$$\begin{aligned}(QB(v_\varepsilon, \theta_t\omega), Qv_\varepsilon)_{\mathcal{H}} &= \int_D \left[f(v_\varepsilon(x,y)) - \frac{1}{2}\int_{-1}^{1} f(v_\varepsilon(x,\xi))d\xi\right] Qv_\varepsilon(x,y)dxdy \\ &\le \frac{L}{2}\int_D\int_{-1}^{1} |v_\varepsilon(x,y) - v_\varepsilon(x,\xi)|\, Qv_\varepsilon(x,y)d\xi dxdy \\ &\le \frac{L}{\sqrt{2}}\left[\int_\Gamma dx\int_{-1}^{1}dy\int_{-1}^{1}d\xi\, |v_\varepsilon(x,y) - v_\varepsilon(x,\xi)|^2\right]^{1/2} \|Qv_\varepsilon\|_{\mathcal{H}}.\end{aligned}$$

If we add and subtract Pv_ε in the expression under the integral, then we easily arrive at the relation

$$(QB(v_\varepsilon, \theta_t\omega), Qv_\varepsilon)_{\mathcal{H}} \le 2L\|Qv_\varepsilon\|^2_{\mathcal{H}}. \tag{66}$$

Thus from (65) we obtain that

$$\frac{1}{2}\frac{d}{dt}\|Qv_\varepsilon\|^2_{\mathcal{H}} + a_\varepsilon(Qv_\varepsilon, Qv_\varepsilon)_{\mathcal{H}} \le 2L\|Qv_\varepsilon\|^2_{\mathcal{H}}. \tag{67}$$

LEMMA 8.2. *Under conditions* (14) *and* (16) *we have that*

$$\lim_{\varepsilon\to 0}\sup\left\{\frac{a_\varepsilon(Qv_\varepsilon, Qv_\varepsilon)_{\mathcal{H}}}{\|Qv_\varepsilon\|^2_{\mathcal{H}}}\ : v \in \mathcal{H}^1_\varepsilon\right\} = +\infty. \tag{68}$$

Proof. Basically we use the same calculations of the spectrum of A_ε as in [11]. ∎

Lemma 8.2 implies that there exists $\varepsilon_0 > 0$ such that

$$\frac{d}{dt}\|Qv_\varepsilon\|^2_{\mathcal{H}} + \gamma_0\|Qv_\varepsilon\|^2_{\mathcal{H}} \le 0$$

for all $0 < \varepsilon \le \varepsilon_0$ and for some $\gamma_0 > 0$. Therefore,

$$\|Qv_\varepsilon(t)\|^2_{\mathcal{H}} \le \|Qv_\varepsilon(0)\|^2_{\mathcal{H}} e^{-\gamma_0 t}, \quad t \ge 0.$$

This implies that the subspace $\mathcal{L}$ attracts all tempered sets (in both the forward and the pullback sense) with exponential (deterministic) speed. Since $\phi_\varepsilon(t,\omega) \equiv \phi_0(t,\omega)$ on $\mathcal{L}$, this implies (63). ∎

Theorem 8.1 implies Theorem 3.1(6).

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