STABILITY OF GRADIENT SEMIGROUPS UNDER PERTURBATIONS

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Abstract. In this paper we prove that gradient-like semigroups (in the sense of [4]) are gradient semigroups (possess a Lyapunov function). This is primarily done to provide conditions under which gradient semigroups, in a general metric space, are stable under perturbation exploiting the known fact (see [4]) that gradient-like semigroups are stable under perturbation. The results presented here were motivated by the work carried out in [6] for groups in compact metric spaces (see also [14] for the Morse Decomposition of an invariant set for a semigroup on a compact metric space).

1. Introduction

The analysis of qualitative properties of semigroups in general phase spaces (infinite-dimensional Banach spaces or general metric spaces) has received much attention throughout the last four decades (see, for instance, [7], [10], [15] or [3]). In particular, the study of compact attracting invariant sets has developed into a large and deep research area, providing vital information for an increasing number of models for phenomena from different areas of Science such as Physics, Biology, Economics, Engineering and others.

When a system is shown to possess a global attractor, all its asymptotic behavior can be described by a detailed analysis of the internal dynamics on this compact invariant set. To this aim, a careful study of the geometrical structure -and its stability under perturbations- of the global attractor arises as a crucial fact. Probably the most general result in this line is what is now known as the *Fundamental Theorem of Dynamical Systems*, suggested in [11] from the results of [6], which describes any flow on a compact metric space as a decomposition of an ordered family of chain recurrent isolated invariant sets and order compatible connections between them. In the terminology of [6], this is called a Morse decomposition

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of a compact invariant set (see Definition 2.10), and has been considered in different frameworks, as in the case of flows ([6]) and semiflows on compact spaces ([14]), or even compact and non-compact topological spaces ([9, 12, 13]).

In [12, 13] the authors obtain (as a corollary of their general theory and using Conley's proof) the existence of a Lyapunov function from a Morse Decomposition for an open semiflow on a compact Hausdorff space or for a completely regular space. We do not assume that the semiflow is open or any additional property of the metric space nor we use Conley's proof.

On the other hand, very recently, it has been introduced in [4] the so-called *gradient-like semigroups* (see Definition 2.8) in Banach spaces (which does not requires the existence of a Lyapunov function) as an intermediate concept between gradient semigroups (i.e., those possessing a Lyapunov function) and semigroup possessing a gradient-like attractor; that is, an attractor that is characterized as the union of unstable sets of associated isolated invariant sets.

In this paper, given a gradient-like nonlinear semigroup in a general metric space, we construct a differentiable Lyapunov function for it proving that gradient-like nonlinear semigroups are in fact gradient semigroups. This is done without any compactness assumption on the associated group or semigroup and any additional assumption on the phase space in which it is defined. Our proofs, in comparison with the classical works as [6, 14], are quite different and considerably extends the results there. We adopt an approach that enables us to use the results on stability of gradient-like (see [4]) semigroups to obtain the stability of gradient semigroups under perturbations.

For the construction of the Lyapunov function we will firstly prove that a disjoint family of isolated invariant sets of a gradient-like semigroup on a general metric space can be reordered in such a way that it becomes a Morse decomposition for the global attractor. Again, the proofs are intuitive, focused on the dynamics of the semigroup and, for instance, not based on chain recurrence and related more classical notions in this theory. A refinement of the results from [6] would lead us to define a Lyapunov function, not only on the attractor but on the whole phase space. Indeed, we will say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathscr{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ is a *gradient semigroup* with respect to Ξ if there exists a continuous function $V : X \to \mathbb{R}$ such that $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$, V is constant in Ξ_i for each $1 \leq i \leq n$ and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$.

Our main result can now be stated as follows

Theorem 1.1. $\{T(t) : t \geq 0\}$ *is a gradient semigroup with respect to* Ξ *if and only if it is a gradient-like semigroup with respect to* Ξ*. In addition the Lyapunov function* $V : X \to \mathbb{R}$ *of a gradient-like semigroup may be chosen in such a way that* $V(\Xi_k) = k$ *for each* $k = 1, 2, \cdots, n$.

One of the straighforward consequences of the previous result is that, if the disjoint family of isolated invariant sets $\mathscr{E} = \{z_1, \cdots, z_n\}$ is made of stationary points, then a nonlinear semigroup is gradient in the sense of [7] (see Definition 3.8.1) if and only if it is gradient-like (see Definition 1.3 in [4]). Then, as gradient-like nonlinear semigroups are stable under perturbation (see [4]), we conclude that **gradient semigroups are stable under perturbation**; that is, the existence of a continuous Lyapunov function is robust under perturbation.

Observe that any Morse decomposition $\boldsymbol{\Xi} = (\Xi_1, \cdots, \Xi_n)$ of a compact invariant set $\mathscr{A}$ leads to a partial order among the isolated invariant sets Ξ_i; that is, we can define an order between two isolated invariant sets Ξ_i and Ξ_j if there is a chain of global solutions $\{\xi_\ell,\ 1 \leq \ell \leq r\}$, with $\lim_{t\to-\infty} \xi_\ell(t) = \Xi_\ell$ and $\lim_{t\to\infty} \xi_\ell(t) = \Xi_{\ell+1}$, $1 \leq \ell \leq \ell - 1$. This defines a partial order and some of the isolated invariant sets in $\boldsymbol{\Xi}$ may not be comparable. In Section 5 we introduce a new Morse decomposition of the attractor of a gradient-like semigroup which improves the construction and dynamical properties of its associated Lyapunov function. Indeed, we prove that, given any gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\boldsymbol{\Xi} = (\Xi_1, \cdots, \Xi_n)$, there exists another Morse decomposition given by the so-called *energy levels* $\mathcal{N} = (\mathscr{N}_1, \mathscr{N}_2, \cdots, \mathscr{N}_p)$ which guarantees a total order in $\mathscr{A}$. Each of the levels $\mathscr{N}_i$, $1 \leq i \leq p$ is made of a finite union of the isolated invariant sets in $\boldsymbol{\Xi}$ and $\mathcal{N}$ is totally ordered. The associated Lyapunov function has different values in any two different sets of $\mathcal{N}$ and any two elements of $\boldsymbol{\Xi}$ which are contained in the same element of $\mathcal{N}$ (same energy level) are not connected.

Before we proceed, let us exhibit several classes of examples that, so far, were not known to be gradient semigroups and that the results in this paper show that they indeed are.

Example 1.2. *Let $A = (a_{i,j})_{i,j=1}^n \in M^{n\times n}(\mathbb{R})$ be a $n \times n$ matrix which satisfies $a_{i,j} = 0$ if $j > i$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ be such that there is a $\xi > 0$ with $a_{i,i}s^2 + f(s)s < 0$ whenever $|s| \geq \xi$ and $1 \leq i \leq n$. It is easy to see that the semigroup $\{T(t) : t \geq 0\}$ associated to the problem*

$$\begin{aligned} \dot{u} &= Au + F(u) \\ u(0) &= u_0 \in \mathbb{R}^n \end{aligned} \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ and $F(u) = (f(u_1), \cdots, f(u_n))^\top$, has a global attractor $\mathscr{A}$ in $\mathbb{R}^n$. Assume that all equilibria of (1.1) are hyperbolic. Following [4] it is not difficult to see that all global bounded solutions of (1.1) are forwards and backwards asymptotic to equilibria and that homoclinic structures are not allowed. Hence the semigroup associated to (1.1) is gradient-like.

We remark that it is not known whether there is a Lyapunov function for (1.1) and that the results proved next ensure that there is a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ which is continuous, decreasing along solutions and with the property that $V(T(t)u) = V(u)$ for all $t \geq 0$ if and only if u is an equilibrium point of (1.1), that is the semigroup $\{T(t) : t \geq 0\}$ (1.1) is gradient (in $\mathbb{R}^n$).

The problem (1.1) *can be changed to a general* cascade *system of the form*

$$\begin{aligned} \dot{u} &= G(u) \\ u(0) &= u_0 \in \mathbb{R}^n \end{aligned} \tag{1.2}$$

with $G \in C^1(\mathbb{R}^n, \mathbb{R}^n)$*,* $G(u) = (G_1(u), \cdots, G_n(u))$ *such that* (1.2) *has a global attractor, the equilibria of* (1.2) *are all hyperbolic and* $G_i(u) = G_1(u_1, \cdots, u_i)$*,* $1 \le i \le n$*. Under these assumptions, the semigroup associated to* (1.2) *is gradient in* $\mathbb{R}^n$*.*

Similar cascade systems for partial differential equations can easily be constructed and we remark that a C^1 *small* perturbation of (1.2) will also be gradient-like (according to the results in [4]). Consequently, a C^1 small perturbation of (1.2) will also give us a gradient semigroup. We present next another example of coupled parabolic partial differential equations with a different nature to show the variety of applicability of our results.

Example 1.3. *If* $f \in C^2(\mathbb{R})$*,* $\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0$*,* $\Omega \subset \mathbb{R}^n$ *is a bounded smooth domain,* $P_0 \in \bar{\Omega}$ *and* $p > n$*, then*

$$\begin{cases} u_t = \Delta u + f(u), \ x \in \Omega, \ t > 0, \\ \partial_n u(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \\ u(\cdot, 0) = u_0 \in W^{1,p}(\Omega) \end{cases}$$

$$\begin{cases} v_t = v_{xx} + f(v), \ x \in (0,1), \ t > 0 \\ v(0,t) = u(P_0, t), \ v_x(1,t) = 0, \\ v(\cdot, 0) = v_0 \in W^{1,p}(0,1) \end{cases}$$

and for $a \in C(\bar{\Omega}, \mathbb{R}^n)$

$$\begin{cases} u_{tt} + \beta u_t = \Delta u + \epsilon a(x) \cdot \nabla u + f(u), \ x \in \Omega, \ t > 0 \\ u(x,t) = 0, \ x \in \partial\Omega, \ t > 0 \\ u(\cdot, 0) = u_0 \in H^1_0(\Omega), \ u_t(\cdot, 0) = v_0 \in L^2(\Omega) \end{cases}$$

correspond to gradient-like semigroups in $W^{1,p}(\Omega) \times W^{1,p}(0,1)$ *and* $H^1_0(\Omega) \times L^2(\Omega)$ *respectively if all the associated equilibria are hyperbolic and* $\epsilon > 0$ *is suitable small (see* [1, 2] *and* [4]*). Hence the associated semigroups are gradient.*

With this, it becomes evident that the class of semigroups **known** to be gradient increases considerably after our result is proved.

In a general metric space X, consider a gradient-like semigroup $\{T(t) : t \ge 0\}$ with respect to the isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ and with a global attractor $\mathscr{A}$. In Section 2 we prove that $\boldsymbol{\Xi}$ can be reordered in such a way that it becomes a Morse decomposition for $\mathscr{A}$. In Section 3 we construct a Lyapunov function in X for $\{T(t) : t \ge 0\}$, showing that a gradient like semigroup is a gradient semigroup. Section 4 is dedicated to a recollection of the results in [4] to conclude that the gradient semigroups are stable under perturbation. In Section 5 we introduce the level grouping isolated invariant sets with same dynamical characteristics. Finally, in Section 6 we make some final comments and present some of our future research on the subject.

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2. Morse decomposition of global attractors for gradient-like semigroups

In this section we present the notions of gradient-like semigroups and of Morse decomposition for a global attractor as well as the relationship between them. In order to do that we will need to introduce some basic notions and results (see [7] for example).

Let X be a metric space with metric $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, and denote by $\mathscr{C}(X)$ the set of continuous maps from X into X. Given a subset $A \subset X$, the ϵ-*neighborhood* of A is the set $\mathcal{O}_\epsilon(A) := \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}$

Next we introduce the notion of semigroups in the metric space X.

Definition 2.1. *A family* $\{T(t) : t \geq 0\} \subset \mathscr{C}(X)$ *is a* semigroup *in* X *if*

- $T(0) = I_X$, *with* I_X *being the identity map in* X,
- $T(t + s) = T(t)T(s)$, *for all* $t, s \in \mathbb{R}^+$ *and*
- $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ *is continuous.*

The notion of invariance plays a fundamental role in the study of the asymptotic behavior of semigroups.

Definition 2.2. *A subset* A *of* X *is said* invariant *under the action semigroup* $\{T(t) : t \geq 0\}$ *if* $T(t)A = A$ *for all* $t \geq 0$.

Now we will introduce the notions of attraction and absorption. For that we recall the definition of *Hausdorff semi-distance*. Given $A, B \subset X$, the Hausdorff semidistance from A to B is given by

$$d_H(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Definition 2.3. *Given two subsets* A, B *of* X *we say that* A attracts B *under the action of the semigroup* $\{T(t) : t \geq 0\}$ *if* $d_H(T(t)B, A) \overset{t\to\infty}{\longrightarrow} 0$ *and we say that* A *absorbs* B *under the action of* $\{T(t) : t \geq 0\}$ *if there is a* $t_B > 0$ *such that* $T(t)B \subset A$ *for all* $t \geq t_B$.

With this we are in the position to define *global attractors.*

Definition 2.4. *A subset* $\mathscr{A}$ *of* X *is a* global attractor *for a semigroup* $\{T(t) : t \geq 0\}$ *if it is compact, invariant under the action of* $\{T(t) : t \geq 0\}$ *and for every bounded subset* B *of* X *we have that* $\mathscr{A}$ *attracts* B *under the action of* $\{T(t) : t \geq 0\}$.

Next we seek to introduce the notion of gradient-like semigroups (see [4]). To that end we first need the definition of isolated invariant set.

Definition 2.5. *Let $\{T(t) : t \geq 0\}$ be a semigroup. We say that an invariant set $\Xi \subset X$ for the semigroup $\{T(t) : t \geq 0\}$ is an* isolated invariant set *if there is an $\epsilon > 0$ such that Ξ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.*

A disjoint *family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that, for some $\epsilon > 0$,*

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing,\ 1 \leq i < j \leq n.$$

Definition 2.6. *A* global solution *for a semigroup $\{T(t) : t \geq 0\}$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $T(t)\xi(s) = \xi(t+s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a* global solution through $x \in X$ *if it is a global solution with $\xi(0) = x$.*

Definition 2.7. *Let $\{T(t) : t \geq 0\}$ be a semigroup which has a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. A* homoclinic structure *associated to $\mathbf{\Xi}$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\mathbf{\Xi}$ $(p \leq n)$ together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that*

$$\Xi_{k_j} \overset{t\to-\infty}{\longleftarrow} \xi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_{k_{j+1}},\ 1 \leq j \leq p$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$ and, if $p = 1$, $\xi_1(\mathbb{R}) \subsetneq \Xi_{k_1}$.

We are now ready to define gradient-like semigroups.

Definition 2.8. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathscr{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T(t) : t \geq 0\}$ is a* gradient-like semigroup relative to $\mathbf{\Xi}$ *if:*

- *For any global solution $\xi : \mathbb{R} \to \mathscr{A}$ there are $1 \leq i, j \leq n$ such that*
$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$
- *There is no homoclinic structure associated to $\mathbf{\Xi}$.*

It is important to notice that Definition 2.8 only uses dynamical properties of the semigroup (that is, only the inner structure of the attractor) and **does not** assume a priori the existence of any kind of Lyapunov function (see Definition 3.1 below). Differently from the concept, with the same name, in [6] and [14] (see Section I.6 in [6] and Definition 5.2 in [14]) that assumes the existence of a Lyapunov function.

Now we will introduce the notion of a Morse decomposition for an attractor $\mathscr{A}$ of a semigroup $\{T(t) : t \geq 0\}$ (see [6] and [14]). We start with the notion of attractor-repeller pairs.

Definition 2.9. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathscr{A}$. We say that a non-empty subset A of $\mathscr{A}$ is a* local attractor *if there is an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(A)) = A$. The* repeller *$A^*$ associated to a local attractor A is the set defined by*

$$A^* := \{x \in \mathscr{A} : \omega(x) \cap A = \varnothing\}.$$

The pair (A, A^*) is called attractor-repeller pair *for $\{T(t) : t \geq 0\}$.*

Note that if A is a local attractor, then A^* is closed and invariant.

Definition 2.10. *Given an increasing family $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathscr{A}$, of $n+1$ local attractors, for $j = 1, \cdots, n$, define $\Xi_j := A_j \cap A^*_{j-1}$. The ordered n-upla $\mathbf{\Xi} := (\Xi_1, \Xi_2, \cdots, \Xi_n)$ is called a* Morse decomposition for $\mathscr{A}$.

Observe that A is a local attractor if and only if it is compact invariant and attracts $\mathcal{O}_\epsilon(A)$ for some $\epsilon > 0$.

We observe that the above definition differs slightly from the usual definition since the local attractor is required to attract a neighborhood of A in X and not only in $\mathscr{A}$ as in [6, 14]. That is needed because, in this way, we can to show the continuity of the Lyapunov function on whole space X (see Proposition 3.3 below) and not only on the attractor, as is done in [6].

We will prove next that local attraction inside $\mathscr{A}$ is equivalent to local attraction in whole space X. To that end, the next result plays a crucial role, actually it is the key result to almost everything that we do in this work. In some sense, it helps us to deal with the absence of reversibility of the semigroup in the proof of Proposition 3.3 and, furthermore, it improves part a) of Proposition 1.3 in [14].

Lemma 2.11. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathscr{A}$. If $A \subset \mathscr{A}$ is a compact invariant set for $\{T(t) : t \geq 0\}$ and there is an $\epsilon > 0$ such that A attracts $\mathcal{O}_\epsilon(A) \cap \mathscr{A}$ then, given $\delta \in (0, \varepsilon)$ there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$, where $\gamma^+(\mathcal{O}_{\delta'}(A)) = \bigcup_{x \in \mathcal{O}_{\delta'}(A)} \bigcup_{t \geq 0} \{T(t)x\}$.*

Proof. If that conclusion is false, there is an $0 < \delta < \epsilon$ such $\gamma^+(\mathcal{O}_{\delta'}(A)) \not\subset \mathcal{O}_\delta(A)$ for each $\delta' \in (0, \delta)$. So there are $x \in A$, $X \ni x_n \stackrel{n\to\infty}{\longrightarrow} x$ and $\mathbb{R} \ni t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ such that $d(T(t_n)x_n, A) = \delta$ and $T(t)x_n \in \mathcal{O}_\delta(A)$, $t \in [0, t_n)$. Since the $\{T(t) : t \geq 0\}$ has a global attractor it is not difficult to see that there is a global solution $\xi : \mathbb{R} \to X$ such that $\xi_n : [-t_n, \infty) \to X$, given by $\xi_n(t) := T(t_n + t)x_n$, satisfies $\xi_n(t) \stackrel{n\to\infty}{\longrightarrow} \xi(t)$ for each $t \in \mathbb{R}$. Clearly $\xi(t) \in \overline{\mathcal{O}_\delta(A)} \cap \mathscr{A} \subset \mathcal{O}_\epsilon(A) \cap \mathscr{A}$ for all $t \leq 0$, and $d(\xi(0), A) = \delta$, and consequently A cannot attract $\mathcal{O}_\epsilon(A) \cap \mathscr{A}$. □

We remark that if A is a local attractor for a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathscr{A}$, then A is in the conditions of the above lemma.

The next result generalizes to semigroups a known result for groups given in [6], and shows that our definition of local attractor is equivalent to that found in [6, 14].

Lemma 2.12. *If $\{T(t) : t \geq 0\}$ is a semigroup in X with a global attractor $\mathscr{A}$ and $S(t) := T(t)|_{\mathscr{A}}$, clearly $\{S(t) : t \geq 0\}$ is a semigroup in the metric space $\mathscr{A}$. If A is a local attractor for $\{S(t) : t \geq 0\}$ in the metric space $\mathscr{A}$ (that is, there is a $\varepsilon > 0$ with $\omega(\mathcal{O}_\epsilon(A) \cap \mathscr{A}) = A$) and K is a compact subset of $\mathscr{A}$ such that $K \cap A^* = \varnothing$, then A attracts K. Furthermore A is a local attractor for $\{T(t) : t \geq 0\}$ in X.*

Proof. Let K be a compact subset of $\mathscr{A}$ such that $K \cap A^* = \varnothing$. From Lemma 2.11, given $0 < \delta < \epsilon$, there is a $0 < \delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$. If $A = \omega(\mathcal{O}_\epsilon(A) \cap \mathscr{A})$ does not attracts K, there are $t_n \overset{n\to\infty}{\longrightarrow} \infty$, $x \in K$ and $K \ni x_n \overset{n\to\infty}{\longrightarrow} x$ such that $d(T(t)x_n, A) \geqslant \delta'$, $0 \leqslant t \leqslant t_n$. Hence $d(T(t)x, A) \geqslant \delta'$ for all $t \geqslant 0$ proving that $\omega(x) \cap A = \varnothing$. It follows that $x \in A^*$ and that is a contradiction.

For the rest of the proof note that $\omega(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\epsilon(A) \cap \mathscr{A}$ and, consequently, $\omega(\mathcal{O}_{\delta'}(A)) \cap A^* = \varnothing$. From the invariance of $\omega(\mathcal{O}_{\delta'}(A))$ and from the property that A attracts $\mathcal{O}_\epsilon(A) \cap \mathscr{A}$, we must have that $\omega(\mathcal{O}_{\delta'}(A)) \subset A$. Since $\omega(\mathcal{O}_{\delta'}(A))$ attracts $\mathcal{O}_{\delta'}(A)$, the result follows. □

Lemma 2.13. *Let $\{T(t) : t \geq 0\}$ be a semigroup in X with a global attractor $\mathscr{A}$ and (A, A^*) an attractor-repeller for $\{T(t) : t \geq 0\}$. Then:*

(i) If $\xi : \mathbb{R} \to X$ is a global bounded solution for $\{T(t) : t \geq 0\}$ through $x \notin A \cup A^*$, then $\xi(t) \overset{t\to\infty}{\longrightarrow} A$ and $\xi(t) \overset{t\to-\infty}{\longrightarrow} A^*$.

(ii) A global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ with the property that $\xi(t) \in \mathcal{O}_\delta(A^*)$ for all $t \leq 0$ for some $\delta > 0$ such that $\mathcal{O}_\delta(A^*) \cap A = \varnothing$ must satisfy $d(\xi(t), A^*) \overset{t\to-\infty}{\longrightarrow} 0$.

(iii) If $x \in X \backslash \mathscr{A}$ then, $T(t)x \overset{t\to\infty}{\longrightarrow} A \cup A^*$.

Proof. Part (i) follows from the Theorem 1.4 in [14].

Part (ii) is divided into two cases ($\overline{\xi(\mathbb{R})} \cap A^* = \varnothing$ and $\overline{\xi(\mathbb{R})} \cap A^* \neq \varnothing$). If $\overline{\xi(\mathbb{R})} \cap A^* = \varnothing$, from Lemma 2.12 we have that $\overline{\xi(\mathbb{R})} \subset A$ which gives a contradiction. On the other hand, if $\overline{\xi(\mathbb{R})} \cap A^* \neq \varnothing$, it follows from (i) that $d(\xi(t), A^*) \overset{t\to-\infty}{\longrightarrow} 0$. This completes the proof of (ii).

Part (iii) is proved as follows. Let $\delta > 0$ be such that $\omega(\mathcal{O}_\delta(A)) = A$ and we chose $\delta' \in (0, \delta)$ with $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$. Then, if there is a $t_0 > 0$ such that $T(t_0)x \in \mathcal{O}_{\delta'}(A)$, we have that $\lim_{t\to\infty} d(T(t)x, A) = 0$. On the other hand, if

$$d(T(t)x, A) \geq \delta' \text{ for all } t \geq 0$$

then $\omega(x) \cap A = \varnothing$ and we must have that $\omega(x) \subset A^*$. Since $\omega(x)$ atracts x we must have that $T(t)x \overset{t\to\infty}{\longrightarrow} A^*$, completing the proof of (iii). □

Part (i) of the previous lemma is done in Theorem 1.4 in [14], the conclusion (iii) (not found in [14, 6]) is needed, in our case, to obtain the continuity of the Lyapunov function in points that do not belong to $\mathscr{A}$.

Corollary 2.14. *If $\{T(t) : t \geq 0\}$ is a semigroup in X with a global attractor $\mathscr{A}$ and (A, A^*) is an attractor-repeller pair for $\{T(t) : t \geq 0\}$, then $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with respect to to the disjoint family of isolated invariant sets $\{A, A^*\}$.*

Next we describe the construction of a Morse decomposition for the attractor of a gradient-like semigroup (relative to the disjoint family of isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$). This is done by explaining how we can obtain the increasing collection of local attractors starting from the collection of isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$. The following lemmas play a fundamental role on that.

Lemma 2.15. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathscr{A}$ and let Ξ be a compact isolated invariant set such that $W^u(\Xi) = \Xi$. Then Ξ is a local attractor; that is, there is a $\delta > 0$ such that $\omega(\mathcal{O}_\delta(\Xi)) = \Xi$.*

Proof. Let $\delta_0 > 0$ be such that Ξ is the maximal invariant set in $\mathcal{O}_{\delta_0}(\Xi)$. Let us prove that, given $\delta \in (0, \delta_0)$, there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$. In fact, if the result was not true, there would exist a $\delta \in (0, \delta_0)$, a sequence $\{x_\ell\}$ in X with $d(x_\ell, \Xi) \stackrel{\ell\to\infty}{\longrightarrow} 0$ and a sequence $\{t_\ell\}$ in $(0, \infty)$ with $t_\ell \stackrel{\ell\to\infty}{\longrightarrow} \infty$ (this follows from the invariance of Ξ and from the continuity of the semigroup) such that $d(T(t_\ell)x_\ell, \Xi) = \delta$ and $d(T(t)x_\ell, \Xi) \leq \delta$ for all $0 \leq t \leq t_\ell$. Since $\{T(t) : t \geq 0\}$ has a global attractor, there is a global solution $\xi : \mathbb{R} \to \mathscr{A}$ such that $\xi_\ell : [-t_\ell, \infty) \to X$, $\xi_\ell(t) := T(t+t_\ell)x_\ell$ for $t \geq -t_\ell$, satisfies $\xi_\ell(t) \to \xi(t)$ for all $t \in \mathbb{R}$ (see Lemma 3.1 in [5] for more details). Clearly $\xi(t) \in \mathcal{O}_\delta(\Xi)$ for all $t \leq 0$ and $d(\xi(0), \Xi) = \delta$. Now, if $z = \xi(0)$, $\alpha_\xi(z) := \{x \in X : \lim_{n\to\infty} \xi(-t_n) = x$, for some sequence $t_n \stackrel{n\to\infty}{\longrightarrow} \infty\}$ is invariant, attracts z and, since Ξ is maximal invariant in $\mathcal{O}_{\delta_0}(\Xi)$, is contained in Ξ. Hence we must have that $\xi(t) \stackrel{t\to-\infty}{\longrightarrow} \alpha_\xi(z) \subset \Xi$ and that is a contradiction with the fact that $W^u(\Xi) = \Xi$.

It follows from the above that, for any $\delta \in (0, \delta_0)$, there is a $\delta' \in (0, \delta)$ such that $\omega(\mathcal{O}_{\delta'}(\Xi)) \subset \overline{\gamma^+(\mathcal{O}_{\delta'}(\Xi))} \subset \overline{\mathcal{O}_\delta(\Xi)} \subset \mathcal{O}_{\delta_0}(\Xi)$. Now, since Ξ is an isolated invariant set, we must have that $\omega(\mathcal{O}_{\delta'}(\Xi)) \subset \Xi$. The other inclusion is trivial and the result follows. ☐

Lemma 2.16. *Let $\{T(t) : t \geq 0\}$ be a gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ and let $\mathscr{A}$ be its global attractor. Then, there is a $k \in \{1, \cdots, n\}$ such that Ξ_k is a local attractor for $\{T(t) : t \geq 0\}$.*

Proof. Assume, by contradiction, that $W^u(\Xi_j) \neq \Xi_j$ for all $1 \leq j \leq n$. Then, for each $1 \leq j \leq n$, there is a global solution $\xi_j : \mathbb{R} \to \mathscr{A}$ such that $\xi_j(t) \stackrel{t\to-\infty}{\longrightarrow} \Xi_j$. From the fact that $\{T(t) : t \geq 0\}$ is gradient-like, $\xi_j(t)$ converges (as $t \to \infty$) to some element of $\mathbf{\Xi}$. This produces a homoclinic structure and gives a contradiction. Hence, there exists $k \in \{1, \cdots, n\}$ such that $W^u(\Xi_k) = \Xi_k$ and, from Lemma 2.15, Ξ_k is a local attractor. ☐

Let $\{T(t) : t \geq 0\}$ be a gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. If (after possible reordering) Ξ_1 is a local attractor for $\{T(t) : t \geq 0\}$ let $\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi_1 = \varnothing\}$ be its repeller, so each Ξ_i, with $i \geq 2$, is contained in Ξ_1^* and more generally the orbit $\xi(\mathbb{R})$ of any global solution $\xi : \mathbb{R} \to \mathscr{A}$ that converges to Ξ_i, $i \geq 2$, when $t \to \infty$, is contained in Ξ_1^*, and considering the restriction $\{T_1(t) : t \geq 0\}$ of $\{T(t) : t \geq 0\}$ to Ξ_1^* we have that $\{T_1(t) : t \geq 0\}$ is a gradient-like semigroup in the space Ξ_1^* with respect to the disjoint family of isolated invariant sets $\{\Xi_2, \cdots, \Xi_n\}$ and we may assume, by the last lemma, that Ξ_2 is a local attractor for the semigroup $\{T_1(t) : t \geq 0\}$ in Ξ_1^*. If $\Xi_{2,1}^*$ is the repeller associated to the local attractor Ξ_2 for $\{T_1(t) : t \geq 0\}$ in Ξ_1^* we may proceed and consider the restriction $\{T_2(t) : t \geq 0\}$ of the semigroup $\{T_1(t) : t \geq 0\}$ to $\Xi_{2,1}^*$ and then $\{T_2(t) : t \geq 0\}$ is a gradient-like semigroup in $\Xi_{2,1}^*$ with respect to the disjoint family of isolated invariant sets $\{\Xi_3, \cdots, \Xi_n\}$.

Proceeding with this until all isolated invariant sets are exhausted we obtain a **reordering** of $\{\Xi_1, \cdots, \Xi_n\}$ in such a way that Ξ_1 is a local attractor for $\{T(t) : t \geq 0\}$. Setting $\mathscr{A} =: \Xi_{0,-1}^*$ and $\Xi_{1,0}^* := \Xi_1^*$, for $j = 2, \cdots, n$, we have that Ξ_j is a local attractor for the restriction of $\{T(t) : t \geq 0\}$ to $\Xi_{j-1,,j-2}^*$ whose repeller will be indicated by $\Xi_{j,j-1}^*$.

With the construction above, if a global solution $\xi : \mathbb{R} \to \mathscr{A}$ satisfies

$$\Xi_i \overset{t \to -\infty}{\longleftarrow} \xi(t) \overset{t \to \infty}{\longrightarrow} \Xi_j \tag{2.1}$$

then $i \geq j$.

At this point, we can use Theorem 1.8 from [14] to conclude that the n-upla $(\Xi_1, \cdots, \Xi_n)$, **ordered** in the way that we have explained above, is a Morse decomposition for the attractor $\mathscr{A}$ of the semigroup $\{T(t) : t \geq 0\}$. We will give another proof of this fact here (more closely related to the gradient-like semigroups and shorter) just for completeness.

Definition 2.17. *Let $\{T(t) : t \geq 0\}$ be a semigroup. The* unstable set *of an invariant set Ξ is defined by*

$$W^{\mathrm{u}}(\Xi) := \{z \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \text{ such that } \xi(0) = z \text{ and } \lim_{t \to -\infty} \mathrm{d}(\xi(t), \Xi) = 0\}.$$

Define $A_0 := \varnothing$, $A_1 := \Xi_1$ and for $j = 2, 3, \cdots, n$

$$A_j := A_{j-1} \cup W^{\mathrm{u}}(\Xi_j). \tag{2.2}$$

It is clear that $A_n = \mathscr{A}$.

Theorem 2.18. *Let $\{T(t) : t \geq 0\}$ be a gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ reordered in such a way that Ξ_j is a local attractor for the restriction of $\{T(t) : t \geq 0\}$ to $\Xi_{j-1,j-2}^*$, as we have explained above. Then A_j defined in* (2.2) *is a local attractor for $\{T(t) : t \geq 0\}$ in X, and*

$$\Xi_j = A_j \cap A_{j-1}^*.$$

As a consequence, $\boldsymbol{\Xi}$ defines a Morse decomposition on $\mathcal{A}$.

Proof. Choose $d > 0$ such that

$$\mathcal{O}_d(A_j) \cap (\cup_{i=j+1}^n \Xi_i) = \varnothing. \tag{2.3}$$

Let us prove that there are $\delta < d$ and $\delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j)) \subset \mathcal{O}_\delta(A_j)$. If that was not the case, there would exist a sequence $\{x_k\}$ in X with $d(x_k, A_j) \stackrel{k\to\infty}{\longrightarrow} 0$, a sequence $\{t_k\}$ in $\mathbb{R}$ with $t_k \stackrel{k\to\infty}{\longrightarrow} \infty$ and, for each x_k, a solution $\xi_k : [-t_k, \infty) \to X$, $\xi_k(t) = T(t+t_k)x_k$, $t \geq -t_k$, $d(\xi_k(t), A_j) \leq \delta$ for all $-t_k \leq t \leq 0$ and $d(\xi_k(0), A_j) = \delta$. Hence, there is a global solution $\xi : \mathbb{R} \to \mathscr{A}$ such that $d(\xi(t), A_j) \leq \delta$ for all $t \leq 0$ and $d(\xi(0), A_j) = \delta$. From (2.3) and the properties of gradient-like semigroups (see Definition 2.8) we must have that $\xi(t) \stackrel{t\to-\infty}{\longrightarrow} \Xi_\ell$, for some $1 \leq \ell \leq j$ and, consequently, $\xi(0) \in W^u(\Xi_\ell) \subset A_j$. This is a contradiction with $d(\xi(0), A_j) = \delta$.

Recall that $\omega(\mathcal{O}_{\delta'}(A_j))$ is compact and invariant. Let us prove that $\omega(\mathcal{O}_{\delta'}(A_j)) = A_j$. In fact, if $\xi : \mathbb{R} \to \omega(\mathcal{O}_{\delta'}(A_j)) \subset \overline{\gamma^+(\mathcal{O}_{\delta'}(A_j))} \subset \mathcal{O}_\delta(A_j)$ is a global solution, it converges backwards to a Ξ_ℓ with $1 \leq \ell \leq j$ (from (2.3)) and, consequently, $\xi(\mathbb{R}) \subset A_j$. This proves that $\omega(\mathcal{O}_{\delta'}(A_j)) \subset A_j$. The other inclusion is obvious and this completes the proof that A_j is a local attractor.

To prove that $\Xi_j = A_j \cap A^*_{j-1}$ note that

$$A_j = \bigcup_{i=1}^{j} W^{\mathrm{u}}(\Xi_i)$$

and $A^*_{j-1} = \{z \in \mathscr{A} : \omega(z) \cap A_{j-1} = \varnothing\}$. Hence, given $z \in A_j \cap A^*_{j-1}$ we have that the global solution $\xi : \mathbb{R} \to \mathscr{A}$ through z must satisfy that

$$\cup_{i=1}^{j} \Xi_i \stackrel{t\to-\infty}{\longleftarrow} \xi(t) \stackrel{t\to\infty}{\longrightarrow} \cup_{i=j}^{n} \Xi_i.$$

As a consequence of that and of the fact that $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$ for which any global solution $\xi : \mathbb{R} \to \mathscr{A}$ satisfies $\Xi_\ell \stackrel{t\to-\infty}{\longleftarrow} \xi(t) \stackrel{t\to-\infty}{\longrightarrow} \Xi_k$ with $\ell \geq k$, we obtain that $z \in \Xi_j$. This shows that $A_j \cap A^*_{j-1} \subset \Xi_j$. The other inclusion is immediate from the definition of A_j and A^*_{j-1}. ☐

The following result plays an important role in the proof of the property that the Lyapunov function is constant along a solution if and only if this solution lies in one of the isolated invariant sets.

Proposition 2.19. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathscr{A}$ and $\mathbf{\Xi} = (\Xi_1, \cdots, \Xi_n)$ a Morse decomposition for $\mathscr{A}$ with family $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathscr{A}$ of local attractors such that $\Xi_j = A_j \cap A^*_{j-1}$ for $j = 1, \cdots, n$. Then,*

$$\bigcap_{j=0}^{n} (A_j \cup A_j^*) = \bigcup_{j=1}^{n} \Xi_j.$$

Proof. Indeed, if $z \in \bigcup_{j=1}^{n} \Xi_j$, let $k \in \{1, 2, \cdots, n\}$ be such that $z \in \Xi_k = A_k \cap A^*_{k-1}$. Hence $z \in A_k \subset A_{k+1} \subset \cdots \subset A_n$ and $z \in A^*_{k-1} \subset A^*_{k-2} \subset \cdots \subset A^*_0$. Thus

$$z \in (\bigcap_{j=k}^{n} A_j) \cap (\bigcap_{j=1}^{k-1} A^*_j) \subset \left[\bigcap_{j=k}^{n}(A_j \cup A^*_j)\right] \cap \left[\bigcap_{j=0}^{k-1}(A_j \cup A^*_j)\right] = \bigcap_{j=0}^{n}(A_j \cup A^*_j),$$

proving the inclusion $\bigcup_{j=1}^{n} \Xi_j \subset \bigcap_{j=0}^{n} (A_j \cup A^*_j)$.

Now, let $z \in \bigcap_{j=0}^{n} (A_j \cup A^*_j)$ and $I := \{i_1, i_2, \cdots, i_k\}$ and $J := \{j_1, j_2, \cdots, j_l\}$ such that $I \cup J = \{0, 1, \cdots, n\}$ with $I \cap J = \varnothing$ and $z \in A_i$ for all $i \in I$ and $z \in A^*_j$ for all $j \in J$. Clearly, if $i := \min I$, necessarily $I = \{i, i + 1, i + 2, \cdots, n\}$ and $J = \{0, 1, \cdots, i - 1\}$, consequently $z \in A_i$ and $z \in A^*_{i-1}$. So, $z \in A_i \cap A^*_{i-1} = \Xi_i$, from which we conclude that $\bigcap_{j=0}^{n} (A_j \cup A^*_j) \subset \bigcup_{j=1}^{n} \Xi_j$ and the proof is completed. □

3. A Lyapunov function for a gradient-like semigroup

Inspired in the work of [6] we will prove in this section the equivalence between gradient semigroups and gradient-like semigroups relative to a disjoint family of isolated invariant sets. The gradient-like semigroups relative to a disjoint family of isolated invariant sets has been defined in Definition 2.8 and a gradient semigroup relative to a disjoint family of isolated invariant sets is defined as follows.

Before we proceed let us fix that, if I, J are subsets of $\mathbb{R}$, a function $w : I \to J$ is said decreasing (increasing) if $w(s) \leq w(t)$ $(w(s) \geq w(t))$ whenever $s \geq t$. If in addition, $w(s) < w(t)$ $(w(s) > w(t))$ whenever $s > t$ we will say that w is strictly decreasing (increasing).

Definition 3.1. *We say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathscr{A}$ and a disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ is a* ***gradient semigroup with respect to*** $\boldsymbol{\Xi}$ *if there is a continuous function $V : X \to \mathbb{R}$ such that:*

(i) The real function $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$,
(ii) V is constant in Ξ_i for each $i = 1, \cdots, n$ and
(iii) $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$.

A function V with the properties above is called a ***Lyapunov function*** *for the gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\boldsymbol{\Xi}$.*

We use some of the ideas found in [6]. Before we start the proof of our main result, let us point out some facts. Note that we work exclusively with semigroups, differently from [6] where the group structure is assumed. Also, the Lyapunov function is constructed and proved to be continuous on the whole space X (which is not, necessarily compact) and not only in the compact invariant set $\mathscr{A}$, as is done in [6] (for the special case of groups).

In [14], all the properties of the Morse decomposition are done for the case of semigroups in compact metric spaces whereas the construction of the Lyapunov function is not done. The construction of the Lyapunov function for a general metric space is what we do in this section.

The following results are the key for the construction of the Lyapunov function.

Lemma 3.2. *If $\{T(t) : t \geq 0\}$ is a semigroup with global attractor $\mathscr{A}$, the map $h : X \to \mathbb{R}$ defined by*

$$h(z) := \sup_{t\geq 0} d(T(t)z, \mathscr{A}), z \in X,$$

is well defined, continuous, decreasing along solutions of $\{T(t) : t \geq 0\}$ and $h^{-1}(0) = \mathscr{A}$.

Proof. Indeed, by Lemma 2.11, given $\varepsilon > 0$ let $0 < \varepsilon' < \varepsilon$ such that $\gamma^+(\mathcal{O}_{\varepsilon'}(\mathscr{A})) \subset \mathcal{O}_\varepsilon(\mathscr{A})$, showing the continuity of h on $\mathcal{A}$. Let $z_0 \in X\backslash\mathscr{A}$ be given, so that $h(z_0) > 0$. Consider $\mathcal{O}_\mu(\mathscr{A})$ for some $0 < \mu < h(z_0)$. From the continuity of the function $X \ni x \mapsto d(x, \mathscr{A}) \in [0,\infty)$ let U be a bounded neighborhood of z_0 such that $d(z, \mathscr{A}) > \mu$ if $z \in U$. Finally, let $\tau > 0$ such that $\gamma^+(T(\tau)U) \subset \mathcal{O}_\mu(\mathscr{A})$ so that it follows the continuity of h en z_0, as for $z \in U$ it holds that $h(z) = \sup_{0\leq s\leq\tau} d(T(s)z, \mathscr{A})$ and, from the continuity properties of the semigroup $\{T(t) : t \geq 0\}$, it follows that $h\,|_U : U \to \mathbb{R}$ is continuous.

To see that h is decreasing along solutions note that, if $z \in X$ and $t_1 > 0$, then

$$h(T(t_1)z) = \sup_{t\geq 0} d(T(t)T(t_1)z, \mathscr{A}) = \sup_{t\geq 0} d(T(t+t_1)z, \mathscr{A}) =$$

$$\sup_{t\geq t_1} d(T(t)z, \mathscr{A}) \leq \sup_{t\geq 0} d(T(t)z, \mathscr{A}) = h(z).$$

□

Proposition 3.3. *Let $\{T(t) : t \geq 0\}$ be a nonlinear semigroup in a metric space (X, d) with global attractor $\mathscr{A}$, and let (A, A^*) an attractor-repeller pair in $\mathscr{A}$. Then, there exists a function $f : X \to \mathbb{R}$ satisfying the following:*

- *(i) $f : X \to \mathbb{R}$ is continuous in X.*
- *(ii) $f : X \to \mathbb{R}$ is decreasing along solutions.*
- *(iii) $f^{-1}(0) = A$ and $f^{-1}(1) \cap \mathscr{A} = A^*$.*
- *(iv) Given $z \in X$, if $f(T(t)z) = f(z)$ for all $t \geq 0$, then $z \in A \cup A^*$.*

Proof. Firstly, observe that A and A^* are disjoint closed subsets of $\mathscr{A}$ and, since $\mathscr{A}$ is a compact subset of X, A and A^* are disjoint closed subsets of X. With the convention that $d(z, \varnothing) = 1$ for each $z \in X$, define the function (the canonical Urysohn function if A and A^* are non-empty) $l : X \to [0, 1]$ associated to (A, A^*) by

$$l(z) := \frac{d(z, A)}{d(z, A) + d(z, A^*)},\ z \in X.$$

Clearly l is well defined, uniformly continuous in X (since, for $d_0 := d(A, A^*) > 0$, it holds that $|l(z) - l(w)| \leq \frac{2}{d_0} d(z, w)$, for any z y w in X). Moreover, $l^{-1}(0) = A$ and $l^{-1}(1) = A^*$.

If we define $k : X \to \mathbb{R}$ by

$$k(z) := \sup_{t \geq 0} \; l(T(t)z),$$

we now show that $k : X \to \mathbb{R}$ is continuous and decreasing along solutions of $\{T(t) : t \geq 0\}$, $k(X) \subset [0, 1]$ (with equality when X is connected and A and A^* are non-empty), $k^{-1}(0) = A$ and $k^{-1}(1) \cap \mathscr{A} = A^*$.

The fact that $k(X) \subset [0, 1]$ follows because $l(T(t)z) \in [0, 1]$ for all $z \in X$ and $t \geq 0$.

To prove that $[0, \infty) \ni t \mapsto k(T(t)z) \in [0, 1]$ is decreasing for each $z \in X$ note that, if $0 \leq t_1 \leq t_2$ we have

$$\begin{aligned} k(T(t_1)z) &= \sup_{t \geq 0} l(T(t)T(t_1)z) = \sup_{t \geq 0} l(T(t + t_1)z) = \sup_{t \geq t_1} l(T(t)z) \\ &\geq \sup_{t \geq t_2} l(T(t)z) = \sup_{t \geq 0} l(T(t + t_2)z) = k(T(t_2)z). \end{aligned}$$

It is clear from the definition of k and from the invariance of A and A^* that $k(A) = \{0\}$ and $k(A^*) = \{1\}$. Now, if $z \in X$ is such that $k(z) = 0$, then $l(T(t)z) = 0$ for all $t \geq 0$. In particular, $0 = l(T(0)z) = l(z)$, and so, $z \in A$, that is, $k^{-1}(0) \subset A$ which shows that $k^{-1}(0) = A$. On the other hand, if $z \in \mathscr{A}$ is such that $k(z) = 1$ and $z \notin A^*$, then $\omega(z) \subset A$. From the continuity of l and the fact that $\omega(z)$ attracts z, we obtain that $\lim_{t \to \infty} l(T(t)z) = 0$. So, there exists a $t_0 > 0$ such that $1 = k(z) = \sup_{0 \leq t \leq t_0} l(T(t)z)$. This implies the existence of a $t' \in [0, t_0]$ such that $l(T(t')z) = 1$; that is, $T(t')z \in A^*$. Consequently $\omega(z) = \omega(T(t')z) \subset A^*$, which contradicts the fact that $\omega(z) \subset A$ and so, if $k(z) = 1$ for some $z \in \mathscr{A}$ we must have that $z \in A^*$. From this we conclude that $k^{-1}(1) \cap \mathscr{A} \subset A^*$ and so $k^{-1}(1) \cap \mathscr{A} = A^*$.

We now prove that, if $z \in \mathscr{A}$ and $k(T(t)z) = k(z)$ for all $t \geq 0$ then $z \in A \cup A^*$. If $z \notin A \cup A^*$, $\omega(z) \subset A$ (note that $z \in \mathscr{A}$) and from the definition of k and the fact that $\omega(z)$ attracts z we have that $k(z) = \lim_{t \to \infty} k(T(t)z) = 0$. Since $k^{-1}(0) = A$, z must belong to A which is a contradiction.

Next we prove the continuity of $k : X \to \mathbb{R}$. We split the proof into three cases:

Case 1) Continuity of $k : X \to \mathbb{R}$ in A^*.

Since $l(z) \leq k(z) \leq 1$, for all $z \in X$, given $z_0 \in A^*$ and $z \in X$ we have that

$$|k(z) - k(z_0)| = 1 - k(z) \leq 1 - l(z).$$

This and the continuity of $l : X \to \mathbb{R}$ in z_0 imply the continuity of $k : X \to \mathbb{R}$ in z_0.

Case 2) Continuity of $k : X \to \mathbb{R}$ in A.

From the continuity of $l : X \to \mathbb{R}$ in A, given $\varepsilon > 0$, there is a $\delta > 0$ such that $l(\mathcal{O}_\delta(A)) \subset [0, \varepsilon)$. Since A is a local attractor ($A = \omega(\mathcal{O}_\epsilon(A))$ for some $\epsilon > 0$) it is invariant and Lemma 2.11 implies that there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$, from which we conclude that $k(\mathcal{O}_{\delta'}(A)) \subset [0, \varepsilon]$.

Case 3) Continuity of $k : X \to \mathbb{R}$ in $X\backslash(A \cup A^*)$.

Given $z_0 \in X\backslash(A \cup A^*)$, from Lemma 2.13 we have that, either $\lim_{t\to\infty} d(T(t)z_0, A) = 0$, or $\lim_{t\to\infty} d(T(t)z_0, A^*) = 0$.

If $\lim_{t\to\infty} d(T(t)z_0, A^*) = 0$ let us prove the continuity of k in z_0. First note that $k(z_0) = 1$. Now, given $\varepsilon > 0$, from the continuity of $l : X \to \mathbb{R}$ in A^* there is an open neighborhood V of A^* in X such that $l(V) \subset (1-\varepsilon, 1]$. If $t_0 > 0$ is such that $T(t_0)z_0 \in V$, from the continuity of $T(t_0) : X \to X$, let U be a neighborhood of z_0 such that $T(t_0)U \subset V$, from which it follows that $k(z) > 1 - \varepsilon$ for all $z \in U$ (for $T(t_0)z \in V$ and then $1 - \varepsilon < l(T(t_0)z) \leq k(z)$). This proves the continuity of k in points z_0 of $X\backslash(A \cup A^*)$ for which $\lim_{t\to\infty} d(T(t)z_0, A^*) = 0$.

If $z_0 \in X\backslash(A \cup A^*)$ and $\lim_{t\to\infty} d(T(t)z_0, A) = 0$, it holds that $l(z_0) > 0$. Choose $\delta > 0$ such that $l(\mathcal{O}_\delta(A)) \subset [0, \frac{l(z_0)}{2})$ and, from Lemma 2.11, there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(A)) \subset \mathcal{O}_\delta(A)$. From this, there is a $t_0 > 0$ with the property that $T(t)z_0 \in \mathcal{O}_\delta(A)$ for all $t \geq t_0$. From the continuity of $T(t_0) : X \to X$, there is a neighborhood U_1 of z_0 in X such that $T(t_0)U_1 \subset \mathcal{O}_{\delta'}(A)$. Then, for all $z \in U_1$ we have that $T(t_0)z \in \mathcal{O}_{\delta'}(A)$ so that $T(t)z \in \mathcal{O}_\delta(A)$ for all $t \geq t_0$. Finally, from the continuity of l, let U_2 be a neighborhood of z_0 in X such that $l(z) > \frac{l(z_0)}{2}$ for all $z \in U_2$ and write $U := U_1 \cap U_2$, so that for all $z \in U$ it holds that $k(z) = \sup_{0\leq t\leq t_0} l(T(t)z)$ and it is really easy to see that the function $U \ni z \mapsto \sup_{0\leq t\leq t_0} l(T(t)z) \in \mathbb{R}$ is continuous, so we obtain the continuity of k in points z_0 of $X\backslash(A \cup A^*)$ for which $\lim_{t\to\infty} d(T(t)z_0, A) = 0$.

Let $h : X \to \mathbb{R}$ be the function defined in Lemma 3.2, that is, $h(z) = \sup_{t\geq 0} d(T(t)z, \mathscr{A})$, $z \in X$, and define $f : X \to \mathbb{R}$ by

$$f(z) := k(z) + h(z),\ z \in X.$$

The continuity of $f : X \to \mathbb{R}$ follows from the continuity of k (proved above) and h (proved in Lemma 3.2). Since k and h are decreasing along solutions of $\{T(t) : t \geq 0\}$ (see above and Lemma 3.2), f also possesses this property.

Clearly $f(A) = \{0\}$. If $f(z) = 0$ for some $z \in X$, then $h(z) = k(z) = 0$ and we must have that $z \in A$. This shows that $f^{-1}(0) = A$.

Also, since $f\,|_{\mathscr{A}} = k\,|_{\mathscr{A}}$ we have that $f^{-1}(1) \cap \mathscr{A} = k^{-1}(1) \cap \mathscr{A} = A^*$.

Finally, (iv) is proved in the following manner. Let $z \in X$ such that $f(T(t)z) = f(z)$ for all $t \geq 0$. If $z \in \mathscr{A}$, we have that $k(T(t)z) = k(z)$ and, consequently, $z \in A \cup A^*$ completing the proof. If $z \in X\backslash\mathscr{A}$, let us show that $\lim_{t\to\infty} d(T(t)z, A^*) = 0$. If that is not the case then, by (iii) in the Lemma 2.13, $\lim_{t\to\infty} d(T(t)z, A) = 0$ and

$$f(z) = \lim_{t\to\infty} f(T(t)z) = \lim_{t\to\infty} k(T(t)z) + \lim_{t\to\infty} h(T(t)z) = 0 + 0 = 0. \tag{3.1}$$

This implies that z is in $A \subset \mathscr{A}$ and that contradicts the fact that $z \in X \backslash \mathscr{A}$. This proves that $\lim_{t\to\infty} d(T(t)z, A^*) = 0$.

Using the same reasoning as in (3.1) we conclude that

$$f(z) = \lim_{t\to\infty} f(T(t)z) = \lim_{t\to\infty} k(T(t)z) + \lim_{t\to\infty} h(T(t)z) = 1 + 0 = 1.$$

This is a contradiction, since $k(z) \geq k(T(t)z)$ for all $t \geq 0$, and $1 = \lim_{t\to\infty} k(T(t)z) \leq k(z) \leq 1$, that is, $k(z) = 1$, but $f(z) = k(z) + h(z)$ and we must have that $h(z) = 0$ which is a contradiction with the fact that $z \notin \mathscr{A}$. Thus, $f(T(t)z) = f(z)$ for all $t \geq 0$ if and only if $z \in A \cup A^*$ and this completes the proof. □

We observe that the proof of continuity of the Lyapunov function for an attractor-repeller pair found in [6] strongly uses that $\{T(t) : t \in \mathbb{R}\}$ is a group and that U is a neighborhood for A then, for all $t \in \mathbb{R}$, $T(t)U$ is also a neighborhood for A. Since we do not assume that the maps $T(t)$'s are homeomorphisms, we use Lemma 2.11 to overcome this difficulty, essentially in the proof of the Case 2 above.

Also, the role of the function $h : X \to \mathbb{R}$, given in Lemma 3.2, is to ensure that, given $z \in X$ the only way to have $f(T(t)z) = f(z)$ for all $t \geq 0$ is that $z \in A \cup A^*$. We must observe that this can not be true, in the general case, for the function k alone.

Finally, the proof of the Case 3) above does not appear in [6], since there the author does not consider points that do not belong to $\mathscr{A}$. That made necessary to prove (iii) in the Lemma 2.13.

Theorem 3.4. *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathscr{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$. Then, $\{T(t) : t \geq 0\}$ is a gradient semigroup with respect to Ξ, in the sense of the Definition 3.1, if and only if it is a gradient-like semigroup with respect to Ξ, in the sense of Definition 2.8. In addition, the corresponding Lyapunov function $V : X \to \mathbb{R}$ may be chosen in such a way that $V(\Xi_k) = k-1$, $k = 1, \cdots, n$.*

Proof. It is clear that a gradient semigroup with respect to Ξ is a gradient-like semigroup with respect to Ξ (See, for example, [7]).

Suppose that $\{T(t) : t \geq 0\}$ is a gradient-like semigroup relatively to Ξ reordered in such a way that it is a Morse decomposition for $\mathscr{A}$. Let $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathscr{A}$ be the sequence of local attractors defined in (2.2) and $\varnothing = A_n^* \subset A_{n-1}^* \subset \cdots \subset A_0^* = \mathscr{A}$ their corresponding repellers such that for each $j = 1, 2, \cdots, n$, we have $\Xi_j = A_j \cap A_{j-1}^*$.

Let $f_j : X \to \mathbb{R}$ be the function constructed in Proposition 3.3 for the attractor-repeller pair (A_j, A_j^*), $j = 1, \cdots, n$ and $f_0 = h$ with h given by Lemma 3.2.

Define the continuous function $V : X \to \mathbb{R}$ by

$$V(z) := \sum_{j=0}^{n} f_j(z),\ z \in X.$$

Then $V : X \to \mathbb{R}$ is a Lyapunov function for the gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\mathbf{\Xi}$.

Indeed, since each $f_j : X \to \mathbb{R}$, $0 \leq j \leq n$, are decreasing along solutions of $\{T(t) : t \geq 0\}$, V is also decreasing along solutions of $\{T(t) : t \geq 0\}$.

Now, if $z \in X$ is such that $V(T(t)z) = V(z)$ for all $t \geq 0$, then, using that each f_j, $0 \leq j \leq n$, are decreasing along solutions of $\{T(t) : t \geq 0\}$, we conclude that $f_j(T(t)z) = f_j(z)$ for all $t \geq 0$, and for each $j = 0, \cdots, n$. It follows that $f_0(z) = 0$ and consequently $z \in \mathscr{A}$. From part (iv) of Proposition 3.3, we have that $z \in (A_j \cup A_j^*)$, for each $j = 0, 1, \cdots, n$; that is, $z \in \bigcap_{j=0}^{n} (A_j \cup A_j^*)$. From Lemma 2.19 we have that

$$\bigcap_{j=0}^{n}(A_j \cup A_j^*) = \bigcup_{j=1}^{n} \Xi_j,$$

and so $z \in \bigcup_{j=1}^{n} \Xi_j$.

If $k \in \{1, \cdots, n\}$ and $z \in \Xi_k = A_k \cap A_{k-1}^*$, it follows that $z \in A_k \subset A_{k+1} \subset \cdots \subset A_n = \mathscr{A}$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \cdots \subset A_0^* = \mathscr{A}$. Hence $f_j(z) = 0$ if $k \leqslant j \leqslant n$, $f_0(z) = 0$ and $f_j(z) = 1$ if $1 \leqslant j \leqslant k-1$. Hence,

$$V(z) = \sum_{j=0}^{n} f_j(z) = \sum_{j=0}^{k-1} f_j(z) + \sum_{j=k}^{n} f_j(z) = \sum_{j=0}^{k-1} 1 + \sum_{j=k}^{n} 0 = k-1.$$

□

Now, we show that it is possible to improve the last result building a Lyapunov function which is strictly decreasing outside the isolated invariant sets and that is differentiable along solutions. Next we present such a construction.

Proposition 3.5. *Let $\{T(t) : t \geq 0\}$ be a semigroup which possesses a global attractor $\mathscr{A}$. Assume that $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. Then, there is a function $W : X \to \mathbb{R}$ which is a Lyapunov function for the gradient-like semigroup $\{T(t) : t \geq 0\}$ with respect to $\mathbf{\Xi}$ and such that*

- (i) *$[0, \infty) \ni t \mapsto W(T(t)z)$ is differentiable for all $z \in X$ and*
- (ii) *$[0, \infty) \ni t \mapsto W(T(t)z)$ is strictly decreasing whenever $z \notin \bigcup_{j=1}^{n} \Xi_j$.*

Proof. Let $V : X \to \mathbb{R}$ be the function defined in Theorem 3.4 and $W : X \to \mathbb{R}$ be the function defined by

$$W(z) := \int_0^\infty e^{-t} V(T(t)z) dt, \ z \in X.$$

It is easy to see that the function is well defined. Next we prove that this function has the desired properties.

We start with the continuity of W. First note that $W(z) \leq V(z)$ for all $z \in X$. Now, from the fact that the semigroup $\{T(t) : t \geq 0\}$ is eventually bounded and the definition of V it is easy to see that for each $z \in X$ there is an $\epsilon_z > 0$ and $t_z > 0$ such that $V(\gamma^+(T(t_z)\mathcal{O}_{\epsilon_z}(z)))$ is bounded. Hence, given $\varepsilon > 0$ and $\overline{z} \in X$ we choose $\bar{t} \geq t_{\overline{z}}$ and a neighborhood B of $\overline{z}$ such that,

$$\int_{\bar{t}}^{\infty} e^{-t} dt < \frac{\varepsilon}{4(M_B + 1)}, \tag{3.2}$$

where $M_B := \sup\{V(T(t)w) : w \in B,\ t \geq t_{\overline{z}}\} \geq 0$.

Now, from the continuity of V and of the mapping $[0, \infty) \times X \ni (t, x) \mapsto T(t)x \in X$, it is easy to see that there exists $\delta > 0$ such that, if $z \in X$ satisfies $d(z, \overline{z}') < \delta$ then,

$$\int_0^{\bar{t}} e^{-s} \left|V(T(t)z) - V(T(t)\overline{z})\right| dt \leq \frac{\varepsilon}{2}.$$

This and (3.2) show that, for $z \in X$ with $d(z, \overline{z}) < \delta$,

$$|W(z) - W(\overline{z})| \leq \int_0^{\bar{t}} e^{-s} |V(T(t)z) - V(T(t)\overline{z})|\, dt + 2M_B \int_{\bar{t}}^{\infty} e^{-t} dt \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Clearly, $W : X \to \mathbb{R}$ is decreasing along solutions of $\{T(t) : t \geq 0\}$. Now, if $z \in \bigcup_{j=1}^{n} \Xi_j$, we have that $T(t)z \in \bigcup_{j=1}^{n} \Xi_j$ for all $t \geq 0$, and $V(T(t)z)$ is constant for all $t \geq 0$, yielding that $W(T(t)z)$ is constant. Conversely, if $z \in X$ is such that $W(T(t)z) = \int_0^{\infty} e^{-s} V(T(t+s)z) dt$ is constant, then $V(T(t)z)$ is constant for all $t \geq 0$ and, consequently, $z \in \bigcup_{j=1}^{n} \Xi_j$ from the properties of V.

Next, given $z \in X \backslash \bigcup_{j=1}^{n} \Xi_j$ let us prove that $[0, \infty) \ni t \mapsto W(T(t)z)$ is strictly decreasing. Indeed, given $t > 0$, we have that

$$W(T(t)z) - W(z) = \int_0^{\infty} e^{-s} [V(T(s+t)z) - V(T(s)z)] ds.$$

From this we see that, if for some $t > 0$ we have that $W(T(t)z) - W(z) = 0$, then $V(T(s + t)z) - V(T(s)z) = 0$ for all $s \geq 0$. In particular, $V(T(t)z) = V(z)$ and, as a consequence of that, $V(T(t)z) = V(T(s)z) = V(z)$ for all $s \in [0, t]$. Repeating this reasoning we conclude that $V(T(s)z) = V(z)$ for all $s \geq 0$, which is in contradiction with the choice of z.

Now, given $z \in X$, $t \geq 0$ and $h \in \mathbb{R}$ we have that

$$\frac{W(T(t+h)z) - W(T(t)z)}{h} = \frac{e^t}{h} \left[(e^h - 1) \int_{t+h}^{\infty} e^{-s} V(T(s)z) ds - \int_t^{t+h} e^{-s} V(T(s)z) ds \right],$$

which converges to

$$e^t \int_t^\infty e^{-s} V(T(s)z)ds - V(T(t)z) \leq 0,$$

proving the differentiability of $[0,\infty) \ni t \mapsto W(T(t)z) \in \mathbb{R}$. □

The idea of the proof for the last result comes from [6]. Note however that our Lyapunov function $V : X \to \mathbb{R}$ is not necessarily bounded as in [6] (the fact that the semigroup is eventually bounded helps to overcome the difficulty that arises). Besides our Lyapunov function is defined on whole (not necessarily compact) space X .

4. Stability under perturbations of gradient semigroups

The equivalence between gradient semigroups and gradient-like semigroups, proved in the previous section, together with the results in [4] prove that gradient semigroups are stable under perturbations. In this section we briefly describe this fact. To that end, we first need to introduce a parameter dependent family of semigroups, as well as the notions of continuity and asymptotic compactness for a parameter dependent family. We start with the notion of continuity for a family of semigroups.

Definition 4.1. *We say that a family of semigroups* $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$*, is continuous at* $\eta = 0$ *if* $T_\eta(t)x \overset{\eta\to 0}{\longrightarrow} T_0(t)x$ *uniformly for* (t, x) *in compact subsets of* $\mathbb{R}^+ \times X$ *as* $\eta \to 0$*.*

The notion of collectively asymptotic compactness which is given next plays a fundamental role in the proof of the main result in [4].

Definition 4.2. *We say that a family of semigroups* $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ *is* collectively asymptotically compact *at* $\eta = 0$ *if, given a sequence* $\{\eta_k\}_{k\in\mathbb{N}}$ *with* $\eta_k \overset{k\to\infty}{\longrightarrow} 0$*, a bounded sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* X *and a sequence* $\{t_k\}_{k\in\mathbb{N}}$ *in* $\mathbb{R}^+$ *with* $t_k \overset{k\to\infty}{\longrightarrow} \infty$*, then* $\{T_{\eta_k}(t_k)x_k\}$ *is relatively compact.*

We are now ready to state the following result from [4].

Theorem 4.3 (Carvalho-Langa)**.** *Let* $\{T_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$*, be a collectively compact family of semigroups which is continuous at* $\eta = 0$*. Assume that*

a) $\{T_\eta(t) : t \geq 0\}$ *possesses a global attractor* $\mathscr{A}_\eta$ *for each* $\eta \in [0, 1]$ *and* $\cup_{\eta\in[0,1]}\mathscr{A}_\eta$ *is bounded.*
b) *There exists* $n \in \mathbb{N}$ *such that* $\mathscr{A}_\eta$ *has* n *isolated invariant sets* $\mathbf{\Xi}_\eta = \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$ *for all* $\eta \in [0, 1]$*, and* $\sup_{1\leqslant i\leqslant n}[\mathrm{dist}_H(\Xi^*_{i,\eta}, \Xi^*_{i,0}) + \mathrm{dist}_H(\Xi^*_{i,0}, \Xi^*_{i,\eta})] \overset{\eta\to 0}{\longrightarrow} 0$*.*
c) $\{T_0(t) : t \geq 0\}$ *is a gradient-like semigroup with respect to* $\mathbf{\Xi}_0$*.*

Then, there exists $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$*,* $\{T_\eta(t) : t \geq 0\}$ *is a gradient-like semigroup and consequently*

$$\mathscr{A}_\eta = \cup_{i=1}^n W^u(\Xi^*_{i,\eta}),\ \ \forall \eta \in [0, \eta_0].$$

As an immediate consequence of this result and the ones in Section 3 we have the following result.

Corollary 4.4. *Under the assumption of Theorem 4.3, there exists* $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a gradient semigroup with respect to* $\mathbf{\Xi}_\eta$.

Corollary 4.5. *Under the assumption of Theorem 4.3, suppose there exists* $n \in \mathbb{N}$ *such that* $\mathscr{A}_\eta$ *has* n *stationary solutions* $\mathbf{\Xi}_\eta = \{\xi_{1,\eta}, \cdots, \xi_{n,\eta}\}$ *for all* $\eta \in [0,1]$ *and* $\sup_{1\leqslant i\leqslant n} \mathrm{dist}(\xi^*_{i,\eta}, \xi^*_{i,0}) \overset{\eta\to 0}{\longrightarrow} 0$. *Then, there exists* $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a gradient semigroup in the sense of* [7].

5. Energy level decomposition of a gradient-like system

Let $\{T(t) : t \geq 0\}$ be a semigroup in X with global attractor $\mathscr{A}$. Next we will give a dynamical description of a gradient-like system by reordering and regrouping the corresponding disjoint isolated invariant subsets to obtain a *totally ordered* family of isolated invariant sets that we will refer to as *energy levels*. This new family of isolated invariant sets is a Morse decomposition of $\mathscr{A}$ with fewer invariant sets but in such a way that it still gives us a Lyapunov function that is constant only in the original isolated invariant sets. In a certain sense this decomposition is the coarsest decomposition which still gives us a Lyapunov function which is constant only in the original isolated invariant sets.

Assume that $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$.

(a) Given Ξ_{l_1} and $\Xi_{l_2} \in \mathbf{\Xi}$, we say that Ξ_{l_1} precedes Ξ_{l_2} (we write $\Xi_{l_1} \prec \Xi_{l_2}$), if there exists a global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ such that $\xi(\mathbb{R}) \subsetneq \Xi_{l_1} \cup \Xi_{l_2}$ and

$$\lim_{t\to-\infty} d(\xi(t), \Xi_{l_2}) = 0 \ \text{ and } \ \lim_{t\to\infty} d(\xi(t), \Xi_{l_1}) = 0.$$

(b) Let us consider

$$\mathcal{M}_1 \ : \ = \{\Xi_\ell \in \mathbf{\Xi} : \text{ there is no element } \Xi \in \mathbf{\Xi} \text{ that precedes } \Xi_\ell\}$$

and, for any integer $k \geq 2$

$$\mathcal{M}_k := \{\Xi_\ell \in \mathbf{\Xi} : \text{ if } \Xi \in \mathbf{\Xi} \text{ and } \Xi \prec \Xi_\ell \text{ then } \Xi \in \mathcal{M}_{k-1}\}.$$

Note that, by definition, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$.

(c) We now define the sets

$$\mathscr{N}_1 := \bigcup_{\Xi\in\mathcal{M}_1} \Xi, \text{ and } \mathscr{N}_k := \bigcup_{\Xi\in\mathcal{M}_k\setminus\mathcal{M}_{k-1}} \Xi, \text{ for all } k \geq 2.$$

Since $\mathbf{\Xi}$ is finite, there exists a positive integer q such that $\mathcal{M}_k = \mathcal{M}_q$ for each $k > q$, so that, $\mathscr{N}_k = \varnothing$ for all $k > q$. Thus, we consider $\mathscr{N}_1, \mathscr{N}_2, \cdots, \mathscr{N}_p$, with $p := \min\{q \in \mathbb{N} : \mathcal{M}_k = \mathcal{M}_q \text{ for each } k > q\}$.

We have the following first result related to this family of sets:

Lemma 5.1. *Let* $\{T(t) : t \geq 0\}$ *be a semigroup with global attractor* $\mathscr{A}$*. Assume that* $\{T(t) : t \geq 0\}$ *is a gradient-like semigroup with respect to the disjoint family of isolated invariant sets* $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$*. Then each element of* $\mathbf{\Xi}$ *belongs to* $\mathscr{N}_k$*, for some* $k \leq p$*.*

Proof. Suppose that $\mathscr{N}_1$ is empty and fix $\Xi \in \mathbf{\Xi}$. Then, since $\Xi \notin \mathscr{N}_1 = \varnothing$, by the definition of $\mathcal{N}_1$, there exists $\Xi^1 \in \mathbf{\Xi}$ with $\Xi^1 \prec \Xi$ and $\Xi^1 \neq \Xi$. Analogously, $\Xi^1 \notin \mathscr{N}_1$, from which we find $\Xi^2 \in \mathbf{\Xi}$ with $\Xi^2 \prec \Xi^2$ and $\Xi^2 \neq \Xi^1$. Also $\Xi^2 \neq \Xi$ since, otherwise, there would be a homoclinic structure $\Xi \prec \Xi^1 \prec \Xi^2 = \Xi$. Following in this way, as $\mathbf{\Xi}$ is finite we arrive at a contradiction in a finite number of steps.

If $\mathbf{\Xi} = \mathscr{N}_1$, we are done. If not, we must have that $\mathscr{N}_2$ is non-empty. Otherwise, given $\Xi \in \mathbf{\Xi} \backslash \mathscr{N}_1$ with $\Xi \notin \mathcal{N}_2$. Then, there exists $\Xi^1 \in \mathbf{\Xi} \backslash \mathscr{N}_1$ with $\Xi \prec \Xi^1$. As before, the fact that $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with respect to $\mathbf{\Xi}$ implies that $\Xi^1 \neq \Xi$. Since $\Xi^1 \notin \mathscr{N}_2$, there is a $\Xi^2 \in \mathbf{\Xi} \backslash \mathscr{N}_1$ with $\Xi^1 \prec \Xi^2$ and, again, $\Xi^2 \neq \Xi^1$ and $\Xi^2 \neq \Xi$. As before, we arrive at a contradiction in a finite number of steps, so that $\mathscr{N}_2 \neq \varnothing$.

If $\mathbf{\Xi} = \mathscr{N}_1 \cup \mathscr{N}_2 = \mathcal{M}_2$, we finish. If not, again $\mathscr{N}_3 \neq \varnothing$. As this argument has to finish in a finite number of steps completing the proof. ☐

The following result will show that $\mathscr{N} = (\mathscr{N}_1, \mathscr{N}_2, \cdots, \mathscr{N}_p)$ is a Morse decomposition for $\mathcal{A}$.

Theorem 5.2. *Let* $\{T(t) : t \geq 0\}$ *be a nonlinear semigroup with global attractor* $\mathscr{A}$*. If* $\{T(t) : t \geq 0\}$ *is a gradient-like semigroup with respect to* $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$*, then* $(\mathscr{N}_1, \mathscr{N}_2, \cdots, \mathscr{N}_p)$ *is a Morse decomposition for* $\mathscr{A}$*.*

Proof. Clearly $\{T(t) : t \geq 0\}$ is a gradient-like semigroup with respect to $\mathscr{N}$. The proof of the result now follows as the proof of Theorem 2.18. ☐

Remark 5.3. *Given a compact invariant set* $\mathscr{A}$*, we define the finest Morse decomposition on* $\mathscr{A}$ *(cf. [13]) as* $(\Xi_1, \ldots, \Xi_n)$ *in which, for each isolated compact invariant set* Ξ_i *the unique Morse decomposition on it is the trivial one. Then, the energy level decomposition of the finest Morse decomposition can be considered as the optimal decomposition of* $\mathscr{A}$*, as it gives a total description of* $\mathscr{A}$ *by a Lyapunov (energy) function which is constant on each level and strictly decreasing on connecting global solutions among all the different levels.*

5.0.1. *Energy level decomposition for a gradient-like system.* All the concepts and results in the previous section can be written in the particular case in which we have a finite set of equilibria:

Definition 5.4. *Let* $\{T(t) : t \geq 0\}$ *be a gradient-like semigroup in* X *with global attractor* $\mathscr{A}$ *with equilibrium points* $\mathscr{E} = \{\zeta_1, \cdots, \zeta_n\}$*.*

(*a*) *Given* ζ_r *and* $\zeta_s \in \mathscr{E}$, *we say that* ζ_r *precedes* ζ_s *(we write* $\zeta_r \prec \zeta_s$*), if there exists a non-constant global solution* $\xi : \mathbb{R} \to X$ *of* $\{T(t) : t \geq 0\}$ *such that*

$$\lim_{t\to-\infty} d(\xi(t), \zeta_s) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t), \zeta_r) = 0.$$

(*b*) *Let us consider*

$$\mathcal{M}_1 \quad : \quad = \{\zeta_\ell \in \mathscr{E} : \text{ there is no element } \zeta \in \mathscr{E} \text{ that preceeds } \zeta_\ell\}$$

and, for any integer $k \geq 2$

$$\mathcal{M}_k := \{\zeta_\ell \in \mathscr{E} : \text{ if } \zeta \in \mathcal{E} \text{ e } \zeta \prec \zeta_\ell \text{ then } \zeta \in \mathcal{M}_{k-1}\}.$$

Note that, by definition, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$.

(*c*) *We now define the sets*

$$\mathscr{N}_1 := \mathcal{M}_1 \text{ and } \mathscr{N}_k := \mathcal{M}_k \backslash \mathcal{M}_{k-1}, \text{ for all } k \geq 2.$$

Since $\mathscr{E}$ *is finite, there exists a positive integer* q *such that* $\mathcal{M}_k = \mathcal{M}_q$ *for each* $k > q$, *so that,* $\mathscr{N}_k = \varnothing$ *for all* $k > q$. *Thus, we consider* $\mathscr{N}_1, \mathscr{N}_2, \cdots, \mathscr{N}_p$, *with* $p := \min\{q \in \mathbb{N} : \mathcal{M}_k = \mathcal{M}_q \text{ for each } k > q\}$.

Lemma 5.5. *If* $\{T(t) : t \geq 0\}$ *is a gradient-like semigroup, then every* $\zeta \in \mathscr{E}$ *is in* $\mathscr{N}_i$ *for some* $i = 1, \cdots, p$, *where* $p \geq 1$ *is the maximum number of non-void energy levels in* $\{T(t) : t \geq 0\}$.

Lemma 5.6. *Let* $\{T(t) : t \geq 0\}$ *be a gradient-like semigroup with global attractor* $\mathscr{A}$ *and a finite set of equilibria* $\mathcal{E} = \{\zeta_1, \cdots, \zeta_n\}$, *and* $(\mathscr{N}_1, \cdots, \mathscr{N}_p)$ *the ordered n-upla of energy levels* $\{T(t) : t \geq 0\}$. *Then,* $(\mathscr{N}_1, \cdots, \mathscr{N}_p)$ *is a Morse decomposition in* $\mathscr{A}$.

6. Final comments and further research

We have proved that a gradient-like semigroup with respect to a disjoint family of isolated invariant sets Ξ is gradient (has a Lyapunov function) with respect to Ξ. In particular, a gradient-like semigroup with respect to a finite set of equilibria is gradient (in the sense of [7]), concluding that gradient semigroups (in the sense of [7]) are stable under perturbation. Note that any global attractor for a semigroup admits at least a Morse decomposition -the trivial one-, being common to have a better description of the internal dynamics of it in several non-trivial examples. Our results apply to any Morse decomposition of a global attractor (which behave continuously with respect to the parameter).

The applications considered in the introduction show that the class of semigroups known to be gradient (in the sense of [7]) increases considerably after the results in this paper have been proved.

On the other hand, it is shown in [4] that a non-autonomous perturbation of a gradient-like system is still gradient-like. Thus, there exists a very natural extension of the results on this paper to a non-autonomous framework.

Finally, we think that the concept of energy levels deserves being exploited, because it is giving a very good description of connections between isolated sets from any given decomposition. In particular, its stability under perturbation seems a proper non-trivial problem to be considered, connected, for instance, to Morse-Smale systems. We plan to present results in these last two lines of research in the near future.

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