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STABILITY AND RANDOM ATTRACTORS FOR A REACTION-DIFFUSION EQUATION WITH MULTIPLICATIVE NOISE

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Abstract. We study the asymptotic behaviour of a reaction-diffusion equation, and prove that the addition of multiplicative white noise (in the sense of Itô) stabilizes the stationary solution $x \equiv 0$. We show in addition that this stochastic equation has a finite-dimensional random attractor, and from our results conjecture a possible bifurcation scenario.

1. **Introduction.** The study of the asymptotic behaviour of evolution equations is one of the most important problems in the field of differential equations, as the vast literature on the subject shows. Perhaps the first step in this programme is to investigate the stability of the stationary solutions, and this is at present a well-developed branch of the theory of stochastic ordinary and partial differential equations (see, among others, Has'minskii [20], Mao [23], [24], Caraballo and Liu [5]). Such a stability analysis is essentially a local study, as we obtain information on the dynamics only around these points. A more complete analysis of the qualitative properties of the system as a whole requires information relating not only to the stationary points but also, for example, to the stability or instability of the associated invariant manifolds of these points. As a further step one can consider the global attractor for the problem (Ladyzhenskaya [22], Hale [19], Temam [32]); this is rapidly becoming one of the main concepts in the theory of infinite-dimensional dynamical systems. Recently, Crauel and Flandoli [9] (see also Schmalfuß [30]) have generalized the theory of deterministic attractors to the stochastic case. This theory of random attractors is turning out to be very fruitful in the study of the long-time dynamics of stochastic ordinary and partial differential equations.

In this paper we study a reaction-diffusion equation perturbed by a multiplicative white noise term, first considering this in the Itô sense,

$$dx(t) = \Delta x(t)\, dt + (ax(t) - x(t)^3)\, dt + \sigma x(t)\, dW_t.$$

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We briefly revise some properties of the deterministic equation ($\sigma = 0$) in section 2. (We choose this particular form of equation for simplicity, but a very similar analysis should give similar results for more general equations, cf. Marion [25].)

There are many results on stability of solutions in the stochastic case, but most treat only ordinary differential equations, and those generalizations which do treat the infinite-dimensional case usually assume a global Lipschitz property for the nonlinear terms of the non-random equation. In our case the nonlinear term is only locally Lipschitz, so that these stochastic stability results cannot be applied directly to our problem. In section 3 we study the stabilizing effect of the stochastic perturbation on the solution $x = 0$, and show that the interval of stability increases as σ is increased, so that large levels of noise simplify the long-time behaviour.

We can write the Itô equation in the alternative Stratonovich form (Stratonovich [28]),

$$dx(t) = \Delta x(t)\,dt + ((a - \tfrac{1}{2}\sigma^2)x(t) - x(t)^3)\,dt + \sigma x(t) \circ dW_t,$$

which makes the stabilization effect of the noise term explicit, and in the remainder of the paper we treat the equation

$$dx(t) = \Delta x(t)\,dt + (\beta x(t) - x(t)^3)\,dt + \sigma x(t) \circ dW_t,$$

investigating its behaviour as β is varied.

As we have noted above, a general study of the deterministic reaction-diffusion equation includes an understanding of its global attractor. In this relatively simple case, this attractor consists of the stationary solutions, which are joined by the stable and unstable manifolds associated with them (see Hale [19] or Henry [21] for example). If we wish to continue our investigation of the stochastic version of this PDE, by analogy it seems sensible to investigate whether or not it has a random attractor. We introduce the relevant theory in section 4, and then in section 5 prove the existence of such a random attractor.

The computations here once again demonstrate the stabilization effect (now seen easily in terms of the parameter β). In particular, if $\beta < \lambda_1$ the random attractor is in fact the deterministic point $\{0\}$, whereas if $\beta \geq \lambda_1$ the attractor could be a much more general (random) set.

One of the most interesting properties of certain attractors of deterministic infinite-dimensional dynamical systems is that they are finite-dimensional subsets of the infinite-dimensional phase space (see Temam [32], for example).To try to get a little more information about the possible complexities of our random attractor when $\beta \geq \lambda_1$, we use the method developed by Debussche in [13] to obtain a bound on its Hausdorff dimension in section 6. (Some tedious computations necessary for the application of Debussche's result are relegated to an appendix.) In particular, we show that if

$$\beta < \frac{1}{d}\sum_{j=1}^{d}\lambda_j,$$

where λ_j are the eigenvalues of the Laplacian arranged in increasing order, then the Hausdorff dimension of the random attractor is bounded by d, P-almost surely. Once again, a little further argument recovers the result that if $\beta < \lambda_1$ then the attractor consists of just one point; but now we can see that increasing β at least allows (within the bound above) for more complexity of the attractor.

To conclude we discuss a possible bifurcation picture near the origin as β passes through λ_1.

2. **Formulation of the problem.** Let $D \subset \mathbf{R}^n$ be an open bounded set with regular boundary. We consider the following partial differential equation of reaction-diffusion type in D perturbed by a linear multiplicative white noise

$$\left\{\begin{array}{rcl} dx(t) & = & \Delta x(t)dt + (ax(t) - x^3(t))dt + \sigma x(t)dW_t \qquad \text{in } D \\ x(t) & = & 0 \text{ on } \partial D \\ x(0) & = & x_0, \end{array}\right. \tag{2.1}$$

where $W_t : \Omega \to \mathbf{R}$, $t \in \mathbf{R}$, is a one dimensional Wiener process.

To pose this problem into a variational form we introduce the following spaces: $H = L^2(D)$ (with $(.,.)$, $|.|$ its scalar product and norm respectively), $V = H_0^1(D)$ $(((.,.)), \|.\|)$. Thus, we can write (2.1) as the following differential equation in H:

$$\begin{array}{rcl} dx & = & -Axdt + (ax - x^3)dt + \sigma x dW_t \qquad \text{in } H \\ x(0) & = & x_0 \in H \end{array} \tag{2.2}$$

where $A : D(A) \subset H \to H$, $Ax = -\Delta x$. The operator A is positive, linear, self-adjoint and with compact inverse A^{-1}. Under these conditions there exist $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$, the sequence of eigenvalues of A, and $w_1, w_2, \ldots$ the associated sequence of eigenfunctions $Aw_i = \lambda_i w_i$, which forms an orthonormal basis in H ([32]).

It is known (Pardoux [26]) that for each $x_0 \in H$ and $T > 0$, there exists a unique strong solution

$$x(t; x_0) \in L^2(\Omega \times (0,T); H_0^1(D)) \cap L^4(\Omega \times (0,T) \times D) \cap L^2(\Omega; C(0,T; L^2(D)).$$

The global attractor (Hale [19], Temam [32], Vishik [33]) is at present one of the main tools in the study of the asymptotic behaviour of infinite-dimensional dynamical systems. A global attractor is a compact set in the phase space, invariant for the semigroup associated to the system and attracting every trajectory as $t \to +\infty$, uniformly on bounded sets. For our particular problem (2.1), the existence of a global attractor $\mathcal{A}$ for the unperturbed system is well known (see, for instance, Marion [25]).

However, when $a < \lambda_1$, notice that the unique solution of the associated elliptic equation is just $x \equiv 0$. Since (2.1), when $\sigma = 0$, is a *gradient system* (Hale [19]), the global attractor consists of all the stationary points and the stable and unstable manifolds joining them (see also Henry [21]). Thus, if there exists just one stationary solution of the elliptic equation we conclude that this is exactly the global attractor. We now prove the existence of one stationary solution if $a < \lambda_1$: for two solutions of $(i = 1, 2)$

$$\left\{\begin{array}{rcl} \Delta x_i(t) + ax_i(t) - x_i^3(t) & = & 0 \qquad \text{in } D \\ x_i(t) & = & 0 \text{ on } \partial D \\ x_i(0) & = & x_i^0 \end{array}\right. \tag{2.3}$$

we have, by denoting $y(t) = x_1(t) - x_2(t)$,

$$\Delta y(t) + ay(t) - (x_1^3(t) - x_2^3(t)) = 0.$$

Multiplying by $y(t)$ in H we obtain

$$-\|y(t)\|^2 + a|y(t)|^2 + (x_2^3(t) - x_1^3(t), x_1(t) - x_2(t)) = 0$$

from where, taking into account that $\|x\|^2 \geq \lambda_1 |x|^2$, for all $x \in V$,

$$-\lambda_1 |y(t)|^2 + a|y(t)|^2 + (x_2^3(t) - x_1^3(t), x_1(t) - x_2(t)) \geq 0.$$

Now note that $(f(u) - f(v))(u - v) \leq 0$ for $u, v \in \mathbf{R}$ and $f(u) = -u^3$ and so $(x_2^3(t) - x_1^3(t), x_1(t) - x_2(t)) \leq 0$. Thus,

$$-\lambda_1 |y(t)|^2 + a|y(t)|^2 \geq 0.$$

On the other hand, as $a < \lambda_1$,

$$(a - \lambda_1)|y(t)|^2 \leq 0,$$

so that we conclude that

$$y(t) = 0 \quad \text{a.e. in } D.$$

This proves that $\{0\}$ is the global attractor for the deterministic system in this case and, consequently, the stationary solution $x \equiv 0$ is asymptotically stable.

3. **Stabilization effect of the multiplicative white noise.** In this section we will prove the effect of stabilization on $x \equiv 0$ produced when we add a multiplicative noise (in the Itô sense) to the deterministic equation.

Before doing this, it is worth noticing that by applying theorem 2.2 in Caraballo and Real [6] one can obtain pathwise exponential stability of the trivial solution of problem (2.1) provided that $\sigma^2 < 2(\lambda_1 - a)$. Indeed, in order to apply this result we need to check a coercivity condition:

$$2(\Delta u + au - u^3, u) + |\sigma u|^2 \leq -\alpha \|u\|^2 + \lambda |u|^2 \tag{3.4}$$

and if $\lambda - \alpha\beta^{-2} < 0$ (with $\beta = \lambda_1^{-1/2}$), then the pathwise exponential stability of the trivial solution holds.

As

$$2(\Delta u + au - u^3, u) + |\sigma u|^2 = -2\|u\|^2 + 2a|u|^2 - 2|u|_{L^4}^4 + \sigma^2 |u|^2,$$

(3.4) is fulfilled by setting $\alpha = 2$ and $\lambda = 2a + \sigma^2$. Thus, the pathwise stability follows provided

$$2a + \sigma^2 - 2\lambda_1 < 0 \Leftrightarrow \sigma^2 < 2(\lambda_1 - a). \tag{3.5}$$

Now, if $\lambda_1 - a > 0$, that is, if $\mathcal{A} = \{0\}$ is exponentially stable for the (deterministic) unperturbed problem, we can assure that if the perturbation is small enough, the trivial solution of the stochastic system remains exponentially stable with probability one. On the other hand, if $\lambda_1 - a < 0$, that is, $\{0\}$ is unstable for the unperturbed equation, this result does not guarantee stability for the stochastic problem as (3.5) does not hold. However, as we shall prove, in the first case one can show not only for small values of σ but for all $\sigma \in \mathbf{R}$ that the null solution of the stochastic equation is pathwise exponentially stable. In the second one ($\lambda_1 - a < 0$), we shall prove that for σ large enough, the trivial solution becomes asymptotically exponentially stable with probability one. This means that the multiplicative noise stabilizes the solution, as we can see in the following result:

Theorem 3.1. *Assume $-\gamma = a - \lambda_1 - \frac{1}{2}\sigma^2 < 0$. Then there exists $N \subset \Omega, P(N) = 0$, such that for $\omega \notin N$ there exists $T(\omega) > 0$ such that for each $x_0 \in H, x_0 \neq 0$,*

$$|x(t, \omega; x_0)|^2 \leq |x_0|^2 e^{-\gamma t}, \quad \forall\ t \geq T(\omega)$$

Proof. Firstly, observe that uniqueness and continuity of solutions of (2.1) immediately implies that for $x_0 \neq 0$, the solution $x(t) = x(t; x_0)$ satisfies $x(t; x_0) \neq 0$ for all $t \geq 0$, $P - a.s.$

Now, Itô's formula for $|x(t)|^2$ yields that

$$\begin{aligned}|x(t)|^2 &= |x_0|^2 + \int_0^t (2x(s), \Delta x(s) + ax(s) - x^3(s))ds \\ &\quad + \int_0^t 2\sigma|x(s)|^2 dW_s + \int_0^t \sigma^2|x(s)|^2 ds \\ &= |x_0|^2 + \int_0^t (-2\|x(s)\|^2 + 2a|x(s)|^2 - 2|x(s)|^4_{L^4} + \sigma^2|x(s)|^2)\, ds \\ &\quad + \int_0^t 2\sigma|x(s)|^2 dW_s.\end{aligned}$$

Denoting $u(t) = |x(t)|^2$ and applying once again Itô's formula to $\log u(t)$ we obtain

$$\begin{aligned}\log|x(t)|^2 &= \log|x_0|^2 + \int_0^t \frac{1}{|x(s)|^2}(-2\|x(s)\|^2 + 2a|x(s)|^2 - 2|x(s)|^4_{L^4} \\ &\quad +\sigma^2|x(s)|^2)\, ds + \int_0^t 2\sigma dW_s - \frac{1}{2}\int_0^t \frac{4\sigma^2|x(s)|^4}{|x(s)|^4} ds \\ &\leq \log|x_0|^2 + \int_0^t (2a + \sigma^2 - 2\lambda_1)ds + 2\sigma W_t - 2\sigma^2 t \\ &\leq \log|x_0|^2 + (2a - \sigma^2 - 2\lambda_1)t + 2\sigma W_t.\end{aligned}$$

As $\lim_{t\to\infty} \dfrac{W_t}{t} = 0 : P - a.s.$, there exists $N \subset \Omega, P(N) = 0$, such that for $\omega \notin N$ there exists $T(\omega)$ such that for all $t \geq T(\omega)$

$$\frac{2\sigma W_t}{t} \leq \frac{1}{2}(-2a + 2\lambda_1 + \sigma^2)$$

so that

$$\log|x(t)|^2 \leq \log|x_0|^2 + \frac{1}{2}(2a - 2\lambda_2 - \sigma^2)t \quad \forall t \geq T(\omega).$$

As $2a - 2\lambda_2 - \sigma^2 < 0$, the proof is complete by taking $\gamma = \dfrac{\sigma^2}{2} + \lambda_1 - a$. □

Note that, in the deterministic case, $\{0\}$ is unstable for $a > \lambda_1$. Thus, the multiplicative noise stabilizes the stationary point $\{0\}$ for a in the interval $[\lambda_1, \frac{1}{2}\sigma^2 + \lambda_1]$. The larger the parameter σ, the longer the stability interval for the zero point. On the other hand, given $a > 0$ we can always choose a value of σ such that the zero point is asymptotically stable for equation (2.1).

If we had considered equation (2.1) with the white noise in the sense of Stratonovich, then we would not have obtained any stabilization effect for the trivial solution, since we would need the condition $a > \lambda_1$ for the stability of both the deterministic and the stochastic equation. This is due to the extra term when applying Itô's formula which does not appear in the Stratonovich case. Thus, it is in fact more sensible to analyse Stratonovich equations in the multiplicative case, highlighting the importance which should be attached to the choice of the sense of the stochastic integrals in such an equation (see Caraballo and Langa [4] for more details on this fact). In particular, in what follows we will consider the Stratonovich equation,

$$dx(t) = \Delta x(t)\, dt + (\beta x(t) - x^3(t))\, dt + \sigma x(t) \circ dW_t, \tag{3.6}$$

where the new parameter β corresponds to $a - \frac{1}{2}\sigma^2$ in our previous (Itô) formulation (Stratonovich [28]).

Since the above argument does not give us any information for values of a greater than $\frac{1}{2}\sigma^2 + \lambda_1$, i.e. for values of $\beta > \lambda_1$, we need a different approach to study the asymptotic behaviour of our problem any further for such values of β. We therefore introduce the theory of random attractors in the next section, and then apply this theory to equation (3.6).

4. **Random attractors.** Recently, Crauel and Flandoli [9] (see also Schmalfuß [30]) have introduced the concept of an attractor for some stochastic partial differential equations, and this has been successfully used in the study of qualitative properties for these equations (see, among others, Schmalfuß [31], Crauel et al. [8], Schenk-Hoppé [29]). This concept has been developed within the framework of the theory of random dynamical systems (Arnold [1]). In this section, we will introduce the concepts of random dynamical system and random attractors and then apply this theory to our particular problem in the remainder of the paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\theta_t : \Omega \to \Omega,\, t \in \mathbf{R}\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t\theta_s$, for all $s, t \in \mathbf{R}$. The flow θ_t together with the probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is called a *(measurable) dynamical system*. Furthermore, we suppose that the shift θ_t is ergodic.

A *random dynamical system* (RDS) on a Polish space (X, d) with Borel σ-algebra $\mathcal{B}$ over θ on $(\Omega, \mathcal{F}, P)$ is a measurable map

$$\begin{array}{rcl} \varphi : \mathbf{R}^+ \times \Omega \times X & \to & X \\ (t, \omega, x) & \mapsto & \varphi(t, \omega)x \end{array}$$

such that $P - a.s.$

i) $\varphi(0, \omega) = \mathrm{id}$ (on X)
ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega),\ \forall\, t, s \in \mathbf{R}^+$ (cocycle property).

A RDS is continuous or differentiable if $\varphi(t, \omega) : X \to X$ is continuous or differentiable.

A random set $K(\omega)$ is said to *absorb* the set $B \subset X$ if $P - a.s.$ there exists $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B \subset K(\omega).$$

Finally, a random set $\mathcal{A}(\omega)$ is said to be a *random attractor* associated to the RDS φ if $P - a.s.$

i) $\mathcal{A}(\omega)$ is a random compact set, that is, $P - a.s.\ \omega \in \Omega$, $\mathcal{A}(\omega)$ is compact and for all $x \in X$ and $P - a.s.$ the map $x \mapsto \mathrm{dis}(x, \mathcal{A}(\omega))$ is measurable.
ii) $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega), : \forall t \geq 0$ (invariance) and
iii) for all $B \subset X$ bounded (and non-random)

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0,$$

where dist(. , .) denotes the Hausdorff semidistance

$$\mathrm{dist(A, B)} = \sup_{\mathrm{a}\in\mathrm{A}} \inf_{\mathrm{b}\in\mathrm{B}} \mathrm{d(a, b)}, \qquad \mathrm{A, B} \subset \mathrm{X}.$$

Note that $\varphi(t, \theta_{-t}\omega)x$ can be interpreted as the position at $t = 0$ of the trajectory which was in x at time $-t$. Thus, the attraction property holds *from* $t = -\infty$.

In this situation, we have the following theorem about existence of random attractors due to Crauel and Flandoli ([9], theorem 3.11):

Theorem 4.1. *Suppose there exists a compact set $D(\omega)$ absorbing every bounded non-random set $B \subset X$. Then, the set*

$$\mathcal{A}(\omega) = \overline{\bigcup_{B\subset X} \Lambda_B(\omega)}$$

is a random attractor for φ, where the union is taken over all $B \subset X$ bounded, and $\Lambda_B(\omega)$ is the omega-limit set of B given by

$$\Lambda_B(\omega) = \bigcap_{n\geq 0} \overline{\bigcup_{t\geq n} \varphi(t, \theta_{-t}\omega)B}.$$

Moreover, Crauel proved in [10] that random attractors are unique and, under the ergodicity assumption on θ_t, there exists a compact set $K \subset X$ such that $P - a.s.$ the random attractor is the omega-limit set of K, that is,

$$\mathcal{A}(\omega) = \bigcap_{n\geq 0} \overline{\bigcup_{t\geq n} \varphi(t, \theta_{-t}\omega)K}.$$

5. **Existence of random attractors.** In this section we will prove that the hypotheses in theorem 4.1 hold, so that there exists a random attractor $\mathcal{A}_{\beta,\sigma}(\omega)$. From now on, we suppose that W_t is a two-sided Wiener process (Arnold [1]). Firstly, we consider equation (2.1) in Stratonovich's sense, that is, for $\beta, \sigma \in \mathbf{R}$, we consider

$$dx(t) = \Delta x(t)dt + (\beta x(t) - x^3(t))dt + \sigma x(t) \circ dW_t. \tag{5.7}$$

By means of the change of variable

$$u(t) = \alpha(t)x(t), \quad \text{with } \alpha(t) = e^{-\sigma W_t}$$

it is easy to check that, formally, $u(t)$ satisfies

$$du(t) = (\Delta u(t) + \beta u(t) - \alpha^{-2}(t)u^3(t))dt, \tag{5.8}$$

and so this equation can be studied ω by ω, as if it were a non-autonomous deterministic system.

In fact, by a proof similar to that in Temam [32], Chap. III, theorem 1.1, one can show that $P - a.s.$ the following holds:

i) for all $t_0 < T$ and all $u_0 \in H$ there exists a unique solution of equation (5.8)

$$u \in \mathcal{C}([t_0, T]); H) \cap L^2(t_0, T; V) \cap L^4([t_0, T]; L^4(D))$$

with $u(t_0) = u_0$,

ii) if $u_0 \in V$, the solution belongs to $\mathcal{C}([t_0, +\infty)); V) \cap L^2_{loc}(t_0, +\infty; D(A))$,

iii) hence, for all $u_0 \in H$, $u \in \mathcal{C}([t_0 + \epsilon, +\infty)); V) \cap L^2_{loc}(t_0 + \epsilon, +\infty; D(A))$ for every $\epsilon > 0$.

iv) Denoting such a solution by $u(t, \omega; t_0, u_0)$, the mapping $u_0 \mapsto u(t, \omega; t_0, u_0)$ is continuous for all $t \geq t_0$.

The corresponding random dynamical system associated to problem (5.7) is thus defined by

$$\varphi(t, \omega)x_0 = \alpha(t, \omega)^{-1}u(t, \omega; t_0, x_0).$$

Theorem 5.1. *Under the preceding assumptions, there exists a random attractor $\mathcal{A}_{\beta,\sigma}(\omega)$ for the dynamical system $\varphi(t, \omega)$ associated to equation (2.1). Moreover, if $\beta - \lambda_1 < 0$, the random attractor becomes $\{0\}$ $P - a.s.$*

We present all the computations below, since not only do they reproduce the stabilization result of section 3, but they also give rise to certain estimates which we will need in the appendix.

Proof. Firstly, we multiply (5.8) by $u(t)$ in H and obtain that $P-a.s.$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t)|^2 &= -\|u(t)\|^2+\beta|u(t)|^2-\alpha^{-2}(t)|u(t)|^4_{L^4}\\ &\le -\lambda_1|u(t)|^2+\beta|u(t)|^2-\alpha^{-2}(t)|u(t)|^4_{L^4}.\end{aligned} \tag{5.9}$$

Thus, and taking into account that $|u|\le c|u|_{L^4}$ (with $c=|D|^{1/4}$, where $|\cdot|$ also denotes Lebesgue measure),

$$\begin{aligned}\frac{d}{dt}|u(t)|^2+\|u(t)\|^2 &\le -\lambda_1|u(t)|^2+2\alpha^{-1}(t)\alpha(t)\beta|u(t)|^2-2\alpha^{-2}(t)|u(t)|^4_{L^4}\\ \frac{d}{dt}|u(t)|^2+\|u(t)\|^2 &\le -\lambda_1|u(t)|^2+c^4\alpha^2(t)\beta^2\\ &\quad +c^{-4}\alpha^{-2}(t)|u(t)|^4-2c^{-4}\alpha^{-2}(t)|u(t)|^4\\ &\le -\lambda_1|u(t)|^2+c^4\alpha^2(t)\beta^2.\end{aligned} \tag{5.10}$$

Integrating in $[t_0,-1]$, with $t_0\le -1$ we obtain

$$\begin{aligned}|u(-1)|^2 &\le e^{-\lambda_1(-1-t_0)}|\alpha(t_0)x_0|^2+\int_{t_0}^{-1}e^{-\lambda_1(-1-s)}c^4\alpha^2(s)\beta^2\,ds\\ &\le e^{\lambda_1}(e^{\lambda_1 t_0}|\alpha(t_0)x_0|^2+c^4\beta^2\int_{t_0}^{-1}e^{\lambda_1 s}\alpha^2(s)ds).\end{aligned} \tag{5.11}$$

Consequently, given $B(0,\rho)\subset H$, $P-a.s.$ there exists $t(\omega,\rho)\le -1$ such that for all $t_0\le t(\omega,\rho)$ and for all $x_0\in B(0,\rho)$

$$|u(-1,\omega;t_0,\alpha(t_0)x_0)|^2\le r_1^2(\omega),$$

with

$$r_1^2(\omega)=e^{\lambda_1}\left(1+c^4\beta^2\int_{-\infty}^{-1}e^{\lambda_1 s}\alpha^2(s)\,ds\right). \tag{5.12}$$

Indeed, it is enough to choose $t(\omega,\rho)$ such that

$$e^{\lambda_1 t_0}\alpha^2(t_0)\rho^2\le 1$$

and take into account (5.11) and the fact that $P-a.s.$ $e^{\lambda_1 s}\alpha^2(s)=e^{\lambda_1 s}e^{-2\sigma W_s}\to 0$ as $s\to-\infty$.

If we now return to (5.9), note that if $\beta<\lambda_1$ we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t)|^2 &\le -\lambda_1|u(t)|^2+\beta|u(t)|^2\\ &= (\beta-\lambda_1)|u(t)|^2 \quad \forall t\ge 0,\end{aligned} \tag{5.13}$$

and so

$$|u(t)|^2\le |u(0)|^2e^{2(\beta-\lambda_1)t},$$

from which

$$\begin{aligned}|x(t,\omega;0,x_0)|^2 &\le |x_0|^2e^{2(\beta-\lambda_1)t}\alpha^{-2}(t)\\ &= |x_0|^2e^{2(\beta-\lambda_1+\sigma\frac{W_t}{t})t}.\end{aligned}$$

Since $P-a.s.$

$$\lim_{t\to+\infty}\frac{W_t}{t}=0, \tag{5.14}$$

we conclude that $P-a.s.$ there exists $t(\omega)$ such that for all $t\ge t(\omega)$

$$(\beta-\lambda_1+\sigma\frac{W_t}{t})<0,$$

that is, we get the exponential asymptotic stability of $x\equiv 0$.

Observe that we also have attraction from $-\infty$, since by integration in (5.13) between t_0 and 0 we obtain

$$|x(0)|^2 \leq \alpha^2(t_0)|x_0|^2 e^{-2(\beta-\lambda_1)t_0},$$

which tends to zero as $t_0 \to -\infty$, so that $B(0,\epsilon) \subset H$, for all $\epsilon \in (0,1]$, is absorbing for equation (5.7), so that the random attractor in this case is the (deterministic) stationary point $x \equiv 0$.

We now make some other estimates before concluding that there exists a compact absorbing set for the trajectories, so that theorem 4.1 can be applied. From now on we suppose that $\beta > \lambda_1$.

¿From (5.10) and for $t \in [-1,0]$ we have

$$|u(t)|^2 \leq e^{-\lambda_1(t+1)}|u(-1)|^2 + c^4\beta^2 \int_{-1}^{t} e^{-\lambda_1(t-s)}\alpha^2(s)\,ds$$

and

$$\int_{-1}^{0} \|u(s)\|^2\,ds \leq |u(-1)|^2 + c^4\beta^2 \int_{-1}^{0} \alpha^2(s)\,ds.$$

Thus, we can conclude that given $B(0,\rho) \subset H$ and $P-a.s.$ there exists $t(\omega,\rho) \leq -1$ such that for all $t_0 \leq t(\omega,\rho)$ and all $u_0 \in B(0,\rho)$

$$|u(t,\omega;t_0,u_0)|^2 \leq e^{-\lambda_1(1+t)}r_1^2(\omega) + c^4\beta_2 \int_{-1}^{t} e^{-\lambda_1(t-s)}\alpha^2(s)\,ds \tag{5.15}$$

$$\int_{-1}^{0} \|u(s,\omega;t_0,u_0)\|^2\,ds \leq r_1^2(\omega) + c^4\beta^2 \int_{-1}^{0} \alpha^2(s)\,ds. \tag{5.16}$$

To get a bound in V we multiply (5.8) by $-\Delta u(t)$ and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= -|\Delta u|^2 + \beta\|u(t)\|^2 + (\alpha^{-2}(t)u^3(t), \Delta u(t)) \\ &\leq -\lambda_1\|u(t)\|^2 + \beta\|u(t)\|^2 - \int_D \alpha^{-2}(t)3u^2(t)|\nabla u(t)|^2\,dx\end{aligned}$$

and note that the last term on the right is non-positive, so if we integrate in $[s,0]$, $s \in [-1,0]$,

$$\|u(0)\|^2 \leq \|u(s)\|^2 + 2(\beta-\lambda_1)\int_s^0 \|u(\sigma)\|^2\,d\sigma.$$

Integrating again in $[0,1]$

$$\|u(0)\|^2 \leq \int_{-1}^{0} \|u(s)\|^2\,ds + 2(\beta-\lambda_1)\int_{-1}^{0} \|u(\sigma)\|^2\,d\sigma.$$

It is now straightforward from (5.16) that

$$\|u(0)\|^2 = \|x(0)\|^2 \leq 2(\beta + \frac{1}{2} - \lambda_1)(r_1^2(\omega) + c^4\beta^2 \int_{-1}^{0} \alpha^2(s)\,ds).$$

Consequently, $P-a.s.$ there exists $r_2(\omega)$ such that given $\rho > 0$ there exists $\hat{t}(\omega) \leq -1$ such that for all $t_0 \leq \hat{t}(\omega)$ and $x_0 \in H$ with $|x_0| \leq \rho$

$$\|x(0,\omega;t_0,x_0)\|^2 \leq r_2^2(\omega),$$

where

$$r_2^2(\omega) = 2(\beta + \frac{1}{2} - \lambda_1)(e^{\lambda_1} + c^4\beta^2 + 1)(\int_{-\infty}^{0} e^{\lambda_1 s}\alpha^2(s)\,ds + \int_{-1}^{0} \alpha^2(s)\,ds)).$$

Thus, we conclude from theorem 4.1 that there exists a random attractor $\mathcal{A}_{\beta,\sigma}(\omega)$ for equation (2.1). $\square$

Note that since the new parameter β corresponds to $a - \frac{1}{2}\sigma^2$ from our original equation (2.2), we once again obtain the stabilization result in theorem 1, since for $\beta < \lambda_1$ we have proved the exponential asymptotic stability of $x \equiv 0$.

Before we continue our study of the asymptotic behaviour of (2.1) by estimating the dimension of the random attractor, there are two points which need to be clarified.

First, the attraction property for the random attractor is *from* $t = -\infty$, while the stabilization result in section 2 treats the behaviour of the solutions as $t \to +\infty$. However, it is not difficult to see (Crauel and Flandoli [9]) that the attraction property to the random attractor implies attraction in probability as $t \to +\infty$, that is, for all $\epsilon > 0$

$$\lim_{t \to +\infty} P(\mathrm{dist}(\varphi(t,\omega)B, \mathcal{A}(\theta_t\omega)) < \epsilon) = 1. \tag{5.17}$$

In our case, we have shown a little more, as we proved that $x \equiv 0$ is the random attractor for $a < \frac{1}{2}\sigma^2 + \lambda_1$, and the attraction as $t \to +\infty$ holds $P - a.s.$ and not only in probability.

The union in ω of $\mathcal{A}_{\beta,\sigma}(\omega)$ is not uniformly bounded in general, so that the union of this family of sets which form the "random attractor" is certainly not a compact set, and can in fact be dense in the phase space. Thus the relationship between such a random attractor and the corresponding deterministic attractor when $\sigma = 0$ is not immediately clear. However, it is not difficult to check that the hypotheses of theorem 2 in Caraballo et al. [3] hold, and this guarantees the upper-semicontinuity of random attractors to the (deterministic) global attractor as $\sigma \to 0$, that is,

$$\lim_{\sigma \to 0} \mathrm{dist}(\mathcal{A}_{\beta,\sigma}(\omega), \mathcal{A}) = 0 \quad \text{with probability one.}$$

Indeed, the main hypothesis in [3] is the existence of a compact set $K \subset H$ such that, for fixed β and $P - a.s.$

$$\lim_{\sigma \to 0} \mathrm{dist}(\mathcal{A}_{\beta,\sigma}(\omega), K) = 0,$$

which can be proved for our problem by taking the limit in $r_2(\omega)$, i.e. $P - a.s.$

$$\lim_{\sigma \to 0} r_2^2(\omega) = 2(\beta + \frac{1}{2} - \lambda_1)(e^{\lambda_1} + \beta^2 + 1)(\lambda^{-1} + 1).$$

Clearly, this relates the random and deterministic attractors in a more satisfactory way.

In the next section, we obtain a bound on the Hausdorff dimension of $\mathcal{A}_{\beta,\sigma}(\omega)$ in terms of β (our bound is independent of σ). In particular, this limits the complexity of the attractor as a function of β, and gives us some clues as to its possible behaviour as β increases through the value λ_1.

6. **Bounds on the Hausdorff dimension of the random attractor.** One of the most surprising properties of global attractors for certain infinite-dimensional dynamical systems (including many important cases) is that these compact sets are in fact finite-dimensional subsets of the original phase space (Temam [32]). This result leads to the fact that the asymptotic behaviour of these systems can be described using a finite number of degrees of freedom (a statement to which one can attach various senses, see, for example Foias and Prodi [18], Foias and Olson [17], Eden et al. [14], Robinson [27]).

Various generalisations of deterministic methods (see e.g. Constantin *et al.* [7]) necessary to perform a similar analysis in the stochastic case can be found in Crauel and Flandoli [11] or Debussche [12], [13] among others. Of these, the most powerful is due to Debussche [13], and gives results which in general agree with the best known deterministic bounds as the level of noise is reduce to zero. We make use of this method below.

Note that although the finite-dimensionality of each of the compact sets $\mathcal{A}_{\beta,\sigma}(\omega)$ indicates that the asymptotic behaviour of the system should be "effectively finite-dimensional", we only have convergence to the (t,ω)-dependent family $\mathcal{A}_{\beta,\sigma}(\theta_t\omega)$ *in probability* as $t \to +\infty$, and it is not clear exactly how to make such a statement rigorous in the stochastic case (although see Flandoli and Langa [16] and Berselli and Flandoli [2] for approaches which emulate the "determining modes" of Foias and Prodi [18]). (Despite the references above for the deterministic case, the problem is still not completely resolved even there, see Robinson [27] for a more lengthy discussion.)

To bound the attractor dimension we use the following result from Debussche [13]. He treats the case of a random attractor $\mathcal{A}(\omega)$ which is invariant under a random map $S(\omega)$: for some measure-preserving ergodic transformation θ on $(\Omega, \mathcal{F}, P)$ we have

$$S(\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta\omega).$$

We need to make the following assumptions about the map $S(\omega)$. Firstly, we need $S(\omega)$ to be *almost surely uniformly differentiable on* $\mathcal{A}(\omega)$, which means that P-almost surely, for every $u \in \mathcal{A}(\omega)$, there exists a bounded linear operator $DS(\omega, u)$ from H into H, such that if $u + h \in \mathcal{A}(\omega)$ also, we have

$$|S(\omega)(u+h) - S(\omega)u - DS(\omega,u)h| \leq K(\omega)|h|^{1+\alpha}, \tag{6.18}$$

where $\alpha > 0$ and $K(\omega)$ is a random variable with $K(\omega) \geq 1$ and

$$E(\ln K) < \infty. \tag{6.19}$$

The result shows (essentially) that if infinitesimal d-volumes are contracted under $S(\omega)$ then the dimension of $A(\omega)$ is almost surely less than d. This infinitesimal expansion is measured by the quantity $\varepsilon_d(S(\omega))$, where

$$\varepsilon_d(L) = \alpha_1(L) \dots \alpha_d(L),$$

with

$$\alpha_j(L) = \sup_P \inf_{\{u \in PH:\ |u|=1\}} |Lu|,$$

where the supremum is taken over all rank d orthogonal projections P. Alternatively, the $\{\alpha_j\}$ are the eigenvalues of L^*L arranged in decreasing order.

To bound the Hausdorff dimension by d, it is sufficient to assume that

$$\varepsilon_d(DS(\omega,u)) \leq \bar{\varepsilon}(\omega),$$

where $\bar{\varepsilon}(\omega)$ is a random variable such that

$$E(\ln \bar{\varepsilon}) < 0,$$

and the additional (relatively easy) condition that, for some random variable $\bar{\alpha} \geq 1$, we have

$$\alpha_1(DS(\omega,u)) \leq \bar{\alpha}(\omega) \qquad \text{with} \qquad E(\ln \bar{\alpha}) < \infty. \tag{6.20}$$

Under the above assumptions, we have $d_H(\mathcal{A}(\omega)) < d$ almost surely.

The main difficulty in applying this result is in checking the differentiability property (6.18). We do this in the appendix, where we work solely with the equation for $u = e^{\sigma W_t}x$,

$$du/dt = \Delta u + \beta u - e^{2\sigma W_t}u^3. \tag{6.21}$$

We prove there the following result, where

$$T(\omega)u_0 = u(1,\omega;0,u_0).$$

Proposition 6.1. *$T(\omega)$ is almost surely differentiable on $\mathcal{A}(\omega)$: P-almost surely, for every $u \in \mathcal{A}(\omega)$, there exists a bounded linear operator $DT(\omega,u)$ such that if $u, u+h \in \mathcal{A}(\omega)$ then*

$$|T(\omega)(u+h) - T(\omega)u - DT(\omega,u)h| \leq K(\omega)|h|^{1+\alpha},$$

where $\alpha > 0$ and $K(\omega)$ satisfies

$$E(\ln K) < \infty. \tag{6.22}$$

Furthermore, $DT(\omega,u)h = U(1)$, where $U(t)$ is the solution of the equation

$$\frac{dU}{dt} = \Delta U + \beta U - 3u^2(t)e^{2\sigma W_t}U \qquad U(0) = h, \tag{6.23}$$

and $u(t)$ solves (6.21) with $u(0) = u$.

Note that in fact we want to verify the differentiability properties for the cocycle generated by the equation for x, not u. We set $\theta = \theta_1$, and consider the random function $S(\omega)$, where

$$S(\omega)x_0 = S(1,\omega;0,x_0),$$

so that

$$S(\omega) = e^{2\sigma W_1}T(\omega).$$

Since

$$E(2\sigma W_1) = 0,$$

(6.19) holds once more, and it follows that S is P-almost surely uniformly differentiable on the attractor, with derivative

$$DS(\omega,x) = e^{2\sigma W_1}DT(\omega,x).$$

Theorem 6.2. *If*

$$\beta < \frac{1}{d}\sum_{j=1}^{d}\lambda_j \leq Cd^{n/2} \tag{6.24}$$

then P-almost surely $d_H(\mathcal{A}(\omega)) < d$. In particular, if $\beta < \lambda_1$ then $\mathcal{A}(\omega)$ consists of just one point.

Proof. We apply Debussche's result. First, to check that $\alpha_1(DS(\omega,x)) \leq \bar{\alpha}(\omega)$ as in (6.20), observe that it follows easily from (6.23) that $\|DT(1,\omega;u)\| \leq e^{\beta}$, and hence that

$$\alpha_1(DS(\omega,x)) \leq \bar{\alpha}(\omega) = e^{2\sigma W_1+\beta}.$$

Clearly, $E(\ln\bar{\alpha}) = \beta < \infty$.

To find a d such that $\varepsilon_d(DS) < 1$, we use the trace formula due to Constantin *et al.* [7] (see also Temam [32], Chapter V). This allows us to write ε_d in another way more dependent on the dynamics. Since $DT(\omega,u)h$ is the solution of the linear equation

$$dU/dt = L(t,u(t))U, \qquad U(0) = h$$

where

$$L(t,u(t)) = \Delta + (\beta - 3u(t)^2e^{2\sigma W_t})I,$$

and $u(t)$ is the solution of (6.21) with $u(0) = u$, we can write

$$DT(\omega,u) = \exp\left(\int_0^1 L(s,u(s))\,ds\right).$$

It follows that

$$DS(\omega, x) = \exp\biggl(2\sigma W_1 + \int_0^1 L(s; x(s))\, ds\biggr),$$

where $x(s)$ is the solution of (6) with $x(0) = x$. We therefore have

$$\varepsilon_d(DS(\omega, x)) = \sup_{P(0)} \exp\biggl(2\sigma W_1 + \mathrm{Tr} \int_0^1 L(s; x(s)) P(s)\, ds\biggr).$$

Here, $P(0)$ is an orthogonal projector of rank d, onto the space spanned by $\{\phi_j\}_{j=1}^d$, and $P(t)$ the projector onto the space spanned by the images of the vectors ϕ_j under the linearised flow $DS(t, \omega; x)$ (the same space as that spanned by their images under the flow $DT(t, \omega; x)$). We look for the smallest d for which we can guarantee that

$$2\sigma W_1 + \sup_{P(0)} \mathrm{Tr} \int_0^1 L(s; x(s)) P(s)\, ds < 0.$$

We concentrate on the second term. For a fixed rank d orthogonal projection P, with range spanned by a set of orthonormal elements in H, $\{\phi_j\}_{j=1}^d$, we have

$$\mathrm{Tr}(L(t)P) = \sum_{j=1}^d (\Delta\phi_j, \phi_j) + \beta \sum_{j=1}^d |\phi_j|^2 - 3e^{2\sigma W_t} \int_D u^2 \phi_j^2\, dy.$$

Since

$$\sum_{j=1}^d (-\Delta\phi_j, \phi_j) \geq \sum_{j=1}^d \lambda_j,$$

where $\{\lambda_j\}$ are the eigenvalues of the Laplacian arranged in increasing order (see Temam [32], chapter VI, section 2.1), we obtain

$$\mathrm{Tr}(L(t)P) \leq -\sum_{j=1}^d \lambda_j + \beta d.$$

Thus we can take $\bar{\varepsilon}(\omega)$ to be the random variable

$$\exp\biggl(2\sigma W_1 - \sum_{j=1}^d \lambda_j + \beta d\biggr),$$

and since $E(W_1) = 0$, it follows that $E(\ln \bar{\varepsilon}) < 0$ if we take

$$\beta < \frac{1}{d} \sum_{j=1}^d \lambda_j.$$

The second part of (vii) follows from the estimate

$$\sum_{j=1}^d \lambda_j \leq c d^{(n+2)/n},$$

see Temam [32](chapter VI, section 2.1, again).

In the case $d = 1$ (6.24) reduces to $\beta < \lambda_1$, and all that remains is to show that $\mathcal{A}(\omega)$ is in fact just one point. For this we make use of Remark 2.6 in Debussche [13], namely that a similar also provides a bound on the fractal dimension of the attractor. For our purposes the exact bound on the fractal dimension is irrelevant. Provided that it is finite, there then exists an orthogonal projection P of finite rank (k, say) such that the map between $\mathcal{A}(\omega)$ and $P\mathcal{A}(\omega)$ is $1-1$ (see Foias and Olson [17], for example). In this way we can identify $\mathcal{A}(\omega)$ with a subset of the

Euclidean space $\mathbf{R}^k$. Since P is Lipschitz, it follows that $d_H(P\mathcal{A}(\omega)) < 1$, and since $\mathcal{A}(\omega)$ is connected so is $P\mathcal{A}(\omega)$. It now follows from lemma 3.12 in Falconer [15] [a connected subset of $\mathbf{R}^k$ with finite one-dimensional Hausdorff measure is arcwise connected] that in fact $P\mathcal{A}(\omega)$, and so $\mathcal{A}(\omega)$, consists of just one point. □

For $\beta < \lambda_1$ we recover something akin to our stabilization result, namely that the attractor $\mathcal{A}(\omega)$ consists of a single point. However, this could be a random point $a(\omega)$, and it is only the analysis in section 3 (or section 5) which guarantees that in fact $a(\omega) = 0$ P-almost surely.

For $\beta > \lambda_1$, the result as it stands does not guarantee that the attractor is any more complicated than $\{0\}$, but rather limits the possible complexity of the attractor "from above".

Finally, note that the behaviour of the attractor dimension is limited by the parameter β, but that σ plays no rôle. In particular, it is possible to have an arbitrarily large level of noise (in the Itô sense) and still have an attractor whose dimension is well controlled.

7. **Conclusions.** In the first part of the paper we investigated the effect of adding a multiplicative Itô white noise term to a well-known deterministic PDE, and saw that this term produced a stabilization of the trivial fixed point $x \equiv 0$. Transforming the equation into its Stratonovich form we saw no corresponding stabilization effect from the noisy term, highlighting the importance of the interpretation of the stochastic integral in such equations. Moreover, this strongly suggests that in the case of a multiplicative noise it is more sensible to consider the equation in the Stratonovich sense.

To investigate the problem further we have proved the existence of a random attractor $\mathcal{A}_{\beta,\sigma}(\omega)$, and shown that this is a finite-dimensional set P-almost surely. In particular, our dimension estimate changes qualitatively when β passes through λ_1.

To conclude, we conjecture that as β is increased through the value λ_1, the fixed point $x \equiv 0$ undergoes a stochastic form of pitchfork bifurcation, giving rise to an attractor which is essentially a 1-dimensional manifold while $\beta < \frac{1}{2}(\lambda_1 + \lambda_2)$. We plan to investigate this further in a future paper.

Appendix. In this appendix we give a proof of Proposition 1. First we need to prove two auxiliary results which give various estimates on the solutions of equation (5.7).

Proposition 7.1. *(Lipschitz property for the solutions)*
Let $x_i(t) = x_i(t, \omega; 0, x_i^0)$, $i = 1, 2$, be two solutions of problem (5.7) for $x_i^0 \in H$. Then, $P - a.s.$

$$|x_1(t) - x_2(t)|^2 \leq e^{2(\beta-\lambda_1)t+2\sigma W_t}|x_1^0 - x_2^0|^2 \quad \forall t \geq 0.$$

In particular, for $t = 1$,

$$|x_1(1, \omega; 0, x_1^0) - x_2(1, \omega; 0, x_2^0)| \leq e^{(\beta-\lambda_1)t+2\sigma W_1}|x_1^0 - x_2^0| \quad \forall t \geq 0.$$

Proof. As $x_i(t)$, $i = 1, 2$, satisfies

$$dx_i(t) = (\Delta x_i(t) + \beta x_i(t) - x_i^3(t))dt + \sigma x_i(t) \circ dW_t,$$

then, for $w(t) = x_1(t) - x_2(t)$ it follows

$$dw(t) = (\Delta w(t) + \beta w(t) - (x_1^3(t) - x_2^3(t)))dt + \sigma w(t) \circ dW_t.$$

By the change of variable

$$v(t) = \alpha(t)w(t), \quad \alpha(t) = e^{-\sigma W_t}$$

we have

$$dv(t) = (\Delta v(t) + \beta v(t) - \alpha(t)(x_1^3(t) - x_2^3(t)))dt.$$

Multiplying by $v(t)$ in H we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v(t)|^2 &= -\|v(t)\|^2 + \beta|v(t)|^2 - \alpha^2(t)\int_D (x_1^3(t) - x_2^3(t), x_1(t) - x_2(t))\, dy \\ &= -\|v(t)\|^2 + \beta|v(t)|^2 + \alpha^2(t)\int_D (x_2^3(t) - x_1^3(t), x_1(t) - x_2(t))\, dy\end{aligned}$$

and note that this last integral is less or equal to zero, so that

$$\frac{d}{dt}|v(t)|^2 \le 2(-\lambda_1 + \beta)|v(t)|^2.$$

Therefore

$$|x_1(t) - x_2(t)|^2 \le e^{2(\beta-\lambda_1)t+2\sigma W_t}|x_1^0 - x_2^0|^2 \quad \forall t \ge 0, \tag{7.25}$$

thus, for $t = 1$,

$$|x_1(1,\omega;0,x_1^0) - x_2(1,\omega;0,x_2^0)| \le e^{(\beta-\lambda_1)+\sigma W_1}|x_1^0 - x_2^0| \tag{7.26}$$

and the proof is complete. □

Proposition 7.2. *If $u(t)$ is the solution of*

$$du/dt = \Delta u + \beta u - e^{2\sigma W_t}u^3, \qquad u(0) \in \mathcal{A}(\omega) \tag{7.27}$$

then for each $p \in Z^+$ there exist random variables $I_{2p}(\omega)$ such that

$$\int_0^1 |u(s)|_{L^{2p}}^{2p}\, dx \le I_{2p}(\omega), \tag{7.28}$$

where, for all $p \in Z^+$ and all $k \ge 0$,

$$E\big(I_{2p}^k\big) < \infty. \tag{7.29}$$

Proof. Since $u(0) \in \mathcal{A}(\omega)$, it follows, using the invariance of the random attractor, that there exists a trajectory $u(t)$ of (7.27) which is defined for all $t \in \Re$ and has $u(t) \in \mathcal{A}(\theta_t\omega)$ for every $t \in \mathbf{R}$. To prove the bound in (7.28), we show first that

$$\int_{t-r}^t |u(s)|_{L^{2p+2}}^{2p+2}\, ds \le C_p\left(\sup_{t-r\le s\le t} e^{-2\sigma W_s}\right)\int_{t-(r+1)}^t |u(s)|_{L^{2p}}^{2p}\, ds.$$

We prove this by induction, using two inequalities derived from (7).

We first convert (7) into a differential inequality involving the norms in various Lebesgue spaces. If we multiply (7) by u^{2k-1} and integrate over D we have

$$\int_D u^{2k-1}\frac{\partial u}{\partial t}\, dy = \frac{1}{2k}\frac{d}{dt}\int_D u^{2k}\, dy,$$

and

$$\begin{aligned}\int_D u^{2k-1}\Delta u\, dy &= \int_D \sum_{j=1}^n u^{2k-1}\partial_j^2 u\, dy \\ &= -\int_D \sum_{j=1}^n (2k-1)u^{2k-2}(\partial_j u)^2\, dy \\ &= -(2k-1)\int_D \sum_{j=1}^n \big(u^{k-1}\partial_j u\big)^2\, dy \le 0,\end{aligned}$$

so we have

$$\frac{1}{2k}\frac{d}{dt}|u|^{2k}_{L^{2k}} + e^{2\sigma W_t}|u|^{2k+2}_{L^{2k+2}} \le \beta|u|^{2k}_{L^{2k}}. \tag{7.30}$$

First we integrate this equation between s and t $(t-1 \le s < t)$ to give

$$\frac{1}{2k}|u(t)|^{2k}_{L^{2k}} \le \beta\int_s^t |u(s)|^{2k}_{L^{2k}}\,ds + \frac{1}{2k}|u(s)|^{2kp}_{L^{2k}},$$

and then integrate again with respect to s between $t-1$ and t to obtain

$$|u(t)|^{2k}_{L^{2k}} \le (1+2k\beta)\int_{t-1}^t |u(s)|^{2k}_{L^{2k}}\,ds. \tag{7.31}$$

Returning to (7.30) and integrating between $t-r$ and t gives

$$\begin{aligned}\int_{t-r}^t e^{2\sigma W_s}|u(s)|^{2k+2}_{L^{2k+2}}\,ds \quad &\le \quad \frac{1}{2k}|u(t-r)|^{2k}_{L^{2k}}\\ &\quad +\beta\int_{t-r}^t |u(s)|^{2k}_{L^{2k}}\,ds.\end{aligned}$$

Now, using the result in (7.31) we have

$$\begin{aligned}\int_{t-r}^t e^{-2\sigma W_s}|u(s)|^{2k+2}_{L^{2k+2}}\,ds \quad &\le \quad \frac{1}{2k}(1+2k\beta)\int_{t-(r+1)}^{t-r} |u(s)|^{2k}_{L^{2k}}\,ds\\ &\quad +\beta\int_{t-r}^t |u(s)|^{2k}_{L^{2k}}\,ds\\ &\le \quad (1+\beta)\int_{t-(r+1)}^t |u(s)|^{2k}_{L^{2k}}\,ds.\end{aligned}$$

Finally, we deduce that

$$\int_{t-r}^t |u(s)|^{2k+2}_{L^{2k+2}}\,ds \le (1+\beta)\left(\sup_{t-r\le s\le t} e^{-2\sigma W_s}\right)\int_{t-(r+1)}^t |u(s)|^{2k}_{L^{2k}}\,ds.$$

To obtain the bound in the statement of the proposition, we apply this estimate repeatedly, writing

$$S_j = \left(\sup_{1-j\le s\le 1} e^{-2\sigma W_s}\right).$$

We have

$$\begin{aligned}\textstyle\int_0^1 |u(s)|^{2k+2}_{L^{2k+2}}\,ds \quad &\le \quad (1+\beta)S_1\textstyle\int_{-1}^1 |u(s)|^{2(k-1)+2}_{L^{2(k-1)+2}}\,ds\\ &\le \quad (1+\beta)^2 S_1 S_2\textstyle\int_{-2}^1 |u(s)|^{2(k-2)+2}_{L^{2(k-2)+2}}\,ds\\ &\le \quad (1+\beta)^k S_1\dots S_k\textstyle\int_{-k}^1 |u(s)|^2\,ds.\end{aligned}$$

Now, since $u(-k) \in \mathcal{A}(\theta_{-k}\omega)$, we have

$$|u(-k)| \le r_1(\theta_{-k}\omega).$$

Now we can use what is essentially (11) once more,

$$|u(s)|^2 \le e^{-\lambda_1 s}\left(e^{-\lambda_1 k}|u(-k)|^2 + c^4\beta^2\int_{-k}^s e^{\lambda_1 s}e^{-2\sigma W_s}\,ds\right),$$

to deduce (7.28) and (7.29). □

We now let $T(t,s;\omega)$ be the random dynamical system generated by the solutions of the transformed equation (7), and show that $T(\omega) = T(1,0;\omega)$ is "almost surely differentiable on $\mathcal{A}(\omega)$" (Proposition 1) as required to apply the result of Debussche.

Proof. (Proposition 1) Thanks to the previous two propositions, the calculations in Debussche [13] can be followed almost exactly, excepting the addition of a term involved an exponentiated white noise. We obtain the equation (cf. p. 987 in [13]),

$$\frac{d}{dt}|r|^2 \leq c_1|r|^2 + c_2 e^{4\sigma W_t}(|u_1|_{L^{q_1}} + |u_2|_{L^{q_1}})^{c_3}|u_1 - u_2|^{2+2\delta}.$$

This yields

$$|r(1)|^2 \leq \left(c \int_0^1 e^{4\sigma W_s}(|u_1|_{L^{q_1}} + |u_2|_{L^{q_1}})^{c_3}\, ds \right) |h|^{2+2\delta}.$$

Setting

$$K(\omega) = \left(\sup_{t\in[0,1]} e^{4\sigma W_t} \right) \max(c\kappa(2r_1, q_1), 1),$$

we have the required differentiability property, using (7.29) to ensure (6.22), since, choosing p so that $2p > \max(r, q)$, we have

$$\int_0^1 |u(s)|_{L^q}^r\, ds \leq C I_p^{r/2p}(\omega).$$

□

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