SKEW PRODUCT SEMIFLOWS AND MORSE DECOMPOSITION

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Abstract. This paper is devoted to the investigation of the dynamics of non-autonomous differential equations. The description of the asymptotic dynamics of non-autonomous equations lies on dynamical structures of some associated limiting non-autonomous - and autonomous - differential equations (one for each global solution in the attractor of the driving semigroup of the associated skew product semi-flow). In some cases, we have infinitely many limiting problems (in contrast with the autonomous - or asymptotically autonomous - case for which we have only one limiting problem; that is, the semigroup itself). We concentrate our attention in the study of the Morse decomposition of attractors for these non-autonomous limiting problems as a mean to understand some of the asymptotics of our non-autonomous differential equations. In particular, we derive a Morse decomposition for the global attractors of skew product semiflows (and thus for pullback attractors of non-autonomous differential equations) from a Morse decomposition of the attractor for the associated driving semigroup. Our theory is well suited to describe the asymptotic dynamics of non-autonomous differential equations defined on the whole line or just for positive times, or for differential equations driven by a general semigroup.

1. Introduction

Recently the analysis of qualitative properties of evolution processes and non-autonomous dynamical systems in general phase spaces - infinite-dimensional Banach spaces or general metric spaces - has received much attention (see, for instance, [4, 5, 6, 7, 10]). In particular, the study of pullback attractors has started to develop into a wide and deep research area, providing qualitative information for the asymptotic dynamics of an increasing number of non-autonomous models of

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phenomena from different areas of science, such as physics, biology, economics, engineering and others.

The pullback attractors are a natural extension for the notion of global attractors of semigroups, in the sense that the global attractors and the pullback attractors are dynamical objects that contain all global bounded solutions. They help us to glance at the inner structure of the asymptotics of an evolution process $\{T(t,s) : t \geqslant s\}$, and enjoy pullback attraction property (attraction at a fixed time $t \in \mathbb{R}$ when $s \to -\infty$). Other kinds of attraction, such as pullback backwards attraction (attraction at time $s \in \mathbb{R}$ when $t \to -\infty$, for the case when $T(s,t)$ is invertible and defining $T(t,s) = T(s,t)^{-1}$, $s > t$), backwards attraction (attraction at $-\infty$ when time $t \to -\infty$) and forwards attraction (attraction at $+\infty$ when $t \to +\infty$) are harder to study and, in general, cannot be obtained from pullback attraction.

Also, as we will see in Theorem 2.7, the uniform attractor, that is, the projection on the first coordinate of the global attractor for the associated skew product semiflow, is given as a union of (possibly) infinite pullback attractors. So, in order to provide a non-autonomous structure for the uniform attractor, which by itself contains no significant information about the asymptotics of our original non-autonomous system, we need to understand the evolution processes enclosed in the non-autonomous dynamical system; more specifically, understand the pullback attractors of each one of these evolution processes.

One of the drawbacks of the theory of pullback attractors is that it requires the vector field to be defined for all times in $\mathbb{R}$, and many models only consider the phenomenon after a given initial time. Of course one can artificially define the vector field for times preceding the given initial one and study the behavior of such system. But then, the pullback attractor would change for each extension and the object "pullback attractor" would lose its importance in the study of the dynamics.

The crucial point here is that, to understand the forwards dynamics of a non-autonomous evolution process (as a general rule), we must understand the dynamics of many (possibly infinite) non-autonomous evolution processes, one for each global solution in the attractor for the driving semigroup of the skew product semiflow. This is in contrast with the autonomous case for which we have only one limiting problem (the semigroup itself) or with the asymptotically autonomous case for which we also have only one limiting problem (the limiting semigroup).

To illustrate the many possible limiting evolution processes that will play a role in the understanding of the dynamics of a given non-autonomous differential equation, consider the initial value problem

$$\begin{cases} \dot{x} = r(t,x), \ t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $r : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is given in Section 3.1. In this case, $r(\cdot + t_n, \cdot)$ with $t_n \to \infty$ may converge to $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ (both independent of t) or to a map $\xi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ ($\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$) which is backwards asymptotic to g (to f) and forwards asymptotic to f (to g). We note that there are infinitely many ξ's and ψ's.

To a limiting vector field η (say $\eta = f, g, \xi$ or ψ) we associate the limiting evolution process $T_\eta(\cdot,\cdot)$ given by $T_\eta(t,s)x_0 = x(t,s,x_0,\eta)$ where $x(t,s,x_0,\eta)$ is the solution of

$$\begin{cases} \dot{x} = \eta(t,x), \ t \geq s \\ x(s) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

at time t. The limiting evolution process $T_\eta(\cdot,\cdot)$ will play an important role in the understanding of the limiting states for the solutions of (1.1) (as seen in Theorem 2.7).

Once one realizes this feature, it becomes clear how rich and difficult is the subject "dynamics of non-autonomous dynamical systems". We already have some insights of this difficulty when we look at a simple concept like hyperbolicity. Indeed, in the autonomous case we choose an equilibrium, linearize it and we can compute the spectrum of the linearized operator to decide whether we have hyperbolicity or not. In the non-autonomous context, we have no way to single out which solutions will be hyperbolic and, if we were able to single out these solutions, how to verify that they actually are hyperbolic is not a trivial task. If we consider a non-autonomous dynamical system given as a non-autonomous perturbation of an autonomous dynamical system which possesses an hyperbolic equilibria, then the simple matter of obtaining that the perturbed non-autonomous dynamical system has a global hyperbolic solution, near the hyperbolic equilibria of the autonomous dynamical system, involves highly non-trivial results on the roughness of exponential dichotomies (see [3],[4]).

Before we proceed, let us recall the notion of global attractor for semigroups. Let Z be a metric space with metric $d : Z \times Z \to \mathbb{R}^+$, and $\mathscr{C}(Z)$ be the space of continuous maps from Z into Z.

Definition 1.1. *A family of mappings* $\{T(t) : t \geq 0\} \subset \mathscr{C}(Z)$ *is a* semigroup *in* Z *if*

- $T(0) = I_Z$, *with* I_Z *being the identity map in* Z,
- $T(t+s) = T(t)T(s)$, *for all* $t, s \in \mathbb{R}^+$, *and*
- $\mathbb{R}^+ \times Z \ni (t,x) \mapsto T(t)x \in Z$ *is continuous.*

Definition 1.2. *A subset* A *of* Z *is said to be* invariant *under the action of the semigroup* $\{T(t) : t \geq 0\}$ *if* $T(t)A = A$ *for all* $t \geq 0$.

Now we will introduce the notions of attraction and absorption. Given $A, B \subset Z$, the Hausdorff semidistance from A to B is given by

$$\mathrm{d}_H(A,B) := \sup_{a \in A} \inf_{b \in B} d(a,b).$$

Definition 1.3. *Given two subsets* A, B *of* Z *we say that* A attracts B *under the action of the semigroup* $\{T(t) : t \geq 0\}$ *if* $\mathrm{d}_H(T(t)B, A) \stackrel{t\to\infty}{\longrightarrow} 0$, *and we say that* A absorbs B *under the action of* $\{T(t) : t \geq 0\}$ *if there is a* $t_B > 0$ *such that* $T(t)B \subset A$ *for all* $t \geq t_B$.

Thanks to these previous definitions, we can now define *global attractors*.

Definition 1.4. *A subset* $\mathcal{A}$ *of* Z *is said to be a* global attractor *for a semigroup* $\{T(t) : t \geq 0\}$ *if it is compact, invariant under the action of* $\{T(t) : t \geq 0\}$, *and for every bounded subset* B *of* Z *we have that* $\mathcal{A}$ *attracts* B *under the action of* $\{T(t) : t \geq 0\}$.

Let us now consider a general non-autonomous differential equation to illustrate the approach we will carry out in this paper. Consider the initial value problem

$$\begin{cases} \dot{x} = f(t,x), \quad t > 0 \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a "*nice*" function which belongs to a metric space $\mathscr{C}$. Assume that, for each $f \in \mathcal{C}$, $x_0 \in \mathbb{R}^n$, the solution of (1.3) is defined for all $t \geq 0$; that is, for each $x_0 \in \mathbb{R}^n$, there is a unique continuous function $\mathbb{R}^+ \ni t \mapsto x(t, f, x_0) \in \mathbb{R}^n$ satisfying (1.3).

Now, following [13] we define the skew product semiflow associated to (1.3) in the following way

$$\Pi(t) : \mathbb{R}^n \times \mathscr{C} \to \mathbb{R}^n \times \mathscr{C}$$

$$\Pi(t)(x_0, f) = (x(t, f, x_0), f_t),$$

where $f_t(s,x) = f(t+s,x)$ for all $t,s \geq 0$ and $x \in \mathbb{R}^n$. Suppose that the map $\mathbb{R}^+ \times \mathscr{C} \times \mathbb{R}^n \ni (t,f,x_0) \mapsto (x(t,f,x_0), f_t) \in \mathbb{R}^n \times \mathscr{C}$ is continuous. It is easy to see that

$$x(t+s,f,x_0) = x(t,f_s,x(s,f,x_0)).$$

From this we have that

$$\Pi(t+s)(x_0,f) = (x(t+s,f,x_0), f_{t+s}) = (x(t,f_s,x(s,f,x_0)), f_t(f_s))$$
$$= \Pi(t)(x(s,f,x_0),f_s) = \Pi(t)\Pi(s)(x_0,f).$$

Assume that $\mathbb{R}^+ \times \mathscr{C} \times \mathbb{R}^n \ni (t,f,x_0) \mapsto \Pi(t)(x_0,f) \in \mathbb{R}^n \times \mathscr{C}$ is continuous ($\{\Pi(t) : t \geqslant 0\}$ is a semigroup in $Z = \mathbb{R}^n \times \mathscr{C}$) and that $\{\Pi(t) : t \geq 0\}$ possesses a global attractor $\mathcal{A}$ in Z. Then, it may seem that we have found a proper way to study the asymptotic dynamics of (1.3). In fact, the set $\mathcal{A}$ possesses dynamics associated to $\{\Pi(t) : t \geq 0\}$ but it does not have any dynamics immediately associated to (1.3). An element of $\mathcal{A}$ is an element of $\mathbb{R}^n \times \mathscr{C}$; that is, an initial condition $y_0 \in \mathbb{R}^n$ and a vector field g (which is not f in general) and $\Pi(t)(y_0,g) = (x(t,g,y_0), g_t)$ has no straightforward relation to (1.3). Let us try to unravel a little the connection of the points in $\mathcal{A}$ with the dynamics of (1.3).

The first step to study the dynamics of (1.3) is to understand the attraction property of $\{\Pi(t) : t \geq 0\}$ as $t \to \infty$ relatively to the solution operator of (1.3).

Given a bounded subset $\mathcal{B}$ of $\mathbb{R}^n \times \mathscr{C}$, $\mathcal{A}$ attracts $\mathcal{B}$ under the action of $\{\Pi(t) : t \geq 0\}$ if

$$\lim_{t\to\infty} \mathrm{d}_H(\Pi(t)\mathcal{B}, \mathcal{A}) = 0.$$

If, for a given bounded subset $B \subset \mathbb{R}^n$, we only consider a bounded subset $\mathcal{B}$ of the form $B \times \{f\}$, this attraction property can be written as

$$\lim_{t\to\infty} \mathrm{d}_H(x(t,f,B) \times \{f_t\}, \mathcal{A}) \geq \lim_{t\to\infty} \mathrm{d}_H(x(t,f,B), A),$$

where $A = \{x \in \mathbb{R}^n : \text{ there exists } g \in \mathcal{C} \text{ such that } (x,g) \in \mathcal{A}\}$. This means that the compact set $A \subset \mathbb{R}^n$ attracts bounded subsets of $\mathbb{R}^n$.

Although the set A does not have any dynamical property relatively to (1.3), we will see that some families in it are crucial to understand the dynamics of (1.3).

Given a non-autonomous differential equation such as (1.3), we can refer to three different but closely related dynamical systems:

- The *driving system* $\{\Theta(t) : t \geqslant 0\}$ associated to the dynamics of the time-dependent terms appearing in the equation, and which is defined by $\Theta(t)f(\cdot,x) = f_t(\cdot,x) = f(t+\cdot,x)$,

- the *skew-product semiflow* $\{\Pi(t):\ t\geqslant 0\}$ defined on the product space $Z=\mathbb{R}^n\times\mathscr{C}$,
- and the associated *non-autonomous dynamical system* $(\varphi,\Theta)_{(\mathbb{R}^n,\mathscr{C})}$ with $\varphi(t,\Theta_s f)x_0=x(t+s,f,x_0)$.

Next, we give the precise definition of the *skew product semiflow* associated to a non-autonomous dynamical system.

Definition 1.5. *Consider two metric spaces (X,d_X) and $(\mathscr{P},d_{\mathscr{P}})$. A* non-autonomous dynamical system *(NDS), denoted by $(\varphi,\Theta)_{(X,\mathscr{P})}$, consists of two ingredients:*

(i) *A* driving semigroup $\{\Theta(t):t\geqslant 0\}$ *in $\mathscr{P}$.*

(ii) *A* cocycle $\varphi:\mathbb{R}^+\times\mathscr{P}\times X\to X$ *over Θ, that is, a continuous map such that the family of mappings $\varphi(t,p):X\to X$ satisfies the cocycle property:*

1) $\varphi(0,p)=I_X$ *for all* $p\in\mathscr{P}$,
2) $\varphi(t+s,p)=\varphi(t,\Theta(s)p)\varphi(s,p)$ *for all* $t,s\geqslant 0$ *and* $p\in\mathscr{P}$.

The associated skew product semiflow *(SPSF)* $\{\Pi(t):t\geq 0\}\subset\mathcal{C}(X\times\mathscr{P})$ *is given by*

$$\Pi(t)(x,p)=(\varphi(t,p)x,\Theta(t)p).$$

It is clear that $\{\Pi(t):\ t\geqslant 0\}$ is a semigroup with phase space[1] $X\times\mathscr{P}$.

The aim of this paper is to describe the internal structure and dynamics (in the sense of Morse decomposition and Lyapunov functions) of the global attractor for the skew product semiflow and its relation with the pullback attractors for the associated (limiting) non-autonomous dynamical systems.

In Section 2 we recall some results, define some basic concepts and establish the notation we will use throughout the paper. Our first result (Theorem 2.7) says that for each global solution η in the attractor of the driving system $\{\Theta(t):t\geqslant 0\}$ we have an associated non-autonomous evolution process $\{T_\eta(t,s):t\geqslant s\}$ with a pullback attractor $\{\mathcal{A}_\eta(t):t\in\mathbb{R}\}$. There are some results in this direction in the literature (see [5] and [2]) which consider that the driving semigroup $\{\Theta(t):t\geqslant 0\}$ is in fact a group defined in a compact metric space.

In Section 4 we construct a Morse decomposition (see Definitions 2.18 and 2.19) for the global attractor of the skew product semiflow $\{\Pi(t):\ t\geqslant 0\}$ from a Morse decomposition for the global attractor of the driving system $\{\Theta(t):\ t\geqslant 0\}$, using the lift of this Morse decomposition (Theorem 4.3); that is, if we have a Morse decomposition $\{\mathscr{M}_1,\cdots,\mathscr{M}_n\}$ for the attractor of $\{\Theta(t):\ t\geqslant 0\}$

[1] We denote the metric in $X\times\mathscr{P}$ simply by d.

then, the family of lifts $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ ($\mathcal{M}_i = \{(x,p) \in \mathcal{A} : p \in \mathscr{M}_i\}$, $i = 1, \cdots, n$), is a Morse decomposition for the attractor $\mathcal{A}$ of the skew product semiflow $\{\Pi(t) : t \geq 0\}$.

We also verify under which conditions, a Morse decomposition in the global attractor of the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$ generates a Morse decomposition in the global attractor of the driving system $\{\Theta(t) : t \geqslant 0\}$ (Theorem 4.6); that is, if $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ is a Morse decomposition of the global attractor $\mathcal{A}$ of $\{\Pi(t) : t \geqslant 0\}$ and $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ (with $\mathscr{M}_i = \{p \in \mathscr{P} :$ there exists $x \in X$ such that $(x,p) \in \mathcal{M}_i\}$, $1 \leq i \leq n$) are disjoint, then they constitute a Morse decomposition for the global attractor $\mathscr{A}$ of $\{\Theta(t) : t \geqslant 0\}$.

In Section 5 we construct a Morse decomposition for the pullback attractor of the non-autonomous dynamical system $(\varphi, \Theta)_{(X,\mathscr{P})}$ and we obtain some dynamical properties of this Morse decomposition. In Theorem 5.6 we prove that, under some hypotheses, each solution of the non-autonomous dynamical system converges forwards to $\underset{p \in \mathscr{M}_i}{\cup} \mathcal{M}_i(p)$, for some $i = 1, \cdots, n$; where $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ is a Morse decomposition of $\{\Theta(t) : t \geqslant 0\}$ and $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ are their lifts. In Theorem 5.7 we prove that a Lyapunov function for $\{\Theta(t) : t \geqslant 0\}$ generates a Lyapunov function for $\{\Pi(t) : t \geqslant 0\}$, which in turn is a non-autonomous Lyapunov function for the non-autonomous dynamical system $(\varphi, \Theta)_{X \times \mathscr{P}}$.

In Section 6 we describe how a Morse decomposition of a pullback attractor is stable under perturbation of the parameter of the associated driving system and particularly, in Theorem 6.6 (Theorem 6.8), under certain stability assumptions, we prove that if ξ is a global solution for $\{S(t,s) : t \geqslant s\}$, where $\{S(t,s) : t \geqslant s\}$ is a forwards (backward) asymptotically autonomous evolution process to a semigroup $\{S_0(t) : t \geqslant 0\}$ with Morse decomposition $\{\Gamma_{1,0}, \cdots, \Gamma_{n,0}\}$, then $\xi(t) \to \Gamma_{i,0}$, for some $i = 1, \cdots, n$, when $t \to \infty$ ($t \to -\infty$).

Finally, in Section 7 we present some applications of our theory that can help us understand a little more of its aspects. An asymptotically autonomous (backward and forwards) non-autonomous differential equation, i.e.

$$\begin{cases} \dot{x} = f(t,x), \quad t \in \mathbb{R}, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where

$$\sup_{x \in \mathbb{R}^n} \|f(t,x) - f_2(x)\|_{\mathbb{R}^n} \overset{t \to -\infty}{\longrightarrow} 0, \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} \|f(t,x) - f_1(x)\|_{\mathbb{R}^n} \overset{t \to \infty}{\longrightarrow} 0,$$

for suitable functions f_1 and f_2, and also a master-slave example, i.e. a system of partially coupled equations

$$\begin{cases} \dot{v} = f(u,v) \quad t > 0 \\ \dot{u} = g(u), \quad t > 0 \\ u(0) = u_0 \in \mathbb{R}^n, \ v(0) = v_0 \in \mathbb{R}^n \end{cases}$$

in which the second equation for $u(t)$ acts as a driving system for the unknown $v(t)$. Finally we present a more concrete example to illustrate the use of the abstract theory, studying the behavior of a planar system of ODE's given by

$$\frac{d}{dt}(x,y) = F(t,(x,y)), \quad t \in \mathbb{R},$$

where $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is forwards and backwards asymptotically autonomous to some special vector fields.

2. Preliminaries

In this section we state the definitions and some known results which will be used throughout the following sections. In particular, we pay special attention to the concept of pullback attractor.

2.1. **Non-autonomous dynamical systems and pullback attractors.** With Definition 1.5 in mind we can see that given an NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$ and a set $\mathscr{R} \subset \mathscr{P}$ which is invariant for $\{\Theta(t) : \ t \geqslant 0\}$, we can consider the restriction $\Theta(t)|_{\mathscr{R}} : \mathscr{R} \to \mathscr{R}$ and the restriction $\varphi|_{\mathbb{R}^+ \times \mathscr{R} \times X} : \mathbb{R}^+ \times \mathscr{R} \times X \to X$, so that we have a new NDS. In this case, the associated skew product semiflow is $\{\Pi(t)|_{X\times\mathscr{R}} : \ t \geqslant 0\}$ in the phase space $X \times \mathscr{R}$.

Definition 2.1. *A family of subsets $\{D(t)\}_{t\in\mathbb{R}}$ of X is called a* non-autonomous set*. If each* fiber *$D(t)$ is closed/compact/open, then $\{D(t)\}_{t\in\mathbb{R}}$ is called a* non-autonomous closed/compact/open set*.*

Definition 2.2. *A* global solution *for a semigroup $\{\Theta(t) : t \geq 0\}$ is a continuous function $\eta : \mathbb{R} \to \mathscr{P}$ such that $\Theta(t)\eta(s) = \eta(t+s)$ for all $s \in \mathbb{R}$ and all $t \in \mathbb{R}^+$. We say that $\eta : \mathbb{R} \to \mathscr{P}$ is a* global solution through *$p \in \mathscr{P}$ if it is a global solution with $\eta(0) = p$.*

Definition 2.3. *Given a global bounded solution $\eta : \mathbb{R} \to \mathscr{P}$ of the driving system $\{\Theta(t) : \ t \geqslant 0\}$, a non-autonomous set $\{D(t)\}_{t\in\mathbb{R}}$ is said to be η-*forwards invariant *under the NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$ if*

*$\varphi(t,\eta(s))D(s) \subset D(t+s)$ for all $s \in \mathbb{R}$ and $t \geqslant 0$. It is said to be η-*invariant *if $\varphi(t,\eta(s))D(s) = D(t+s)$ for all $s \in \mathbb{R}$ and $t \geqslant 0$.*

Definition 2.4. *Given a global bounded solution $\eta : \mathbb{R} \to \mathscr{P}$ of the driving system $\{\Theta(t) :\ t \geqslant 0\}$ and two non-autonomous sets $\{D(t)\}_{t\in\mathbb{R}}$ and $\{A(t)\}_{t\in\mathbb{R}}$, we say that $\{A(t)\}_{t\in\mathbb{R}}$ η-*pullback attracts *$\{D(t)\}_{t\in\mathbb{R}}$ if*

$$\lim_{t\to\infty} \mathrm{d}_H(\varphi(t,\eta(s-t))D(s-t), A(s)) = 0, \text{ for each } s \in \mathbb{R}.$$

Definition 2.5. *A* universe *$\mathfrak{D}$ is a collection of nonempty non-autonomous sets which is closed with respect to set inclusion, i.e. if $\{D^1(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}_\eta$ and $D^2(t) \subset D^1(t)$ for all $t \in \mathbb{R}$, then $\{D^2(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$. A non-autonomous compact set $\{A(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}_\eta$ is called a $(\mathfrak{D},\eta)$-*pullback attractor *of $(\varphi,\Theta)_{(X,\mathscr{P})}$ if*

- **(i)** *$\{A(t)\}_{t\in\mathbb{R}}$ is η-invariant;*
- **(ii)** *$\{A(t)\}_{t\in\mathbb{R}}$ η-pullback attracts all families $\{D(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$.*

Remark 2.6. *The above definitions are a simple rewriting of the known definitions for the non-autonomous setting given in* [4] *for the case of a non-injective driving system $\{\Theta(t) :\ t \geqslant 0\}$, where there may be more than one global solution through a given point $p \in \mathscr{P}$.*

Another important fact is the relationship between the global attractor of a skew product semiflow and the pullback attractors of the evolution processes it may contain. Such a relation is expressed in our next result.

Theorem 2.7. *Assume that the skew product semiflow $\{\Pi(t) : t \geq 0\}$ possesses a global attractor $\mathcal{A}$, consequently, the driving system $\{\Theta(t) : t \geq 0\}$ has a global attractor $\mathscr{A}$. If $\eta(\cdot) : \mathbb{R} \to \mathscr{P}$ is a global bounded solution for $\{\Theta(t) : t \geq 0\}$ then, the evolution process $\{T_\eta(t,s) : t \geq s\}$ given by*

$$T_\eta(t,s)x = \varphi(t-s,\eta(s))x,\ x \in X,$$

possesses a $(\mathfrak{D},\eta)$-pullback attractor $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ with the property that $\mathcal{A}_\eta(t) = \{x \in X : (x,\eta(t)) \in \mathcal{A}\}$, where $\mathfrak{D}$ is the collection of all non-autonomous sets $\{D(t)\}_{t\in\mathbb{R}}$ such that $\underset{t\in\mathbb{R}}{\cup} D(t)$ is bounded in X. Of course,

$$\mathcal{A} = \left\{ \bigcup_{t\in\mathbb{R}} \mathcal{A}_\eta(t) \times \{\eta(t)\},\ \eta(\cdot) \text{ is a global bounded solution for } \{\Theta(t) : t \geq 0\} \right\}$$

Proof: Define $\mathscr{K} = \overline{\eta(\mathbb{R})} \subset \mathcal{A}$, which is a compact set in $\mathscr{P}$ and invariant under the action of $\{\Theta(t) : t \geqslant 0\}$. Thus the semigroup $\{\Pi_{\mathscr{K}}(t) : t \geqslant 0\}$ given by the restriction $\Pi_{\mathscr{K}}(t) = \Pi(t)|_{X \times \mathscr{K}} : X \times \mathscr{K} \to X \times \mathscr{K}$ is well defined and has a global attractor $\mathcal{A}_K$. By Theorem 3.3 in [2], the non-autonomous set $\{\mathcal{A}_\eta(t)\}_{t\in\mathbb{R}}$, given by $\mathcal{A}_\eta(t) = \{x \in X : (x, \eta(t)) \in \mathcal{A}\}$, is the pullback attractor for the evolution process $\{T_\eta(t,s) : t \geqslant s\}$. The last assertion is straightforward. ■

2.2. **Morse decomposition for gradient-like semigroups.** We now recall the notions of gradient-like semigroup (see [3]) and Morse decomposition (see, for instance, [13]) for a global attractor. We first define isolated invariant sets.

<u>Definition</u> 2.8. *Let $\{T(t) : t \geq 0\}$ be a semigroup on Z. We say that an invariant set $\Xi \subset Z$ for the semigroup $\{T(t) : t \geq 0\}$ is an* isolated invariant set *if there is an $\epsilon > 0$ such that Ξ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.*

A disjoint *family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that,*

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing,\ 1 \leq i < j \leq n,$$

for some $\epsilon > 0$.

Recalling Definition 2.2, we define a global solution for a semigroup $\{T(t) : t \geqslant 0\}$ in a metric space Z as follows:

<u>Definition</u> 2.9. *A* global solution *for a semigroup $\{T(t) : t \geq 0\}$ is a continuous function $\xi : \mathbb{R} \to Z$ such that $T(t)\xi(s) = \xi(t+s)$ for all $s \in \mathbb{R}$ and all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to Z$ is a* global solution through *$z \in Z$ if it is a global solution with $\xi(0) = z$.*

<u>Definition</u> 2.10. *Consider a semigroup $\{T(t) : t \geqslant 0\}$ with a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. Let*

$$\delta_0 = \frac{1}{2} \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} d(\Xi_i, \Xi_j) > 0$$

*Let $\epsilon_0 < \delta_0$, $\Xi \in \mathbf{\Xi}$ and $\epsilon \in (0, \epsilon_0)$. An $\epsilon-$*chain *from Ξ to Ξ is a subset $\{\Xi_{\ell_1}, \ldots, \Xi_{\ell_k}\}$ of $\mathbf{\Xi}$, together with points $\{y_1, \cdots, y_k\}$ in Z and $\{t_1, \sigma_1, \ldots, t_k, \sigma_k\}$ in $\mathbb{R}$ such that, $0 < \sigma_i < t_i$, $1 \leqslant i \leqslant k$, $k \leqslant n$, $d(y_i, \Xi_{\ell_i}) < \epsilon$, $1 \leqslant i \leqslant k$, $\Xi = \Xi_{\ell_1} = \Xi_{\ell_{k+1}}$, $d(T(\sigma_i)y_i, \cup_{i=1}^n \Xi_i) > \epsilon_0$ and $d(T(t_i)y_i, \Xi_{\ell_{i+1}}) < \epsilon$, $1 \leqslant i \leqslant k$. We say that $\Xi \in \mathbf{\Xi}$ is* chain recurrent *if there exist a fixed $\epsilon_0 > 0$ and an $\epsilon-$chain from Ξ to Ξ, for each $\epsilon \in (0, \epsilon_0)$.*

Remark 2.11. *We can define an ϵ-chain between two different sets $\Xi_i, \Xi_j \in \mathbf{\Xi}$ analogously, but we will not use this concept here.*

We are now ready to define (in the terminology of [3]), gradient-like semigroups (or dynamically gradient semigroups in the terminology of [4]).

Definition 2.12. *Let $\{T(t) : t \geqslant 0\}$ be a semigroup with a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$, and assume that it possesses a global attractor $\mathcal{A}$. We say that $\{T(t) : t \geqslant 0\}$ is a* generalized gradient-like semigroup relative to $\mathbf{\Xi}$ *if the following conditions are satisfied:*

($G1$) *For any global solution $\xi : \mathbb{R} \to \mathcal{A}$, there are $1 \leq i, j \leq n$ such that*

$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$

($G2$) $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_{\mathfrak{n}}\}$ *has no chain recurrent sets.*

Remark 2.13. *When each Ξ_i consists only of a single stationary point, we say that the semigroup is a* gradient-like semigroup.

Definition 2.14. *Let $\{T(t) : t \geq 0\}$ be a semigroup which possesses a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. A* homoclinic structure *associated to $\mathbf{\Xi}$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\mathbf{\Xi}$ ($p \leq n$) together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that*

$$\Xi_{k_j} \overset{t\to-\infty}{\longleftarrow} \xi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_{k_{j+1}},\ 1 \leq j \leq p,$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$.

We now refer to the following result in [3], which relates the properties ($G1$) and ($G2$) to the non-existence of homoclinic structures.

Lemma 2.15. *Let $\{T(t) : t \geqslant 0\}$ be a semigroup with a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ and a global attractor $\mathcal{A}$. If $\{T(t) : t \geqslant 0\}$ satisfies ($G1$), then ($G2$) is satisfied if and only if $\mathcal{A}$ has no homoclinic structures.*

With this result we can redefine the concept of generalized gradient-like semigroups in the following equivalent way:

Definition 2.16. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T(t) : t \geq 0\}$ is a generalized* gradient-like semigroup relative to $\mathbf{\Xi}$ *if:*

$(G1)$ *For any global solution* $\xi : \mathbb{R} \to \mathcal{A}$ *there are* $1 \leq i, j \leq n$ *such that*

$$\Xi_i \stackrel{t\to-\infty}{\longleftarrow} \xi(t) \stackrel{t\to\infty}{\longrightarrow} \Xi_j.$$

$(G2')$ *There is no homoclinic structure associated to* $\mathbf{\Xi}$.

Next we introduce the notion of a Morse decomposition for the attractor $\mathcal{A}$ of a semigroup $\{T(t) : t \geq 0\}$ (see [8], [12] or [13]). We start with the notion of attractor-repeller pair.

Definition 2.17. *Let* $\{T(t) : t \geq 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$. *We say that a non-empty subset* A *of* $\mathcal{A}$ *is a* local attractor *if there is an* $\epsilon > 0$ *such that* $\omega(\mathcal{O}_\epsilon(A)) = A$. *The* repeller A^* *associated to a local attractor* A *is the set defined by*

$$A^* := \{x \in \mathcal{A} : \omega(x) \cap A = \varnothing\}.$$

The pair (A, A^*) *is called* attractor-repeller pair *for* $\{T(t) : t \geq 0\}$.

Note that if A is a local attractor, then A^* is closed and invariant.

Definition 2.18. *Given an increasing family* $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$, *of* $n + 1$ *local attractors, for* $j = 1, \cdots, n$, *define* $\Xi_j := A_j \cap A^*_{j-1}$. *The ordered* n*- tuple* $\mathbf{\Xi} := \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ *is called a* Morse decomposition for $\mathcal{A}$.

An equivalent definition of Morse decomposition (see [12]) for the attractor $\mathcal{A}$ of a semigroup $\{T(t) : t \geqslant 0\}$ is the following:

Definition 2.19. *Let* $\{T(t) : t \geqslant 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$. *Assume that there exists a collection* $\mathbf{\Xi} := \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ *of disjoint, compact and invariant subsets of* $\mathcal{A}$ *satisfying the following: for a given global solution* $\xi : \mathbb{R} \to \mathcal{A}$ *of* $\{T(t) : t \geqslant 0\}$

1. *either* $\xi(t) \in \Xi_i$, *for all* $t \in \mathbb{R}$ *and some* $i = 1, \cdots, n$;
2. *or there exist* $1 \leqslant i < j \leqslant n$ *such that* $\Xi_j \stackrel{t\to-\infty}{\leftarrow} \xi(t) \stackrel{t\to\infty}{\to} \Xi_i$.

Remark 2.20. *Note that the local attractors are ordered by inclusion, differently from the obtained Morse sets, which are disjoint. With a Morse decomposition* $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ *we can construct a sequence of local attractors setting*

$$A_i = \Xi_i \cup \left[\cup_{j=1}^{i-1} W^u(\Xi_j) \right].$$

As it was proved in [1], the local attraction in $\mathcal{A}$ is equivalent to the local attraction in Z.

Definition 2.21. *We will say that a semigroup* $\{T(t) : t \geq 0\}$ *with a global attractor* $\mathcal{A}$ *and a disjoint family of isolated invariant sets* $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ *is a* gradient semigroup *with respect to* $\mathbf{\Xi}$*, if there exists a continuous function* $V : Z \to \mathbb{R}$ *such that* $[0,\infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ *is decreasing for each* $x \in Z$*,* V *is constant in* Ξ_i *for each* $1 \leq i \leq n$*, and* $V(T(t)x) = V(x)$ *for all* $t \geq 0$ *if and only if* $x \in \bigcup_{i=1}^{n} \Xi_i$*.*

It has been also proved in [1] that a semigroup $\{T(t) : t \geq 0\}$ is a gradient semigroup with respect to $\mathbf{\Xi}$ if and only if it is a gradient-like semigroup with respect to $\mathbf{\Xi}$. Essentially, this important result says that, given a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ for a semigroup $T(t)$, the dynamical property of being gradient-like, the existence of an associated ordered family of local attractor-repellers, and the existence of a Lyapunov functional related to $\mathbf{\Xi}$, are equivalent properties. Since, from the results in [3], gradient-like nonlinear semigroups are stable under perturbation, we conclude that gradient semigroups are stable under perturbation as well; that is, the existence of a continuous Lyapunov function is robust under perturbation.

2.2.1. *Homoclinic structures in ω-limit sets.* At light of Example 3.1 below, we can state a more general result concerning ω-limit sets and homoclinic structures of a given gradient-like semigroup with a finite number of stationary points.

Proposition 2.22. *Let* $\{T(t) : t \geqslant 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$*, and a finite set of stationary points* $\mathcal{E}$*. Let* $z \in Z$ *and assume that there exist* $x, y \in \omega(z)$ *such that* $x \in \mathcal{E}$ *and* $x \neq y$*. Then* x *is a chain recurrent point.*

Proof: It is straightforward from the definition of the ω-limit set. ■

Proposition 2.23. *Let* $\{T(t) : t \geqslant 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$*, a finite set of stationary points* $\mathcal{E}$ *and assume that it satisfies (G1). Let* $z \in Z$ *and assume that there exist* $x, y \in \omega(z) \cap \mathcal{E}$ *with* $x \neq y$*. Then, there is a finite collection* $\{\xi_1, \cdots, \xi_n\}$ *of global solutions and points* $\{z_1, \cdots, z_{n+1}\} \subset \mathcal{E}$*, with* $z_1 = x$ *and* $z_{n+1} = y$ *such that*

$$z_i \overset{t\to-\infty}{\longleftarrow} \xi_i(t) \overset{t\to\infty}{\longrightarrow} z_{i+1}, \text{ for } i = 1, \cdots, n.$$

Proof: See [3, Lemma 2.2]. ■

3. Time dependence and translations

We now describe the set of functions which lead to the phase space $\mathscr{P}$ for the driving system $\Theta(t)$. For a more detailed approach, we refer to [13].

For any two Banach spaces V,W we will let $C(I,W)$, $C(V,W)$ and $C(I\times V,W)$ denote the spaces of continuous functions defined on, respectively, I, V and $I\times V$, and taking values on W, where either $I=\mathbb{R}^+$ or $I=\mathbb{R}$. In addition to these spaces, we define $C_b(V,W)$ (or $C_b(I\times V,W)$) as the collection of all $f\in C(V,W)$ (or $C(I\times V,W)$) such that for every bounded set $B\subset V$ (or bounded set $B\subset V$ and compact set $J\subset I$), there is a $K_0\geqslant 0$, such that $\|f(u)\|_W\leqslant K_0$ (or $\|f(t,u)\|_W\leqslant K_0$), for all $u\in B$ (or $(t,u)\in J\times B$).

These spaces of continuous functions are Frechét spaces with a metric topology which is described by the *uniform convergence on bounded sets.* The metric in this case is generated by a countable family of pseudonorms $\|\cdot\|_k$ as follows: Let B_k be the closed neighborhood of the origin in V of radius k, and set $I_k=I\cap[-k,k]$. Define

$$\|f\|_k \doteq \sup_{u\in B_k, t\in I_k}\|f(t,u)\|_W$$

A sequence f_n is said to converge to f, i.e. $f=\lim_{n\to\infty} f_n$, whenever

$$\lim_{n\to\infty}\|f-f_n\|_k=0,\ \text{for all } k\geqslant 1.$$

It turns out that $C_b(I\times V,W)$ is a complete metric space with this metric.

For each $f\in C_b(I\times V,W)$, we define the *translate* f_τ by

$$f_\tau(t,u)\doteq f(t+\tau,u),\quad u\in V \text{ and } t,\tau\in I.$$

Note that $f_\tau\in C_b(I\times V,W)\doteq\mathscr{C}$ for all $\tau\in I$. Furthermore, the mapping $(f,\tau)\to f_\tau$ is a continuous map of $\mathscr{C}\times I$ into $\mathscr{C}$, where $\mathscr{C}$ has the topology defined by the uniform convergence on compact sets. Let $\{\Theta(\tau):\tau\geq 0\}$ be the semigroup defined by $\Theta(\tau)f=f_\tau$, $\tau\geq 0$. The *positive orbit* of f is the set

$$\gamma^+(f)\doteq\{\Theta(\tau)f:\ \tau\in\mathbb{R}^+\},$$

and the ω-limit $\omega(f)$ of $f\in\mathscr{C}$ is

$$\omega(f)=\{g\in\mathscr{C}:\ \text{there exists a sequence } \tau_n\overset{n\to\infty}{\longrightarrow}\infty \text{ such that } f_{\tau_n}\overset{n\to\infty}{\longrightarrow} g \text{ in } \mathscr{C}\}.$$

A global solution through f is a function $\xi:\mathbb{R}\to\mathcal{C}$ such that $\Theta(\tau)\xi(t)=\xi(t+\tau)$ for all $\tau\geq 0$ and $t\in\mathbb{R}$ and, if there is a global solution ξ through f, the $\alpha-$limit of f associated to ξ is

$$\alpha_\xi(f)=\{g\in\mathscr{C}:\text{ there exists at sequence }\tau_n\overset{n\to\infty}{\longrightarrow}-\infty\text{ such that }\xi(\tau_n)\overset{n\to\infty}{\longrightarrow}g\text{ in }\mathscr{C}\}.$$

3.1. **Example.** In this subsection we develop an example for which we can describe part of the structure of the attractor for a driving semigroup of translations. The description of the asymptotic dynamics of the driving semigroup will play a fundamental role in the description of the dynamics of the NDS. To emphasize how rich can be the dynamics of the driving semigroup, we consider the following example:

Given two continuous functions $f,g:\mathbb{R}^n\to\mathbb{R}^n$, we define $r:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}$ by

$$r(t,x)=h(t)f(x)+(1-h(t))g(x),$$

where $h:\mathbb{R}^+\to[0,1]$ is defined as follows:

Let $\{a_n^i\}$, $1\leq i\leq 5$, be sequences of real numbers with

- **(i)** $a_0^1=1$ and $a_n^5=a_{n+1}^1$, for all $n\in\mathbb{N}$,
- **(ii)** $a_n^i<a_n^{i+1}$, $1\leq i\leq 4$,
- **(iii)** If $\tau_n^i\doteq a_n^{i+1}-a_n^i$, $1\leq i\leq 4$, then $\tau_n^i\to\infty$ as $n\to\infty$

and $h_1,h_2:\mathbb{R}\to[0,1]$ be continuous functions satisfying $1\overset{t\to-\infty}{\longleftarrow}h_1(t)\overset{t\to\infty}{\longrightarrow}0$ and $0\overset{t\to-\infty}{\longleftarrow}h_2(t)\overset{t\to\infty}{\longrightarrow}1$. Now, we define the function h in such a way that is uniformly continuous, $0\leqslant h\leqslant 1$, $h(t)\equiv 1$ in $[a_n^1,a_n^2]$ for all $n\in\mathbb{N}$, $h(t)\equiv 0$ in $[a_n^3,a_n^4]$ for all $n\in\mathbb{N}$, $h(t-t_n^2)\,\chi_{[-\tau_n^2,\tau_n^2]}\overset{n\to\infty}{\longrightarrow}h_1(t)$ uniformly for t in compact subsets of $\mathbb{R}$ and $h(t-t_n^4)\,\chi_{[-\tau_n^4,\tau_n^4]}\overset{n\to\infty}{\longrightarrow}h_2(t)$ uniformly for t in compact subsets of $\mathbb{R}$.

Consider the semigroup of translations $\{\Theta(t):t\geqslant 0\}$ defined in $P\doteq\overline{\gamma^+(r)}^{\mathscr{C}}$ by $\Theta(t)r_1(\cdot,x)=r_1(t+\cdot,x)$ for each $t\geqslant 0$, $x\in X$ and $r_1\in\overline{\gamma^+(r)}^{\mathscr{C}}$.

First, we choose the sequence $t_n^1=a_n^1+\dfrac{\tau_n^1}{2}$, thus

$$\Theta(t_n^1)r(s,x)=r(s+t_n^1,x)=f(x),$$

if $s\in[0,\frac{\tau_n^1}{2}]$. Hence $\Theta(t_n^1)r\to f$ as $n\to\infty$ in the uniform convergence on bounded sets, which shows that $f\in\omega(r)$, the omega-limit set of r. Choosing $t_n^3=a_n^3+\frac{\tau_n^3}{2}$ we can see that $\Theta(t_n^3)r\to g$ as $n\to\infty$, which shows that $g\in\omega(r)$ in a similar way.

Choosing $t_n^2=a_n^2+\frac{\tau_n^2}{2}$ we see that

$$\Theta(t_n^2)r(t,x)=r(t+t_n^2,x).$$

We can also see that $\Theta(t_n^2)r \to \xi_1(t,x) := h_1(t)f(x) + (1-h_1(t))g(x)$ uniformly in compact subsets of $\mathbb{R}\times\mathbb{R}^n$ and $\Theta(t_n^4)r \to \xi_2(t,x) := h_2(t)f(x) + (1-h_2(t))g(x)$ uniformly in compact subsets of $\mathbb{R}\times\mathbb{R}^n$. It is clear that ξ_1 and ξ_2 are global solutions for the driving system $\{\Theta(t) :\ t \geqslant 0\}$, ξ_1 is a connection between f and g and ξ_2 is a connection between g and f.

We note that $\Theta(\tau)\xi_1$ ($\Theta(\tau)\xi_2$) is also a connection between f and g (g and f), for any $\tau \in \mathbb{R}$.

In this way we have found some of the possible limiting vector fields that may arise in the closure of the positive orbit (in the attractor of the driving semigroup). As one can see, these limiting vector fields may have completely different structure from that of r. In particular in what was described above, there are four different vector fields involved; that is, f, g, a non-autonomous map that is backwards asymptotic to f and forwards asymptotic to g and a map that is backwards asymptotic to g and forwards asymptotic to f.

After the analysis done in Section 2.2, it will become clear that f and g belong to the same Morse set for any given Morse Decomposition of the global attractor of the driving system $\{\Theta(t) : t \geq 0\}$ in $\overline{\gamma^+(r)}$.

Remark 3.1.

1. *This example gives us a first understanding of how rich (and difficult) becomes the behaviour of non-autonomous dynamical systems; that is, if the non-autonomous dynamical system associated to* (1.1) *has a uniform attractor and* $\{t_n\}_{n\in\mathbb{N}}$ *is a sequence with the property that* $t_n \overset{n\to\infty}{\longrightarrow} \infty$, *there is a subsequence* $\{t_{n_k}\}_{k\in\mathbb{N}}$ *and* $\psi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$r(\cdot + t_{n_k}, x) \overset{k\to\infty}{\longrightarrow} \psi(\cdot, x),$$

uniformly for compact subsets of $\mathbb{R}\times\mathbb{R}^n$. *Therefore, the forwards behavior of the pullback attractor of this equation (when restricted to this sequence, as in Theorem 2.7)* **is not** *related to the initial equation. It is related to the equation*

$$\begin{cases} \dot{x} = \psi(t,x), t \in \mathbb{R}, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

which is the limiting problem along this subsequence (that is, a point in the ω*-limit of* $r(\cdot,x)$ *under the action of the driving semigroup of translations). In the same way, for* $r : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ *and sequences converging to* $-\infty$, *we can see that the backward behavior of the non-autonomous equation is related to another non-autonomous equation (also a limiting problem associated with a point in the* α*-limit of the driving semigroup of translations).*

But this analysis works only for this subsequence, so that we could have another subsequence of $\{t_n\}_{n\in\mathbb{N}}$ *converging to a different non-autonomous function. This is in contrast to the autonomous or forwards (backwards) asymptotically autonomous cases for which the forwards attractor (backwards attractor) is a single point.*

Also, we can see that the behavior of a non-autonomous equation (both forwards and backwards) are related to points in the ω*-limit and* α*-limit of points* $r(\cdot,x)$ *under the action of the semigroup of translations; which in general are a lot smaller than the whole hull* $\overline{\gamma(r)}$ *of the function* r*. Therefore, contrary to what we frequently see in the literature, to deal with the asymptotic dynamics of the non-autonomous differential equation we just have to take into account the global attractor of the driving semigroup of translations defined on the hull* $\overline{\gamma(r)}$*.*

2. *In Example 3.1, we could have that* $h_1(t) = 1$ *for* $t \leq -1$ *and* $h_1(t) = 0$ *for* $t \geq 1$ *and* $h_2 = 1 - h_1$ *(for instance) that would give rise to a semigroup for which the connections reach the equilibria in finite time (a feature that does not occur when we are working with semigroups generated by ODE's).*

3. *Also in Example 3.1, if we construct the function* h *in a different way, different phenomena may arise, for instance, taking* h *continuous,* $h \equiv 1$ *in the intervals* $[a_n^1, a_n^2]$*,* $h \equiv 0$ *in the intervals* $[a_n^3, a_n^4]$ *and linear in the intervals* $[a_n^2, a_n^3]$ *and* $[a_n^4, a_n^5]$*,* $n \in \mathbb{N}$*, the points in* $\omega(r)$ *will include the continuum of equilibria for* $\{\Theta(\tau) : \tau \geq 0\}$ *given by the segment connecting* f *and* g*; that is,* $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ *(see Figure* 1*(b)). Indeed, with the function* h *constructed in this manner, we would have*

$$h(s) = \frac{s - a_n^3}{a_n^2 - a_n^3}, \text{ if } s \in [a_n^2, a_n^3],$$

thus if $\alpha \in [0,1]$ *and* $t_n^2 = a_n^2 + (1-\alpha)(a_n^3 - a_n^2)$*, we have*

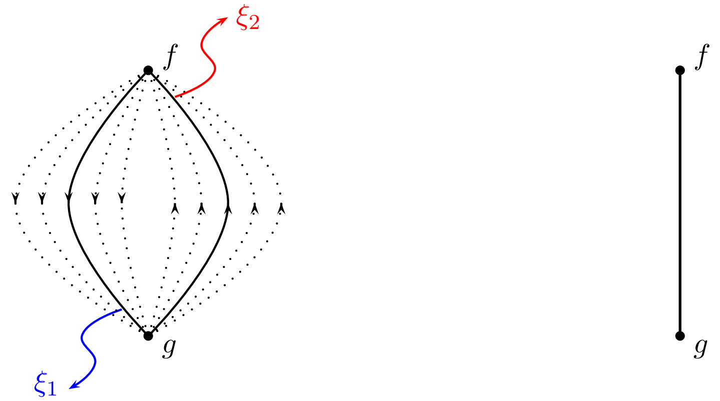
$$h(t + t_n^2) = \frac{t}{a_n^2 - a_n^3} + \alpha,$$

for as long as $t + t_n^2 \leqslant a_n^3$*. Therefore, for* t *in a compact set of* $\mathbb{R}^+$ *we have that*

$$h(t + t_n^2) \to \alpha \text{ as } n \to \infty,$$

which in turn implies that $\Theta(t_n^2)r(t,x) \to \alpha f(x) + (1-\alpha)g(x)$ *as* $n \to \infty$*.*

4. *In Example 3.1, the semigroup* $\{\Theta(\tau) : \tau \geq 0\}$ *in* $Z = \overline{\gamma^+(r)}$ *has two invariant sets, namely* $\Xi_1 = \{f\}$ *and* $\Xi_2 = \{g\}$*, and both of them are chain recurrent (see Figure* 1*(a)). In the* *context of part* **3** *above,* $\{f\}$ *and* $\{g\}$ *are not isolated and there is a continuum of equilibria for the semigroup* $\{\Theta(\tau) : \tau \geq 0\}$ *in* $Z = \overline{\gamma^+(r)}$ *(the segment whose endpoints are* f *and* g*, see Figure* 1*(b)).*



(a) Chain recurrent equilibria *(b) Continuum of equilibria*

Figure 1

4. Morse decomposition for a skew product semiflow

In this section we will describe the relationship between the Morse decompositions of the skew product semiflow, the driving semiflow and the pullback attractors associated to evolution processes related to the global solutions of the driving semiflow.

Indeed, our primary interest is to obtain a Morse decomposition for the global attractor $\mathcal{A}$ of the skew product semiflow $\{\Pi(t) : \ t \geqslant 0\}$ in terms of a Morse decomposition of the global attractor $\mathscr{A}$ of the driving system $\{\Theta(t) : \ t \geqslant 0\}$.

4.1. The lift of a Morse decomposition from $\mathscr{P}$ to $X \times \mathscr{P}$.

Definition 4.1. *Given any* $\mathscr{R} \subset \mathscr{P}$ *and* $\mathcal{D} \subset X \times \mathscr{P}$*, we define the subset* $\mathcal{L}^{\mathcal{D}}_{\mathscr{R}} \subset X \times \mathscr{P}$ *by*

$$\mathcal{L}^{\mathcal{D}}_{\mathscr{R}} = \{(x,p) \in \mathcal{D} : \ p \in \mathscr{R}\}.$$

The set $\mathcal{L}^{\mathcal{D}}_{\mathscr{R}}$ *is called the* **lift** *of* $\mathscr{R}$ *in* $\mathcal{D}$*.*

Remark 4.2. *If* $\psi_2 : X \times \mathscr{P} \to \mathscr{P}$ *is the projection on the second coordinate, that is,* $\psi_2(x,p) = p$ *for all* $(x,p) \in X \times \mathscr{P}$*, then we can see that* $\mathcal{L}^{\mathcal{D}}_{\mathscr{R}} = \psi_2^{-1}(\mathscr{R}) \cap \mathcal{D}$*.*

We can now prove the main theorem of this section:

Theorem 4.3. *Let $(\varphi,\Theta)_{(X,P)}$ be a non-autonomous dynamical system and $\{\Pi(t) : t \geqslant 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t) : t \geqslant 0\}$ has a global attractor $\mathcal{A}$ (hence, $\{\Theta(t) : t \geqslant 0\}$ has a global attractor) and that the global attractor $\mathscr{A}$ of the driving semigroup $\{\Theta(t) : t \geqslant 0\}$ possesses a Morse decomposition $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$.*

Define, for each $i = 1, \cdots, n$, the set $\mathcal{M}_i \doteq \mathcal{L}^{\mathcal{A}}_{\mathscr{M}_i}$. Then, the family $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ is a Morse decomposition for the global attractor $\mathcal{A}$ of $\{\Pi(t) : t \geqslant 0\}$. Moreover, the set $\mathcal{M}_i$ coincides with the global attractor of the semigroup $\{\Pi_i(t) : t \geqslant 0\}$ defined on $X \times \mathscr{M}_i$ by $\Pi_i(t) = \Pi(t)|_{X\times\mathscr{M}_i}$ for each $i = 1, \cdots, n$.

Proof: First we prove the non-emptiness of each $\mathcal{M}_i$. Let $p \in \mathscr{M}_i$ (since it is non-empty, such p exists) and take $x \in X$. Since $\mathcal{A}$ attracts points of $X \times \mathscr{P}$ under the action of $\{\Pi(t) : t \geqslant 0\}$, we have that

$$\lim_{t\to\infty} \mathrm{d}_H(\Pi(t)(x,p), \mathcal{A}) = 0.$$

Since $\mathcal{A}$ is compact and non-empty, we know that there exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ and a point $(x_0, p_0) \in \mathcal{A}$ such that $t_n \to \infty$ and $\Pi(t_n)(x,p) \to (x_0,p_0)$. But $\Pi(t_n)(x,p) = (\varphi(t_n,p)x, \Theta(t_n)p)$, and thus $\varphi(t_n,p)x \to x_0$ and $\Theta(t_n)p \to p_0$. Since $\mathscr{M}_i$ is closed and invariant, $p_0 \in \mathscr{M}_i$. Hence $(x_0,p_0) \in \mathcal{A}$ and $p_0 \in \mathscr{M}_i$, which shows that $(x_0,p_0) \in \mathcal{M}_i$.

Now we prove the invariance of each $\mathcal{M}_i$ under the action of $\{\Pi(t) : t \geqslant 0\}$. Let $(x,p) \in \mathcal{M}_i$. Then

$$\Pi(t)(x,p) = (\varphi(t,p)x, \Theta(t)p),$$

and, since $\mathcal{A}$ is invariant by Π, $(\varphi(t,p)x, \Theta(t)p) \in \mathcal{A}$. From the invariance of $\mathscr{M}_i$, $\Theta(t)p \in \mathscr{M}_i$ and thus $\Pi(t)(x,p) \in \mathcal{M}_i$.

Now, if $(y,q) \in \mathcal{M}_i$, from the invariance of $\mathscr{M}_i$ under $\{\Theta(t) : t \geq 0\}$, there exists $p \in \mathscr{M}_i$ such that $\Theta(t)p = q$. Let $\eta : \mathbb{R} \to \mathscr{A}$ be a global solution of $\{\Theta(t) : t \geq 0\}$ through p at $t = 0$. It follows that $\eta(t) = q$. Define the process $\{T_\eta(t,s) : t \geq s\}$ in X as in Theorem 2.7 and let $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ be its pullback attractor. Since $(y,q) \in \mathcal{A}$ we have that $y \in \mathcal{A}_\eta(t)$. Choose $x \in \mathcal{A}_\eta(0)$ such that $T_\eta(t,0)x = y$, consequently, $\Pi(t)(x,p) = (y,q)$. Since $(x,p) \in \mathcal{A}_\eta(0) \times \{\eta(0)\} \subset \mathcal{A}$ and $p \in \mathscr{M}_i$, we have that $(x,p) \in \mathcal{M}_i$.

Since $\mathcal{M}_i \subset \mathcal{A}$, to show that it is compact it remains to show that it is closed. Assume that $\{(x_n,p_n)\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}_i$ such that $(x_n,p_n) \to (x,p)$ as $n \to \infty$. Clearly $(x,p) \in \mathcal{A}$ and since $\{p_n\}_{n\in\mathbb{N}} \subset \mathscr{M}_i$, $p \in \mathscr{M}_i$ and thus $(x,p) \in \mathcal{M}_i$.

Now, if $(x,p) \in \mathcal{M}_i \cap \mathcal{M}_j$, with $i \neq j$, then it implies that $p \in \mathscr{M}_i \cap \mathscr{M}_j$, which is a contradiction since they are disjoint. Therefore $\mathcal{M}_i$ and $\mathcal{M}_j$ are disjoint if $i \neq j$.

Now, given a global solution $\xi : \mathbb{R} \to \mathcal{A}$ for $\{\Pi(t) : \ t \geqslant 0\}$ we have that

$$\xi(t) = (x(t), \eta(t)), \text{ for all } t \in \mathbb{R},$$

where $\eta : \mathbb{R} \to \mathscr{P}$ is a global solution for $\{\Theta(t) : t \geqslant 0\}$ and $x : \mathbb{R} \to X$ satisfies $\varphi(t, \eta(s))x(s) = x(t+s)$ for all $t \geqslant 0$ and $s \in \mathbb{R}$. Since $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ is a Morse decomposition for $\mathscr{P}$, there exist $1 \leqslant i < j \leqslant n$ such that

$$\mathscr{M}_j \overset{t\to-\infty}{\longleftarrow} \eta(t) \overset{t\to\infty}{\longrightarrow} \mathscr{M}_i.$$

We will show now that $\xi(t) \to \mathcal{M}_i$. For this purpose, assume that this is not the case, that is, assume that there exist $\epsilon_0 > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ and

$$d(\xi(t_n), \mathcal{M}_i) \geqslant \epsilon_0, \text{ for all } n \in \mathbb{N}.$$

But $\{\xi(t_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and we can also assume that $\xi(t_n) \to (x,p) \in \mathcal{A}$. Since $\eta(t_n) \to \mathscr{M}_i$, it follows that $p \in \mathscr{M}_i$ and thus $(x,p) \in \mathcal{M}_i$, but

$$d((x,p), \mathcal{M}_i) \geqslant \epsilon_0,$$

and since $\mathcal{M}_i$ is compact, this is a contradiction.

Analogously, $\xi(t) \to \mathcal{M}_j$ as $t \to -\infty$. In a similar way, we can prove that there are no homoclinic structures in $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$, as there are no homoclinic structures in $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$.

To prove the second statement, fix $i = 1, \cdots, n$ and let $\mathcal{A}_i$ be the global attractor of $\{\Pi_i(t) : t \geqslant 0\}$. It is easy to see that $\mathcal{A}_i \subset \mathcal{A}$. Now if $(x,p) \in \mathcal{A}_i$, then $p \in \mathscr{M}_i$ and thus $(x,p) \in \mathcal{M}_i$. Conversely, if $(x,p) \in \mathcal{M}_i$, then $(x,p) \in \mathcal{A}$, and thus there exists a global solution $\xi : \mathbb{R} \to \mathcal{A}$. But, by the invariance of $\mathscr{M}_i$, we have that $\xi(t) \in \mathcal{A}_i$ for all $t \in \mathbb{R}$, and hence $(x,p) \in \mathcal{A}_i$. ■

4.2. **The projection of a Morse decomposition from $X \times \mathscr{P}$ to $\mathscr{P}$.**

We are now interested in the opposite problem. Indeed, we investigate when a given Morse decomposition on the global attractor $\mathcal{A}$ of the skew product semiflow $\{\Pi(t) : \ t \geqslant 0\}$ generates a Morse decomposition in the global attractor $\mathscr{A}$ of the driving system $\{\Theta(t) : \ t \geqslant 0\}$.

Definition 4.4. *Given any $\mathcal{D} \subset X \times \mathscr{P}$ and $\mathscr{R} \subset \mathscr{P}$, we define the subset $\mathscr{Q}_{\mathcal{D}}^{\mathscr{R}}$ by*

$$\mathscr{Q}_{\mathcal{D}}^{\mathscr{R}} = \{p \in \mathscr{R} : \ (x,p) \in \mathcal{D} \text{ for some } x \in X\}.$$

*The set $\mathscr{Q}_{\mathcal{D}}^{\mathscr{R}}$ is called the $\mathscr{P}$-**projection** of $\mathcal{D}$ over $\mathscr{R}$.*

Remark 4.5. *Notice again that if* $\psi_2 : X\times\mathscr{P} \to \mathscr{P}$ *denotes the projection on the second coordinate, then it is clear that* $\mathscr{Q}_{\mathcal{D}}^{\mathscr{R}} = \psi_2(\mathcal{D}) \cap \mathscr{R}$.

Theorem 4.6. *Assume that* $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ *is a Morse decomposition for the global attractor* $\mathcal{A}$ *of the skew product semiflow* $\{\Pi(t): \ t \geqslant 0\}$. *Let* $\mathscr{A}$ *be the global attractor of the driving system* $\{\Theta(t): \ t \geqslant 0\}$.

Define $\mathscr{M}_i \doteq \mathscr{Q}_{\mathcal{M}_i}^{\mathscr{A}}$, *for each* $i = 1, \cdots, n$ *and assume that the family* $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ *is disjoint. Then,* $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ *is a Morse decomposition for* $\mathscr{A}$.

Proof: We know that $\mathscr{M}_i$ is compact and non-empty for each $i = 1, \cdots, n$.

To see that $\mathscr{M}_i$ is invariant under the action of $\{\Theta(t) : t \geqslant 0\}$ let $p \in \mathscr{M}_i$ and $x \in X$ such that $(x, p) \in \mathcal{M}_i$. Since $\mathcal{M}_i$ is invariant for $\{\Pi(t) : t \geqslant 0\}$, we have that $\Pi(t)(x, p) = (\varphi(t, p)x, \Theta(t)p) \in \mathcal{M}_i$, therefore $\Theta(t)p \in \mathscr{M}_i$.

Now, let $q \in \mathscr{M}_i$ and $y \in X$ such that $(y, q) \in \mathcal{M}_i$ and from the invariance of $\mathcal{M}_i$, there exists $(x, p) \in \mathcal{M}_i$ such that $\Pi(t)(x, p) = (y, q)$. Thus $p \in \mathscr{M}_i$ and $\Theta(t)p = q$ and concludes the invariance of $\mathscr{M}_i$ under the action of $\{\Theta(t) : t \geqslant 0\}$.

Now, given a global solution $\eta : \mathbb{R} \to \mathscr{A}$ of $\{\Theta(t) : \ t \geqslant 0\}$, consider the associated evolution process $\{T_\eta(t, s) : t \geqslant s\}$, given by Theorem 2.7, and $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ its pullback attractor. Let $x : \mathbb{R} \to X$ be a global solution for this evolution process; that is, $x(t) \in \mathcal{A}_\eta(t)$ for all $t \in \mathbb{R}$ and $T_\eta(t, s)x(s) = x(t)$ for all $t \geqslant s$. Define $\xi : \mathbb{R} \to \mathcal{A}$ by

$$\xi(t) = (x(t), \eta(t)), \text{ for all } t \in \mathbb{R}.$$

In this way, ξ is a global solution for the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$ and since $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ is a Morse decomposition for $\mathcal{A}$, we have that, for some $i < j$,

$$\mathcal{M}_j \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \mathcal{M}_i,$$

which means that

$$\mathscr{M}_j \overset{t\to-\infty}{\longleftarrow} \eta(t) \overset{t\to\infty}{\longrightarrow} \mathscr{M}_i.$$

Again, there are no homoclinic structures in $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ since there are no homoclinic structures in $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$. ∎

5. Morse decomposition for pullback attractors

We begin this section stating some known results about the Morse decomposition for pullback attractors of NDS in [2], where the approach used there requires that the set $\mathscr{P}$ be compact and that $\{\Theta(t) : t \geqslant 0\}$ be in fact a group over $\mathscr{P}$, i.e. $\Theta(t) : \mathscr{P} \to \mathscr{P}$ is invertible, with its continuous inverse given by $\Theta(t)^{-1} = \Theta(-t)$ (in particular $\mathscr{P} = \mathscr{A}$ and there is a unique global solution through each point $p \in \mathscr{P}$).

Given a set $\mathcal{D} \subset X \times \mathscr{P}$, we define the p-section of $\mathcal{D}$ as the set $\mathcal{D}(p) \subset X$ as

$$\mathcal{D}(p) = \{x \in X : (x,p) \in \mathcal{D}\}.$$

In the same way, given a non-autonomous set $\{\mathcal{D}(p)\}_{p\in\mathscr{P}}$, with $\mathcal{D}(p) \subset X$ for all $p \in \mathscr{P}$, we define $\mathcal{D} \subset X \times \mathscr{P}$ as

$$\mathcal{D} = \bigcup_{p\in\mathscr{P}} \mathcal{D}(p) \times \{p\}.$$

Also, given $\mathcal{D} \subset X \times \mathscr{P}$ or a non-autonomous set $\{\mathcal{D}(p)\}_{p\in\mathscr{P}}$ and $\mathscr{R} \subset \mathscr{P}$ we define the **X-projection** of $\mathcal{D}$ (or the **X-projection** of $\{\mathcal{D}(p)\}_{p\in\mathscr{P}}$) along $\mathscr{R}$ by

$$P_{\mathscr{R}}(\mathcal{D}) = P_{\mathscr{R}}(\{\mathcal{D}(p)\}_{p\in\mathscr{P}}) = \underset{p\in\mathscr{R}}{\cup} \mathcal{D}(p).$$

In this setting, we can define the *pullback attractor* for the NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$ in the following way:

Definition 5.1. *Let $\mathfrak{B}$ be the collection of all bounded sets in X. A non-autonomous set $\{\mathcal{S}(p)\}_{p\in\mathscr{P}}$ is a* pullback attractor *of $(\varphi,\Theta)_{(X,\mathscr{P})}$ if:*

- **(i)** *$P_{\mathscr{P}}(\{\mathcal{S}(p)\}_{p\in\mathscr{P}}) \in \mathcal{B}$ and each $\mathcal{S}(p)$ is compact;*
- **(ii)** *$\{\mathcal{S}(p)\}_{p\in\mathscr{P}}$ is invariant under the NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$, i.e.*

$$\varphi(t,p)\mathcal{S}(p) = \mathcal{S}(\Theta(t)p), \text{ for all } t \geqslant 0 \text{ and } p \in \mathscr{P};$$

- **(iii)** *$\{\mathcal{S}(p)\}_{p\in\mathscr{P}}$ pullback attracts all non-autonomous sets $\{\mathcal{D}(p)\}_{p\in\mathscr{P}}$ such that $P_{\mathscr{P}}(\{\mathcal{D}(p)\}_{p\in\mathscr{P}}) \in \mathfrak{B}$, i.e.*

$$\lim_{t\to\infty} \mathrm{d}_H(\varphi(t,\Theta(-t)p)\mathcal{D}(\Theta(-t)p), \mathcal{S}(p)) = 0, \text{ for all } p \in \mathscr{P}.$$

Theorem 5.2 (Theorem 3.3 in [2]; see also Proposition 3.31 in [10])**.** *Assume that $\mathcal{A} = \bigcup_{p\in\mathscr{P}} \mathcal{A}(p) \times \{p\}$ is the global attractor of the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$. Then, the associated non-autonomous set $\{\mathcal{A}(p)\}_{p\in\mathscr{P}}$ is the pullback attractor of the NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$.*

<u>Definition</u> 5.3. *Let $\mathcal{A}$ be the global attractor of the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$. A non-autonomous compact pair $(\{\mathcal{H}(p)\}_{p\in\mathscr{P}}, \{\mathcal{H}^*(p)\}_{p\in\mathscr{P}})$ is called a pullback attractor-repeller pair in $\{\mathcal{A}(p)\}_{p\in\mathscr{P}}$ if the associated pair $(\mathcal{H}, \mathcal{H}^*)$ is an attractor-repeller of the global attractor $\mathcal{A}$ of $\Pi(t)$.*

<u>Definition</u> 5.4. *Let $\mathcal{A}$ be the global attractor of the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$. Let $(\{\mathcal{H}_i(p)\}_{p\in\mathscr{P}}, \{\mathcal{H}_i^*(p)\}_{p\in\mathscr{P}})$, $i = 0, \cdots, n$, be pullback attractor-repeller pairs in $\{\mathcal{A}(p)\}_{p\in\mathscr{P}}$ with*

$$\emptyset = \mathcal{H}_0(p) \subsetneq \mathcal{H}_1(p) \subsetneq \cdots \subsetneq \mathcal{H}_n(p) = \mathcal{A}(p)$$

and

$$\mathcal{A}(p) = \mathcal{H}_0^*(p) \supsetneq \mathcal{H}_1^*(p) \supsetneq \cdots \supsetneq \mathcal{H}_n^*(p) = \emptyset$$

for all $p \in \mathscr{P}$. Then, the family $\big\{\{\mathcal{M}_1(p)\}_{p\in\mathscr{P}}, \cdots, \{\mathcal{M}_n(p)\}_{p\in\mathscr{P}}\big\}$ of invariant non-autonomous compact sets, defined by

$$\mathcal{M}_i(p) = \mathcal{H}_i(p) \cap \mathcal{H}_{i-1}^*(p), \text{ for all } p \in \mathscr{P} \text{ and } 1 \leq i \leq n,$$

is called a pullback Morse decomposition of $\{\mathcal{A}(p)\}_{p\in\mathscr{P}}$, *and each* $\{\mathcal{M}_i(p)\}_{p\in\mathscr{P}}$ *is called* pullback Morse set.

By Theorem 4.3 and Theorem 5.1 in [2] we can now prove the following result describing the internal asymptotic dynamics of a pullback Morse decomposition for the pullback attractor $\{\mathcal{A}(p)\}_{p\in\mathscr{P}}$:

<u>Theorem</u> 5.5 (Theorem 5.1 in [2])**.** *Assume that the driving system $\{\Theta(t) : t \geqslant 0\}$ has a global attractor $\mathscr{A}$ with a Morse decomposition $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$. Define, for each $i = 1, \cdots, n$ as in Theorem 4.3, the set $\mathcal{M}_i \doteq \mathcal{L}^{\mathcal{A}}_{\mathscr{M}_i}$. The family $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$, is then a Morse decomposition for the global attractor $\mathcal{A}$ of $\{\Pi(t) : t \geqslant 0\}$. Then, the NDS $(\varphi, \Theta)_{(X,\mathscr{A})}$ has a pullback Morse decomposition $\big\{\{\mathcal{M}_1(p)\}_{p\in\mathscr{A}}, \cdots, \{\mathcal{M}_n(p)\}_{p\in\mathscr{A}}\big\}$, where*

$$\mathcal{M}_i(p) = \{x \in X : (x, p) \in \mathcal{M}_i\},$$

is the p-section of $\mathcal{M}_i$, for all $p \in \mathscr{A}$ and $i = 1, \cdots, n$. Assume that $\big\{\{\mathcal{M}_1(p)\}_{p\in\mathscr{A}}, \cdots, \{\mathcal{M}_n(p)\}_{p\in\mathscr{A}}\big\}$ is described by pullback attractor-repeller pairs $(\{\mathcal{H}_i(p)\}_{p\in\mathscr{A}}, \{\mathcal{H}_i^*(p)\}_{p\in\mathscr{A}})$, $i = 1, \ldots, n$ such that $P_{\mathscr{A}}(\{\mathcal{H}_i(p)\}_{p\in\mathscr{A}}) \cap P_{\mathscr{A}}(\{\mathcal{H}_i^*(p)\}_{p\in\mathscr{A}}) = \emptyset$ for $i = 1, \ldots, n$. Then, the collection of pullback Morse sets determines the limiting behavior of NDS φ on $\{\mathcal{A}(p)\}_{p\in\mathscr{A}}$. More precisely, we have:

(i) *For any singleton non-autonomous set* $\{x(p)\}_{p\in\mathscr{A}}$ *in* $\{\mathcal{A}(p)\}_{p\in\mathscr{A}}$, *by writing* $x=\bigcup_{p\in\mathscr{A}} x(p)\times\{p\}$, *if*

$$\mathrm{d}_H(x,\bigcup_{i=1}^n \mathcal{H}_i^*)>0 \text{ or } x\in\bigcup_{i=1}^n \mathcal{H}_i^*,$$

we have

$$\lim_{t\to+\infty}\mathrm{d}_H(\varphi(t,\Theta(-t)p)x(\Theta(-t)p),\bigcup_{i=1}^n \mathcal{M}_i(p))=0,$$

for each $p\in\mathscr{A}$.

(ii) *If* φ *is invertible on* $\{\mathcal{A}(p)\}_{p\in\mathscr{A}}$ *then, for any singleton non-autonomous set* $\{x(p)\}_{p\in\mathscr{A}}$ *in* $\{\mathcal{A}(p)\}_{p\in\mathscr{A}}$ *with*

$$\mathrm{d}_H(x,\bigcup_{i=1}^n \mathcal{H}_i)>0 \text{ or } x\in\bigcup_{i=1}^n \mathcal{H}_i$$

we have

$$\lim_{t\to+\infty}\mathrm{d}_H(\varphi(-t,\Theta(t)p)x(\Theta(t)p),\bigcup_{i=1}^n \mathcal{M}_i(p))=0,$$

for each $p\in\mathscr{A}$ *and* $\varphi(-t,\Theta(t)p)\doteq(\varphi(t,p))^{-1}$.

(iii) *Moreover, under the hypotheses of Theorems 5.2 and 5.5, the family* $\{\mathcal{M}_i(p)\}_{p\in\mathscr{M}_i}$ *is the pullback attractor of the NDS* $(\varphi,\Theta)_{(X,\mathscr{M}_i)}$. *In particular, we have that*

$$\lim_{t\to\infty}\sup_{p\in\mathscr{M}_i}\mathrm{d}_H(\varphi(t,p)B,P_{\mathscr{M}_i}(\{\mathcal{M}_i(p)\}_{p\in\mathscr{M}_i}))=0,$$

for all $B\subset X$ *bounded.*

We can now state the main new result of this section, concerning the asymptotic behavior of the solutions for the NDS $(\varphi,\Theta)_{(X,\mathscr{P})}$ as $t\to\infty$ and for any $p\in\mathscr{P}$.

Theorem 5.6. *Let* $(\varphi,\Theta)_{(X,\mathscr{P})}$ *be a non-autonomous dynamical system and* $\{\Pi(t):t\geqslant 0\}$ *the associated skew product semiflow. Assume that* $\{\Pi(t):\ t\geqslant 0\}$ *possesses a global attractor* $\mathcal{A}$ *(hence the driving system* $\{\Theta(t):t\geqslant 0\}$ *has a global attractor* $\mathscr{A}$*) and let* $\{\mathscr{M}_1,\cdots,\mathscr{M}_n\}$ *be a Morse decomposition for* $\mathscr{A}$. *Let* $\{\mathcal{M}_1,\cdots,\mathcal{M}_n\}$ *be the corresponding Morse decomposition for the global attractor* $\mathcal{A}$ *of* $\{\Pi(t):\ t\geqslant 0\}$, *given by Theorem 4.3. Then, given* $(x,p)\in X\times\mathscr{P}$, *there exists* $i=1,\cdots,n$ *such that*

$$\lim_{t\to\infty}\mathrm{d}_H(\varphi(t,p)x,P_{\mathscr{M}_i}(\mathcal{M}_i))=0.$$

Proof: Let $(x, p) \in X \times \mathscr{P}$, and consider the solution $\xi : \mathbb{R}^+ \to X \times \mathscr{P}$ for the skew product semiflow $\{\Pi(t) : t \geqslant 0\}$ starting at (x, p). We know that

$$\xi(t) = \Pi(x, p) = (\varphi(t, p)x, \Theta(t)p), \text{ for all } t \geqslant 0.$$

Since $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ is a Morse decomposition for the global attractor $\mathscr{A}$ of the driving system $\{\Theta(t) : t \geqslant 0\}$, there exists $i = 1, \cdots, n$ such that $\Theta(t)p \to \mathscr{M}_i$ as $t \to \infty$. We claim that $\xi(t) \to \mathcal{M}_i$. Assume, by contradiction, that this is not the case, i.e. there exist $\epsilon_0 > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$d(\xi(t_n), \mathcal{M}_i) \geqslant \epsilon_0, \text{ for all } n \in \mathbb{N}.$$

But $d(\xi(t_n), \mathcal{A}) \overset{n\to\infty}{\longrightarrow} 0$ and we can assume that there exists $(x_0, p_0) \in \mathcal{A}$ such that $\xi(t_n) \to (x_0, p_0)$. But this implies that $p_0 \in \mathscr{M}_i$, which in turn implies that $(x_0, p_0) \in \mathcal{M}_i$ and gives a contradiction, since

$$0 = d((x_0, p_0), \mathcal{M}_i) \overset{n\to\infty}{\longleftarrow} d(\xi(t_n), \mathcal{M}_i) \geqslant \epsilon_0 > 0.$$

Thus, $\xi(t) \to \mathcal{M}_i$, and then $\varphi(t, p)x \to P_{\mathscr{M}_i}(\mathcal{M}_i)$. ■

Our last result in this section provides the equivalence between the existence of Morse decomposition and Lyapunov function for the skew product semiflow $\{\Pi(t) : t \geq 0\}$ from the existence of a Lyapunov function (and thus a Morse decomposition) for the global attractor of the driving system $\{\Theta(t) : t \geq 0\}$.

Theorem 5.7. *Let $(\varphi, \Theta)_{(X,\mathscr{P})}$ be an NDS and $\{\Pi(t) : t \geqslant 0\}$ the associated skewproduct semiflow. Assume that $\{\Pi(t) : t \geqslant 0\}$ has a global attractor $\mathcal{A}$ and let $\mathscr{A}$ be the global attractor for the driving semigroup $\{\Theta(t) : t \geqslant 0\}$. If there is a Lyapunov function for $\{\Theta(t) : t \geqslant 0\}$, then there is a non-autonomous Morse decomposition $\big\{\{\mathcal{M}_1(p)\}_{p\in\mathscr{P}}, \cdots, \{\mathcal{M}_n(p)\}_{p\in\mathscr{P}}\big\}$ and a continuous Lyapunov function $L : X \times \mathscr{P} \to \mathbb{R}^+$ with the following properties:*

- **(i)** *$L(\varphi(t, p)x, \Theta(t)p) \leqslant L(x, p)$ for any $(x, p) \in X \times \mathscr{P}$ and $t \geqslant 0$.*
- **(ii)** *$L(\varphi(t, p)x, \Theta(t)p) = L(x, p)$ when $x \in \cup_{i=1}^n \mathcal{M}_i(p)$ for all $t \geqslant 0$, and L takes different constant values on different Morse sets.*
- **(iii)** *$L(\varphi(t, p)x, \Theta(t)p) < L(x, p)$ when $x \in X \setminus \cup_{i=1}^n \mathcal{M}_i(p)$ for all $t > 0$.*

Proof: The Lyapunov function for $\{\Theta(t) : t \geqslant 0\}$ generates a Morse decomposition for its global attractor $\mathscr{A}$. This Morse decomposition generates a Morse decomposition for the global attractor $\mathcal{A}$ in the phase space $X \times \mathscr{P}$. So the result follows from [1, Theorem 3.4 and Proposition 3.5]. ■

6. Dynamics under perturbation

6.1. **Small non-autonomous perturbations.** We can also see how these structures behave under small non-autonomous perturbations. The results below follow from [3]:

Theorem 6.1. *Let $(\varphi,\Theta)_{(X,\mathscr{P})}$ be a NDS and $\{\Pi(t): t \geqslant 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t): t \geqslant 0\}$ has a global attractor $\mathcal{A}$ and let $\mathscr{A}$ be the global attractor for the driving semigroup $\{\Theta(t): t \geqslant 0\}$. Let also $p_0 \in \mathscr{P}$ be a fixed point of Θ. Assume that the following conditions hold:*

- (a) *$\{\varphi(t,p_0): t \geq 0\}$ is a gradient-like semigroup relatively to the set of equilibria $\mathcal{E}_0 = \{e_{1,0},\cdots,e_{n,0}\}$.*
- (b) *For each p sufficiently close to p_0, there exists a global solution $\eta_p : \mathbb{R}^+ \to \mathscr{P}$ of Θ through p such that $\{\varphi(t-s,\eta_p(s)): t \geq s\}$ possesses n isolated global solutions $\xi_{i,p}: \mathbb{R} \to X$ $i = 1,2,\cdots,n$, and $\sup_{1\leq i\leq n}\sup_{t\in\mathbb{R}} d_X(\xi_{i,p}(t), e_{i,0}) \xrightarrow{p\to p_0} 0$.*
- (c) *For each compact set $K \subset \mathbb{R}^+ \times X$ and the global solution $\eta_p : \mathbb{R}^+ \to \mathscr{P}$ of Θ through p,*
$$\sup_{s\in\mathbb{R}}\sup_{(t,x)\in K} d_X(\varphi(t-s,\eta_p(s))x, \varphi(t-s,p_0)x) \xrightarrow{p\to p_0} 0.$$
- (d) *There exists $\mu > 0$ such that, if $\phi : \mathbb{R} \to X$ is a bounded solution of $\{\varphi(t-s,\eta_p(s)): t \geq s\}$ and there are $t_0 \in \mathbb{R}$ and $i \in \{1,\cdots,n\}$ with $\sup_{t\leq t_0} \mathrm{d}_H(\phi(t), \xi_{i,p}(\mathbb{R})) < \mu$ (resp. $\sup_{t\geq t_0} \mathrm{d}_H(\phi(t), \xi_{i,p}(\mathbb{R})) < \mu$), then $\lim_{t\to-\infty} d_X(\phi(t), \xi_{i,p}(t)) = 0$ (resp. $\lim_{t\to\infty} d_X(\phi(t), \xi_{i,p}(t)) = 0$).*

Then, for all p sufficiently close to p_0, $\{\varphi(t-s,\eta_p(s)): t \geq s\}$ is a non-autonomous gradient-like evolution process with respect to the disjoint set of isolated invariant families $\boldsymbol{\Xi}_p = \{\xi_{1,p},\cdots,\xi_{n,p}\}$.

Theorem 6.2. *Let $(\varphi,\Theta)_{(X,\mathscr{P})}$ be a NDS and $\{\Pi(t): t \geqslant 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t): t \geqslant 0\}$ has a global attractor $\mathcal{A}$ and let $\mathscr{A}$ be the global attractor for the driving semigroup $\{\Theta(t): t \geqslant 0\}$. Let also $p_0 \in \mathscr{P}$ be a fixed point of Θ. Assume that the following conditions hold:*

- (a) *$\{\varphi(t,p_0): t \geq 0\}$ is a generalized gradient-like semigroup with disjoint family of isolated invariant sets $\{\Gamma_{1,0},\cdots\Gamma_{n,0}\}$.*
- (b) *For each p sufficiently close to p_0, there exists a global solution $\eta_p : \mathbb{R}^+ \to \mathscr{P}$ of Θ through p such that $\{\varphi(t-s,\eta_p(s)): t \geq s\}$ possesses $n \in \mathbb{N}$ isolated invariant families*
$$\boldsymbol{\Xi}_p = \{\Xi_{1,p}(\cdot),\cdots,\Xi_{n,p}(\cdot)\},$$

with traces $\{\Gamma_{1,p},\cdots,\Gamma_{n,p}\}$*, where* $\Gamma_{i,p}=\cup_{t\in\mathbb{R}}\Xi_{i,p}(t)$*, which behave upper e lower semi-continuously as* p *goes to* p_0 $(\sup_{1\leqslant i\leqslant n}[\mathrm{dist}(\Gamma_{i,p},\Gamma_{i,0})+\mathrm{dist}(\Gamma_{i,0},\Gamma_{i,p})]\stackrel{p\to p_0}{\longrightarrow}0)$.

(c) *For each compact set* $K\subset\mathbb{R}^+\times X$ *and the global solution* $\eta_p:\mathbb{R}^+\to\mathscr{P}$ *of* Θ *through* p,

$$\sup_{s\in\mathbb{R}}\sup_{(t,x)\in K} d_X(\varphi(t-s,\eta_p(s))x,\varphi(t-s,p_0)x)\stackrel{p\to p_0}{\longrightarrow}0.$$

d) *There exists* $\mu>0$ *such that, if* $\phi:\mathbb{R}\to X$ *is a bounded solution of* $\{\varphi(t-s,\eta_p(s)):t\geq s\}$ *and there are* $t_0\in\mathbb{R}$ *and* $i\in\{1,\cdots,n\}$ *with* $\sup_{t\leq t_0}\mathrm{d}_H(\phi(t),\Gamma_{i,p})<\mu$ *(resp.* $\sup_{t\geq t_0}\mathrm{d}_H(\phi(t),\Gamma_{i,p})<\mu$*), then* $\lim_{t\to-\infty}\mathrm{d}_H(\phi(t),\Xi_{i,p}(t))=0$ *(resp.* $\lim_{t\to\infty}\mathrm{d}_H(\phi(t),\Xi_{i,p}(t))=0$*).*

Then, for all p *sufficiently close to* p_0*,* $\{\varphi(t-s,\eta_p(s)):t\geq s\}$ *is a generalized gradient-like evolution process with isolated invariant families* $\mathbf{\Xi}_p=\{\Xi_{1,p}(\cdot),\cdots,\Xi_{n,p}(\cdot)\}$*.*

6.2. **Asymptotically autonomous evolution processes.** In this section we will consider asymptotically autonomous evolution processes. Loosely speaking, an evolution process is asymptotically autonomous if it is very close to an autonomous evolution process when the initial times are very large. This idea leads to the following definition (for a similar definition see [11]).

Definition 6.3. *Let* $\{S(t,s):t\geqslant s\}$ *be an evolution process and* $\{S_0(t):t\geqslant 0\}$ *be a semigroup in a metric space* Z*. We say that*

- $\{S(t,s):t\geqslant s\}$ *is asymptotically autonomous at* $-\infty$ *if*

$$S(t+s,s)u_0\stackrel{s\to-\infty}{\longrightarrow}S_0(t)u_0$$

- $\{S(t,s):t\geqslant s\}$ *is asymptotically autonomous at* $+\infty$ *if*

$$S(t+s,s)u_0\stackrel{s\to+\infty}{\longrightarrow}S_0(t)u_0$$

uniformly for t *in bounded intervals of* $[0,\infty)$ *and for* u_0 *in compact subsets of* Z*.*

In order to obtain information about an asymptotically autonomous evolution process using the results of the previous sections it is convenient to introduce a new evolution process which is close, for all initial times, to an autonomous evolution process.

Let $\tau\in\mathbb{R}$, $\{S(t,s):t\geqslant s\}$ be an evolution process and $\{T(t):t\geqslant 0\}$ be a semigroup, and construct the following truncated evolution processes:

- forwards truncation at time τ

$$S_\tau(t,s) = \begin{cases} S(t,s), & \text{if } s \leqslant t \leqslant \tau, \\ T(t-\tau)S(\tau,s), & \text{if } s \leqslant \tau \leqslant t, \\ T(t-s), & \text{if } \tau \leqslant s \leqslant t. \end{cases}$$

- Backward truncation at time τ

$$S_\tau(t,s) = \begin{cases} T(t-s), & \text{if } s \leqslant t \leqslant \tau, \\ S(t,\tau)T(\tau-s), & \text{if } s \leqslant \tau \leqslant t \\ S(t,s), & \text{if } \tau \leqslant s \leqslant t. \end{cases}$$

We have the following property for the truncations:

Theorem 6.4. *Let $\{S(t,s) : t \geqslant s\}$ be an asymptotically autonomous evolution process at $-\infty$ (at $+\infty$) and $\{S_0(t) : t \geqslant 0\}$ the associated semigroup. Assume that the semigroup satisfies a uniform continuity condition, i.e. given $\epsilon > 0$, a bounded interval $I \subset \mathbb{R}^+$ and a compact set $K \subset Z$ there exists $\delta = \delta(\epsilon, I, K)$ such that $\|S_0(t)u - S_0(t)v\| < \epsilon$, if $\|u - v\| < \delta$, $u, v \in K$, for all $t \in I$. Then the forwards (backward) truncation of $\{S(t,s) : t \geqslant s\}$ satisfies $\|S_\tau(t+r,r)u - S_0(t+r,r)u\|_Z \longrightarrow 0$ as $\tau \to -\infty$ ($\tau \to +\infty$) uniformly for $r \in \mathbb{R}$ and for (t,u) in compact subsets of $\mathbb{R}^+ \times Z$.*

Proof: We first deal with the case of the forwards truncation evolution process. We know that for each $\tau \in \mathbb{R}$, the forwards truncation at τ is given by

$$S_\tau(t,s) = \begin{cases} S(t,s), & \text{if } s \leqslant t \leqslant \tau, \\ S_0(t-\tau)S(\tau,s), & \text{if } s \leqslant \tau \leqslant t, \\ S_0(t-s), & \text{if } \tau \leqslant s \leqslant t. \end{cases}$$

Now given $\epsilon > 0$, a bounded interval $I \subset \mathbb{R}^+$ and a compact set $K \subset Z$, from the uniform continuity of $\{S_0(t) : t \geqslant 0\}$, there exists $0 < \delta < \epsilon$ such that

$$\|S_0(t)u - S_0(t)v\|_Z < \epsilon, \text{ if } \|u - v\|_Z < \delta, u, v \in K, \text{ for all } t \in J,$$

where $J = [0, 2M]$ and $M = \sup I$.

Now, since the process $\{S(t,s) : t \geqslant s\}$ is asymptotically autonomous at $-\infty$, for the $\delta > 0$ above, there exists $r_0 < 0$ such that, if $r \leqslant r_0$,

$$\|S(t+r,r)u - S_0(t)u\|_Z < \delta, \text{ for all } t \in I \text{ and } u \in K.$$

Now we have for $\tau \leqslant r_0$,

$$S_\tau(t+r,r)u - S_0(t)u = \begin{cases} S(t+r,r)u - S_0(t)u, & \text{if } t+r \leqslant \tau, \\ S_0(t+r-\tau)S(\tau,r)u - S_0(t+r-\tau)S_0(\tau-r)u, & \text{if } r \leqslant \tau \leqslant t+r, \\ 0, & \text{if } \tau \leqslant r, \end{cases}$$

which implies[2] that

$$\sup_{r\in\mathbb{R}} \|S_\tau(t+r,r)u - S_0(t)u\|_Z < \epsilon, \text{ for all } t \in I \text{ and } u \in K.$$

The case for asymptotically autonomous evolution process at $+\infty$ follows analogously, just reminding that for each bounded interval $I \subset \mathbb{R}^+$ and each compact set $K \subset Z$, the set $\overline{\underset{t\in I}{\cup} S(t)K}$ is also compact, since we have the uniform continuity property. ∎

We can state the following result:

<u>Theorem</u> 6.5. *Let $(\varphi,\Theta)_{(X,\mathscr{P})}$ be a NDS and $\{\Pi(t) : t \geqslant 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t) : t \geqslant 0\}$ has a global attractor $\mathcal{A}$ and let $\mathscr{A}$ be the global attractor for the driving semigroup $\{\Theta(t) : t \geqslant 0\}$. Let also $p_0 \in \mathscr{P}$ be a fixed point of Θ and assume that there exists a bounded global solution $\eta : \mathbb{R} \to \mathscr{P}$ such that $\eta(s) \to p_0$ as $s \to -\infty$ and that*

(a') *If $T(t) \doteq \varphi(t,p_0)$, for all $t \geqslant 0$ then $\{T(t) : t \geqslant 0\}$ is a generalized gradient-like semigroup with isolated invariant sets $\{\Gamma_{1,0}, \cdots \Gamma_{n,0}\}$.*

(b') *If $S(t,s) \doteq \varphi(t-s,\eta(s))$ for all $t \geqslant s$, the evolution process $\{\varphi(t-s,\eta(s)) : t \geq s\}$ possesses $n \in \mathbb{N}$ isolated invariant families*

$$\boldsymbol{\Xi} = \{\Xi_1(\cdot), \cdots, \Xi_n(\cdot)\},$$

which behave upper and lower semi-continuously as $s \to -\infty$, that is

$$\sup_{1\leqslant i\leqslant n} [d_H(\Xi_i(s),\Gamma_{i,0}) + d_H(\Gamma_{i,0},\Xi_i(s))] \overset{s\to-\infty}{\longrightarrow} 0.$$

(c') *There exist $\mu > 0$ such that, if $\xi : \mathbb{R} \to X$ is a bounded solution of $\{\varphi(t-s,\eta(s)) : t \geq s\}$ and there are $t_0 \in \mathbb{R}$ and $i \in \{1,\cdots,n\}$ with $\sup_{t\leq t_0} d_X(\xi(t),\Xi_i^*(t)) < \mu$, then $\lim_{t\to-\infty} d_X(\xi(t),\Xi_i^*(t)) = 0$.*

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[2]Here, for the estimate of the term $\|S_0(t+r-\tau)S(\tau,r)u - S_0(t+r-\tau)S_0(\tau-r)u\|_Z$, is where we have the need to use the interval J for the uniform continuity property, since $t+r-\tau$ does not need to be in I, but $t+r-\tau \in J$.

Then, there exists $\tau_0 < 0$ such that, for all $\tau \leqslant \tau_0$, the forwards truncated evolution process $\{S_\tau(t,s) : t \geqslant s\}$ is a generalized gradient-like evolution process.

Proof: Since $\eta(s) \to p_0$ as $s \to -\infty$, the evolution process $\{S(t,s) : t \geqslant s\}$ is asymptotically autonomous at $-\infty$, and thus the truncated process satisfies also item (d) of Theorem 6.2, which gives us the result. ■

With this result, we can look more closely to the behavior of solutions $\xi : \mathbb{R} \to X$ of the evolution process $\{S(t,s) : t \geqslant s\}$ at $-\infty$.

Theorem 6.6. *Under the hypotheses and notations of Theorem 6.5, if $\xi : \mathbb{R} \to X$ is a bounded global solution for the evolution process $\{S(t,s) : t \geqslant s\}$, then there exists $i = 1, \cdots, n$ such that $\xi(s) \to \Gamma_{i,0}$ as $s \to -\infty$.*

Proof: Let $\xi : \mathbb{R} \to X$ be a bounded global solution for the evolution process $\{S(t,s) : t \geqslant 0\}$. Then, for each $\tau \in \mathbb{R}$, the function $\xi_\tau : \mathbb{R} \to X$ defined by

$$\xi_\tau(t) = \begin{cases} \xi(t), & \text{if } t \leqslant \tau \\ T(t-\tau)\xi(\tau), & \text{if } t \geqslant \tau \end{cases},$$

is a bounded global solution for the forwards truncated evolution process $\{S_\tau(t,s) : t \geqslant s\}$. By Theorem 6.5, there exists $\tau_0 < 0$ and $i = 1, \cdots, n$ such that $\mathrm{d}_H(\xi_{\tau_0}(s), \Xi_i(s)) \to 0$ as $s \to -\infty$ and since $\mathrm{d}_H(\Xi_i^*(s), \Gamma_{i,0}) \to 0$ as $s \to -\infty$ and $\xi_{\tau_0}(s) = \xi(s)$ for $s \leqslant \tau_0$, the result follows. ■

Analogously, we can state the result:

Theorem 6.7. *Let $(\varphi, \Theta)_{(X,\mathscr{P})}$ be a NDS and $\{\Pi(t) : t \geqslant 0\}$ the associated skew product semiflow. Assume that $\{\Pi(t) : t \geqslant 0\}$ has a global attractor $\mathcal{A}$ and let $\mathscr{A}$ be the global attractor for the driving semigroup $\{\Theta(t) : t \geqslant 0\}$. Let also $p_0 \in \mathscr{P}$ be a fixed point of Θ and assume that there exists a bounded global solution $\eta : \mathbb{R} \to \mathscr{P}$ such that $\eta(s) \to p_0$ as $s \to \infty$ and that:*

(a') *If $T(t) \doteq \varphi(t, p_0)$, for all $t \geqslant 0$ then $\{T(t) : t \geqslant 0\}$ is a generalized gradient-like semigroup with isolated invariant sets $\{\Gamma_{1,0}, \cdots \Gamma_{n,0}\}$.*

(b') *If $S(t,s) \doteq \varphi(t-s, \eta(s))$ for all $t \geqslant s$, the evolution process $\{\varphi(t-s, \eta(s)) : t \geq s\}$ possesses $n \in \mathbb{N}$ isolated invariant families*

$$\mathbf{\Xi} = \{\Xi_1(\cdot), \cdots, \Xi_n(\cdot)\},$$

which behave upper e lower semi-continuously as $s \to \infty$, that is

$$\sup_{1 \leqslant i \leqslant n} [d_H(\Xi_i(s), \Gamma_{i,0}) + d_H(\Gamma_{i,0}, \Xi_i(s))] \overset{s\to\infty}{\longrightarrow} 0.$$

(c') *There exist $\mu > 0$ such that, if $\xi : \mathbb{R} \to X$ is a bounded solution of $\{\varphi(t-s, \eta(s)) : t \geq s\}$ and there are $t_0 \in \mathbb{R}$ and $i \in \{1, \cdots, n\}$ with $\sup_{t \geq t_0} d_X(\xi(t), \Xi_i(t)) < \mu$, then $\lim_{t\to\infty} d_X(\xi(t), \Xi_i(t)) = 0$.*

Then, there exists $\tau_0 > 0$ such that for all $\tau \geqslant \tau_0$ the backward truncated evolution process $\{S_\tau(t,s) : t \geqslant s\}$ is a generalized gradient-like evolution process.

Theorem 6.8. *Under the hypotheses and notations of Theorem 6.7, if $\xi : \mathbb{R} \to X$ is a bounded global solution for the evolution process $\{S(t,s) : t \geqslant s\}$, then there exists $i = 1, \cdots, n$ such that $\xi(s) \to \Gamma_{i,0}$ as $s \to \infty$.*

7. Applications

We now give useful examples for applications which help us to understand different aspects of our theory. Remember that in Subsection 3.1 we have already given an example of a non-autonomous differential equation, defined only for positive times, where our theory can be applied.

Let us start with an example of a driving system.

Example 7.1. *Consider the system of autonomous differential equations*

$$\begin{cases} \dot{v} = f(u,v) \quad t > 0 \\ \dot{u} = g(u), \quad t > 0 \\ u(0) = u_0 \in \mathbb{R}^n, \ v(0) = v_0 \in \mathbb{R}^n, \end{cases} \tag{7.1}$$

where the u component is decoupled, so the system (7.1) generates a skew product semiflow. The u-component here may be considered to represent an independent system that drives the v-component of the system in the sense that

$$\dot{v} = f(u(t), v)$$

for any given solution $u(t)$ of $\dot{u} = g(u)$. Assume that the system $\dot{u} = g(u)$ generates a semigroup $\{\Theta(t) : t \geqslant 0\}$ in $\mathbb{R}^n$, that is, $\Theta(t)u_0 = u(t, u_0)$, where $u(\cdot, u_0)$ is the unique solution for $t > 0$ of the problem

$$\begin{cases} \dot{u} = g(u), \ t > 0 \\ u(0) = u_0. \end{cases}$$

Assume also that $\{\Theta(t) : t \geqslant 0\}$ *has a global attractor* $\mathscr{A}$ *with a Morse decomposition* $\{\mathscr{M}_1, \cdots, \mathscr{M}_n\}$ *and also that the generated skew product semiflow* $\{\Pi(t) : \ t \geqslant 0\}$ *has a global attractor* $\mathcal{A}$*. Then we have that for every pair of points* $(u_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n$*, the solution* $v(t, v_0, u_0)$ *of the problem*

$$\begin{cases} \dot{v}(t) = f(\Theta(t)u_0, v(t)), \ t > 0 \\ v(0) = v_0, \end{cases}$$

satisfies $v(t, v_0, u_0) \overset{t\to\infty}{\longrightarrow} P_{\mathscr{M}_{i_0}}(\mathcal{M}_{i_0})$*, for some* $i_0 = 1, \cdots, n$*, where* i_0 *is such that* $\Theta(t)u_0 \overset{t\to\infty}{\longrightarrow} \mathscr{M}_{i_0}$*.*

Example 7.2. *Consider a function* $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *and the non-autonomous dynamical system*

$$\begin{cases} \dot{x} = f(t, x), \quad t \in \mathbb{R} \\ x(0) = x_0. \end{cases} \tag{7.2}$$

Assume that there are functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\sup_{x\in\mathbb{R}^n} \|f(t,x) - f_2(x)\|_{\mathbb{R}^n} \overset{t\to-\infty}{\longrightarrow} 0, \qquad \textit{and} \qquad \sup_{x\in\mathbb{R}^n} \|f(t,x) - f_1(x)\|_{\mathbb{R}^n} \overset{t\to\infty}{\longrightarrow} 0.$$

Now considering the space $\mathscr{C} = C_b(I \times \mathbb{R}^n, \mathbb{R}^n)$*, we can see that*

$$\|\Theta(t)f(s,x) - f_2(x)\|_k = \sup_{x\in B_k, s\in I_k} \|f(t+s,x) - f_2(x)\|_{\mathbb{R}^n} \overset{t\to-\infty}{\longrightarrow} 0,$$

and

$$\|\Theta(t)f(s,x) - f_1(x)\|_k = \sup_{x\in B_k, s\in I_k} \|f(t+s,x) - f_1(x)\|_{\mathbb{R}^n} \overset{t\to\infty}{\longrightarrow} 0,$$

for all $k \in \mathbb{N}$*, which means that* $\Theta(t)f \overset{t\to-\infty}{\longrightarrow} f_2$ *and* $\Theta(t)f \overset{t\to\infty}{\longrightarrow} f_1$ *in the uniform convergence on bounded sets, thus* $\mathscr{K} \doteq \overline{\gamma(f)}^{\mathscr{C}} = \{\Theta(t)f\}_{t\geqslant 0} \cup \{f_1, f_2\}$ *and this set has a Morse decomposition* $\{\mathscr{M}_1, \mathscr{M}_2\}$ *given by*

$$\mathscr{M}_1 = \{f_1\} \quad \textit{ and } \mathscr{M}_2 = \{f_2\}.$$

If the skew product semiflow $\{\Pi(t) : \ t \geqslant 0\}$ *given by* $\Pi(t)(x_0, g) = (x(t, g, x_0), g_t)$ *in the phase state* $\mathbb{R}^n \times \mathscr{K}$ *has a global attractor* $\mathcal{A}$ *then* $\{\mathcal{M}_1, \mathcal{M}_2\}$ *is a Morse decomposition for* $\mathcal{A}$*, where*

$$\mathcal{M}_1 = \mathcal{A}(f_1) \times \{f_1\} \textit{ and } \mathcal{M}_1 = \mathcal{A}(f_2) \times \{f_2\}.$$

Therefore, the solution $x(t, f, x_0)$ *of the problem (7.2) converges to* $\mathcal{A}(f_1)$ *as* $t \to \infty$ *and* $\mathcal{A}(f_2)$ *as* $t \to -\infty$*. Moreover, we know that* $\mathcal{A}(f_1)$ $(\mathcal{A}(f_2))$ *is the global attractor of the problem* $\dot{x} = f_1(x)$ $(\dot{x} = f_2(x))$*, and if* $\mathcal{A}(f_1)$ $(\mathcal{A}(f_2))$ *has a Morse decomposition* $\{M_1^1, \cdots, M_m^1\}$ $(\{M_1^2, \cdots, M_p^2\})$ *then, under the hypothesis* (b') *and* (c') *of Theorem 6.8 (Theorem 6.6) we conclude that there exists* $i = 1, \cdots, m$ $(j = 1, \cdots, p)$ *such that* $x(t, f, x_0)$ *converges to* M_i^1 *as* $t \to \infty$ $(M_j^2$ *as* $t \to -\infty)$*.*

Remark 7.3. *Hypothesis (b') of Theorem 6.6 (Theorem 6.8) can be verified, for example, using the roughness of hyperbolic points (exponential dichotomy of global solutions or normally hyperbolic periodic orbits) under small non-autonomous perturbations. Hypothesis (c') is checked with hyperbolicity for hyperbolic points (exponential dichotomy for global solutions or normal hyperbolicity for periodic orbits) (see, for instance,* [9, 13, 4]*).*

Example 7.4. *This last example illustrates how we can use the general theory in a more concrete case. Let us consider the planar system*

$$\frac{d}{dt}(x,y) = F\left(t,(x,y)\right), \quad t \in \mathbb{R}. \tag{7.3}$$

Assume that $F(t,(x,y)) \to F_1(x,y)$ *as* $t \to -\infty$ *and that* $F(t,(x,y)) \to F_2(x,y)$ *as* $t \to \infty$*, where* $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfy*

1. $F_1(x,y) = (f(x), g(x,y))$*, where* $f(x) = x - x^3$ *and* $g(x,y) = (1-x^2)y - y^3$*. Clearly they satisfy the conditions of Example 7.1;*

2. F_2 *is given in polar coordinates by*

$$F_2(r\cos\theta, r\sin\theta) = G(r,\theta) = (-r(r-1)(r-2), 1).$$

From Example 7.1, we know the Morse decomposition $\{\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2\}$ *for the global attractor* $\mathcal{A}$ *of the planar system*

$$\begin{cases} \dot{x} = f(x) \\ \dot{y} = g(x,y) \end{cases},$$

given a Morse decomposition $\{\mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2\}$ *for the global attractor* $\mathscr{A} = [-1,1] \subset \mathbb{R}$ *of the equation* $\dot{x} = f(x)$*, where* $\mathscr{M}_0 = \{-1\}$*,* $\mathscr{M}_1 = \{1\}$*,* $\mathscr{M}_2 = \{0\}$*. Indeed, it is not hard to see that the Morse decomposition is this case is given by* $\mathcal{M}_0 = \{(-1,0)\}$*,* $\mathcal{M}_1 = \{(1,0)\}$ *and* $\mathcal{M}_2 = \{(0,y) : y \in [-1,1]\}$*.*

Furthermore, we already know that the system

$$\frac{d}{dt}(x,y) = F_2(x,y), \quad t > 0, \tag{7.4}$$

generates a generalized gradient-like system, with invariant sets given by $\Xi_0 = \{0\}$*,* $\Xi_1 = \{(1,\theta) : \theta \in [0,2\pi]\}$ *and* $\Xi_2 = \{(2,\theta) : \theta \in [0,2\pi]\}$*.*

Thus, by Example 7.2, we know that every solution $\xi : \mathbb{R} \to \mathbb{R}^2$ *of the system (7.3) satisfies (see Figure 2)*

(a) $\xi(t) \to \mathcal{M}_i$ *for some* $i = 0, 1, 2$ *as* $t \to -\infty$*,*

(b) $\xi(t) \to \Xi_j$ *for some* $j = 0, 1, 2$ *as* $t \to \infty$.

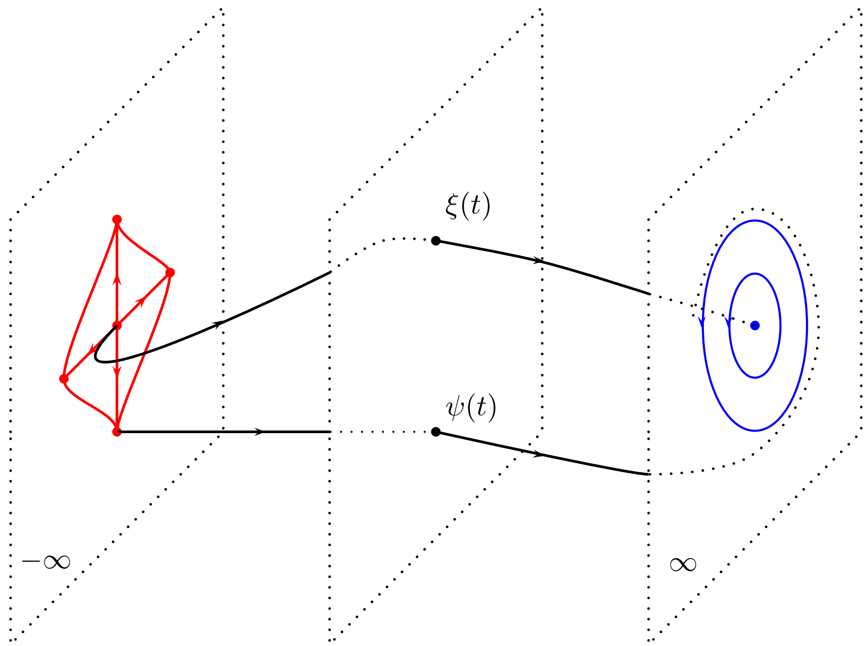

Figure 2: Asymptotic behaviors for solutions ξ, ψ *of* (7.3).

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