

**PULLBACK ATTRACTORS FOR REACTION-DIFFUSION
EQUATIONS IN SOME UNBOUNDED DOMAINS WITH AN
 H^{-1} -VALUED NON-AUTONOMOUS FORCING TERM AND
WITHOUT UNIQUENESS OF SOLUTIONS**

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ABSTRACT. The existence of a pullback attractor for a reaction-diffusion equations in an unbounded domain containing a non-autonomous forcing term taking values in the space H^{-1} , and with a continuous nonlinearity which does not ensure uniqueness of solutions, is proved in this paper. The theory of set-valued non-autonomous dynamical systems is applied to the problem.

Dedicated to Peter E. Kloeden on his 60th birthday.

1. Introduction and setting of the problem. Let $\Omega \subset \mathbb{R}^N$ be a nonempty open set, not necessarily bounded, and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_{\Omega} |u(x)|^2 dx \leq \lambda_1^{-1} \int_{\Omega} |\nabla u(x)|^2 dx \quad \forall u \in H_0^1(\Omega). \quad (1)$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω ,

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(x, u) + h(t) & \text{in } \Omega \times (\tau, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, +\infty), \\ u(x, \tau) = u_{\tau}(x), & x \in \Omega, \end{cases} \quad (2)$$

where $\tau \in \mathbb{R}$, $u_{\tau} \in L^2(\Omega)$, $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that $f(x, \cdot) \in C(\mathbb{R})$ for almost every $x \in \Omega$, and satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $p \geq 2$ and positive functions $C_1(x)$, $C_2(x) \in L^1(\Omega)$ such that

$$|f(x, s)|^{\frac{p}{p-1}} \leq \alpha_1 |s|^p + C_1(x) \quad \forall s \in \mathbb{R}, x \in \Omega, \quad (3)$$

$$f(x, s)s \leq -\alpha_2 |s|^p + C_2(x) \quad \forall s \in \mathbb{R}, x \in \Omega. \quad (4)$$

Several aspects of reaction-diffusion equations are being analyzed over the last years, particularly, their asymptotic behaviour. The motivations for the study of this kind of evolution equation are out of any doubt (see the cited references in

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this paper as well as the references cited in those). However, we will mention below some papers which are significant contributions in any of the cases considered when uniqueness of solutions cannot be ensured or do not hold (see [1] and the references therein for the case of uniqueness of solutions). Therefore, the dynamical system generated by our problem will be a set-valued (or multi-valued) one.

The study of autonomous reaction-diffusion equations without uniqueness of solutions in a bounded domain Ω in the autonomous case (i.e. h does not depend on the time t), or in the non-autonomous case but with strong uniformity properties on the time dependent terms, can be found in [17], [20], [21], [22], [23], [24], [41], [44], [46], where the classical theory of global attractor is adapted to handle this set-valued case. Nevertheless, the theory of trajectory attractors is used in [13], [14] to investigate the problem.

In the autonomous case, when the domain Ω is unbounded, but we have uniqueness of solutions, several studies on our problem can be found in [2], [3], [4], [5], [16], [18], [19], [25], [26], [32], [33], [37], [39], [43], [47], [48], while in the case of non-uniqueness (but still being Ω unbounded, and the problem autonomous), some results on the existence of attractors have been obtained in [34], [35], [36].

However, due to the non-autonomous character of our problem in this paper, we have to use an appropriate framework. Being possible to choose amongst several theories (skew-product flows, uniform attractors, trajectory attractors, pullback attractors) we will use the theory of pullback attractors since this allows for more generality in the non-autonomous terms (see [12], [9], [10], [11], [29], [30], [1] for some results concerning pullback attractors and several reasons justifying the interest of using this theory).

Concerning existence of pullback attractors for reaction-diffusion equations with uniqueness of solutions in bounded or unbounded domains several results are given in [1], [8], [27], [42], [49], [50].

Finally, it is also worth mentioning that the existence of random (pullback) attractor for a stochastic reaction-diffusion equation in an unbounded domain has been proved in [6] in the case of uniqueness of solutions.

Now, our aim in this paper is to consider a much more general problem: a reaction-diffusion equation in an unbounded domain, with a continuous nonlinearity and a non-autonomous forcing term with values in the space H^{-1} which does not have uniqueness of solutions, and we will use the theory of multi-valued non-autonomous (pullback) dynamical systems to prove the existence of a pullback attractor for our problem.

In Section 2 we establish a result ensuring existence of solution of our reaction-diffusion problem. Some preliminaries on the theory of multi-valued (or set-valued) non-autonomous dynamical systems are stated in Section 3. Finally, the existence of a pullback attractor for our model is proved in Section 4.

2. Existence of solution. We prove in this section a result on existence of solutions of problem (2).

By $|\cdot|$ we denote the norm in $L^2(\Omega)$, by $\|\cdot\| = |\nabla\cdot|$ the norm in $H_0^1(\Omega)$ and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$. We will use (\cdot, \cdot) to denote the scalar product in $L^2(\Omega)$ or $[L^2(\Omega)]^N$, and $\langle \cdot, \cdot \rangle$ to denote either the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $L^{p'}(\Omega)$ and $L^p(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p .

Definition 1. A weak solution of (2) is any function $u : (\tau, +\infty) \rightarrow L^p(\Omega) \cap H_0^1(\Omega)$, such that $u \in L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega))$ for all $T > \tau$, and

$$(u(t), w) + \int_{\tau}^t (\nabla u(s), w) ds = (u_{\tau}, w) + \int_{\tau}^t \langle f(x, u(s)) + h(s), w \rangle ds \quad \forall t \geq \tau, \quad (5)$$

for all $w \in L^p(\Omega) \cap H_0^1(\Omega)$.

It is well known [14, p.285] under the above assumptions on u_{τ} , f and h , if u is a weak solution of (2), then $u \in C([\tau, +\infty); L^2(\Omega))$, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous on every interval $[\tau, T]$ and $\frac{d}{dt}|u(t)|^2 = 2 \left\langle \frac{du}{dt}, u \right\rangle$ for a.a. $t \in (\tau, T)$. Hence, it satisfies the energy equality

$$|u(t)|^2 + 2 \int_{\tau}^t \|u(s)\|^2 ds = |u_{\tau}|^2 + 2 \int_{\tau}^t \langle f(x, u(s)) + h(s), u(s) \rangle ds \quad \forall t \geq \tau.$$

From now on, for all $m \geq 1$, we denote

$$\Omega_m = \Omega \cap \{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in \mathbb{R}^N .

We will denote by " \rightharpoonup " the weak convergence in the corresponding indicated space, while " \rightarrow " will denote the strong convergence, as usual.

Theorem 2. Assume that Ω satisfies (1), $h \in L_{loc}^2(\mathbb{R}; H^{-1}(\Omega))$ and $f \in C(\mathbb{R})$ satisfies (3) and (4). Then, for all $\tau \in \mathbb{R}$, $u_{\tau} \in L^2(\Omega)$, there exists at least a weak solution u of (2).

Proof. (Sketch) For each integer $n \geq 1$, we denote by

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t) w_j,$$

a solution of

$$\begin{cases} \frac{d}{dt}(u_n(t), w_j) = -(\nabla u_n(t), \nabla w_j) + \langle f(x, u_n(t)), w_j \rangle + \langle h(t), w_j \rangle & t > \tau, \\ (u_n(\tau), w_j) = (u_{\tau}, w_j) & j = 1, \dots, n, \end{cases}$$

where $\{w_j : j \geq 1\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ is a Hilbert basis of $L^2(\Omega)$ such that $\text{span}\{w_j\}_{j \geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$.

It is a standard matter to deduce that

$$\{u_n\} \text{ is bounded in } L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega)), \quad (6)$$

and

$$f(x, u_n) \text{ is bounded in } L^{p'}(\tau, T; L^{p'}(\Omega)),$$

for all $T > \tau$ (we note that the above estimates allow to extend every solution to a global one).

Then, there exists a subsequence $\{u_{\mu}\} \subset \{u_n\}$ such that

$$u_{\mu} \overset{*}{\rightharpoonup} u \text{ weak-star in } L^{\infty}(\tau, T; L^2(\Omega)),$$

$$u_{\mu} \rightharpoonup u \text{ in } L^p(\tau, T; L^p(\Omega)),$$

$$u_{\mu} \rightharpoonup u \text{ in } L^2(\tau, T; H_0^1(\Omega)), \quad (7)$$

$$f(x, u_{\mu}) \rightharpoonup \chi \text{ in } L^{p'}(\tau, T; L^{p'}(\Omega)), \quad (8)$$

for all $T > \tau$. Now (7) implies that

$$\Delta u_\mu \rightharpoonup \Delta u \text{ in } L^2(\tau, T; H^{-1}(\Omega)).$$

On the other hand, to prove that $\chi(t) = f(x, u(t))$, we argue similarly to [40]. Also, arguing in a similar way as in [40, p.75] we first deduce

$$\limsup_{a \rightarrow 0} \sup_{\mu} \int_{\tau}^{T-a} |u_\mu(t+a) - u_\mu(t)|^2 dt = 0, \quad (9)$$

for all $T > \tau$.

Let $\phi \in C^1([0, +\infty))$ be a function such that

$$\begin{aligned} 0 &\leq \phi(s) \leq 1, \\ \phi(s) &= 1 \quad \forall s \in [0, 1], \\ \phi(s) &= 0 \quad \forall s \geq 2. \end{aligned}$$

For each μ and $m \geq 1$, we define

$$v_{\mu,m}(x, t) = \phi\left(\frac{|x|_{\mathbb{R}^N}^2}{m^2}\right) u_\mu(x, t) \quad \forall x \in \Omega_{2m}, \forall \mu, \forall m \geq 1. \quad (10)$$

We obtain from (6) that, for all $m \geq 1$, the sequence $\{v_{\mu,m}\}_{\mu \geq 1}$ is bounded in $L^\infty(\tau, T; L^2(\Omega_{2m})) \cap L^p(\tau, T; L^p(\Omega_{2m})) \cap L^2(\tau, T; H_0^1(\Omega_{2m}))$, for all $T > \tau$.

In particular, it follows that

$$\limsup_{a \rightarrow 0} \sup_{\mu} \left(\int_{\tau}^{\tau+a} |v_{\mu,m}(x, t)|_{L^2(\Omega_{2m})}^2 dt + \int_{T-a}^T |v_{\mu,m}(x, t)|_{L^2(\Omega_{2m})}^2 dt \right) = 0.$$

On the other hand, from (9) we deduce that for $m \geq 1$,

$$\limsup_{a \rightarrow 0} \sup_{\mu} \left(\int_{\tau}^{T-a} |v_{\mu,m}(x, t+a) - v_{\mu,m}(x, t)|_{L^2(\Omega_{2m})}^2 dt \right) = 0.$$

Moreover, as Ω_{2m} is a bounded set, then $H_0^1(\Omega_{2m})$ is included in $L^2(\Omega_{2m})$ with compact injection.

Then, by the compactness Theorem 13.3 and Remark 13.1 of [45] with $X = L^2(\Omega_{2m})$, $Y = H_0^1(\Omega_{2m})$, $r = 2$ and $\mathcal{G} = \{v_{\mu,m}\}_{\mu \geq 1}$, we obtain that

$$\{v_{\mu,m}\}_{\mu \geq 1} \text{ is relatively compact in } L^2(\tau, T; L^2(\Omega_{2m})),$$

and thus, taking into account that $v_{\mu,m}(x, t) = u_\mu(x, t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$

$$\{u_{\mu|\Omega_m}\}_{\mu \geq 1} \text{ is pre-compact in } L^2(\tau, T; L^2(\Omega_m)). \quad (11)$$

It is not difficult to conclude from (11), (7) and (1), via a diagonal procedure, the existence of a subsequence $\{u_\mu^\mu\}_{\mu \geq 1} \subset \{u_\mu\}_{\mu \geq 1}$ such that

$$u_\mu^\mu \rightarrow u \text{ a.e. in } \Omega_m \times (\tau, +\infty) \text{ as } \mu \rightarrow \infty, \quad \forall m \geq 1.$$

Then, as f is continuous,

$$f(x, u_\mu^\mu) \rightarrow f(x, u) \text{ a.e. in } \Omega_m \times (\tau, +\infty),$$

and as $\{f(x, u_\mu^\mu)\}$ is bounded in $L^{p'}(\Omega_m \times (\tau, T))$, by Lemma 1.3, Chapter 1 in [28], we obtain

$$f(x, u_\mu^\mu) \rightharpoonup f(x, u) \text{ in } L^{p'}(\tau, T; L^{p'}(\Omega_m)) \quad \forall T > \tau.$$

From (8)

$$f(x, u_\mu) \rightharpoonup \chi|_{\Omega_m \times (\tau, T)} \text{ in } L^{p'} \left(\tau, T; L^{p'}(\Omega_m) \right).$$

By the uniqueness of the weak limit, we have

$$\chi = f(x, u) \text{ a.e. in } \Omega_m \times (\tau, T) \quad \forall T > \tau, \forall m \geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$, we obtain

$$\chi = f(x, u) \text{ a.e. in } \Omega \times (\tau, +\infty). \quad (12)$$

Then, (12) and (8) yield that

$$f(x, u_\mu) \rightharpoonup f(x, u(t)) \text{ in } L^{p'}(\tau, T; L^{p'}(\Omega)) \quad \forall T > \tau, \quad (13)$$

and thanks to the equation satisfied by u'_μ and the fact that $\text{span} \{w_j\}_{j \geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$, it is a standard matter to prove that we can pick an element in the equivalence class of u satisfying

$$(u(t), w) = (u_\tau, w) + \int_\tau^t \langle \Delta u(s) + f(x, u(s)) + h(s), w \rangle ds, \quad (14)$$

for all $t \geq \tau$, for any $w \in H_0^1(\Omega) \cap L^p(\Omega)$. \square

Remark 3. *Observe that the conditions on the function f do not allow to obtain the uniqueness of the Cauchy problem (see [24] for a counterexample in the autonomous case).*

3. Preliminaries on the theory of pullback attractors. As the uniqueness of the Cauchy problem fails to be true for our equation, we have to work with set-valued non-autonomous dynamical systems.

First we recall some basic definitions for set-valued non-autonomous dynamical systems and establish a sufficient condition for the existence of a pullback attractor for these systems (see [10], [11], [12], [30] and [31] for more details).

Let $X = (X, d_X)$ be a metric space, and let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X , and let us denote $\mathbb{R}_d^2 := \{(t, s) \in \mathbb{R}^2 : t \geq s\}$.

Definition 4. *A multi-valued map $U : \mathbb{R}_d^2 \times X \rightarrow \mathcal{P}(X)$ is called a multi-valued non-autonomous dynamical system (MNDS) on X (also named a multi-valued process on X) if*

$$U(s, s, \cdot) = \text{id}_X(\cdot) \text{ for all } s \in \mathbb{R},$$

$$U(t, \tau, x) \subset U(t, s, U(s, \tau, x)) \text{ for all } \tau \leq s \leq t, x \in X,$$

where $U(t, \tau, V) := \bigcup_{x_0 \in V} U(t, \tau, x_0)$ for any non-empty set $V \subset X$.

An MNDS is said to be strict if

$$U(t, \tau, x) = U(t, s, U(s, \tau, x)) \text{ for all } \tau \leq s \leq t, x \in X.$$

Definition 5. *An MNDS U on X is said to be **upper-semicontinuous** if for all $t \geq \tau$ the mapping $U(t, \tau, \cdot)$ is upper-semicontinuous from X into $\mathcal{P}(X)$, i.e., for any $x_0 \in X$ and for every neighborhood \mathcal{N} in X of the set $U(t, \tau, x_0)$, there exists $\delta > 0$ such that $U(t, \tau, y) \subset \mathcal{N}$ whenever $d_X(x_0, y) < \delta$.*

Let \mathcal{D} be a class of sets parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

We will say that the class \mathcal{D} is inclusion-closed, if $\widehat{D} \in \mathcal{D}$ and $\emptyset \neq D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, imply that $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\}$ belongs to \mathcal{D} .

Definition 6. We say that a family $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is *pullback \mathcal{D} -absorbing* if for every $\widehat{D} \in \mathcal{D}$ and every $t \in \mathbb{R}$, there exists $\tau(t, \widehat{D}) \leq t$ such that

$$U(t, \tau, D(\tau)) \subset B(t) \text{ for all } \tau \leq \tau(t, \widehat{D}).$$

Definition 7. The MNDS U is *asymptotically compact* with respect to a family $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ if for all $t \in \mathbb{R}$ and every sequence $\tau_n \leq t$ tending to $-\infty$, any sequence $y_n \in U(t, \tau_n, B(\tau_n))$ is pre-compact.

Let $dist_X(\cdot, \cdot)$ denote the Hausdorff semidistance, defined by

$$dist_X(C_1, C_2) := \sup_{x \in C_1} \inf_{y \in C_2} d_X(x, y) \text{ for } C_1, C_2 \subset X.$$

Definition 8. A family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a *global pullback \mathcal{D} -attractor* for the MNDS U if it satisfies

1. $A(t)$ is compact for any $t \in \mathbb{R}$,
2. \widehat{A} is pullback \mathcal{D} -attracting, i.e.

$$\lim_{\tau \rightarrow -\infty} dist_X(U(t, \tau, D(\tau)), A(t)) = 0 \quad \forall t \in \mathbb{R},$$

for all $\widehat{D} \in \mathcal{D}$,

3. \widehat{A} is negatively invariant, i.e.,

$$A(t) \subset U(t, \tau, A(\tau)), \text{ for any } (t, \tau) \in \mathbb{R}_d^2.$$

\widehat{A} is said to be a *strict global pullback \mathcal{D} -attractor* if the invariance property in the third item is strict, i.e.,

$$A(t) = U(t, \tau, A(\tau)), \text{ for } (t, \tau) \in \mathbb{R}_d^2.$$

Theorem 9. Assume that the MNDS U is upper-semicontinuous, and let $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ be pullback \mathcal{D} -absorbing and such that U is asymptotically compact with respect to \widehat{B} .

Then, the set \widehat{A} given by

$$A(t) := \Lambda(\widehat{B}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau, B(\tau))} \quad t \in \mathbb{R}, \quad (15)$$

is a pullback \mathcal{D} -attractor for the MNDS U .

Moreover, suppose that \mathcal{D} is inclusion closed, $\widehat{B} \in \mathcal{D}$, and $B(t)$ is closed in X for any $t \in \mathbb{R}$. Then the family \widehat{A} defined by (15) belongs to \mathcal{D} , and is the unique pullback \mathcal{D} -attractor with this property. In addition, in this case, if U is a strict MNDS, then \widehat{A} is strictly invariant.

Proof. See [12] and [30]. □

Remark 10. For some discussions on the relationship between the concept of \mathcal{D} -attractor and the notion of attractor for fixed bounded subsets of X , see [29] and [30].

4. The pullback attractor for system (2). In this section we prove our main result in this paper. First, we need a priori estimates and a continuity result which are established in the next subsections.

4.1. A priori estimates. For each $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$, let us denote $S(\tau, u_\tau)$ the set of all weak solutions of (2) defined for all $t \geq \tau$.

We define a multi-valued map $U : \mathbb{R}_d^2 \times L^2(\Omega) \rightarrow \mathcal{P}(L^2(\Omega))$ by

$$U(t, \tau, u_\tau) = \{u(t) : u \in S(\tau, u_\tau)\}, \quad \tau \leq t, \quad u_\tau \in L^2(\Omega). \quad (16)$$

Lemma 11. *Under the assumptions of Theorem 2, the multi-valued mapping U defined by (16) is a strict MNDS on $L^2(\Omega)$.*

Proof. It is easy to check that U is well defined. Moreover, U satisfies the first part in Definition 4.

Let us now prove that $U(t, \tau, u_\tau) \subset U(t, s, U(s, \tau, u_\tau))$ also holds for all $\tau \leq s \leq t$, $u_\tau \in L^2(\Omega)$. Consider $\phi \in U(t, \tau, u_\tau)$. Then from the definition of U , there exists a solution $u \in S(\tau, u_\tau)$ such that $u(t) = \phi$. If $\tau \leq s$, then $u(s) \in U(s, \tau, u_\tau)$, and as

$$U(t, s, u(s)) = \{z(t) : z \in S(s, u(s)) \text{ such that } z(s) = u(s)\},$$

obviously,

$$u(t) = \phi \in U(t, s, u(s)) \subset U(t, s, U(s, \tau, u_\tau)).$$

Thus,

$$U(t, \tau, u_\tau) \subset U(t, s, U(s, \tau, u_\tau)) \quad \forall \tau \leq s \leq t.$$

To prove that the MNDS is strict, let us consider $\phi \in U(t, s, U(s, \tau, u_\tau))$. Then there exists a solution u to (2) such that $u(s) = y(s)$, where y is another solution to (2) with initial value $y(\tau) = u_\tau$. We now define

$$z(r) = \begin{cases} y(r) & \text{if } \tau \leq r \leq s, \\ u(r) & \text{if } s \leq r \leq t. \end{cases}$$

It is clear that $z(\cdot)$ is solution to (2) (see [35]), and it is also holds that $z(\tau) = y(\tau) = u_\tau$, and $z(t) = u(t) = \phi$, i.e., $\phi \in U(t, \tau, u_\tau)$. Which means that

$$U(t, s, U(s, \tau, u_\tau)) \subset U(t, \tau, u_\tau) \quad \forall \tau \leq s \leq t.$$

□

Let \mathcal{R}_{λ_1} be the set of all functions $r : \mathbb{R} \rightarrow (0, +\infty)$ such that

$$\lim_{t \rightarrow -\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by \mathcal{D}_{λ_1} the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$ for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$, where $\overline{B}(0, r_{\widehat{D}}(t))$ denotes the closed ball in $L^2(\Omega)$ centered at zero with radius $r_{\widehat{D}}(t)$. Observe that the class \mathcal{D}_{λ_1} is inclusion-closed.

Lemma 12. *Suppose that Ω satisfies (1) and suppose that $f \in C(\mathbb{R})$ satisfies (3) and (4). Let $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, with $h_i \in L_{loc}^2(\mathbb{R}; L^2(\Omega))$ for all $1 \leq i \leq N$, such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds < +\infty \quad \forall t \in \mathbb{R}. \quad (17)$$

Then, the balls $B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$, where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t \in \mathbb{R}$ by

$$R_{\lambda_1}^2(t) = 2e^{-\lambda_1 t} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds + 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)} + 1, \quad (18)$$

form a family $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ which is pullback \mathcal{D}_{λ_1} -absorbing for the MNDS U defined by (16).

Proof. As a consequence of (18), it is evident that $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$. On the other hand, taking into account the energy equality, (1) and (4), if $u \in S(\tau, u_\tau)$ we obtain

$$\frac{d}{dt} \left(e^{\lambda_1 t} |u(t)|^2 \right) + \frac{\lambda_1}{2} e^{\lambda_1 t} |u(t)|^2 \leq 2e^{\lambda_1 t} \sum_{i=1}^N |h_i(t)|^2 + 2e^{\lambda_1 t} \|C_2\|_{L^1(\Omega)}, \quad (19)$$

for $t \geq \tau$.

In particular, integrating between τ and t , we have

$$e^{\lambda_1 t} |u(t)|^2 \leq e^{\lambda_1 \tau} |u_\tau|^2 + 2 \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds + 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)} e^{\lambda_1 t},$$

for all $t \geq \tau$.

From this inequality, we deduce that if $\widehat{D} \in \mathcal{D}_{\lambda_1}$ and $y \in U(t, \tau, D(\tau))$, then

$$|y|^2 \leq e^{\lambda_1(\tau-t)} r_{\widehat{D}}^2(\tau) + 2e^{-\lambda_1 t} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds + 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)}.$$

Consequently the family \widehat{B}_{λ_1} is pullback \mathcal{D}_{λ_1} -absorbing for U . \square

Lemma 13. *Under the assumptions in Lemma 12, for any real numbers $t_1 \leq t_2$ and any $\varepsilon > 0$, there exist $T = T(t_1, t_2, \varepsilon, \widehat{B}_{\lambda_1}) \leq t_1$ and $M = M(t_1, t_2, \varepsilon, \widehat{B}_{\lambda_1}) \geq 1$ verifying*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x, t) dx \leq \varepsilon, \quad \forall \tau \leq T, \quad t \in [t_1, t_2], \quad m \geq M,$$

for any weak solution $u \in S(\tau, u_\tau)$ where $u_\tau \in B_{\lambda_1}(\tau)$.

Proof. Let $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$ and $u \in S(\tau, u_\tau)$ be fixed. We take a smooth function $\theta \in C^1([0, +\infty))$ verifying

$$\begin{aligned} 0 &\leq \theta(s) \leq 1, \\ \theta(s) &= 0 \quad \forall s \in [0, 1], \\ \theta(s) &= 1 \quad \forall s \geq 2. \end{aligned}$$

Under the above assumptions on u_τ , f and h , if u is a weak solution of (2), the function $\|\theta u(t)\|^2 = \int_{\mathbb{R}^N} \theta^2 \left(\frac{|x|^2}{m^2} \right) |u(x, t)|^2 dx$ is absolutely continuous and

$\frac{d}{dt} \|\theta u\|^2 = 2 \left\langle \frac{du}{dt}, \theta^2 u \right\rangle$ for a.a. t (see [35, Lemma 3] or [34, Lemma 32]). On

the other hand (see for example [7, propositions IX.4 and IX.5]) observe that

$\theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(\cdot, t) \in H_0^1(\Omega)$, a.e. in (τ, ∞) , with

$$\partial_i \left(\theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) \right) = \theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \partial_i u(x, t) + \frac{2x_i}{m^2} \theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t), \quad (20)$$

and the same is true replacing θ by θ^2 .

Hence, we obtain for every $t \geq \tau$,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx + \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \\
& + \frac{4}{m^2} \int_{\Omega} \theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) x \cdot \nabla u(x, t) dx \\
& = \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) f(x, u(x, t)) u(x, t) dx - \sum_{i=1}^N \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i(x, t) \partial_i u(x, t) dx \\
& \quad - \sum_{i=1}^N \int_{\Omega} \frac{4x_i}{m^2} \theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) h_i(x, t) dx \\
& = I_1 + I_2 + I_3.
\end{aligned} \tag{21}$$

From (4), we obtain

$$I_1 \leq \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx. \tag{22}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
I_2 & \leq \frac{1}{4} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \\
& \quad + \sum_{i=1}^N \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, t) dx.
\end{aligned} \tag{23}$$

Using that $\theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) = 0$ if $|x|_{\mathbb{R}^N} > \sqrt{2}m$, $\theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \leq C_{\theta'}$ for all x , and the Cauchy-Schwarz inequality, we obtain

$$|I_3| \leq \frac{16}{m^2} C_{\theta'}^2 N \int_{\Omega} |u(x, t)|^2 dx + \sum_{i=1}^N \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, t) dx. \tag{24}$$

Moreover, we have

$$\begin{aligned}
& \left| \frac{4}{m^2} \int_{\Omega} \theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) x \cdot \nabla u(x, t) dx \right| \\
& \leq \frac{4}{m} C_{\theta'} \int_{\Omega} |u(x, t)|^2 dx + \frac{4}{m} C_{\theta'} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx.
\end{aligned} \tag{25}$$

From (21)-(25) we deduce

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx + \left(\frac{3}{4} - \frac{4}{m} C_{\theta'} \right) \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \\
& \leq \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx + \frac{4}{m} C_{\theta'} \int_{\Omega} u^2(x, t) dx \\
& \quad + \frac{16}{m^2} C_{\theta'}^2 N \int_{\Omega} u^2(x, t) dx + 2 \sum_{i=1}^N \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, t) dx.
\end{aligned} \tag{26}$$

Thus, by (20) we have

$$\begin{aligned} \left| \nabla \left(\theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) \right) \right|^2 &= \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 \\ &\quad + \frac{4|x|_{\mathbb{R}^N}^2}{m^4} \left(\theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) \right)^2 u^2(x, t) \\ &\quad + \frac{4}{m^2} \theta' \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) \theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) x \cdot \nabla u(x, t), \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\Omega} \left| \nabla \left(\theta \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u(x, t) \right) \right|^2 dx &\leq \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \\ &\quad + \frac{8}{m^2} C_{\theta'}^2 \int_{\Omega} u^2(x, t) dx \\ &\quad + \frac{4}{m} C_{\theta'} \left(\int_{\Omega} u^2(x, t) dx + \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \right) \\ &= \left(1 + \frac{4}{m} C_{\theta'} \right) \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx \\ &\quad + \left(\frac{8}{m^2} C_{\theta'}^2 + \frac{4}{m} C_{\theta'} \right) \int_{\Omega} u^2(x, t) dx. \end{aligned}$$

From this inequality and (1) we obtain

$$\begin{aligned} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) |\nabla u(x, t)|^2 dx &\geq \left(\frac{m}{m + 4C_{\theta'}} \right) \lambda_1 \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx \quad (27) \\ &\quad - \left(\frac{m}{m + 4C_{\theta'}} \right) \left(\frac{8}{m^2} C_{\theta'}^2 + \frac{4}{m} C_{\theta'} \right) \int_{\Omega} u^2(x, t) dx. \end{aligned}$$

Assume that $\frac{3}{4} - \frac{4}{m} C_{\theta'} > 0$ (and this is true for m large enough). Then, from (26) and (27), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx &\quad (28) \\ &+ \left(\frac{3}{4} - \frac{4}{m} C_{\theta'} \right) \left(\frac{m}{m + 4C_{\theta'}} \right) \lambda_1 \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx \\ &\leq \left(\frac{4C_{\theta'}}{m} + \frac{16C_{\theta'}^2 N}{m^2} + \left(\frac{m}{m + 4C_{\theta'}} \right) \left(\frac{8C_{\theta'}^2}{m^2} + \frac{4C_{\theta'}}{m} \right) \left(\frac{3}{4} - \frac{4C_{\theta'}}{m} \right) \right) \int_{\Omega} u^2(x, t) dx \\ &+ 2 \sum_{i=1}^N \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, t) dx + \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx. \end{aligned}$$

Evidently, there exists m_0 such that for all $m \geq m_0$ we have

$$\left(\frac{3}{4} - \frac{4}{m} C_{\theta'} \right) \left(\frac{m}{m + 4C_{\theta'}} \right) > \frac{1}{2}.$$

Then from (28), if we denote $\widehat{C} = 14C_{\theta'} + 44C_{\theta'}^2 N$, and multiplying by $e^{\lambda_1 t}$, we obtain

$$\begin{aligned} \frac{d}{dt} \left(e^{\lambda_1 t} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u^2(x, t) dx \right) &\leq \frac{\widehat{C}}{m} e^{\lambda_1 t} \int_{\Omega} u^2(x, t) dx \\ &+ 4 \sum_{i=1}^N e^{\lambda_1 t} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, t) dx \\ &+ 2e^{\lambda_1 t} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx. \end{aligned}$$

Integrating now between τ and t , and using the properties of θ , we have

$$\begin{aligned} \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x, t) dx &\leq e^{-\lambda_1 t} e^{\lambda_1 \tau} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u_{\tau}^2(x) dx \\ &+ \frac{\widehat{C}}{m} e^{-\lambda_1 t} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds \\ &+ 4 \sum_{i=1}^N e^{-\lambda_1 t} \int_{-\infty}^t e^{\lambda_1 s} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, s) dx ds \\ &+ 2\lambda_1^{-1} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx, \end{aligned} \quad (29)$$

for all $m \geq m_0$, $t \geq \tau$.

On the other hand, from (19), integrating between τ and t , we have

$$\begin{aligned} \frac{\lambda_1}{2} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds &\leq e^{\lambda_1 \tau} |u_{\tau}|^2 + 2 \sum_{i=1}^N \int_{\tau}^t e^{\lambda_1 s} |h_i(s)|^2 ds \\ &+ 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)} e^{\lambda_1 t}. \end{aligned} \quad (30)$$

Thus, if we take $u_{\tau} \in B_{\lambda_1}(\tau)$, we obtain

$$\begin{aligned} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds &\leq 2\lambda_1^{-1} e^{\lambda_1 \tau} R_{\lambda_1}^2(\tau) \\ &+ 4\lambda_1^{-1} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds \\ &+ 4\lambda_1^{-2} \|C_2\|_{L^1(\Omega)} e^{\lambda_1 t}. \end{aligned} \quad (31)$$

Let us fix $t_1 \leq t_2 \in \mathbb{R}$. Observing that $\lim_{\tau \rightarrow -\infty} e^{\lambda_1 \tau} R_{\lambda_1}^2(\tau) = 0$, from (17) and (31), we deduce that there exists a constant $C(t_1, t_2)$ such that

$$e^{-\lambda_1 t} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds \leq C(t_1, t_2) \quad \forall t \in [t_1, t_2], \tau \leq t_1,$$

and therefore, by (29),

$$\begin{aligned} \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x, t) dx &\leq e^{-\lambda_1 t} e^{\lambda_1 \tau} R_{\lambda_1}^2(\tau) + \frac{\widehat{C}}{m} C(t_1, t_2) \\ &+ 4 \sum_{i=1}^N e^{-\lambda_1 t} \int_{-\infty}^t e^{\lambda_1 s} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, s) dx ds \\ &+ 2\lambda_1^{-1} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx, \end{aligned} \quad (32)$$

for all $m \geq m_0$ and $t \in [t_1, t_2]$, for every $u \in S(\tau, u_\tau)$, where $\tau \leq t_1$ and $u_\tau \in B_{\lambda_1}(\tau)$.

On the other hand, from (17) and Lebesgue's Dominated Convergence Theorem, for every $t \in [t_1, t_2]$ we obtain

$$\begin{aligned} \int_{-\infty}^t e^{\lambda_1 s} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, s) dx ds \\ \leq \int_{-\infty}^{t_2} \int_{\Omega} \chi_{\{|x|_{\mathbb{R}^N} \geq m\}} e^{\lambda_1 s} h_i^2(x, s) dx ds \longrightarrow 0 \text{ as } m \rightarrow \infty, \end{aligned} \quad (33)$$

for all $i = 1, \dots, N$, where χ is the indicator function.

Analogously,

$$\int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx \leq \int_{\Omega} \chi_{\{|x|_{\mathbb{R}^N} \geq m\}} C_2(x) dx \longrightarrow 0 \text{ as } m \rightarrow \infty. \quad (34)$$

From (32), (33) and (34) we deduce our lemma. \square

Remark 14. *It is clear from the proof that Lemma 13 above also holds for any $\widehat{D} \in \mathcal{D}_{\lambda_1}$ instead of \widehat{B}_{λ_1} .*

Lemma 15. *Under the assumptions in Lemma 12, let K be a relatively compact set in $L^2(\Omega)$. Then, for all $\tau \leq T$ and $\varepsilon > 0$ there exists $M = M(\tau, T, \varepsilon, K)$ such that*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x, t) dx \leq \varepsilon, \quad \forall t \in [\tau, T], \quad \forall m \geq M,$$

for any $u \in S(\tau, u_\tau)$, where $u_\tau \in K$ is arbitrary.

Proof. We note that, as shown in Lemma 13, we have

$$\begin{aligned} \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2(x, t) dx &\leq e^{-\lambda_1 t} e^{\lambda_1 \tau} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u_\tau^2(x) dx \\ &+ \frac{\widehat{C}}{m} e^{-\lambda_1 t} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds \\ &+ 4 \sum_{i=1}^N e^{-\lambda_1 t} \int_{-\infty}^t e^{\lambda_1 s} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) h_i^2(x, s) dx ds \\ &+ 2\lambda_1^{-1} \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) C_2(x) dx, \end{aligned} \quad (35)$$

for all $m \geq m_0$, and any $u \in S(\tau, u_\tau)$, where $\tau \leq t$ and $u_\tau \in L^2(\Omega)$ are arbitrary, and where m_0 and \widehat{C} are defined in Lemma 13. On the other hand, as K is a bounded subset of $L^2(\Omega)$, from (30) we deduce that for some constant $c > 0$,

$$\begin{aligned} \int_{\tau}^t e^{\lambda_1 s} |u(s)|^2 ds &\leq 2\lambda_1^{-1} e^{\lambda_1 \tau} c^2 + 4\lambda_1^{-1} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds \\ &\quad + 4\lambda_1^{-2} \|C_2\|_{L^1(\Omega)} e^{\lambda_1 t}, \end{aligned}$$

and thus there exists a constant $C(\tau, T)$ such that

$$e^{-\lambda_1 t} \int_{\tau}^t e^{\lambda_1 s} \int_{\Omega} u^2(x, s) dx ds \leq C(\tau, T), \quad \forall t \in [\tau, T], \quad (36)$$

for any $u \in S(\tau, u_\tau)$, where $u_\tau \in K$ is arbitrary. Finally, as K is a relatively compact subset of $L^2(\Omega)$, then for all $\varepsilon > 0$ there exists m_ε such that

$$\int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{m^2} \right) u_\tau^2(x) dx < \varepsilon \quad \forall u_\tau \in K, \quad \forall m \geq m_\varepsilon. \quad (37)$$

In the contrary case, there would exist an $\varepsilon > 0$ and a sequence $\{u_n\} \subset K$ such that

$$\int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{n^2} \right) u_n^2(x) dx \geq \varepsilon, \quad \forall n \geq 1.$$

But then, there would exist a convergent subsequence $\{u_\mu\} \subset \{u_n\}$, with $u_\mu \rightarrow u$ in $L^2(\Omega)$ as $\mu \rightarrow \infty$. And thus we would have

$$\begin{aligned} \varepsilon &\leq \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{\mu^2} \right) u_\mu^2(x) dx \\ &\leq 2 \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{\mu^2} \right) (u_\mu(x) - u(x))^2 dx + 2 \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{\mu^2} \right) u^2(x) dx \\ &\leq 2 \int_{\Omega} (u_\mu(x) - u(x))^2 dx + 2 \int_{\Omega} \theta^2 \left(\frac{|x|_{\mathbb{R}^N}^2}{\mu^2} \right) u^2(x) dx, \end{aligned}$$

and therefore, making $\mu \rightarrow \infty$, we would have $\varepsilon \leq 0$, which is a contradiction. From (35)-(37), and taking into account (33) and (34), we deduce the assertion of the lemma. \square

4.2. A continuity result. Further, we obtain a continuity result leading to the upper semicontinuity of the MNDS U .

Proposition 16. *Under the assumptions in Lemma 12, let $\tau \in \mathbb{R}$ and $\{u_\tau^n\} \subset L^2(\Omega)$ be a sequence converging weakly in $L^2(\Omega)$ to an element $u_\tau \in L^2(\Omega)$. For each $n \geq 1$ let us fix $u_n \in S(\tau, u_\tau^n)$. Then there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ satisfying that there exists $u \in S(\tau, u_\tau)$ such that*

$$u_\mu(t) \rightharpoonup u(t) \text{ in } L^2(\Omega) \quad \forall t \geq \tau, \quad (38)$$

$$u_\mu \rightharpoonup u \text{ in } L^2(\tau, T; H_0^1(\Omega)) \quad \forall T > \tau, \quad (39)$$

$$u_\mu \rightharpoonup u \text{ in } L^p(\tau, T; L^p(\Omega)) \quad \forall T > \tau, \quad (40)$$

$$f(x, u_\mu) \rightharpoonup f(x, u) \text{ in } L^{p'}(\tau, T; L^{p'}(\Omega)) \quad \forall T > \tau, \quad (41)$$

$$u_{\mu|_{\Omega_m}} \rightarrow u_{|_{\Omega_m}} \text{ in } L^2(\tau, T; L^2(\Omega_m)) \quad \forall T > \tau, \quad \forall m \geq 1. \quad (42)$$

Finally, if the sequence $\{u_\tau^n\}$ converges strongly in $L^2(\Omega)$ to u_τ , then

$$u_\mu \rightarrow u \text{ in } L^2(\tau, T; L^2(\Omega)) \quad \forall T > \tau, \quad (43)$$

and

$$u_\mu(t) \rightarrow u(t) \text{ in } L^2(\Omega) \quad \forall t \geq \tau. \quad (44)$$

Proof. Taking into account the energy equality

$$\frac{1}{2} \frac{d}{dt} |u_n(t)|^2 + |\nabla u_n(t)|^2 = \langle f(x, u_n(t)), u_n(t) \rangle + \langle h(t), u_n(t) \rangle,$$

if we argue similarly to the proof of Theorem 2, we obtain that there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ such that

$$u_\mu \rightharpoonup u \text{ in } L^2(\tau, T; H_0^1(\Omega)), \quad (45)$$

$$u_\mu \rightharpoonup u \text{ in } L^p(\tau, T; L^p(\Omega)),$$

$$f(x, u_\mu) \rightharpoonup f(x, u) \text{ in } L^{p'}(\tau, T; L^{p'}(\Omega)),$$

for all $T > \tau$, and $u \in S(\tau, u_\tau)$.

On the other hand, in particular, for a fixed $T > \tau$, the sequence $\{u_\mu(T)\}$ is bounded in $L^2(\Omega)$, then there exists a subsequence $\{u_{\mu'}\} \subset \{u_\mu\}$ such that

$$u_{\mu'}(T) \rightharpoonup \xi \text{ in } L^2(\Omega). \quad (46)$$

Let $w \in H_0^1(\Omega) \cap L^p(\Omega)$. From the equation satisfied by $u_{\mu'}$, we obtain

$$(u_{\mu'}(T), w) = \left(u_\tau^{\mu'}, w\right) + \int_\tau^T \langle \Delta u_{\mu'}(s) + f(x, u_{\mu'}(s)) + h(s), w \rangle ds,$$

and thus, making $\mu' \rightarrow \infty$,

$$(\xi, w) = (u_\tau, w) + \int_\tau^T \langle \Delta u(s) + f(x, u(s)) + h(s), w \rangle ds.$$

Consequently, as $u \in S(\tau, u_\tau)$, we obtain

$$(\xi, w) = (u(T), w) \quad \forall w \in H_0^1(\Omega) \cap L^p(\Omega),$$

and therefore, by density, it follows

$$\xi = u(T). \quad (47)$$

Then, from (46), (47), we can deduce that the whole sequence $\{u_\mu(T)\}$ satisfies

$$u_\mu(T) \rightharpoonup u(T) \text{ in } L^2(\Omega).$$

As $T > \tau$ has been taken arbitrarily, we see that (38) holds.

On the other hand, reasoning as in the proof of (11) in Theorem 2, we can deduce that for all $m \geq 1$,

$$\{u_{\mu|\Omega_m}\}_{\mu \geq 1} \text{ is pre-compact in } L^2(\tau, T; L^2(\Omega_m)) \quad \forall T > \tau. \quad (48)$$

From (39) and (48), we deduce (42).

Assume now that the sequence $\{u_\tau^n\}$ converges strongly in $L^2(\Omega)$ to u_τ , and let us fix $T > \tau$.

Then, by Lemma 15, we have that for all $\varepsilon > 0$ there exists $M_\varepsilon \geq 1$ such that

$$\begin{aligned} \int_\tau^T \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} (u_n - u)^2 dx ds &\leq 2 \int_\tau^T \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u_n^2 dx ds \\ &\quad + 2 \int_\tau^T \int_{\Omega \cap \{|x|_{\mathbb{R}^N} \geq 2m\}} u^2 dx ds \\ &\leq 4\varepsilon(T - \tau), \forall n \geq 1, \forall m \geq M_\varepsilon. \end{aligned} \quad (49)$$

Moreover, by (42),

$$\int_\tau^T \int_{\Omega_{2m}} (u_\mu - u)^2 dx ds \rightarrow 0, \text{ as } \mu \rightarrow \infty, \text{ for all } m \geq 1. \quad (50)$$

From (49) and (50) we obtain (43).

From (43) we deduce that from every subsequence of $\{u_\mu\}$ we can extract a subsequence that we will denote by $\{u_\nu\}$, such that

$$|u_\nu(t)| \rightarrow |u(t)| \quad a.e. \text{ in } (\tau, T). \quad (51)$$

Let us define

$$J_\nu(t) = \frac{1}{2} |u_\nu(t)|^2 - \int_\tau^t \langle h(s), u_\nu(s) \rangle ds - \int_\tau^t \int_\Omega C_2(x) dx ds,$$

and

$$J(t) = \frac{1}{2} |u(t)|^2 - \int_\tau^t \langle h(s), u(s) \rangle ds - \int_\tau^t \int_\Omega C_2(x) dx ds,$$

for all $t \geq \tau$.

It is clear that J_ν and J are continuous functions. Also, from (39) and (51) we see that

$$J_\nu(t) \rightarrow J(t) \quad a.e. \ t \in (\tau, T) \quad \text{as } \nu \rightarrow \infty. \quad (52)$$

On the other hand, taking into account the energy equality and (4), we obtain

$$\frac{1}{2} \frac{d}{dt} |u_\nu(t)|^2 \leq \int_\Omega C_2(x) dx + \langle h(t), u_\nu(t) \rangle \quad t > \tau. \quad (53)$$

Thus, for every ν , the function J_ν is a non-increasing function of t .

We are now in position to show that

$$J_\nu(t) \rightarrow J(t) \quad \text{for all } t \in (\tau, T). \quad (54)$$

Let $t \in (\tau, T)$ and $\varepsilon > 0$ be fixed. From (52) and the continuity of J , we can take $t' > t$ and $t'' < t$ such that

$$J_\nu(t') \rightarrow J(t') \quad \text{as } \nu \rightarrow \infty, \quad (55)$$

$$J_\nu(t'') \rightarrow J(t'') \quad \text{as } \nu \rightarrow \infty, \quad (56)$$

$$|J(t'') - J(t)| \leq \varepsilon, \quad (57)$$

and

$$|J(t) - J(t')| \leq \varepsilon. \quad (58)$$

As J_ν is a non-increasing function of t , we obtain

$$J_\nu(t') - J_\nu(t) \leq 0, \quad (59)$$

and

$$J_\nu(t'') - J_\nu(t) \geq 0, \quad (60)$$

for every ν . Using (57) and (60) we have

$$\begin{aligned} J_\nu(t) - J(t) &= J_\nu(t) - J_\nu(t'') + J_\nu(t'') - J(t'') \\ &\quad + J(t'') - J(t) \\ &\leq |J_\nu(t'') - J(t'')| + \varepsilon. \end{aligned} \quad (61)$$

Analogously, using (58) and (59) we obtain

$$\begin{aligned} J(t) - J_\nu(t) &= J(t) - J(t') + J(t') - J_\nu(t') \\ &\quad + J_\nu(t') - J_\nu(t) \\ &\leq |J(t') - J_\nu(t')| + \varepsilon. \end{aligned} \quad (62)$$

From (55), (56), (61) and (62), we have

$$\limsup_{\nu \rightarrow \infty} |J(t) - J_\nu(t)| \leq \varepsilon, \quad (63)$$

and therefore, as $\varepsilon > 0$ is arbitrary, (54) follows from (63). Thanks to (63), and taking into account (39), we deduce that

$$|u_\nu(t)| \rightarrow |u(t)| \quad \forall t \in (\tau, T),$$

and then, by (38), we obtain

$$u_\nu(t) \rightarrow u(t) \text{ in } L^2(\Omega) \quad \forall t \in (\tau, T).$$

Then from a standard contradiction argument combined with the fact that $T > \tau$ has been taken arbitrarily, we deduce that (44) holds. \square

4.3. Existence of the global pullback attractor. Now, we are ready to obtain the main result of this paper, that is, the existence of the global pullback attractor.

Lemma 17. *Under the assumptions in Lemma 12, the MNDS U defined by (16) is upper semicontinuous.*

Proof. If U is not upper semicontinuous, then there exist $\tau \leq t$, a point $u_\tau \in L^2(\Omega)$, a neighborhood \mathcal{N} of $U(t, \tau, u_\tau)$ and a sequence $y_n \in U(t, \tau, u_\tau^n)$ with $u_\tau^n \rightarrow u_\tau$ in $L^2(\Omega)$, such that $y_n \notin \mathcal{N}$ for all n . Proposition 16 implies that there exists a subsequence $\{y_\mu\} \subset \{y_n\}$ and $y \in U(t, \tau, u_\tau)$ such that $y_\mu \rightarrow y$ in $L^2(\Omega)$, which is a contradiction. \square

Lemma 18. *Under the assumptions in Lemma 12, the MNDS U defined by (16) is asymptotically compact with respect to the family \widehat{B}_{λ_1} defined in that lemma.*

Proof. Let us fix a sequence $\tau_n \rightarrow -\infty$, a sequence $u_{\tau_n} \in B_{\lambda_1}(\tau_n)$ and $t \in \mathbb{R}$. We have to prove that from any sequence $y_n \in U(t, \tau_n, u_{\tau_n})$ we can extract a subsequence that converges in $L^2(\Omega)$. As $y_n \in U(t, \tau_n, u_{\tau_n})$, there exists $u_n \in S(\tau_n, u_{\tau_n})$ such that $u_n(t) = y_n$. As the family \widehat{B}_{λ_1} is pullback \mathcal{D}_{λ_1} -absorbing and $\tau_n \rightarrow -\infty$, there exists $n_0(t) \geq 1$ such that $\tau_n \leq t - 1$ and

$$u_n(t - 1) \in U(t - 1, \tau_n, u_{\tau_n}) \subset U(t - 1, \tau_n, B_{\lambda_1}(\tau_n)) \subset B_{\lambda_1}(t - 1), \quad (64)$$

for all $n \geq n_0(t)$. From (64), we deduce that there exists a subsequence $\{u_\mu\} \subset \{u_n\}$ and $\zeta_0 \in B_{\lambda_1}(t - 1)$, such that

$$u_\mu(t - 1) \rightarrow \zeta_0 \text{ in } L^2(\Omega). \quad (65)$$

As $u_\mu \in S(t-1, u_\mu(t-1))$, by Proposition 16 we have that there exists a subsequence $\{u_{n'}\} \subset \{u_\mu\}$, such that there exists $u \in S(t-1, \zeta_0)$ satisfying in particular

$$y_{n'} = u_{n'}(t) \rightharpoonup u(t) \text{ in } L^2(\Omega), \quad (66)$$

and

$$u_{n'}|_{\Omega_{2m}} \rightarrow u|_{\Omega_{2m}} \text{ in } L^2(t-1, t, L^2(\Omega_{2m})), \quad \forall m \geq 1. \quad (67)$$

By Lemma 13, for any $\varepsilon > 0$ there exists $T = T(t-1, t, \varepsilon, \widehat{B}_{\lambda_1}) \leq t-1$, and $M = M(t-1, t, \varepsilon, \widehat{B}_{\lambda_1}) \geq 1$, such that

$$\begin{aligned} & \int_{t-1}^t \int_{|x|_{\mathbb{R}^N} \geq 2m} (u_{n'}(x, s) - u(x, s))^2 dx ds \\ & \leq \int_{t-1}^t \int_{|x|_{\mathbb{R}^N} \geq 2m} u_{n'}^2(x, s) dx ds \\ & + \int_{t-1}^t \int_{|x|_{\mathbb{R}^N} \geq 2m} u^2(x, s) dx ds \leq 2\varepsilon, \end{aligned} \quad (68)$$

for all $m \geq M$ and any n' such that $\tau_{n'} \leq T$. From (67) and (68) we have

$$u_{n'} \longrightarrow u \text{ in } L^2(t-1, t; L^2(\Omega)).$$

Now, if we argue similarly to Proposition 16 we obtain

$$y_{n'} = u_{n'}(t) \rightarrow u(t) \text{ in } L^2(\Omega).$$

□

Now, as a direct consequence of the preceding results, we have the existence of the pullback attractor for the MNDS U defined by (16).

Theorem 19. *Under the assumptions in Lemma 12, the MNDS U defined by (16) has a unique pullback \mathcal{D}_{λ_1} -attractor \widehat{A} belonging to \mathcal{D}_{λ_1} , which is given by*

$$A(t) := \Lambda(\widehat{B}_{\lambda_1}, t) = \overline{\bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau, B_{\lambda_1}(\tau))}, \quad (69)$$

where \widehat{B}_{λ_1} was defined in Lemma 12, and the closure is taken in $L^2(\Omega)$. Moreover, \widehat{A} is strictly invariant.

Remark 20. *Let us denote by \mathcal{B} the family of all nonempty bounded subsets of $L^2(\Omega)$. For each $t \in \mathbb{R}$, let us define*

$$A(t) := \overline{\bigcup_{B \in \mathcal{B}} \Lambda(B, t)}, \quad (70)$$

where

$$\Lambda(B, t) := \overline{\bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau, B)},$$

and the closures are taken in $L^2(\Omega)$.

According to the results in [29] and [30], under the assumptions in Theorem 19, the family $\widehat{A} = \{A(t) : t \in \mathbb{R}\}$ belongs to \mathcal{D}_{λ_1} and is a pullback attractor of \mathcal{B} in the sense of [9] (see [15] for the single-valued case), and more exactly satisfies:

- a) $A(t)$ is compact for any $t \in \mathbb{R}$,

b) $\widehat{\mathcal{A}}$ is pullback \mathcal{B} -attracting, i.e.

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{L^2(\Omega)}(U(t, \tau, B), A(t)) = 0 \quad \forall t \in \mathbb{R},$$

for all $B \in \mathcal{B}$,

c) $\widehat{\mathcal{A}}$ is negatively invariant, i.e.,

$$\mathcal{A}(t) \subset U(t, \tau, \mathcal{A}(\tau)), \text{ for any } t \geq \tau,$$

d) $\mathcal{A}(t) \subset A(t)$ for any $t \in \mathbb{R}$, where $A(t)$ is defined by (69).

e) If moreover

$$\sup_{t \leq 0} e^{-\lambda_1 t} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 ds < \infty,$$

then in fact

$$\mathcal{A}(t) = A(t) \quad \text{for any } t \in \mathbb{R}.$$

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