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ON THE STRUCTURE OF THE GLOBAL ATTRACTOR FOR INFINITE-DIMENSIONAL NON-AUTONOMOUS DYNAMICAL SYSTEMS WITH WEAK CONVERGENCE

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Dedicated to Prof. Dr. José Real on occasion of his 60th birthday

Abstract. The aim of this paper is to describe the structure of global attractors for infinite-dimensional non-autonomous dynamical systems with recurrent coefficients. We consider a special class of this type of systems (the so–called weak convergent systems). We study this problem in the framework of general non-autonomous dynamical systems (cocycles). In particular, we apply the general results obtained in our previous paper [6] to study the almost periodic (almost automorphic, recurrent, pseudo recurrent) and asymptotically almost periodic (asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent) solutions of different classes of differential equations (functional-differential equations, evolution equation with monotone operator, semi-linear parabolic equations).

1. **Introduction.** The objective of this paper is to analyze the well-known Seifert's problem for several types of infinite-dimensional non-autonomous dynamical systems with weak convergence. To be more precise, consider a differential equation

$$x' = f(t, x), \tag{1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Assume that the right-hand side of (1) satisfies hypotheses ensuring existence, uniqueness and extendability of solutions of (1), i.e., for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $x(t; t_0, x_0)$ of equation (1) with initial data t_0, x_0, and defined for all $t \geq t_0$.

Then, we can establish the following interesting problem.

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Seifert's Problem (see [16] for more details): Suppose that equation (1) is dissipative and the function f is almost periodic (with respect to time). Does equation (1) possess an almost periodic solution?

Fink and Fredericson [16] and Zhikov [29] established that, in general, even when equation (1) is scalar, the answer to Seifert's question is negative.

Denote by $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ the space of all continuous functions $f:\mathbb{R}\times\mathbb{R}^n\mapsto\mathbb{R}^n$ equipped with the compact-open topology. Equation (1) (respectively, the function f) is called regular if, for all $x_0\in\mathbb{R}^n$ and $g\in H(f):=\overline{\{f_\tau:\ \tau\in\mathbb{R}\}}$ (where the bar denotes the closure in the space $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $f_\tau(t,x):=f(t+\tau,x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^n$), the equation

$$x'=g(t,x)$$

has a unique solution $\varphi(t,x,g)$ passing through the point x at the initial moment $t=0$, and defined on $\mathbb{R}_+:=\{t\in\mathbb{R}|\ t\ge 0\}$.

In our previous paper [6], we included several comments concerning some aspects related to this problem, and some relevant references dealing with it. In addition, we showed that if equation (1) is weak convergent (i.e., there exists a positive number L such that $\lim\limits_{t\to+\infty}|\varphi(t,x_1,g)-\varphi(t,x_2,g)|=0$ for all $|x_i|\le L$ $(i=1,2)$ and $g\in H(f)$), and f is pseudo recurrent with respect to the time variable (in particular, f is recurrent, almost automorphic, Bohr almost periodic or quasi periodic), then, equation (1) admits a unique pseudo recurrent (respectively, recurrent, almost automorphic, Bohr almost periodic, quasi periodic) solution. If this solution is Lyapunov stable, then the Levinson center (the compact global attractor) is a minimal almost periodic set. If it is not Lyapunov stable, then the Levinson center contains a minimal almost periodic set, but it is not minimal (this means, in particular, that equation (1) admits a family (more than one) of solutions which are bounded on $\mathbb{R}$). In [7] we generalize this result to the case of difference equations.

In this paper we will carry out a similar analysis to prove analogous results for the following three classes of differential equations:

- Functional differential equations (FDEs)
$$x'=f(t,x_t) \tag{2}$$
with finite delay.
- Evolution equations $x'+Ax=f(t)$ with monotone (generally speaking nonlinear) operator A.
- Semi-linear parabolic equations $x'+Ax=F(t,x)$ with linear (unbounded) operator A.

We present our results in the framework of general non-autonomous dynamical systems (cocycles) and we apply our abstract theory mainly developed in [6] to the three classes of differential equations mentioned previously.

In order not to be repetitive with our previous papers on this topic, especially [6, 7], we will skip to recall preliminary definitions and results which are necessary for our analysis and refer the reader to these already published papers.

The paper is organized as follows.

Section 2 is devoted to the study of asymptotic behavior of non-autonomous FDEs with finite delay. In particular, we give a description of the structure of the compact global attractor for weak convergent FDEs (Theorem 2.5). We study the almost periodic and asymptotically almost periodic solutions (Subsection 2.1), uniformly compatible (by the character of recurrence with the right–hand side) solutions of strict dissipative equations (Subsection 2.2), convergence and weak convergence for functional-differential equations (FDEs) with finite delay, and also the problem of existence of almost periodic solutions of uniformly dissipative FDEs are studied (Subsection 2.3).

In Sections 3 and 4 we present some results about convergence and/or weak convergence of two classes of infinite-dimensional differential equations with unbounded operators: evolution equations $x' + Ax = f(t)$ with monotone operator (generally speaking non-linear) A, and semi–linear equation $x' + Ax = F(t, x)$ with linear (unbounded) part A, respectively.

2. **Functional differential equations (FDEs) with finite delay.** Let us first recall some notions and notations concerning functional differential equations (see [17] for more details). Let $r > 0$, $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a, b] = [-r, 0]$, then we set $C_r := C([-r, 0], \mathbb{R}^n)$. Let $\tau \in \mathbb{R}$, $A \geq 0$ and $u \in C([\tau - r, \tau + A], \mathbb{R}^n)$. We will define $u_t \in C_r$ for all $t \in [\tau, \tau + A]$ by the equality $u_t(\theta) := u(t + \theta)$, $-r \leq \theta \leq 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{3}$$

where $f : \mathbb{R} \times C_r \to \mathbb{R}^n$ is continuous.

Let us set $H(f) := \overline{\{f_s : s \in \mathbb{R}\}}$, where $f_s(t, \cdot) = f(t + s, \cdot)$ and by bar we denote the closure in the compact-open topology on $C(\mathbb{R} \times C_r, \mathbb{R}^n)$.

Along with equation (3) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{4}$$

where $g \in H(f)$.

Below, in this section, we suppose that equation (3) is regular.

Remark 2.1. 1. Denote by $\tilde{\varphi}(t, u, f)$ the solution of equation (3) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $u \in C_r$, i.e., $\tilde{\varphi}(s, u, f) = u(s)$ for all $s \in [-r, 0]$. By $\varphi(t, u, f)$ we will denote below the trajectory of equation (3), corresponding to the solution $\tilde{\varphi}(t, u, f)$, i.e., the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into C_r, defined by $\varphi(t, u, f)(s) := \tilde{\varphi}(t + s, u, f)$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r, 0]$.

2. Taking into account item 1. in this remark, we will use below the notions of “solution” and “trajectory” for equation (3) as synonymous concepts.

2.1. **Weak convergent FDEs with finite delay.** Consider a differential equation

$$u' = f(\sigma(t, y), u_t) \quad (y \in Y), \tag{5}$$

where $f \in C(Y \times C_r, \mathbb{R}^n)$, and $(Y, \mathbb{R}, \sigma)$ is a dynamical system.

It is well known [3, 27] that the mapping $\varphi : \mathbb{R}_+ \times C_r \times Y \mapsto \mathbb{R}^n$ possesses the following properties:

(i) $\varphi(0, u, y) = u$ for all $u \in C_r$ and $y \in Y$;
(ii) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{R}_+$, $u \in C_r$ and $y \in Y$;
(iii) the mapping φ is continuous.

Thus, the triplet $\langle C_r, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ is a cocycle (non-autonomous dynamical system) which is associated to (generated by) equation (5). In this case the dynamical system $(Y, \mathbb{R}, \sigma)$ is called *base dynamical system* (or driving system). Denote by $X := C_r \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generated by the cocycle φ, i.e., $\pi(t,(u,g)) = (\varphi(t,u,g),\sigma(t,g))$ for all $t \in \mathbb{R}_+$ and $(u,g) \in C_r \times H(f) = X$.

Example 2.2. We consider equation (3). Along with equation (3) consider the family of equations (4), where $g \in H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ and f_τ is the τ-shift of f with respect to time, i.e., $f_\tau(t,u) := f(t+\tau,u)$ for all $(t,u) \in \mathbb{R} \times C_r$. Suppose that the function f is regular [27], i.e. for all $g \in H(f)$ and $u \in \mathbb{R}^n$ there exists a unique solution $\varphi(t,u,g)$ of equation (4). Denote by $Y = H(f)$ and $(Y,\mathbb{R},\sigma)$ a shift dynamical system on Y induced by the Bebutov dynamical system $(C(\mathbb{R} \times C_r,\mathbb{R}^n),\mathbb{R},\sigma)$. Now the family of equations (4) can be written as

$$u' = F(\sigma(t,y),u_t) \quad (y \in Y)$$

if we define $F \in C(Y \times C_r,\mathbb{R}^n)$ by the equality $F(g,u) := g(0,u)$ for all $g \in H(f)$ and $u \in C_r$.

Below we suppose that equation (5) is regular. Equation (5) is called *dissipative* (see [8]), if there exists a positive number r such that

$$\limsup_{t\to+\infty} ||\varphi(t,u,y)|| < r \tag{6}$$

for all $u \in C_r$ and $y \in Y$, where $||\cdot||$ is the norm in C_r.

In this section we give a simple geometric condition which guarantees existence of a unique almost periodic solution and this solution, generally speaking, is not the unique solution of equation (5) which is bounded on $\mathbb{R}$.

A function $f \in C(Y \times C_r,\mathbb{R}^n)$ is said to be *completely continuous* if for any bounded subset $A \subset C_r$ the set $f(Y \times A) \subset \mathbb{R}^n$ is relatively compact.

Lemma 2.3. *Let $H(f)$ be compact. The following statements hold:*

(i) *for any point $x \in X := C_r \times H(f)$ there exist a neighborhood U_x of the point x and a positive number $l_x > 0$ such that $\pi(l_x,U_x)$ is relatively compact, i.e., the dynamical system $(X,\mathbb{R}_+,\pi)$ is locally compact;*

(ii) *if the function f is completely continuous, then for any bounded and positively invariant subset $A \subset X$ there exists a positive number $t_0 = t_0(A)$ such that $\pi(t_0,A)$ is a relatively compact subset of X.*

Proof. This assertion follows from Lemma 6.1 and Corollary 6.3 in [17, Ch. III] and from the compactness of $H(f)$. ☐

Corollary 2.4. *Under the conditions of Lemma 2.3 the dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotically compact.*

We can now state the main results in this section.

Theorem 2.5. *Suppose that the following conditions are fulfilled:*

(i) *the function f is completely continuous;*

(ii) *equation (5) is regular and dissipative;*

(iii) *the space Y is compact, and the dynamical system $(Y,\mathbb{R},\sigma)$ is minimal;*

(iv) *for all $y \in Y$*

$$\lim_{t\to+\infty} ||\varphi(t,u_1,y) - \varphi(t,u_2,y)|| = 0, \tag{7}$$

where $\varphi(t,u_i,y)$ $(i = 1,2)$ is a solution of equation (5) which is bounded on $\mathbb{R}$.

Then,

(i) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then equation (5) admits a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution $\varphi(t, u_y, y)$ ($u_y \in C_r$);*

(ii) *every solution $\varphi(t, u, y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent)*

Proof. Let $\langle C_r, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be the cocycle associated to equation (5). Denote by $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system, where $X := C_r \times Y$ and $\pi := (\varphi, \sigma)$. Consider the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by the cocycle φ (respectively, by equation (5)), where $h := pr_2 : X \mapsto Y$. Since Y is compact, it is evident that the dynamical system $(Y, \mathbb{R}, \sigma)$ is compact dissipative and its Levinson center J_Y coincides with Y. Now we will show that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is point dissipative. Indeed. Let $x := (u, y) \in C_r \times Y = X$ be an arbitrary point. Notice that the set $\sum_x^+ := \bigcup\{\pi(t, x)|\ t \in \mathbb{R}_+\}$ is relatively compact. To this end, it is sufficient to show that the set $A := pr_1(\sum_x^+) = \bigcup\{\varphi(t, u, y)|\ t \in \mathbb{R}_+\}$ is relatively compact in the phase space C_r. But the last statement follows from the completely continuity of f, the boundedness of $\varphi(t, u, y)$ on $\mathbb{R}_+$, and the Arzelá-Ascoli Theorem. Thus, the ω–limit set ω_x of the point x is a nonempty, compact and invariant set of $(X, \mathbb{R}_+, \pi)$. Denote by $\Omega_X := \overline{\bigcup\{\omega_x|\ x \in X\}}$. It is easy to see from our assumptions that Ω_X is a compact set. Indeed, it is sufficient to note that the set $pr_1(\omega_x) = \{v \in C_r|\ (v, y) \in \omega_x\}$ is a bounded set because, according to the dissipativity of equation (5), we have

$$||v|| \le r \tag{8}$$

for all $v \in pr_1(\omega_x)$ and $x \in X$, where r is the positive number appearing in (6). Taking into account (8), the invariance of the set Ω_X, and the complete continuity of f, we conclude that the set $\mathcal{A} = pr_1(\Omega_X)$ is relatively compact in C_r and, consequently, the set Ω_X is relatively compact in X. Thus, the dynamical system is point dissipative. Since, thanks to Lemma 2.3, $(X, \mathbb{R}_+, \pi)$ is locally dissipative, then by Theorem 1.10 in [8, Ch. 1], it is compactly dissipative. Denote by J_X its Levinson center and $I_y := pr_1(J_X \bigcap X_y)$ for all $y \in Y$, where $X_y := \{x \in X :\ h(x) = y\}$. According to the definition of the set $I_y \subseteq C_r$ and Theorem 2.24 in [8, Ch. 2, p. 95], $u \in I_y$ if and only if the solution $\varphi(t, u, y)$ is defined on $\mathbb{R}$ and relatively compact (i.e., the set $\overline{\varphi(\mathbb{R}, u, y)} \subseteq C_r$ is compact). Thus $I_y = \{u \in C_r :$ such that $(u, y) \in J_X\}$. It is easy to see that condition (7) means that the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is weak convergent. Now, to finish the proof of the theorem, it is sufficient to apply Theorem 3.5 in [6], Theorem 2.2.2 [10, Ch.II,p.21], Lemma 6.5.19 and Corollary 6.5.20 from [11, Ch.VI,p.195] to the non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (5). □

Remark 2.6. 1. Taking into account Remark 2.1 (item 1) it is easy to see that condition (7) is equivalent to

$$\lim_{t \to +\infty} |\tilde{\varphi}(t, u_1, y) - \tilde{\varphi}(t, u_2, y)| = 0. \tag{9}$$

2. Under the assumptions in Theorem 2.5, there exists a unique almost periodic solution of equation (5), but equation (5) may have more than one solution defined on $\mathbb{R}$ and relatively compact.

2.2. **Convergent FDEs with finite delay.** Let $\varphi(\cdot,\phi,g)$ denote the solution of (4) passing through the point $\phi \in C_r$ for $t=0$ defined for all $t \geq 0$.

Let $Y := H(f)$ and denote by $(Y,\mathbb{R},\sigma)$ the dynamical system of translations on $H(f)$. Let $X := C_r \times Y$, $(X,\mathbb{R}_+,\pi)$ be the dynamical system on X defined in the following way: $\pi((\phi,g),\tau) := (\varphi(\tau,\phi,g),g_\tau)$. We prove now a very important property of the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\sigma),\ (Y,\mathbb{R},\sigma),h\rangle$, where $h = pr_2\ : X \mapsto Y$. Namely, we can establish the following result.

Theorem 2.7. *Suppose that the following conditions are fulfilled:*
- (i) *equation (3) is regular;*
- (ii) *for every bounded subset $A \subset C_r$, the set $f(\mathbb{R}\times A)$ is bounded in $\mathbb{R}^n$;*
- (iii) *the function f is pseudo recurrent, i.e., the shift dynamical system $(H(f)$, $\mathbb{R}$, $\sigma))$ is pseudo recurrent;*
- (iv) *equation (3) is strictly dissipative, i.e.,*
$$\langle g(t,\phi_1) - g(t,\phi_2), \phi_1(0) - \phi_2(0)\rangle < 0 \tag{10}$$
for all $g \in H(f)$ and $\phi_i \in C_r$ $(i=1,2)$ with $\phi_1(0) \neq \phi_2(0)$;
- (v) *equation (3) admits a solution $\varphi(t,u_0,f)$ which is bounded on $\mathbb{R}_+$.*

Then,
- (i) *equation (3) is convergent, i.e., the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\sigma),\ (Y,\mathbb{R},\sigma),h\rangle$ generated by equation (3) is convergent;*
- (ii) *if the function f is τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent), then the equation (3) admits a unique τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent) solution.*

Proof. Let $\tilde{\varphi}(t,u_i,g)$ $(i=1,2)$ be two solutions of equation (4) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) and denote by $\tilde{\alpha}(t) := |\tilde{\varphi}(t,u_1,g) - \tilde{\varphi}(t,u_2,g)|^2$ for all $t \in \mathbb{R}_+$ (respectively, on $\mathbb{R}$), then by (10) we have
$$\frac{d\tilde{\alpha}(t)}{dt} = 2\langle g(t,\varphi(t,u_1,g)) - g(t,\varphi(t,u_2,g),\tilde{\varphi}(t,u_1,g) - \tilde{\varphi}(t,u_2,g)\rangle \leq 0 \tag{11}$$
for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and consequently we obtain
$$\tilde{\alpha}(t_2) \leq \tilde{\alpha}(t_1) \tag{12}$$
for all $t_1,t_2 \in \mathbb{R}_+$ (respectively, $t_1,t_2 \in \mathbb{R}$) with $t_2 \geq t_1$. From (12) it follows that
$$|\tilde{\varphi}(t,u_1,g) - \tilde{\varphi}(t,u_2,g)| \leq |u_1(0) - u_2(0)| \tag{13}$$
for all $u_1,u_2 \in C_r$, $g \in H(f)$ and $t \geq 0$.

Notice that, under our assumptions, every equation (4) admits at least one solution which is defined and bounded on $\mathbb{R}$. Indeed. Since f is pseudo recurrent then, in particular, f is Poisson stable and, consequently, $\omega_f = H(f)$, where ω_f is ω–limit set of the function f in the Bebutov dynamical system $(C(\mathbb{R}\times C_r,\mathbb{R}^n),\mathbb{R},\sigma)$. Thus for every $g \in H(f)$ there exists a sequence $t_n \to +\infty$ such that $f_{t_n} \to g$ as $n \to +\infty$. Since the solution $\varphi(t,u_0,f)$ is bounded on $\mathbb{R}_+$, without loss of generality, we can assume that the sequence $\{\varphi(\tau_n,u_0,g)\}$ is convergent and denote by v its limit. Then, we have $\varphi(t+\tau_n,u_0,f) = \varphi(t,\varphi(\tau_n,u_0,f),f_{\tau_n}) \to \varphi(t,v,g)$. It is clear that the solution $\varphi(t,v,g)$ of equation (4) is defined and bounded on $\mathbb{R}$. From this fact and inequality (13) it follows that every solution of every equation (4) is bounded on $\mathbb{R}_+$. From Corollary 2.4 it follows that every positively semi-trajectory of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ is relatively compact.

Consider the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (3). We define the function $V : X \dot{\times} X \mapsto \mathbb{R}_+$ as follows:

$$V((u_1, g), (u_2, g)) := ||u_1 - u_2||. \tag{14}$$

Note that under the conditions of the theorem, and by the facts established above, the following conditions are fulfilled:

(i) by Corollary 2.4, the dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotically compact;

(ii) by (13), the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ associated to equation (3) is $V-$monotone, where $V : X \dot{\times} X \mapsto \mathbb{R}_+$ is defined by (14);

(iii) if $\tilde{\varphi}(t, u_i, g)$ $(i = 1, 2)$ are two solutions of equation (4) which are bounded on $\mathbb{R}$, then, by Theorem 4.10 in [9, p. 677], the trajectories $\varphi(t, u_i, g)$ (respectively, the solutions $\tilde{\varphi}(t, u_i, g)$) $(i = 1, 2)$ are jointly Poisson stable. Since the function $\alpha(t) := ||\varphi(t, u_1, g) - \varphi(t, u_2, g)||$ (for all $t \in \mathbb{R}$) (respectively, the function $\tilde{\alpha}$) is Poisson stable and monotone, then, it is a constant, i.e.,

$$|\tilde{\varphi}(t, u_1, g) - \varphi(t, u_2, g)| = |u_1(0) - u_2(0)| \quad (\forall\ t \in \mathbb{R})$$

(respectively,

$$||\varphi(t, u_1, g) - \varphi(t, u_2, g)|| = ||u_1 - u_2|| \quad (\forall\ t \in \mathbb{R})\ ); \tag{15}$$

(iv) the positive semi-trajectory $\sum_{x_0}^{+}$, where $x_0 := (u_0, f) \in X_f = \{(u, f) :\ u \in C_r\}$, is relatively compact in X;

(v) the dynamical system $(Y, \mathbb{R}, \sigma)$ if pseudo recurrent.

If $u_1 \neq u_2$, then it follows from (15) that $u_1(0) \neq u_2(0)$.

Now we will establish that, for $u_1, u_2 \in C_r$, $(u_1(0) \neq u_2(0)$ and $(u_i, g) \in L_X$ $(i = 1, 2))$

$$||\varphi(t, u_1, g) - \varphi(t, u_2, g)|| < ||u_1 - u_2||,$$

for all $t > 0$, where $|| \cdot ||$ is the norm on the space C_r. Indeed, consider the function $\tilde{\alpha} : \mathbb{R} \mapsto \mathbb{R}_+$ defined above. Since $u_1 \neq u_2$, then from (15) it follows that $u_1(0) \neq u_2(0)$ and $\tilde{\varphi}(t, u_1, g) \neq \tilde{\varphi}(t, u_1, g)$ for all $t \in \mathbb{R}$. Then, from (10) and (11) it follows that

$$\frac{d\tilde{\alpha}(t)}{dt} = 2\langle g(t, \varphi(t, u_1, g)) - g(t, \varphi(t, u_2, g), \tilde{\varphi}(t, u_1, g) - \tilde{\varphi}(t, u_2, g)\rangle < 0$$

for all $t \in \mathbb{R}$ and, consequently the function $\tilde{\alpha}$ is strictly monotone decreasing on $\mathbb{R}$. Note that

$$||\varphi(t, u_1, g) - \varphi(t, u_2, g)|| = \max_{-r \leq s \leq 0} |\tilde{\varphi}(t + s, u_1, g) - \tilde{\varphi}(t + s, u_2, g)| =$$

$$|\tilde{\varphi}(t + s_t, u_1, g) - \tilde{\varphi}(t + s_t, u_2, g)| < |\tilde{\varphi}(s_t, u_1, g) - \tilde{\varphi}(s_t, u_2, g)| \leq ||u_1 - u_2||,$$

for all $t > 0$, $g \in H(f)$ and $u_1, u_2 \in C_r$ $(u_1 \neq u_2)$, where s_t is some number (depending on t) in the segment $[-r, 0]$.

Now, to finish the proof, it is sufficient to apply Corollary 3.12 in [6]. □

Remark 2.8. Theorem 2.7 remains true if we replace the standard scalar product $\langle \cdot, \cdot \rangle$ on the space $\mathbb{R}^n$ by an arbitrary scalar product $\langle u, u\rangle_W := \langle Wu, u\rangle$, where $W = (w_{ij})_{i,j=1}^n$ $(w_{ij} \in \mathbb{R})$ is a symmetric and positive defined $n \times n$–matrix.

2.3. **Uniform dissipative FDEs with finite delay.** Below we will show that if we replace assumption (10) by a stronger condition, then Theorem 2.7 is true without the requirement that there exists at least one solution which is bounded on $\mathbb{R}_+$. Namely, we will establish the following theorem.

Denote by $C(Y, C_r)$ the Banach space of all continuous mappings $\gamma : Y \to C_r$ endowed with the norm $||\gamma|| := \max_{y \in Y} ||\gamma(y)||_{C_r}$.

Theorem 2.9. *Suppose that the following conditions are fulfilled:*

(i) *equation (3) is regular;*
(ii) *for every bounded subset $A \subset C_r$ the set $f(\mathbb{R} \times A)$ is bounded in $\mathbb{R}^n$;*
(iii) *the function f is pseudo recurrent, i.e., the shift dynamical system $(H(f)$, $\mathbb{R}$, $\sigma))$ is pseudo recurrent;*
(iv) *equation (3) is uniformly strictly dissipative, i.e., there exists a number β such that*

$$\langle g(t, \phi_1) - g(t, \phi_2), \phi_1(0) - \phi_2(0)\rangle \leq -\beta|\phi_1(0) - \phi_2(0)|^2 \tag{16}$$

for all $t \in \mathbb{R}_+$, $g \in H(f)$ and $\phi_i \in C_r$ $(i = 1, 2)$ with $\phi_1(0) \neq \phi_2(0)$.

Then, the following statements hold:

(i) *there exists a unique mapping $\gamma \in C(Y, C_r)$ such that $\gamma(\sigma(t, g)) = \varphi(t, \gamma(g), g)$ for all $g \in H(f)$ and $t \in \mathbb{R}_+$;*
(ii) *the equality*

$$\lim_{t \to +\infty} ||\varphi(t, u, g) - \varphi(t, \gamma(g), g)|| = 0$$

holds for all $g \in H(f)$ and $v \in C_r$.

Proof. According to (16) we have

$$\begin{aligned}\frac{d\tilde{\alpha(t)}}{dt} &= 2\langle g(t, \varphi(t, u_1, g)) - g(t, \varphi(t, u_2, g), \tilde{\varphi}(t, u_1, g) - \tilde{\varphi}(t, u_2, g)\rangle \\ &\leq -2\beta|\tilde{\varphi}_1(t) - \tilde{\varphi}_2(t)|^2\end{aligned}$$

for all $t \in \mathbb{R}_+$ and, consequently,

$$\tilde{\alpha(t)} \leq |u_1(0) - u_2(0)|^2 \exp(-\beta t) \tag{17}$$

for all $t \in \mathbb{R}_+$. From (17) we obtain

$$\begin{aligned}||\varphi(t, u_1, g) - \varphi(t, u_2, g)|| &= \max_{-r \leq s \leq 0} |\tilde{\varphi}(t + s, u_1, g) - \tilde{\varphi}(t + s, u_2, g)| \\ &= |\tilde{\varphi}(t + s_t, u_1, g) - \tilde{\varphi}(t + s_t, u_2, g)| \\ &\leq |\tilde{\varphi}(s_t, u_1, g) - \tilde{\varphi}(s_t, u_2, g)| \exp(-\beta t) \\ &\leq ||u_1 - u_2|| \exp(-\beta t),\end{aligned} \tag{18}$$

for all $t \geq 0$, $g \in H(f)$ and $u_1, u_2 \in C_r$, where s_t is some number (depending on t) in the segment $[-r, 0]$.

Consider the cocycle $\langle C_r, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ generated by equation (3), where $Y = H(f)$ and $\varphi(t, v, g)$ is a unique solution of equation (4) passing through $v \in C_r$ at the initial moment $t = 0$. For all $t \in \mathbb{R}_+$ we define a mapping $S^t : C(Y, C_r) \mapsto C(Y, C_r)$ by the equality

$$(S^t\eta)(g) := \varphi(t, \eta(g), g_{-t}) \tag{19}$$

for all $\eta \in C(Y, C_r)$, $g \in H(f) = Y$ and $t \in \mathbb{R}_+$. It is clear that, under the conditions of our theorem and thanks to (19), we can define correctly a continuous mapping S^t $(t \in \mathbb{R}_+)$ from $C(Y, C_r)$ into itself and the equality

$$S^t \circ S^\tau = S^{t+\tau} \tag{20}$$

holds for all $t, \tau \in \mathbb{R}_+$, where $\circ$ is the composition of mappings S^t and S^τ. Equality (20) means that the family of nonlinear operators $\{S^t\}_{t\in\mathbb{R}_+}$ forms a commutative semigroup. Let now $\gamma_i \in C(Y, C_r)$ $(i = 1, 2)$. Then, according to inequality (18), we have

$$\begin{aligned} ||S^t\gamma_1 - S^t\gamma_2|| &= \max_{g\in H(f)} ||\varphi(t, \gamma_1(g), g_{-t}) - \varphi(t, \gamma_2(g), g_{-t})|| \\ &\le \exp(-\beta t) \max_{g\in H(f)} ||\gamma_1(g) - \gamma_2(g)||_{C_r} \\ &= \exp(-\beta t)||\gamma_1 - \gamma_2|| \end{aligned} \tag{21}$$

for all $t \in \mathbb{R}_+$ and $\gamma_1, \gamma_2 \in C(Y, C_r)$. From (21) it follows that $Lip(S^t) \le \exp(-\beta t)$ ($Lip(F)$ is the Lipschitz constant of F) and, consequently, for $t > 0$ the mapping S^t is a contraction. Since the semigroup $\{S^t\}_{t\in\mathbb{R}_+}$ is commutative, then it admits a unique fixe point γ, i.e., $\gamma(\sigma(t, g)) = \varphi(t, \gamma(g), g)$ for all $g \in H(f)$ and $t \in \mathbb{R}_+$. Thus the first statement of our theorem is proved.

The second statement follows from the inequality (18). In fact, we have

$$||\varphi(t, u, g) - \varphi(t, \gamma(g), g)|| \le ||u - \gamma(g)|| \exp(-\beta t) \tag{22}$$

for all $g \in H(f)$, $t \in \mathbb{R}_+$ and $u \in C_r$. Passing to the limit in (22) we obtain the necessary statement. The result is completely proved. □

Corollary 2.10. *Under the conditions of Theorem 2.9 the following statements hold:*

(i) *equation (3) is convergent;*
(ii) *if the function f is τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent), the equation (3) admits a unique τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent) solution and every solution of equation (3) is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent).*

Proof. This statement follows from Theorem 2.9. □

Remark 2.11. 1. Actually Theorem 2.9 establishes the convergence of equation (3).

2. Theorem 2.9 remains true if we replace (16) by a more general condition: there are numbers $\beta > 0$ and $\delta \ge 0$ such that

$$\langle g(t, \phi_1) - g(t, \phi_2), \phi_1(0) - \phi_2(0)\rangle \le -\beta|\phi_1(0) - \phi_2(0)|^{2+2\delta}$$

for all $g \in H(f)$, $t \in \mathbb{R}_+$ and $\phi_1, \phi_2 \in C_r$. More information about different generalizations of this type can be found in the work [12]. Below we will prove this fact which is not based on the ideas used in the proof of Theorem 2.9.

Theorem 2.12. *Suppose that the following conditions are fulfilled:*

(i) *equation (3) is regular;*
(ii) *for every bounded subset $A \subset C_r$ the set $f(\mathbb{R} \times A)$ is bounded in $\mathbb{R}^n$;*

(iii) *the function f is pseudo recurrent, i.e., the shift dynamical system* $(H(f), \mathbb{R}, \sigma))$ *is pseudo recurrent;*

(iv) *equation (3) is uniformly strictly dissipative, i.e., there exist numbers* $\beta > 0$ *and* $\delta \geq 0$ *such that*

$$\langle g(t,\phi_1) - g(t,\phi_2), \phi_1(0) - \phi_2(0)\rangle \leq -\beta|\phi_1(0) - \phi_2(0)|^{2+2\delta} \tag{23}$$

for all $t \in \mathbb{R}_+$, $g \in H(f)$ *and* $\phi_i \in C_r$ $(i = 1,2)$ *with* $\phi_1(0) \neq \phi_2(0)$.

Then,

(i) *equation (3) is dissipative;*

(ii) *there exists a unique mapping* $\gamma \in C(Y, C_r)$ *such that* $\gamma(\sigma(t,g)) = \varphi(t, \gamma(g), g)$ *for all* $g \in H(f)$ *and* $t \in \mathbb{R}_+$;

(iii) *the equality*

$$\lim_{t\to+\infty} ||\varphi(t,u,g) - \varphi(t,\gamma(g),g)|| = 0$$

holds for all $g \in H(f)$ *and* $v \in C_r$.

Proof. First, we will show that equation (3) is dissipative. Indeed, denote by $w(t) := |\tilde{\varphi}(t,u,g)|^2$. Then, according to (23), we have

$$\begin{aligned} \frac{dw(t)}{dt} &= 2\langle g(t,\varphi(t,u,g)) - g(t,0), \tilde{\varphi}(t,u,g)\rangle + 2\langle g(t,0), \tilde{\varphi}(t,u,g)\rangle \\ &\leq -2\beta|\tilde{\varphi}(t,u,g)|^{2+2\delta} + 2M|\tilde{\varphi}(t,u,g)| \end{aligned} \tag{24}$$

for all $t \in \mathbb{R}_+$, where $M := \sup\limits_{t\in\mathbb{R}} |f(t,0)| \geq \sup\limits_{t\in\mathbb{R}} |g(t,0)|$ (for all $g \in H(f)$). Consider the scalar differential equation

$$x' = -2\beta x^{1+\beta} + 2Mx^{1/2} \tag{25}$$

on the semi-axis $\mathbb{R}_+$. It is easy to check that this equation possesses two fixed points $x_0 = 0$, $x_1 = (\frac{M}{\beta})^{2/(1+2\beta)}$ and the segment $[x_0, x_1]$ is the global attractor for (25). This means, in particular, that

$$\limsup_{t\to+\infty} \phi(t,x) \leq r_0 \tag{26}$$

for all $x \in \mathbb{R}_+$, where $r_0 := x_1$ and by $\phi(t,x)$ we denote the unique solution of equation (25) with initial condition $\phi(0,x) = x$ $(x \in \mathbb{R}_+)$. Note that from (24) and (25) it follows that

$$|\tilde{\varphi}(t,u,g)| \leq \sqrt{\phi(t,|u(0)|^2)}$$

for all $t \in \mathbb{R}_+$ and, consequently,

$$\limsup_{t\to+\infty} |\tilde{\varphi}(t,u,g)| \leq \big(\frac{M}{\beta}\big)^{1/(1+2\beta)}. \tag{27}$$

From (27) we obtain

$$\limsup_{t\to+\infty} ||\varphi(t,u,g)|| = \limsup_{t\to+\infty} |\tilde{\varphi}(t+s_t,u,g)| \leq \big(\frac{M}{\beta}\big)^{1/(1+2\beta)}, \tag{28}$$

where $s_t \in [-r, 0]$ is some number depending on t. Taking into account (28) and the fact that $\big(\frac{M}{\beta}\big)^{1/(1+2\beta)}$ is an absolute constant, we conclude that (3) is dissipative.

Consider the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (3). Note that, owing to our assumptions and the facts established above, the following conditions are fulfilled:

(i)
$$\langle g(t,\phi_1)-g(t,\phi_2),\phi_1(0)-\phi_2(0)\rangle < 0$$
for all $t\in\mathbb{R}_+$, $g\in H(f)$ and $\phi_i\in C_r$ $(i=1,2)$ with $\phi_1(0)\neq\phi_2(0)$.

(ii) by Lemma 2.3 the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ associated to equation (3) is locally compact;

(iii) by Corollary 2.4 the dynamical system $(X,\mathbb{R}_+,\pi)$ is asymptotically compact;

(iv) every positive semi-trajectory $\sum_x^+$, where $x:=(u,g)\in X_g=\{(u,g):\ u\in C_r\}$, is relatively compact in X;

(v) the dynamical system $(Y,\mathbb{R},\sigma)$ if pseudo recurrent.

Now to finish the proof of our theorem it is sufficient to apply Corollary 3.12 in [6] and Theorem 2.7. □

Remark 2.13. Theorem 2.12 remains true if we replace condition (23) by
$$\langle g(t,\phi_1)-g(t,\phi_2),\phi_1(0)-\phi_2(0)\rangle \le -\zeta(|\phi_1(0)-\phi_2(0)|^2),$$
where $\zeta\in\mathcal{K}$ possessing the following properties:

(i) $x^{-1/2}\zeta(x)\to+\infty$ as $x\to+\infty$;

(ii) the differential equation $x'=-2\zeta(x)+Mx^{1/2}$ defines a semi-flow on $\mathbb{R}_+$ (M is a constant defined in the proof of Theorem 2.12).

This statement can be proved using the same reasoning as that in the proof of Theorem 2.12.

3. **Convergent evolution equations with monotone operators.** Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|=\sqrt{\langle\cdot,\cdot\rangle}$, and E be a reflexive Banach space contained in H algebraically and topologically. Furthermore, let E be dense in H, and here H can be identified with a subspace of the dual E' of E and $\langle\cdot,\cdot\rangle$ can be extended by continuity to $E'\times E$.

Let A be an operator (generally speaking, nonlinear) with the domain of definition $D(A)\subseteq H$.

Recall (see [2, 24]) that the operator A is said to be

- monotone, if
$$\langle Au_1-Au_2,u_1-u_2\rangle\ge 0$$
for all $u_1,u_2\in D(A)$;
- strictly monotone, if
$$\langle Au_1-Au_2,u_1-u_2\rangle> 0$$
for all $u_1,u_2\in D(A)$ $(u_1\neq u_2)$;
- semi-continuous, if for each $u,v\in D(A)$ and $w\in H$ the function $\varphi:\mathbb{R}\to\mathbb{R}$ defined by the equality $\varphi(t):=\langle A(u+tv),w\rangle$ (for all $t\in\mathbb{R}$) is continuous;
- uniformly monotone, if there exist positive numbers α and $p\ge 2$ such that
$$\langle Au_1-Au_2,u_1-u_2\rangle\ge \alpha|u-v|^p$$
for all $u,v\in D(A)$.

Note that the family of monotone operators can be partially ordered by including graphics. A monotone operator is called maximal, if it is maximal among the monotone operators.

Let $(Y,\mathbb{R},\sigma)$ be a dynamical system on the metric space Y. In this subsection we suppose that Y is a compact space. We consider the initial value problem
$$u'(t)+Au(t)=f(\sigma(t,y))\quad (y\in Y) \tag{29}$$

$$u(0) = u, \tag{30}$$

where $A : E \to E'$ is bounded (generally non-linear),

$$|Au|_{E'} \leq C|u|_E^{p-1} + K, u \in E, p > 1,$$

coercive,

$$\langle Au, u\rangle \geq a|u|_E^p, u \in E, a > 0,$$

monotone,

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geq 0, u_1, u_2 \in E,$$

and semi-continuous (see [25]).

A nonlinear "elliptic" operator given by

$$Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i}) \quad \text{in } D \subset \mathbb{R}^n$$

$$u = 0 \text{ on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^n$, $\phi(\cdot)$ is an increasing function satisfying

$$\phi|_{[-1,1]} = 0,\ c|\xi|^p \leq \sum_{i=1}^{n} \xi_i\phi(\xi_i) \leq C|\xi|^p \ (\text{ for all } |\xi| \geq 2\ ),$$

provides an example of such kind of operator with $H = L^2(D), E = W_0^{1,p}(D),\ E' = W^{-1,p'}(D),\ p' = \frac{p}{p-1}$.

The following result is established in [25] (Ch. 2 and Ch. 4). If $x \in H$ and $f \in C(\Omega, E'),\ p' = \frac{p}{p-1}$, then there exists a unique solution $\varphi \in C(\mathbb{R}_+, H)$ of (29) – (30).

Let $(\mathbb{R}, \mathfrak{B}; \mu)$ be a space where μ is a Radon measure and $\mathfrak{B}$ is a Banach space with norm $|\cdot|$.

Let $1 \leq p \leq +\infty$. By $L^p(\mathbb{R}; \mathfrak{B}, \mu)$ we denote the space of all measurable functions (classes of functions) $f : \mathbb{R} \to \mathfrak{B}$ such that $|f| \in L^p(\mathbb{R}; \mathbb{R}; \mu)$, where $|f|(s) = |f(s)|$. The space $L^p(\mathbb{R}; \mathfrak{B}; \mu)$ is endowed with the norm

$$||f||_{L^p} = (\int_{\mathbb{R}} |f(s)|^p d\mu(s))^{1/p} \quad \text{and} \quad ||f||_\infty = \text{ess sup}_{s\in\mathbb{R}}|f(s)|. \tag{31}$$

$L^p(\mathbb{R}; \mathfrak{B}; \mu)$ with norm (31) is a Banach space.

Denote by $L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$ the set of all function $f : \mathbb{R} \to \mathfrak{B}$ such that $f_l \in L^p([-l, l] \cap \mathbb{R}; \mathfrak{B}; \mu)$ for every $l > 0$, where f_l is the restriction of the function f onto $[-l, l] \cap \mathbb{R}$.

In the space $L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$ we define the following family of semi-norms $||\cdot||_{l,p}$:

$$||f||_{l,p} = ||f_l||_{L^p([-l,l]\cap\mathbb{R};\mathfrak{B};\mu)} \quad (l > 0). \tag{32}$$

These semi-norms in (32) define a metrizable topology on $L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$. The metric given by this topology can be defined, for instance, by

$$d_p(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||\varphi - \psi||_{n,p}}{1 + ||\varphi - \psi||_{n,p}}.$$

Let us define a mapping $\sigma : L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu) \times \mathbb{R} \to L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$ as follows: $\sigma(f, \tau) = f_{(\tau)}$ for all $f \in L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$ and $\tau \in \mathbb{R}$, where $f_{(\tau)}(s) := f(s + \tau)$ $(s \in \mathbb{R})$.

Lemma 3.1. [10, Ch. 1] $(L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu), \mathbb{R}, \sigma)$ *is a dynamical system.*

Let $Y := H(f) = \overline{\{f_{(\tau)} | \ \tau \in \mathbb{R}\}}$, where by bar it is denoted the closure in $L^1(\mathbb{R}, H)$. By $(Y, \mathbb{R}, \sigma)$ we denote the dynamical system of shifts on Y induced by the dynamical system $(L^1_{loc}(\mathbb{R}, H), \mathbb{R}, \sigma)$. Put $X := \overline{D(A)} \times Y$ and define $\pi : \mathbb{R}_+ \times \overline{D(A)} \times Y \to \overline{D(A)} \times Y$ by the equality $\pi(t, (v, g)) := (\varphi(t, v, g), g_t)$ and $h := pr_2 : X \to Y$. As it is shown in the work [19], the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a non-autonomous dynamical system.

Applying the general theory developed in [6] to the constructed non-autonomous dynamical systems, we obtain the corresponding statements for equation (29). Let us establish some of them.

Theorem 3.2. *Suppose that the following conditions are fulfilled:*

(i) *equation (29) is compact dissipative, i.e., the cocycle φ (or equivalently, the skew-product dynamical system generated by equation (29)) generated by equation (29) is compact dissipative;*

(ii) *the space Y is compact, and the dynamical system $(Y, \mathbb{R}, \sigma)$ is minimal;*

(iii) *for all $y \in Y$*
$$\lim_{t \to +\infty} |\varphi(t, u_1, y) - \varphi(t, u_2, y)| = 0, \tag{33}$$
where $\varphi(t, u_i, y)$ $(i = 1, 2)$ is solution of equation (29) passing through u_i at the initial moment $t = 0$ which is relatively compact on $\mathbb{R}$.

Then,

(i) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then equation (29) admits a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution $\varphi(t, u_y, y)$ $(u_y \in \overline{D(A)})$;*

(ii) *every solution $\varphi(t, x, y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent)*

Theorem 3.3. *Let $(Y, \mathbb{R}, \sigma)$ be pseudo recurrent, operator A be strictly monotone, and there exists at least one solution $\varphi(t, x_0, y)$ of equation (29) which is relatively compact on $\mathbb{R}_+$.*

Then,

(i) *equation (29) is convergent, i.e., the cocycle φ associated to equation (29) is convergent;*

(ii) *for all $y \in Y$, equation (29) admits a unique solution $\varphi(t, x_y, y)$ which is relatively compact on $\mathbb{R}$ and uniformly compatible, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot, x_y, y)}$;*

(iii) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then*

   (a) *equation (29) has a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution;*

   (b) *every solution $\varphi(t, x, y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent);*

   (c) $\lim_{t \to \infty} |\varphi(t, x, y) - \varphi(t, x_y, y)| = 0$ *for all $x \in \overline{D(A)}$ and $y \in Y$.*

Remark 3.4. *If we suppose that operator A is uniformly monotone, then Theorem 3.3 is also true without the requirement that there exists at least one solution which is relatively compact on $\mathbb{R}_+$. Below we will prove this statement.*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Denote by $\varphi(t_0 + 0) := \lim_{t \to t_0, t > t_0} \varphi(t)$ if the last limit exists.

The mapping φ is called upper semi-continuous from the right at the point $t_0 \in \mathbb{R}_+$, if there exists $\limsup_{t \to t_0, t > t_0} \varphi(t) \le \varphi(t_0)$.

The mapping $f : X \to X$ is called a φ-contraction, if $\rho(f(x_1), f(x_2)) \le \varphi(\rho(x_1, x_2))$ for all $x_1, x_2 \in X$, where φ is some mapping from $\mathbb{R}_+$ to itself.

Then, we recall the following well-known result which will be useful in our proofs.

Theorem 3.5. [1, 4, 22] *Let $f : X \to X$ be a φ-contraction. Suppose that the mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:*

(G1) $\varphi(t) < t$ *for all* $t > 0$;

(G2) φ *s monotonically increasing, i.e.* $t_1 \le t_2$ *implies* $\varphi(t_1) \le \varphi(t_2)$*;*

(G3) φ *is right continuous on* $\mathbb{R}_+$*, i.e.* $\varphi(t_0 + 0) = \varphi(t_0)$ *for all* $t_0 \in \mathbb{R}_+$.

Then f has a unique fixed point x_0 and $\lim_{n \to \infty} f^n(x) = x_0$ for all $x \in X$.

We can now establish the following result.

Theorem 3.6. *Let $(Y, \mathbb{R}, \sigma)$ be pseudo recurrent and operator A be uniformly monotone.*

Then,

(i) *equation (29) is convergent, i.e., the cocycle φ associated to equation (29) is convergent;*

(ii) *for all $y \in Y$, equation (29) admits a unique solution $\varphi(t, x_y, y)$ which is relatively compact on $\mathbb{R}$ and uniformly compatible, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot, x_y, y)}$;*

(iii) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then*

  (a) *equation (29) has a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution;*

  (b) *every solution $\varphi(t, x, y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent);*

  (c) $\lim_{t \to \infty} |\varphi(t, x, y) - \varphi(t, x_y, y)| = 0$ *for all* $x \in \overline{D(A)}$ *and* $y \in Y$.

Proof. Let $u_i \in \overline{D(A)}$ $(i = 1, 2)$ and $\varphi(t, u_i, y)$ be a unique solution of equation (29). By uniform monotony of operator A we have

$$\begin{aligned} &\frac{d|\varphi(t, u_1, y) - \varphi(t, u_2, y)|^2}{dt} \\ &\quad = -2\langle A(\varphi(t, u_1, y)) - A(\varphi(t, u_2, y), \varphi(t, u_1, y) - \varphi(t, u_2, y)\rangle \\ &\quad \le -2\alpha|\varphi(t, u_1, y) - \varphi(t, u_2, y)|^p \end{aligned} \tag{34}$$

for all $t \in \mathbb{R}_+$. Denote by $\omega(t) := |\varphi(t, u_1, y) - \varphi(t, u_2, y)|^2$, then from (34) we obtain

$$\omega'(t) \le -2\alpha\omega(t)^{p/2}. \tag{35}$$

We will consider two cases.

1. If $p = 2$, then from (35) we have $|\varphi(t, u_1, y) - \varphi(t, u_2, y)| \le e^{-\alpha t}|u_1 - u_2|$ for all $t \in \mathbb{R}_+$, $u_1.u_2 \in \overline{D(A)}$ and $y \in Y$. To finish the proof in this case it is necessary to use the same reasoning as in the proof of Theorem 2.9.

2. Let now $p > 2$. Thanks to inequality (35) we obtain

$$|\varphi(t, u_1, y) - \varphi(t, u_2, y)| \le \frac{|u_1 - u_2|}{(1 + |u_1 - u_2|^{\frac{p-2}{p}}\alpha(p-2)t)^{\frac{2}{p-2}}} \tag{36}$$

for all $t \in \mathbb{R}_+$, $u_1, u_2 \in \overline{D(A)}$ and $y \in Y$. Thus we have

$$|\varphi(t,u_1,y) - \varphi(t,u_2,y)| \le \omega(t,|u_1-u_2|) \tag{37}$$

for all $t \in \mathbb{R}_+$, $u_1, u_2 \in \overline{D(A)}$ and $y \in Y$, where

$$\omega(t,r) := r(1+\alpha(p-2)t|u_1-u_2|^{\frac{p-2}{p}})^{-\frac{2}{p-2}} \tag{38}$$

is the function with the following properties:

(i) $\omega(0,r) = r$ for all $r \in \mathbb{R}_+$;
(ii) $\omega'(t) = -2\alpha\omega(t)$;
(iii) the mapping $\omega(t,\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strict increasing;
(iv) $\omega(t,r) < r$ for all $t > 0$ and $r > 0$.

Let $C(Y,\overline{D(A)})$ be the Banach space of all continuous $\nu : Y \to \overline{D(A)}$ with the sup-norm. Now we define for all $t \in \mathbb{R}_+$ a mapping S^t from $C(Y,\overline{D(A)})$ into itself by following rule $(S^t\nu)(y) := \varphi(t,\nu(y),\sigma(-t,y))$ for all $y \in Y$. It easy to check that the family of maps $\{S^t\}_{t\ge0}$ forms a semigroup with respect to composition (more exactly $S^tS^\tau = S^{t+\tau}$ for all $t,\tau \in \mathbb{R}_+$). Notice that from (37) and the fact that $\omega(t,\cdot)$ is increasing we have

$$\begin{aligned} d(S^t\nu_1,S^t\nu_2) &:= \max_{y\in Y}|\varphi(t,\nu_1(y),\sigma(-t,y)) - \varphi(t,\nu_2(y),\sigma(-t,y))| \\ &\le \max_{y\in Y}\omega(t,|\nu_1(y)-\nu 2(y)|) \le \omega(t,d(\nu_1,\nu_2)) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $\nu_1,\nu_2 \in C(Y,\overline{D(A)})$.

Note that for all $t > 0$ the operator S^t acting on the complete metric space $(C(Y,\overline{D(A)}),d)$ is a φ–contraction possessing the properties $(G1)-(G3)$, where $\varphi := \omega(t,\cdot)$. Let $t_0 > 0$. According to Theorem 3.5 S^{t_0} has a unique fixed point $\gamma \in C(Y,\overline{D(A)})$. Since the semi-group $\{S^t\}_{t\ge0}$ is commutative, then γ is a unique common fixed point of this semi-group. This means, in particular, that $\varphi(t,\gamma(y),y) = \gamma(\sigma(t,y))$ for all $t \in \mathbb{R}_+$. Thus, equation (29) possesses at least one relatively compact on $\mathbb{R}_+$ solution $\varphi(t,\gamma(y),y)$. In addition we have

$$\sup_{|u|\le r, y\in Y}|\varphi(t,u,y) - \varphi(t,\gamma(y),y)| \le e^{-\alpha t}\sup_{|u|\le r,y\in Y}|u-\gamma(y)| \to 0 \tag{39}$$

as $t \to +\infty$ for every $r > 0$. From (39) it follows that the cocycle φ is compact dissipative. Now to finish the proof it is sufficient to apply Theorem 3.2. □

Remark 3.7. *Theorem 3.6 generalizes and make precise Theorem 7.10 in* [12].

4. **Semi-linear parabolic equations.** Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and associated norm $|\cdot| := \langle\cdot,\cdot\rangle^{1/2}$, and A be a self-adjoint operator with domain $D(A)$.

An operator is said (see, for example, [13, Ch. II]) to have a discrete spectrum in the space H, if there exists an orthonormal basis $\{e_k\}$ of eigenvectors, such that $\langle e_k,e_j\rangle = \delta_{kj}$, $Ae_k = \lambda_k e_k$ $(k,j = 1,2,\dots)$ and $0 < \lambda_1 \le \lambda_2 \le \dots, \lambda_k \le \dots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

One can define an operator $f(A)$ for a wide class of functions f defined on the positive semi-axis as follows:

$$\begin{aligned} D(f(A)) &:= \{h = \textstyle\sum_{k=1}^\infty c_ke_k \in H : \sum_{k=1}^\infty c_k[f(\lambda_k)]^2 < +\infty\}, \\ f(A)h &:= \textstyle\sum_{k=1}^\infty c_kf(\lambda_k)e_k, \quad h \in D(f(A)). \end{aligned} \tag{40}$$

In particular, we can define operators A^α for all $\alpha \in \mathbb{R}$. For $\alpha = -\beta < 0$ this operator is bounded. The space $D(A^{-\beta})$ can be regarded as the completion of the space H with respect to the norm $|\cdot|_\beta := |A^{-\beta}\cdot|$.

The following statements hold [13, Ch. II]:

(i) The space $\mathcal{F}_{-\beta} := D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum_{k=1}^{\infty} c_k e_k$ such that

$$\sum_{k=1}^{\infty} c_k \lambda_k^{-2\beta} < +\infty;$$

(ii) For any $\beta \in \mathbb{R}$, the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded operator mapping $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that

$$A^\beta D(A^\alpha) = D(A^{\alpha-\beta}),\ A^{\beta_1+\beta_2} = A^{\beta_1}A^{\beta_2}.$$

(iii) For all $\alpha \in \mathbb{R}$, the space $\mathcal{F} := D(A^\alpha)$ is a separable Hilbert space with the inner product $\langle \cdot,\cdot \rangle_\alpha := \langle A^\alpha \cdot, A^\alpha \cdot \rangle$ and the norm $|\cdot|_\alpha := |A^\alpha \cdot|$.

(iv) The operator A with the domain $\mathcal{F}_{1+\alpha}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\alpha$.

(v) The embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$ is continuous, i.e., $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and there exists a positive constant $C = C(\alpha,\beta)$ such that $|\cdot|_\beta \le C|\cdot|_\alpha$.

(vi) $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.

(vii) Let $\alpha_1 > \alpha_2$, then the space $\mathcal{F}_{\alpha_1}$ is compactly embedded into $\mathcal{F}_{\alpha_2}$, i.e., every sequence bounded in $\mathcal{F}_{\alpha_1}$ is relatively compact in $\mathcal{F}_{\alpha_2}$.

(viii) The resolvent $\mathcal{R}_\lambda(A) := (A - \lambda I)^{-1}$, $\lambda \ne \lambda_k$ is a compact operator in each space $\mathcal{F}_\alpha$, where I is the identity operator.

According to (40) we can define an exponential operator e^{-tA}, $t \ge 0$, in the scale spaces $\{\mathcal{F}_\alpha\}$. Note some of its properties [13, Ch. II]:

a. For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator e^{-tA} maps $\mathcal{F}_\alpha$ into $\bigcap\limits_{\beta \ge 0} \mathcal{F}_\beta$ and

$$|e^{-tA}x|_\alpha \le e^{-\lambda_1 t}|x|_\alpha \tag{41}$$

for all $x \in \mathcal{F}_\alpha$.

b. $e^{-t_1A}e^{-t_2A} = e^{-(t_1+t_2)A}$ for all $t_1, t_2 \in \mathbb{R}_+$;

c.

$$|e^{-tA}x - e^{-\tau A}x|_\beta \to 0 \tag{42}$$

as $t \to \tau$ for every $x \in \mathcal{F}_\beta$ and $\beta \in \mathbb{R}$;

d. For any $\beta \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\beta, e^{-tA})$;

e.

$$|A^\alpha e^{-tA}h| \le \Big[\big(\tfrac{\alpha-\beta}{t}\big)^{\alpha-\beta} + \lambda_1^{\alpha-\beta}\Big] e^{-t\lambda_1}|A^\beta h|,\ \alpha \ge \beta$$
$$||A^\alpha e^{-tA}|| \le \big(\tfrac{\alpha}{t}\big)^\alpha e^{-\alpha},\ t > 0,\ \alpha > 0. \tag{43}$$

Let (Y,ρ) be a compact complete metric space and $(Y,\mathbb{R},\sigma)$ be a dynamical system on Y. Consider an evolutionary differential equation

$$u' + Au = F(\sigma(t,y),u) \quad (y \in Y) \tag{44}$$

in the separable Hilbert space H, where A is a linear (generally speaking unbounded) positive operator with discrete spectrum, and F is a non-linear continuous mapping acting from $Y \times \mathcal{F}_\theta$ into H, $0 \le \theta < 1$, possessing the property

$$|F(y,u_1) - F(y,u_2)| \le L(r)|A^\theta(u_1-u_2)| \tag{45}$$

for all $u_1, u_2 \in B_\theta(0,r) := \{u \in \mathcal{F}_\theta : \ |u|_\theta \le r\}$. Here $L(r)$ denotes the Lipschitz constant of F on the set $B_\theta(0,r)$.

A function $u : [0,a) \mapsto \mathcal{F}_\theta$ is said to be a mild solution (in $\mathcal{F}_\theta$) of equation (44) passing through the point $x \in \mathcal{F}_\theta$ at the initial moment $t = 0$ (notation $\varphi(t,x,y)$) if $u \in C([0,T],\mathcal{F}_\theta)$ and satisfies the integral equation

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A}F(\sigma(\tau,y),u(\tau))d\tau \tag{46}$$

for all $t \in [0,T]$ and $0 < T < a$.

In the book [13, Ch. II], it is proved that, under the conditions listed above, there exists a unique solution $\varphi(t,x,y)$ of equation (45) passing through the point x at the initial moment $t = 0$, and it is defined on a maximal interval $[0,a)$, where a is some positive number depending on $(x,y) \in \mathcal{F}_\theta \times Y$. Below we will generalize this result.

Theorem 4.1. *Let $x_0 \in \mathcal{F}_\theta$, $r > 0$ and the conditions listed above be fulfilled. Then, there exist positive numbers $\delta = \delta(x_0,r)$ and $T = T(x_0,r)$ such that equation (44) admits a unique solution $\varphi(t,x,y)$ ($x \in B_\theta[x_0,\delta] = \{x \in \mathcal{F}_\theta \mid |x-x_0|_\theta \le \delta\}$) defined on the interval $[0,T]$ with the conditions: $\varphi(0,x,y) = x$, $|\varphi(t,x,y) - x_0|_\theta \le r$ for all $t \in [0,T]$ and the mapping $\varphi : [0,T] \times B[x_0,\delta] \times Y \to \mathcal{F}_\theta$ ($(t,x,y) \mapsto \varphi(t,x,y)$) is continuous.*

Proof. Let $x_0 \in \mathcal{F}_\theta$, $r > 0$, $\delta > 0$ and $T > 0$. We consider the space $C_{x_0,r,\delta,T}$ of all continuous functions $\psi : [0,T] \times B_\theta[x_0,\delta] \times Y \to B_\theta[x_0,r]$ equipped with the distance

$$d(\psi_1,\psi_2) := \sup\{|\psi_1(t,x,y) - \psi(t,x,y)|_\theta \ : \ 0 \le t \le T, x \in B_\theta[x_0,\delta], y \in Y\}$$

which is a complete metric space.

We define the operator Φ acting onto $C_{x_0,r,\delta,T}$ by the equality

$$(\Phi\psi)(t,x,\omega) = e^{-At}x + \int_0^t e^{-A(t-s)}F(\sigma(\tau,y),\psi(s,x,y)))ds.$$

There exist $\delta_1 = \delta_1(x_0,r) > 0$ and $T_1 = T_1(x_0,r) > 0$ such that $\Phi C_{x_0,r,\delta,T} \subseteq C_{x_0,r,\delta,T}$ for all $\delta \in (0,\delta_1]$ and $T \in (0,T_1]$. In fact,

$$\begin{aligned} |(\Phi\psi)(t,x,\omega) - x_0|_{\mathcal{F}_\theta} &\le |e^{-At}x - x_0|_{\mathcal{F}_\theta} \\ &\quad + |\int_0^t e^{-A(t-s)}F(\sigma(\tau,y),\psi(s,x,y)))ds|_{\mathcal{F}_\theta} \\ &\le m(\delta,T) \\ &\quad + \int_0^t [(\frac{\theta}{t-\tau})^\theta + \lambda_1^\theta]d\tau \max_{0\le\tau\le t}|F(\sigma(\tau,y),\psi(\tau,x,y))|. \end{aligned} \tag{47}$$

where $m(\delta,T) := \sup\{|e^{-tA}x - x_0|_{\mathcal{F}_\theta} \ : \ t \in [0,T], x \in B_\theta[x_0,r]\}$.

Note that

$$
\begin{aligned}
m(\delta,T) &:= \sup\{|e^{-tA}x - x_0|_{\mathcal{F}_\theta} \ : \ t\in[0,T], x\in B_\theta[x_0,r]\}\\
&\le \sup\{|e^{-tA}x - e^{-tA}x_0|_{\mathcal{F}_\theta} \ : \ t\in[0,T], x\in B_\theta[x_0,r]\}\\
&\quad + |e^{-tA}x_0 - x_0|_{\mathcal{F}_\theta}\\
&\le \delta \max_{0\le t\le T}||e^{-tA}||_\theta + \max_{0\le t\le T}|e^{-tA}x_0 - x_0|, \qquad (48)
\end{aligned}
$$

and by properties (41),(42), and from (48) we obtain $m(\delta,T)\to 0$ as $T+\delta\to 0$.

Now we will estimate the second term in inequality (47). Notice that

$$
\begin{aligned}
|F(\sigma(\tau,y),\psi(\tau,x,y))| &\le |F(\sigma(\tau,y),\psi(\tau,x,y)) - F(\sigma(\tau,y),x_0)|\\
&\quad + |F(\sigma(\tau,y),x_0)|\\
&\le L(|x_0|+r)|\psi(\tau,x,y)-x_0|_\theta + M_{x_0}\\
&\le L(|x_0|+\delta)\delta + M_{x_0} \qquad (49)
\end{aligned}
$$

for all $\tau\in[0,T]$, $x\in B_\theta[0,\delta]$ and $y\in Y$, where $M_{x_0} := \max\limits_{y\in Y}|F(y,x_0)|_\theta$. Thus, it follows from (49) that the second term of the right–hand side of inequality (47) tends to zero as well, as $\delta+T\to 0$ and, consequently, the necessary statement is proved.

Let now $\psi_1,\psi_2\in C_{x_0,r,\delta,T}$, then

$$
\begin{aligned}
&|(\Phi\psi_1)(t,x,\omega)) - (\Phi\psi_2)(t,x,\omega))|_\theta\\
&= |\int_0^t e^{-(t-\tau)A}[F(\sigma\tau,\psi_1(\tau,x,y)) - F(\sigma\tau,\psi_2(\tau,x,y))]d\tau|_\theta\\
&\le L(\delta+|x_0|)\sup_{0\le t\le T, x\in B_\theta[x_0,\delta], y\in Y}|\psi_1(t,x,y)-\psi(t,x,y)|_\theta\int_0^t[(\frac{\theta}{t-\tau})^\theta + \lambda_1^\theta]d\tau
\end{aligned}
$$

and, consequently, $d(\Phi\psi_1,\Phi\psi_2)\le L(x_0,\delta,T)d(\psi_1,\psi_2)$, where

$$
L(x_0,\delta,T) = L(|x_0|+\delta)\max_{0\le t\le T}\int_0^t[(\frac{\theta}{t-\tau})^\theta + \lambda_1^\theta]d\tau
$$

and $L(x_0,\delta,T)\to 0$ as $T\to 0$. Thus there exists $T_2 = T_2(x_0,\delta)>0$ such that $L(x_0,\delta,T)<1$ for all $T\in(0,T_2]$. Denote by $\delta(x_0,r) := \delta_1(x_0,r)$ and $T(x_0,r) := min(T_1(x_0,r),T_2(x_0,r))$, then the mapping $\Phi : C_{x_0,r,\delta,T}\to C_{x_0,r,\delta,T}$ is a contraction and, consequently, there exists a unique function $\varphi\in C_{x_0,r,\delta,T}$ satisfying equation (44) on the interval $[0,T]$. The theorem is proved. □

Remark 4.2. *Theorem 4.1 holds true for the following equation*

$$
u' + Au = F(\sigma(t,y),u)
$$

if the continuous function $F : Y\times\mathcal{F}_\theta\to H$ satisfies the following conditions:

(i)
$$
\sup\{|F(y,0)|_{\mathcal{F}_\theta} \ : \ y\in Y\} < \infty
$$
(Y, generally speaking, is not compact);

(ii) *F is locally Lipschitz, i.e., for every $r>0$ there exists $L(r)>0$ such that*
$$
|F(y,u_1) - F(y,u_2)|_{\mathcal{F}_\theta} \le L(r)|u_1-u_2|_{\mathcal{F}_\theta}
$$
for all $u_1,u_2\in\mathcal{F}_\theta$ with the condition that $|u_i|_{\mathcal{F}_\theta}\le r$ $(i=1,2)$.

In the sequel, we suppose that the function $F \in C(Y \times \mathcal{F}_\theta, H)$ is regular, i.e., for any $u \in \mathcal{F}_\theta$ and $y \in Y$ there exists a unique solution $\varphi(t,u,y)$ of equation (44) passing through the point u at the initial moment $t=0$, is defined on $\mathbb{R}_+$ and the mapping $\varphi : \mathbb{R}_+ \times \mathcal{F}_\theta \times Y \mapsto \mathcal{F}_\theta$ is continuous

Lemma 4.3. *Let $\langle \mathcal{F}_\theta, \varphi, (Y,\mathbb{R},\sigma)\rangle$ be the cocycle generated by equation (44) and $M \subseteq X_\theta := \mathcal{F}_\theta \times Y$ positively invariant (with respect to the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$, where $\pi := (\varphi,\sigma)$) and bounded. Then, there exists a relatively compact set $K \subseteq X_\alpha$ ($\alpha \in (\theta,1)$) such that*

$$\lim_{t\to+\infty} \beta(\pi(t,M),K) = 0,$$

where $\beta(A,B) := \sup_{a\in B} \rho_\alpha(a,B)$, $\rho_\alpha(a,B) := \inf_{b\in B} \rho_\alpha(a,b)$, $\rho_\alpha(a,b) := \rho(y_a,y_b)+|x_a - x_b|_\alpha$, $a := (x_a,y_a)$ and $b := (x_b,y_b)$.

Proof. Let $M \subseteq X_\theta$ be a positively invariant and bounded set in $(X_\theta,\mathbb{R}_+,\pi)$, then there exists a positive number R_0 such that

$$|\varphi(t,x,y)|_\theta \le R_0$$

for all $t \in \mathbb{R}_+$ and $(x,y) \in M$. Let l be a positive number. Since $\varphi(t+l,x,y) = \varphi(l,\varphi(t,x,y),\sigma(t,y))$ for all $(x,y) \in M$ and $t \in \mathbb{R}_+$, then from (46) we obtain

$$\varphi(t+l,x,y) = e^{-lA}\varphi(t,x,y) + \int_0^l e^{-(l-\tau)A} F(\sigma(t+\tau,y),\varphi(t+\tau,x,y)))d\tau. \quad (50)$$

From (50) and (43) we obtain

$$\begin{aligned} |A^\alpha \varphi(t+l,x,y)| &\le |A^\theta \varphi(t,x,y)| \quad (51)\\ &\quad + \int_0^l |e^{-(l-\tau)A} F(\sigma(t+\tau,y),\varphi(t+\tau,x,y)))| d\tau \\ &\le (\alpha-\theta)^{\alpha-\theta} e^{-(\alpha-\theta)} |\varphi(t,x,y)|_\theta \\ &\quad + \int_0^l (\frac{\alpha}{1-\tau})^\alpha e^{-\alpha} |F(\sigma(t+\tau,y),\varphi(t+\tau,x,y)))| d\tau. \end{aligned}$$

Note that

$$\begin{aligned} |F(\sigma(t,y),\varphi(t,x,y))| &\le |F(\sigma(t,y),\varphi(t,x,y))) - F(\sigma(t,y),0)| \\ &\quad + |F(\sigma(t,y),0)| \\ &\le L(R_0)R_0 + M_0 \quad (52) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $(x,y) \in M$ and, consequently, from (51) and (52) we obtain

$$|A^\alpha \varphi(t+l,x,y)| \le R_\alpha$$

for all $t \in \mathbb{R}_+$ and $(x,y) \in M$, where

$$R_\alpha := (\alpha-\theta)^{\alpha-\theta} e^{-(\alpha-\theta)} R_0 + \alpha^\alpha \frac{e^{1-2\alpha}}{1-\alpha}.$$

Since the space $\mathcal{F}_\alpha$ is compactly embedded in $\mathcal{F}_\theta$ ($\alpha \in (\theta,1)$), then the set $M_l := \{\pi(t,(x,y)) : \ t \ge l, (x,y) \in M\} \subseteq M$ is a relatively compact set in $\mathcal{F}_\theta$ and, consequently, the omega limit set $\omega(M)$ is a nonempty, compact and invariant set of the dynamical system $(X_\theta,\mathbb{R}_+,\pi)$ and

$$\lim_{t\to+\infty} \beta(\pi(t,M),\omega(M)) = 0.$$

Our lemma is completely proved now. ☐

Equation (44) (equivalently, the cocycle φ generated by equation (44)) is said to be *dissipative* if there exists a positive number R_0 such that for all $r > 0$ there exists a positive number $l = l(r)$ such that

$$|\varphi(t, x, y)|_\theta \le R_0$$

for all $t \ge l(r)$, $||x||_\theta \le r$ and $y \in Y$.

Theorem 4.4. *If equation (44) is dissipative, then it admits a compact global attractor, i.e., there exists a nonempty, compact and invariant subset $J \subseteq X_\theta = \mathcal{F}_\theta \times Y$ which attracts every bounded subset $M \subseteq X_\theta$. This means that*

$$\lim_{t\to+\infty} \beta(\pi(t, M), J) = 0$$

for all bounded subset M from X_θ.

Proof. Let (44) be dissipative and R_0 be the positive number appearing in (52). Denote by $M_0 := \{(x, y) \in X_\theta : \ |x|_\theta \le R_0 \text{ and } y \in Y\}$. Then, by the dissipativity (44) and the choice of R_0, there exists a positive number l_0 such that $\bigcup\{\pi(t, M_0) : t \ge l_0\} \subseteq M_0$, i.e., the set $\mathcal{M}_0 := \bigcup\{\pi(t, M_0) : \ t \ge l_0\}$ is bounded and positively invariant. According to Lemma 4.3 there exists a nonempty and compact subset X_θ which attract the set $\mathcal{M}_0$. Denote by $J := \omega(\mathcal{M}_0)$. The set J is nonempty, compact, invariant and attract the set $\mathcal{M}_0$.

Now, let M be an arbitrary bounded subset of X_θ. Then, there exists a positive number $r = r(M)$ such that $M \subseteq B_\theta[0, r] \times Y$. By the dissipativity of (44) there exists a positive number $l = l(r)$ such that $\pi(t, M) \subseteq B_\theta[0, r] \times Y$ for all $t \ge l(r)$ and, consequently, the set M is also attracted by J. □

The following result follows directly from Theorem 4.4 and Theorem 2.24 in [8, Ch. 2, p. 95].

Corollary 4.5. *If equation (44) if dissipative, then the following statements hold:*

(i) *the set*

$$I_y := \{x \in \mathcal{F}_\theta | \text{ the solution of equation (44) } \varphi(t, x, y) \\ \text{is defined on } \mathbb{R} \text{ and } \sup_{t\in\mathbb{R}} |\varphi(t, x, y)|_\theta < +\infty\}$$

is not empty, compact and connected for each $y \in Y$;

(ii) $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ *for all $t \in \mathbb{R}_+$ and $y \in Y$;*

(iii) $\mathbb{I} := \bigcup\{I_y | \ y \in Y\}$ *is compact and connected if Y is compact and connected as well;*

(iv) *the equalities*

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \sigma(-t, y)), I_y) = 0$$

and

$$\lim_{t\to+\infty} \beta(\varphi(t, M, y), \mathbb{I}) = 0$$

take place for all $y \in Y$ and bounded subset $M \subseteq \mathcal{F}_\theta$.

Finally, we can establish the next result.

Theorem 4.6. *Suppose that the following conditions are fulfilled:*

(i) *Y is minimal, i.e., $H(y) = Y$ for all $y \in Y$, where $H(y) := \overline{\{\pi(t, y)| \ t \in \mathbb{R}\}}$;*

(ii) *equation (44) is dissipative;*

(iii) *for all pair of solutions $\varphi(t, x_i, y)$ $(i = 1, 2)$ of equation (44) defined and bounded on $\mathbb{R}$ we have*

$$\lim_{t\to+\infty} |\varphi(t, x_1, y) - \varphi(t, x_2, y)|_\theta = 0. \tag{53}$$

Then,

(i) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then equation (44) admits a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution $\varphi(t, x_y, y)$ $(x_y \in \mathcal{F}_\theta)$;*

(ii) *every solution $\varphi(t, u, y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent)*

Proof. Let $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be the cocycle associated to equation (44). Denote by $(X_\theta, \mathbb{R}_+, \pi)$ the skew-product dynamical system, where $X_\theta := \mathcal{F}_\theta \times Y$ and $\pi := (\varphi, \sigma)$ (i.e., $\pi(t, (x, y)) := (\varphi(t, x, y), \sigma(t, y))$ for all $(x, y) \in \mathcal{F}_\theta \times Y$ and $t \in \mathbb{R}_+$). Consider a non-autonomous dynamical system $\langle (X_\theta, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by the cocycle φ (respectively, by equation (44)), where $h := pr_2 : X \mapsto Y$. Since Y is compact, it is evident that the dynamical system $(Y, \mathbb{R}, \sigma)$ is compact dissipative and its Levinson center J_Y coincides with Y. According to Theorem 4.4, the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is compact dissipative. Denote by J_X its Levinson center and $I_y := pr_1(J_X \bigcap X_y)$ for all $y \in Y$, where $X_y := h^{-1}(y)$. According to the definition of the set $I_y \subseteq \mathcal{F}_\theta$ and Theorem 2.24 in [8, Ch. 2, p. 95], $u \in I_y$ if and only if the solution $\varphi(t, u, y)$ is defined on $\mathbb{R}$ and relatively compact (i.e., the set $\overline{\varphi(\mathbb{R}, u, y)} \subseteq \mathcal{F}_\theta$ is compact). Thus $I_y = \{u \in \mathcal{F}_\theta :$ if and only if $(x, y) \in J_X\}$. It is easy to see that condition (53) means that the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is weak convergent. Now, to finish the proof of the theorem, it is sufficient to apply Theorem 3.5 in [6] for the non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by equation (44). □

Remark 4.7. *Some interesting ideas and results related to the theory developed in our paper can be found in* [23].

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REFERENCES

[1] D. W. Boyd and J. S. W. Wong , *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

[2] H. Brezis, "Operateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert", Vol. 5 of Math. Studies, North Holland, 1973.

[3] I. U. Bronsteyn, "Extensions of Minimal Transformation Group", Noordhoff, 1979.

[4] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A 71, Indag. Math. **30**, (1968), 27–35.

[5] T. Caraballo and D.N. Cheban, *Levitan/Bohr Almost Periodic and Almost Automorphic Solutions of Second-Order Monotone Differential Equations*, Journal of Differential Equations **251** (2011), 708-727.

[6] T. Caraballo and D.N. Cheban, *On the Structure of the Global Attractor for Non-autonomous Dynamical Systems with Weak Convergence*, Communications in Pure and Applied Analysis (2011) (to appear).

[7] T. Caraballo and D.N. Cheban, *On the Structure of the Global Attractor for Non-autonomous Difference Equations with Weak Convergence*, Journal of Difference Equations and Applications, (2011) (to appear).

[8] D.N. Cheban, "Global Attractors of Non-Autonomous Dissipative Dynamical Systems", Interdisciplinary Mathematical Sciences 1. River Edge, NJ: World Scientific, 2004, 528pp.

[9] D.N. Cheban, *Levitan Almost Periodic and Almost Automorphic Solutions of V-monotone Differential Equations*, J. Dynamics and Differential Equations **20** (2008), no. 3, 669–697.

[10] D.N. Cheban, "Asymptotically Almost Periodic Solutions of Differential Equations", Hindawi Publishing Corporation, New York, 2009, 203 pp.

[11] D. N. Cheban, "Global Attractors of Set-Valued Dynamical and Control Systems, Nova Science Publishers, New York, 2010, xvii+269 p.

[12] D.N. Cheban and B. Schmalfuß, *Invariant Manifolds, Global Attractors, Almost Automorphic and Almost Periodic Solutions of Non-Autonomous Differential Equations*, J. Math. Anal. Appl. **340** (2008), no. 1, 374–393.

[13] I. D. Chueshov, "Vvedenie v teoriyu beskonechnomernykh dissipativnykh sistem. Universitetskie Lektsii po Sovremennoi Matematike", AKTA, Kharkiv, 1999. 436 pp. (in Russian) [English translation: Introduction to the theory of infinite-dimensional dissipative systems. University Lectures in Contemporary Mathematics. AKTA, Kharkiv, 1999. 436 pp.]

[14] C. Conley, "Isolated Invariant Sets and the Morse Index", Region. Conf. Ser. Math., No.38, 1978. Am. Math. Soc., Providence, RI.

[15] B. P. Demidovich, "Lectures on Mathematical Theory of Stability", Moscow, Nauka, 1967. (in Russian)

[16] A. M. Fink and P. O. Fredericson, *Ultimate Boundedness Does not Imply Almost Periodicity,* Journal of Differential Equations **9** (1971), 280–284.

[17] J. K. Hale, "Theory of Functional-Differential Equations". Springer-Verlag, New York-Heidelberg-Berlin, 1977.

[18] J. K. Hale, "Asymptotic Behaviour of Dissipative Systems", Amer. Math. Soc., Providence, RI, 1988.

[19] N. Hassani, Systems Dynamiques Nonautonomes Contractants et leur Applications. Theése de magister. Algerie, USTHB, 1983.

[20] M. W. Hirsch, H. L. Smith and X.-Q. Zhao, *Chain Transitivity, Attractivity, and Strong Repellers for Semidynamical Systems*, J. Dyn. Diff. Eqns **13** (2001), no. 1, 107–131.

[21] D. Husemoller, "Fibre Bundles", Springer–Verlag, Berlin–Heidelberg–New York, 1994.

[22] W. A. Kirk and B. Sims, "Handbook of Metric Fixed Point Theory", Kluwer Academic Publishers, Dodrecht/Boston/London/2001, 703 pp.

[23] P. E. Kloeden and H. M. Rodrigues, *Dynamics of a class of ODEs more general than almost periodic*, Nonlinear Analysis TMA, **74** (2011), 2695-2719.

[24] B.M. Levitan, V.V. Zhikov, "Almost Periodic Functions and Differential Equations", Cambridge Univ. Press, London, 1982.

[25] J. L. Lions, "Quelques Methodes de Résolution des Problèmes aux Limites non Linéaires", Dunod, Paris, 1969.

[26] B.A. Shcherbakov, *The comparability of the motions of dynamical systems with regard to the nature of their recurrence*, Differential Equations **11** (1975), no. 7, 1246–1255.

[27] G. R. Sell, "Topological Dynamics and Ordinary Differential Equations", Van Nostrand-Reinhold, London, 1971.

[28] Yoshizawa T., "Stability theory and the existence of periodic solutions and almost periodic solutions.", Applied Mathematical Sciences, Vol. 14, Springer-Verlag, New York-Heidelberg, 1975. vii+233 pp.

[29] V. V. Zhikov, *On Stability and Unstability of Levinson's centre*, Differentsial'nye Uravneniya **8** (1972), no. 12, 2167–2170.

[30] V.V. Zhikov, *Monotonicity in the Theory of Almost Periodic Solutions of Non-Linear operator Equations*, Mat. Sbornik **90** (1973), 214–228; English transl., Math. USSR-Sb. **19** (1974), 209-223.

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