Stochastics and Dynamics

ON THE STOCHASTIC 3D-LAGRANGIAN AVERAGED NAVIER-STOKES α-MODEL WITH FINITE DELAY *

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Existence and uniqueness of solutions for a stochastic version of the 3D-Lagrangian averaged Navier-Stokes (LANS-α) equation in a bounded domain and containing some hereditary characteristics are proved.

Keywords: 3D-Lagrangian averaged Navier-Stokes equations, variational solutions, finite delays, cylindrical Wiener process

AMS Subject Classification: 35K55, 60H15

1. Introduction

In this paper we study existence and uniqueness of solution for the 3D-Lagrangian averaged Navier-Stokes (LANS-α) equations, with homogeneous Dirichlet boundary condition in a bounded domain, in the case in which random perturbations and terms containing finite delays appear. To be more precise, let D be a connected and bounded open subset of $\mathbb{R}^3$, with a Lipschitz boundary ∂D, a final time $T > 0$, and a time lag $h > 0$. We denote by A the Stokes operator, and consider the system

$$\begin{cases} \partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u\cdot\nabla)(u - \alpha\Delta u) \\ -\alpha\nabla u^* \cdot \Delta u + \nabla p = F(t, u_t) + G(t, u_t)\dot{W}(t), \quad \text{in } \ D \times (0,T), \\ \nabla \cdot u = 0, \quad \text{in } \ D \times (0,T), \\ u = 0, \quad Au = 0, \quad \text{on } \partial D \times (0,T), \\ u(0) = u^0, \quad \text{in } \ D, \\ u(t) = \phi(t), \ \text{in } \ D \times (-h, 0), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown random fields on $D \times [0,T]$, representing, respectively, the large-scale (or averaged) velocity and the pressure, in each point of $D \times [0,T]$, of an incompressible viscous fluid with constant density filling the

*Partly supported by Ministerio de Ciencia y Tecnología (Spain) and FEDER (European Community) under the project BFM2002-03068

domain D. The constants $\nu > 0$ and $\alpha > 0$ represent respectively the kinematic viscosity of the fluid, and the square of the spatial scale at which fluid motion is filtered. The terms $F(t, u_t)$ and $G(t, u_t)\dot{W}(t)$ are random external forces depending eventually on u and containing information about the history of the problem, where $\dot{W}(t)$ denotes the time derivative of a cylindrical Wiener process. Finally, u^0 is a given initial velocity field, and ϕ is an initial velocity field defined in $(-h, 0)$, where $h > 0$ is fixed.

Problem (1.1) in the deterministic case, i.e. when $G = 0$, and when F does not depend on u, has been studied in [3] (see also [10]). In the stochastic case but without delay, it has been analysed in [2]. The consideration of some sort of delay terms in the evolution models is important, since it is sensible to think that the future state of certain systems may not depend only on their actual state, but also on their history. We aim in this paper to follow a similar scheme in order to treat this delay version.

The content of the paper is as follows. In Section 2 we first establish a result on the existence and uniqueness of solutions for an abstract stochastic partial functional differential equation with finite delay. The rigorous statement of our problem as well as the main results are included in Section 3.

2. Notations and preliminary results on an abstract model

Assume that $\{\Omega, \mathcal{F}, P\}$ is a complete probability space, and let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, where we consider fixed $T, h > 0$. For $t < 0$ we set $\mathcal{F}_t = \mathcal{F}_0$. Let $\{\beta_t^j,\ t \geq 0,\ j = 1, 2, ...\}$ be a given sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on this space, and suppose given K, a separable Hilbert space, and $\{e_j;\ j = 1, 2, ...\}$, an orthonormal basis of K. We denote by $\{W(t);\ t \geq 0\}$, the cylindrical Wiener process with values in K defined formally as

$$W(t) = \sum_{j=1}^{\infty} \beta^j(t) e_j.$$

It is well known that this series does not converge in K, but rather in any Hilbert space $\widetilde{K}$ such that $K \subset \widetilde{K}$, being the injection of K in $\widetilde{K}$ Hilbert-Schmidt (see e.g. [4] for more details).

For any separable Banach space X, $a, b \in \mathbb{R}$ with $-h \leq a < b \leq T$, and $p \in [1, \infty]$, we will denote by $M^p_{\mathcal{F}_t}(a, b; X)$ the space of all processes $\varphi \in L^p(\Omega \times (a, b), dP \times dt; X)$ that are $\mathcal{F}_t$-progressively measurable. The space $M^p_{\mathcal{F}_t}(a, b; X)$ is a Banach subspace of $L^p(\Omega \times (a, b), dP \times dt; X)$.

We will write $L^p_{\mathcal{F}_t}(\Omega; C([a, b]; X))$, for $1 \leq p < \infty$, to denote the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes $\{\varphi(t);\ a \leq t \leq b\}$ satisfying

$$E\left(\sup_{a\leq t\leq b} \|\varphi(t)\|_X^p\right) < \infty.$$

Given another separable Hilbert space $\widetilde{H}$, with scalar product $(\cdot,\cdot)_{\widetilde{H}}$, let us denote by $\mathcal{L}^2(K;\widetilde{H})$ the separable Hilbert space of Hilbert-Schmidt operators from K into $\widetilde{H}$, and by $((\cdot,\cdot))_{\mathcal{L}^2(K;\widetilde{H})}$ and $\|\cdot\|_{\mathcal{L}^2(K;\widetilde{H})}$ the scalar product and its associated norm in $\mathcal{L}^2(K;\widetilde{H})$, where for all R and S in $\mathcal{L}^2(K;\widetilde{H})$,

$$((R,S))_{\mathcal{L}^2(K;\widetilde{H})} = \sum_{j=1}^{\infty}(Re_j, Se_j)_{\widetilde{H}}.$$

For any process $\Psi \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\widetilde{H}))$, one can define the stochastic integral of Ψ with respect to the cylindrical Wiener process W_t, denoted by

$$\int_0^t \Psi(s)\,dW(s),\ 0 \le t \le T,$$

as the unique continuous $\widetilde{H}$-valued $\mathcal{F}_t$-martingale such that for all $h \in \widetilde{H}$,

$$\Big(\int_0^t \Psi(s)\,dW(s), h\Big)_{\widetilde{H}} = \sum_{j=1}^{\infty}\int_0^t (\Psi(s)e_j, h)_{\widetilde{H}}\, d\beta^j(s), \quad 0 \le t \le T,$$

where the integral with respect to $\beta^j(s)$ is understood in the sense of Itô, and the series converges in $L^2(\Omega; C([0,T]))$. See e.g. [4] for the properties of the stochastic integral so defined.

Let X be a Banach space. Given a function $u : [-h,T] \longrightarrow X$, for each $t \in (0,T)$ we denote by u_t the mapping defined as $u_t(s) = u(t+s)$, for any $s \in [-h,0]$.

Let $\mathcal{H}$ and $\mathcal{U}$ be two separable real Hilbert spaces, such that $\mathcal{U} \subset \mathcal{H}$ with compact injection, and $\mathcal{U}$ is dense in $\mathcal{H}$.

We denote by $(\cdot,\cdot)_{\mathcal{H}}$ and $((\cdot,\cdot))_{\mathcal{U}}$ the scalar product in $\mathcal{H}$ and $\mathcal{U}$ respectively, and we use $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{U}}$ to denote their corresponding associated norms.
We identify $\mathcal{H}$ with its topological dual $\mathcal{H}^*$, but we consider $\mathcal{U}$ as a subspace of $\mathcal{H}^*$, identifying $v \in \mathcal{U}$ with the element $f_v \in \mathcal{H}^*$, defined by

$$f_v(h) = (v,h)_{\mathcal{H}}, \quad \forall\, h \in \mathcal{H}.$$

We will denote by $\|\cdot\|_{\mathcal{U}^*}$ the norm in $\mathcal{U}^*$, and by $\langle\cdot,\cdot\rangle$ the duality product between $\mathcal{U}^*$ and $\mathcal{U}$.

We suppose given:

a) An operator $\widetilde{A} \in \mathcal{L}(\mathcal{U},\mathcal{U}^*)$, such that
  - a1) $\widetilde{A}$ is self adjoint,
  - a2) there exists $\widetilde{\alpha} > 0$, such that

$$2\langle\widetilde{A}v, v\rangle \ge \widetilde{\alpha}\|v\|^2_{\mathcal{U}},\ \forall\, v \in \mathcal{U}. \tag{2.2}$$

Observe that there exist a Hilbert basis $\{v_k;\, k \ge 1\} \subset \mathcal{U}$ of $\mathcal{H}$ and an increasing sequence $\{\lambda_k;\, k \ge 1\} \subset (0,\infty)$ such that

$$\widetilde{A}v_k = \lambda_k v_k, \quad \forall\, k \ge 1. \tag{2.3}$$

b) A bilinear mapping $\widetilde{B} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}^*$, such that

b1) $\langle \widetilde{B}(u,v), u \rangle = 0$, for all $u, v \in \mathcal{U}$,
and there exists a constant $\widetilde{c} > 0$ such that

b2) $\|\widetilde{B}(u,v)\|_{\mathcal{U}^*} \leq \widetilde{c}|u|_{\mathcal{H}}\|v\|_{\mathcal{U}}$, for all $(u,v) \in \mathcal{U} \times \mathcal{U}$,

b3) $\langle \widetilde{B}(u,v), w \rangle \leq \widetilde{c}\|u\|_{\mathcal{U}}\|v\|_{\mathcal{U}}|w|_{\mathcal{H}}$, for all $u, v, w \in \mathcal{U}$.

c) A positive constant $h > 0$, and a measurable random mapping $\widetilde{F} : \Omega \times [0,T] \times C^0([-h,0];\mathcal{H}) \longrightarrow \mathcal{U}^*$ such that, for all $\xi \in C^0([-h,0];\mathcal{H})$, $\widetilde{F}(\cdot,\xi)$ is $\mathcal{F}_t-$progressively measurable,

c1) $\widetilde{F}(\cdot,0) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{U}^*)$,

c2) there exists a constant $L_{\widetilde{F}} > 0$ such that, for all $\xi, \mu \in C^0([-h,0];\mathcal{H})$,

$$\|\widetilde{F}(t,\xi) - \widetilde{F}(t,\mu)\|_{\mathcal{U}^*} \leq L_{\widetilde{F}}\|\xi - \mu\|_{C^0([-h,0];\mathcal{H})}, dP \times dt - \text{a.e.}$$

c3) there exists $C_{\widetilde{F}} > 0$ such that, for all $t \in (0,T)$, and all $u, v \in C^0([-h,T];\mathcal{H})$,

$$\int_0^t \|\widetilde{F}(s,u_s) - \widetilde{F}(s,v_s)\|^2_{\mathcal{U}^*} ds \leq C_{\widetilde{F}} \int_{-h}^t |u(s) - v(s)|^2_{\mathcal{H}} ds;$$

c4) there exists $\widetilde{C}_{\widetilde{F}} > 0$ such that, for all $t \in [0,T]$, all decreasing $\rho \in C^0([0,T])$, and all $u, v \in C^0([-h,T];\mathcal{H})$ such that $u = v$ in $[-h,0]$,

$$\int_0^t \rho(s)\|\widetilde{F}(s,u_s) - \widetilde{F}(s,v_s)\|^2_{\mathcal{U}^*} ds \leq \widetilde{C}_{\widetilde{F}} \int_0^t \rho(s)|u(s) - v(s)|^2_{\mathcal{H}} ds;$$

d) A measurable random mapping $\widetilde{G} : \Omega \times (0,T) \times C^0([-h,0];\mathcal{H}) \to \mathcal{L}^2(K;\mathcal{H})$, such that for all $\xi \in C^0([-h,0];\mathcal{H})$, $\widetilde{G}(\cdot,\xi)$ is $\mathcal{F}_t-$progressively measurable,

d1) $\widetilde{G}(\cdot,0) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\mathcal{H}))$,

d2) there exists $L_{\widetilde{G}} > 0$ such that

$$\|\widetilde{G}(t,\xi) - \widetilde{G}(t,\mu)\|_{\mathcal{L}^2(K;\mathcal{H})} \leq L_{\widetilde{G}}|\xi - \mu|_{C^0([-h,0];\mathcal{H})}, \ dP \times dt - \text{a.e.};$$

for all $\xi, \mu \in C^0([-h,0];\mathcal{H})$,

d3) there exists $C_{\widetilde{G}} > 0$ such that for all $t \in [0,T]$, and all $u, v \in C^0([-h,T];\mathcal{H})$,

$$\int_0^t \|\widetilde{G}(s,u_s) - \widetilde{G}(s,v_s)\|^2_{\mathcal{L}^2(K;\mathcal{H})} ds \leq C_{\widetilde{G}} \int_{-h}^t |u(s) - v(s)|^2_{\mathcal{H}} ds;$$

d4) there exists $\widetilde{C}_{\widetilde{G}} > 0$ such that for all $t \in [0,T]$, for all decreasing $\rho \in C^0([0,T])$, and all $u, v \in C^0([-h,T];\mathcal{H})$ such that $u = v$ in $[-h,0]$,

$$\int_0^t \rho(s)\|\widetilde{G}(s,u_s) - \widetilde{G}(s,v_s)\|^2_{\mathcal{L}^2(K;\mathcal{H})} ds \leq \widetilde{C}_{\widetilde{G}} \int_0^t \rho(s)|u(s) - v(s)|^2_{\mathcal{H}} ds;$$

e) An initial value $u^0 \in L^2(\Omega,\mathcal{F}_0,P;\mathcal{H})$, and an initial function $\phi \in M^2_{\mathcal{F}_t}(-h,0;\mathcal{H}))$.

Remark 2.1. See [7] for some examples of functions $\widetilde{F}$ and $\widetilde{G}$.

We consider the equation

$$u(t) + \int_0^t \widetilde{A}u(s)\,ds + \int_0^t \widetilde{B}(u(s),u(s))\,ds$$
$$= u^0 + \int_0^t \widetilde{F}(s,u_s)\,ds + \int_0^t \widetilde{G}(s,u_s)\,dW(s),\ P-\text{a.s.},\ \forall\, t \in [0,T]. \qquad (2.5)$$

Definition 2.1. A solution of (2.5), corresponding to the initial data u^0 and ϕ, is a process

$$u \in M^2_{\mathcal{F}_t}(0,T;\mathcal{U}) \cap M^2_{\mathcal{F}_t}(-h,T;\mathcal{H}) \cap L^2(\Omega; L^\infty(0,T;\mathcal{H})),$$

such that the equation (2.5) is satisfied in $\mathcal{U}^*$, $P-$a.s. for all $t \in [0,T]$, and which coincides with ϕ in $(-h,0)$.

We can now establish the following results.

Proposition 2.1. *If u is a solution of (2.5), then $u \in L^2(\Omega; C^0([0,T];\mathcal{H}))$, and satisfies for all $t \in [0,T]$*

$$|u(t)|^2_{\mathcal{H}} + 2\int_0^t \langle \widetilde{A}u(s), u(s)\rangle\,ds = |u^0|^2_{\mathcal{H}} + 2\int_0^t \langle \widetilde{F}(s,u_s), u(s)\rangle\,ds \qquad (2.6)$$
$$+2\int_0^t (u(s), \widetilde{G}(s,u_s)dW(s)) + \int_0^t \|\widetilde{G}(s,u_s)\|^2_{\mathcal{L}^2(K;\mathcal{H})}\,ds,$$

and

$$E\int_0^t (u(s), \widetilde{G}(s,u_s)dW(s)) = 0,\ \forall\, t \in [0,T]. \qquad (2.7)$$

Proof. If u is a solution of (2.5), then $\widetilde{B}(u,u) \in M^1_{\mathcal{F}_t}(0,T;\mathcal{U}^*)$, $\widetilde{A}u - \widetilde{F}(t,u_t) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{U}^*)$, and $\widetilde{G}(t,u_t) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\mathcal{H}))$.

Consequently, Theorem 3.2 in [11] (p. 58) implies that u is $P-$a.s. continuous with values in $\mathcal{H}$, and by means of b1)- b3), it holds (2.6). Finally, (2.7) follows from the facts that $\widetilde{G}(t,u_t) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;\mathcal{H}))$ and $u \in L^2(\Omega; L^\infty(0,T;\mathcal{H}))$. □

Theorem 2.1. *Suppose all the above hypotheses and that, moreover, $\widetilde{F}(\cdot,0) \in L^4(\Omega; L^2(0,T;\mathcal{U}^*))$, $\widetilde{G}(\cdot,0) \in L^4(\Omega; L^2(0,T;\mathcal{L}^2(K;\mathcal{H})))$, $u^0 \in L^4(\Omega,\mathcal{F}_0,P;\mathcal{H})$ and $\phi \in L^4(\Omega; L^2(-h,T;\mathcal{H}))$. Then, there exists a unique solution u to (2.5), which satisfies in addition,*

$$u \in L^4(\Omega; C^0([0,T];\mathcal{H})) \cap L^4(\Omega; L^2(0,T;\mathcal{U})) \cap L^4(\Omega; L^2(-h,T;\mathcal{H})).$$

In fact, there exists $C > 0$, depending only on $\widetilde{\alpha}, T, C_{\widetilde{F}}$ and $C_{\widetilde{G}}$ such that

$$\begin{aligned} & E\left(\sup_{[0,T]} |u(t)|^4_{\mathcal{H}}\right) + E\left[\left(\int_0^T \|u(t)\|^2_{\mathcal{U}} dt\right)^2\right] \\ & \leq C\left\{E\left(|u^0|^4_{\mathcal{H}}\right) + E\left[\left(\int_{-h}^0 |\phi(t)|^2_{\mathcal{H}} dt\right)^2\right]\right. \\ & \left. + E\left[\left(\int_0^T \|\widetilde{F}(t,0)\|^2_{\mathcal{U}^*} dt\right)^2\right] + E\left[\left(\int_0^T \|\widetilde{G}(t,0)\|^2_{\mathcal{L}^2(K;\mathcal{H})} dt\right)^2\right]\right\}. \end{aligned} \tag{2.8}$$

Proof. The uniqueness of solutions is proved in a standard way. Let u^1 and u^2 be two solutions of (2.5), and denote $\bar{u} = u^1 - u^2$. Then, by applying the Ito formula to the process $\sigma(t)|\bar{u}(t)|^2$, where

$$\sigma(t) = \exp(-\mu_1 \int_0^t \|u^1(s)\|^2_{\mathcal{U}} \, ds - \mu_2 \int_0^t \|u^2(s)\|^2_{\mathcal{U}} \, ds), \; 0 \leq t \leq T,$$

(μ_1, μ_2 are appropriate positive constants) the Gronwall lemma, and taking into account our assumptions, we can prove that $\bar{u}(t) = 0$ $P-$a.s. for all $t \in [-h, T]$.

As for the existence, despite the fact that the nonlinear operator $\widetilde{B}$ does not satisfy the same assumptions than the one appearing in the generalized 2D-Navier-Stokes model considered in [1], it is possible to follow a similar scheme to that one in [1] (see also [2] for the proof in the nondelay case), but with the necessary technical changes because of the delay appearing in our model (see [9] for a more complete and detailed description). □

3. Statement of the problem and the main results

3.1. *Notations and properties of the nonlinear term*

We first establish some notations and recall some properties regarding the nonlinear term $(u \cdot \nabla)(u - \alpha \Delta u) - \alpha \nabla u^* \cdot \Delta u$ appearing in (1.1).

We will denote $(\cdot, \cdot)$ and $|\cdot|$, respectively, the scalar product and associated norm in $(L^2(D))^3$, and by $(\nabla u, \nabla v)$ the scalar product in $((L^2(D))^3)^3$ of the gradients of u and v. We consider the scalar product in $(H^1_0(D))^3$ defined by

$$((u, v)) = (u, v) + \alpha(\nabla u, \nabla v), \quad \forall \, u, v \in (H^1_0(D))^3, \tag{3.9}$$

where its associated norm $\|\cdot\|$ is, in fact, equivalent to the usual gradient norm. Let us denote by H the closure in $(L^2(D))^3$ of the set

$$\mathcal{V} = \{v \in (\mathcal{D}(D))^3 : \nabla \cdot v = 0 \text{ in } D\},$$

and by V the closure of $\mathcal{V}$ in $(H^1_0(D))^3$. Then, H is a Hilbert space equipped with the inner product of $(L^2(D))^3$, and V is a Hilbert subspace of $(H^1_0(D))^3$.

Denote by A the Stokes operator, with domain $D(A) = (H^2(D))^3 \cap V$, defined by

$$Aw = -\mathcal{P}(\Delta w), \quad \forall\, w \in D(A),$$

where $\mathcal{P}$ is the projection operator from $(L^2(D))^3$ onto H. Recall that as ∂D is Lipschitz, $|Aw|$ defines in $D(A)$ a norm which is equivalent to the $(H^2(D))^3$-norm, i.e., there exists a constant $c_1 > 0$, depending only of D, such that

$$\|w\|_{(H^2(D))^3} \le c_1 |Aw|, \quad \forall\, w \in D(A), \tag{3.10}$$

and so $D(A)$ is a Hilbert space with respect to the scalar product

$$(v, w)_{D(A)} = (Av, Aw).$$

For $u \in D(A)$ and $v \in (L^2(D))^3$, we define $(u \cdot \nabla)v$ as the element of $(H^{-1}(D))^3$ given by

$$\langle (u \cdot \nabla)v, w\rangle = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j\rangle, \quad \forall\, w \in (H_0^1(D))^3. \tag{3.11}$$

Observe that (3.11) is meaningful, since $H^2(D) \subset L^\infty(D)$, and $H_0^1(D) \subset L^6(D)$, with continuous injections. This implies that $u_i w_j \in H_0^1(D)$, and there exists a constant $c_2 > 0$, depending only on D, such that

$$|\langle (u \cdot \nabla)v, w\rangle| \le c_2 |Au||v|\|w\|, \quad \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{3.12}$$

Now, if $u \in D(A)$, then $\nabla u^* \in (H^1(D))^{3\times 3} \subset (L^6(D))^{3\times 3}$, and consequently, for $v \in (L^2(D))^3$, we have that $\nabla u^* \cdot v \in (L^{3/2}(D))^3 \subset (H^{-1}(D))^3$, with

$$\langle \nabla u^* \cdot v, w\rangle = \sum_{i,j=1}^{3} \int_D (\partial_j u_i) v_i w_j \, dx, \quad \forall\, w \in (H_0^1(D))^3. \tag{3.13}$$

It follows that there exists a constant $c_3 > 0$, depending only on D, such that

$$|\langle \nabla u^* \cdot v, w\rangle| \le c_3 |Au||v|\|w\|, \quad \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{3.14}$$

We have the following results (see [2] for the proofs).

Proposition 3.1. *For all $(u, w) \in D(A) \times D(A)$ and all $v \in (L^2(D))^3$, it holds*

$$\langle (u \cdot \nabla)v, w\rangle = -\langle \nabla w^* \cdot v, u\rangle. \tag{3.15}$$

Consider now the trilinear form defined by

$$b^\#(u, v, w) = \langle (u \cdot \nabla)v, w\rangle + \langle \nabla u^* \cdot v, w\rangle, \quad \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3.$$

Proposition 3.2. *The trilinear form $b^\#$ satisfies*

$$b^\#(u, v, w) = -b^\#(w, v, u), \quad \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times D(A), \tag{3.16}$$

and consequently,

$$b^\#(u, v, u) = 0, \quad \forall\, (u, v) \in D(A) \times (L^2(D))^3. \tag{3.17}$$

Moreover, there exists a constant $c > 0$, depending only on D, such that

$$|b^{\#}(u,v,w)| \leq c|Au||v|\|w\|, \quad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3, \quad (3.18)$$

$$|b^{\#}(u,v,w)| \leq c\|u\||v||Aw|, \quad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times D(A). \quad (3.19)$$

Thus, in particular, $b^{\#}$ is continuous on $D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.

3.2. *Statement of the problem*

Assume that F and G are measurable, Lipschitz and sublinear mappings from $\Omega \times (0,T) \times C^0([-h,0];V)$ into $(H^{-1}(D))^3$ and from $\Omega \times (0,T) \times C^0([-h,0];V)$ into $\mathcal{L}^2(K;(L^2(D))^3)$, respectively. More precisely, for all $\xi, \mu \in C^0([-h,0];V)$ and $u, v \in C^0([-h,T];V)$, $F(\cdot,\xi)$ and $G(\cdot,\xi)$ are $\mathcal{F}_t-$progressively measurable, and

$$\|F(t,\xi) - F(t,\mu)\|_{(H^{-1}(D))^3} \leq L_F \|\xi - \mu\|_{C^0([-h,0];V)}, \quad dP \times dt - \text{a.e.}, \quad (3.20)$$

$$\|G(t,\xi) - G(t,\mu)\|_{\mathcal{L}^2(K;(L^2(D))^3)} \leq L_G \|\xi - \mu\|_{C^0([-h,0];V)}, \ dP \times dt - \text{a.e.}, \quad (3.21)$$

$$F(\cdot,0) \in M^2_{\mathcal{F}_t}(0,T;(H^{-1}(D))^3), G(\cdot,0) \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2(K;(L^2(D))^3)), \quad (3.22)$$

$$\int_0^t \|F(s,u_s) - F(s,v_s)\|^2_{(H^{-1}(D))^3}\, ds \leq C_F \int_{-h}^t \|u(s) - v(s)\|_V^2\, ds, \quad (3.23)$$

$$\int_0^t \|G(s,u_s) - G(s,v_s)\|^2_{\mathcal{L}^2(K;(L^2(D))^3)}\, ds \leq C_G \int_{-h}^t \|u(s) - v(s)\|_V^2\, ds, \quad (3.24)$$

and for all decreasing $\rho \in C^0([0,T])$, and all $u, v \in C^0([-h,T];V)$ such that $u = v$ in $[-h,0]$.

$$\int_0^t \rho(s) \|F(s,u_s) - F(s,v_s)\|^2_{(H^{-1}(D))^3}\, ds \leq \widetilde{C}_F \int_0^t \rho(s) \|u(s) - v(s)\|_V^2\, ds, \quad (3.25)$$

$$\int_0^t \rho(s) \|G(s,u_s) - G(s,v_s)\|^2_{\mathcal{L}^2(K;(L^2(D))^3)}\, ds \leq \widetilde{C}_G \int_0^t \rho(s) \|u(s) - v(s)\|_V^2\, ds. \quad (3.26)$$

Finally, assume that

$$u^0 \in L^2(\Omega, \mathcal{F}_0, P; V) \ \text{ and } \ \phi \in M^2_{\mathcal{F}_t}(-h,0;V). \quad (3.27)$$

Definition 3.1. A variational solution to problem (1.1) is a stochastic process $u \in M^2_{\mathcal{F}_t}(0,T;D(A)) \cap M^2_{\mathcal{F}_t}(-h,T;V) \cap L^2(\Omega; L^\infty(0,T;V))$, weakly continuous with values in V, such that for all $w \in D(A)$,

$$((u(t),w)) + \nu \int_0^t (u(s) + \alpha Au(s), Aw)\, ds + \int_0^t b^{\#}(u(s), u(s) - \alpha\Delta u(s), w)\, ds$$
$$= ((u^0,w)) + \int_0^t \langle F(s,u_s), w\rangle\, ds + (\int_0^t G(s,u_s)\, dW(s), w), \quad t \in [0,T], \quad (3.28)$$

and coincides with ϕ in $(-h,0)$.

Observe that (3.28) can be easily deduced from (1.1) by multiplying the first equation in (1.1) by $w \in D(A)$, taking into account the definition of the scalar product $((\cdot,\cdot))$, the definition of $b^{\#}$, and the equality (3.15).

3.3. *The main results*

Our two major results are the following.

Proposition 3.3. *Under the hypotheses (3.20)-(3.27), there exists at most a variational solution of (1.1). Moreover, if u is the variational solution of (1.1), then $u \in L^2(\Omega; C([0,T]; V))$ and satisfies*

$$\begin{aligned}
&\|u(t)\|^2 + 2\nu \int_0^t (u(s) + \alpha A u(s), Au(s))\, ds \\
&\leq \|u^0\|^2 + 2\int_0^t \langle F(s, u_s), u(s)\rangle\, ds \qquad (3.29)\\
&+2\int_0^t (u(s), G(s,u_s) dW(s)) + \frac{1}{1+\alpha\mu_1}\int_0^t \|G(s,u_s)\|^2_{\mathcal{L}^2(K;(L^2(D))^3)}\, ds,
\end{aligned}$$

and

$$E\int_0^t (u(s), G(s,u_s)dW(s)) = 0,$$

for all $t \in [0,T]$, where μ_1 denotes the first eigenvalue of A.

Theorem 3.1. *In addition to the assumptions in Theorem 3.3, suppose that*

$$F(t,0) \in L^4(L^2(0,T;(H^{-1}(D))^3)), \quad G(t,0) \in L^4(L^2(0,T;\mathcal{L}^2(K;(L^2(D))^3))),$$

and, $u_0 \in L^4(\Omega, \mathcal{F}_0, P; V)$ and $\phi \in L^4(\Omega; L^2(-h,0;V))$ are $\mathcal{F}_t$-progressively measurable. Then, there exists a unique variational solution u of (1.1), and moreover,

$$u \in L^4(\Omega; C([0,T]; V)) \cap L^4(\Omega; L^2(0,T; D(A))) \cap L^4(\Omega; L^2(-h,T;V)).$$

In fact, there exists $C > 0$, depending only on α, ν, T, C_F and C_G, such that

$$\begin{aligned}
&E\left(\sup_{t\in[0,T]} \|u(t)\|^4\right) + E\left[\left(\int_0^T |Au(t)|^2\, dt\right)^2\right] \\
&\leq C\left[E(\|u^0\|^4) + E\left(\int_{-h}^0 \|\phi(t)\|^2\, dt\right)^2 + E\left[\left(\int_0^T \|F(t,0)\|^2_{(H^{-1}(D))^3}\, dt\right)^2\right]\right. \\
&\left. +E\left[\left(\int_0^T \|G(t,0)\|^2_{\mathcal{L}^2(K;(L^2(D))^3)}\, dt\right)^2\right]\right].
\end{aligned}$$

Moreover, associated to the variational solution u, there exists a unique $p \in L^1(\Omega, \mathcal{F}_t, P; H^{-1}(0,t;H^{-1}(D)))$, for all $t \in (0,T]$, such that P-a.s.,

$$\begin{aligned}
&\partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u\cdot\nabla)(u - \alpha\Delta u) \\
&-\alpha\nabla u^* \cdot \Delta u + \nabla p = F(t,u_t) + G(t,u_t)\dot{W}(t), \text{ in } (\mathcal{D}'((0,T)\times D))^3, \\
&\int_D p\, dx = 0, \text{ in } \mathcal{D}'(0,T),
\end{aligned}$$

where $G(t,u_t)\dot{W}(t)$ denotes the time derivative of $\int_0^t G(s,u_s)\, dW(s)$.

3.4. *Proofs of Proposition 3.3 and Theorem 3.1*

To prove these results we will check that Proposition 2.1 and Theorem 2.1 can be applied.

Let us consider $\mathcal{H} = V$, with $(u,v)_{\mathcal{H}} = ((u,v))$, and $\mathcal{U} = D(A)$, with $((u,v))_{\mathcal{U}} = (Au, Av)$. Let us define

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av), \quad u, v \in D(A).$$

It is clear that $\widetilde{A}$ satisfies a) and a1). Moreover, $\forall v \in D(A)$ it holds

$$2\langle \widetilde{A}v, v\rangle = 2\nu(Av, v) + 2\nu\alpha(Av, Av) = 2\nu \left\|\nabla v\right\|^2_{(L^2(D))^3} + 2\nu\alpha \left\|v\right\|^2_{D(A)} \geq 2\nu\alpha \left\|v\right\|^2_{D(A)},$$

and, if we denote by μ_k and w_k, $k \geq 1$, the eigenvalues and their corresponding eigenvectors associated to A,

$$\begin{aligned}\langle \widetilde{A}w_k, v\rangle &= \nu(Aw_k, v) + \nu\alpha(Aw_k, Av) = \nu(\mu_k w_k, v) + \nu\alpha(\mu_k w_k, Av)\\ &= \nu\mu_k\left[(w_k, v) + \nu\alpha(Aw_k, v)\right] = \nu\mu_k (w_k, v)_{\mathcal{H}}\end{aligned}$$

then $\widetilde{A}$ also satisfies (2.2) with $\widetilde{\alpha} = 2\nu\alpha$ and (2.3) with

$$\lambda_k = \nu\mu_k, \quad v_k = \frac{w_k}{\|w_k\|_{\mathcal{H}}} = \frac{w_k}{\sqrt{1+\alpha\mu_k}}.$$

On the other hand, consider $\widetilde{B}(u,v)$ and $\widetilde{F}(t,\xi)$ given by

$$\langle \widetilde{B}(u,v), w\rangle = b^{\#}(u, v - \alpha\Delta v, w), \quad \forall\,(u,v,w) \in D(A)\times D(A)\times D(A), \tag{3.30}$$

$$\langle \widetilde{F}(t,\xi), w\rangle = \langle F(t,\xi), w\rangle, \quad \forall\,(\xi, w) \in C^0([-h,0];V)\times D(A).$$

Then, it is straightforward to check that b), b1)-b3), c), c1)-c4) hold.
Finally, let I denote the identity operator in H and define $\widetilde{G}(t,\xi)$ as

$$\widetilde{G}(t,\xi) = (I+\alpha A)^{-1}\circ \mathcal{P}\circ G(t,\xi), \quad \forall\,(t,\xi) \in (0,T)\times C^0([-h,0];V).$$

First, observe that $I + \alpha A$ is bijective from $D(A)$ into H, and

$$(((I+\alpha A)^{-1} f, w)) = (f, w), \quad \forall\, f \in H, \quad \forall\, w \in V. \tag{3.31}$$

Thus, for each $f \in H$,

$$\|(I+\alpha A)^{-1} f\|^2 = (f, u) \leq |f|\,|u|, \tag{3.32}$$

where $u = (I+\alpha A)^{-1} f$, i.e., $(u, w_k) + \alpha(Au, w_k) = (f, w_k)$, $\forall k \geq 1$ so, $(1+\alpha\mu_k)(u, w_k) = (f, w_k)$, which implies

$$(u, w_k) = \frac{1}{(1+\alpha\mu_k)}(f, w_k) \leq \frac{1}{(1+\alpha\mu_1)}(f, w_k)$$

and

$$|u|^2 = \sum_{k=1}^{\infty}(u, w_k)^2 \leq \frac{1}{(1+\alpha\mu_1)^2}\sum_{k=1}^{\infty}(f, w_k) = \frac{1}{(1+\alpha\mu_1)^2}|f|^2,$$

and, therefore,

$$\|(I+\alpha A)^{-1}f\|^2 \leq \frac{1}{1+\alpha\mu_1}|f|^2.$$

Next, for each $j \geq 1$, and all $(t,u,w) \in (0,T) \times C^0([-h,T];V) \times D(A)$, we have

$$(G(t,u_t)e_j, w) = ((I+\alpha A)(\widetilde{G}(t,u_t)e_j), w) = ((\widetilde{G}(t,u_t)e_j, w)),$$

and, for all $(t,w) \in (0,T) \times D(A)$, it follows

$$(\int_0^t G(s,u_s)\, dW(s), w) = \sum_{j=1}^{\infty} \int_0^t (G(s,u_s)e_j, w)\, d\beta^j(s)$$

$$= \sum_{j=1}^{\infty} \int_0^t ((\widetilde{G}(t,u_t)e_j, w))\, d\beta^j(s) = ((\int_0^t \widetilde{G}(s,u_s)\, dW(s), w)).$$

Now, Proposition 2.1 and Theorem 2.1 implies Proposition 3.3 and the first part in Theorem 3.1. As for the existence of the pressure p, notice that from (3.12) and (3.14), we can deduce for $u \in M^2_{\mathcal{F}_t}(0,T;D(A))$ that

$$(u\cdot\nabla)(\Delta u) + \nabla u^* \cdot \Delta u \in M^1_{\mathcal{F}_t}(0,T;(H^{-1}(D))^3),$$

with

$$E\int_0^T \|(u\cdot\nabla)(\Delta u) + \nabla u(t)^* \cdot \Delta u(t)\|_{(H^{-1}(D))^3}\, dt \leq cE\int_0^T |Au(t)|^2\, dt.$$

On the other hand, if $u \in M^2_{\mathcal{F}_t}(0,T;D(A))$, then $u - \alpha\Delta u \in M^2_{\mathcal{F}_t}(0,T;(L^2(D))^3)$, and, as a consequence, $\partial_t(u-\alpha\Delta u) \in L^2(\Omega,\mathcal{F}_t,P;H^{-1}(0,T;(L^2(D))^3)$.
Also, if $u \in L^4(\Omega,\mathcal{F},P;C([0,T];V))$, and it is $\mathcal{F}_t$-progressively measurable, then $F(t,u_t) \in M^4_{\mathcal{F}_t}(0,T;(H^{-1}(D))^3)$, and arguing as in [8], if follows

$$G(t,u_t) \in L^4(\Omega,\mathcal{F}_t,P;W^{-1,\infty}(0,t;(L^2(D)^3))), \quad \forall\, t \in [0,T].$$

Finally, concerning the term $Au - \alpha\Delta(Au)$, we have that if $u \in M^2_{\mathcal{F}_t}(0,T;D(A))$, then $Au - \alpha\Delta(Au) \in M^2_{\mathcal{F}_t}(0,T;(H^{-2}(D))^3)$.

Reasoning again as in [8], and more precisely, by using Remark 4.3 from [8], it follows the uniqueness of p.

Acknowledgements. We would like to express our thanks to the anonymous referee for his/her useful suggestions and comments.

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