Nonlinear Partial Functional Differential Equations: Existence and Stability

Tomás Caraballo
Dpto. Ecuaciones Diferenciales y Análisis Numérico.
Universidad de Sevilla. Apdo. Correos 1160.
41080-SEVILLA. Spain.
e-mail: caraball@cica.es

Dedicated to Pepi, my wife.

Abstract

Existence and uniqueness of solutions for a class of nonlinear functional differential equations in Hilbert spaces are established. Sufficient conditions which guarantee the transference of exponential stability from partial differential equations to partial functional differential equations are studied. The stability results derived are also applied to ordinary differential equations with hereditary characteristics.

Keywords: Partial differential equation; partial functional differential equation; exponential stability.

AMS 2000 Classifications: 35R10.

1 Introduction

The study of functional differential equations is motivated by the fact that when one wants to model some evolution phenomena arising in Physics, Biology, Engineering, etc., some hereditary characteristics such as aftereffect, time lag and time delay can appear in the variables. Typical examples arise from the researches of materials with termal memory, biochemical reactions, population models, etc. (see, for instance, Hale and Lunel [8], Ruess [16]-[17], Webb [20], Wu [23] and the references therein). On the other hand, one important and interesting problem in the analysis of functional differential equations is the stability, the theory of which has been greatly developed over the last years.

As is well known, in the case without any hereditary features, Lyapunov's technique is available to obtain sufficient conditions for the stability of solutions of (partial) differential equations. However, in the case of differential equations with hereditary properties, for instance, even in the case of constant time delays, Lyapunov's method becomes difficult to apply effectively as Krasovskii [12] pointed out. The main reason is that it is much more difficult (or even impossible in some cases) to construct proper Lyapunov functions (or functionals) for functional differential equations than for those without any hereditary characteristics. As a consequence, a comparison technique has been developed by various authors such as Krasovskii [12] and Mao [14] (among others). Let us illustrate this point in more detail.

Consider the following one-dimensional delay differential equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-h)), \quad t > 0, \tag{1}$$

where $h > 0$, or equivalently,

$$\frac{dx(t)}{dt} = f(t, x(t), x(t)) + [f(t, x(t), x(t-h)) - f(t, x(t), x(t))] \tag{2}$$

Clearly if $h > 0$ is small enough, the perturbation term $f(t, x(t), x(t-h)) - f(t, x(t), x(t))$ could be expected to be so small that the perturbed equation (2) would behave asymptotically as equation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t)). \tag{3}$$

In particular, it can be proved that, under some circumstances, exponential stability is transferred from the nondelay equation (3) to the delay one (1) if the constant time lag $h > 0$ appearing in the problem is sufficiently small (see, e.g. [14]). So, in order to find out whether the functional equation (1) is exponentially stable, one can check the exponential stability of the equation (2) and then compute whether the time lag $h > 0$ is sufficiently small.

Nevertheless, it is worth pointing out that this kind of results can be somewhat restrictive for many practical applications. In fact, the situation turns out to be rather complicated when one considers the general functional differential equations, even the usual delay differential systems. To this respect, we should mention that in a wide variety of problems, the history of the phenomenon has a decisive influence on the future behaviour of the system, and, in some cases, not only a short period of the past has to be taken into account, but a large one. So it seems rather unnatural to look for results which hold only for small values of the deviating arguments. In this work, we shall carry out an investigation in this direction.

One of the main aims of this paper is to give sufficient conditions (which, in particular, may contain the corresponding results in finite dimension) in order to transfer the exponential stability of partial differential equations to partial functional differential equations. The problem we are referring to is devoted to the consideration of an infinite dimensional version of (1) in which f has the following form:

$$f(t, x, y) = A(t, x) + f_1(t, y),$$

with the family of (non-linear) operators $A(t, \cdot)$ satisfying some kinds of coercivity conditions (see Section 3) as well as f_1 satisfying Lipschitz continuous ones. We would also like to mention that, in some sense, a suitable coercivity condition implies the (exponential) stability of solutions in nondelay cases. In addition to this, we will be able to assure exponential stability for a great number of finite dimensional functional differential equations where the results in [14] only guarantee this kind of stability for constant and sufficiently small delays.

In Section 2, we begin with some preliminary results. Section 3 is devoted to establish some results on the existence and uniqueness of solutions for a class of partial functional differential equations in a variational context. Some results on exponential stability are studied in Section 4. Finally, several examples are given in Section 5 to illustrate the theory derived in the preceding sections.

2 Preliminaries

First of all, we introduce the framework in which our analysis is going to be carried out. Let V be a separable Banach space and H be a real separable Hilbert one such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

where V' is the dual of V and the injections are continuous and dense. We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V' respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V', V, and by $(\cdot,\cdot)$ the scalar product in H.

Let us take $h \geq 0$, $p \geq 1$ and $T > 0$, let $C_H = C(-h, 0; H)$ be the space of all continuous functions from $[-h, 0]$ into H with sup-norm $\|\psi\|_{C_H} = \sup_{-h\leq s\leq 0} |\psi(s)|$, $\psi \in C_H$, similarly, let $C_V = C(-h, 0; V)$, $L^p_V = L^p(-h, 0; V)$ and $L^p_H = L^p(-h, 0; H)$. Given a function $x(\cdot) \in L^p(-h, T; V) \cap C(-h, T; H)$, we associate with an $L^p_V \cap C_H$-valued function x_t, $t \geq 0$, by setting $x_t(s) = x(t + s)$, $s \in [-h, 0]$. The first purpose of this paper is to establish existence and uniqueness results for a class of nonlinear partial functional differential equations of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t, x(t)) + f(t, x_t), \quad t \in [0, T], \\ x(t) = \psi(t), \quad t \in [-h, 0], \end{cases} \tag{4}$$

where, in general, the operators are assumed to be nonlinear. In fact, we are interested in the case in which $A(t,\cdot) : V \to V'$ is a family of nonlinear monotone and coercive operators and $f(t,\cdot) : X \to H$ is Lipschitz continuous, where X will denote C_H, C_V, L^p_H or L^p_V. On the one hand, it is worth pointing out that, in many applications, A usually denotes a partial differential operator (linear or nonlinear), while f uses to be a first order partial differential one. On the other, we want to mention that the problem of existence of solutions to (4) and its stability have been previously analyzed by Travis and Webb in [18] for linear autonomous operator A generating a strongly continuous semigroup, and in [19] when A is the generator of an analytic semigroup by using the tools of the semigroup theory of operators. Also Webb [21] considers a similar problem for autonomous and accretive operator A by the same theory of (linear or not) semigroups of operators. Moreover, there exits a wide literature on the existence of different classes of solutions (strong, mild, integral, etc.) to functional differential equations even in the more general context of differential inclusions. It is well worth reading the work by Ruess [16] (see also [17]), where we can find a description of some of the different techniques used to handle with this question, in addition to a large list of references concerning these methods. Among others, let us mention, for the univalued case, the method of lines (cf. Kartsatos [9]), the Galerkin approximations (cf. Kartsatos and Parrott [10]), the Kato approximants (cf. Kartsatos and Parrott [11]), etc. Extensions to the multivalued framework can be found in several works included in the references in [16]. However, on the one hand, the case involving unbounded (linear or not) operators f (i.e. when f is a partial differential operator of first order), despite of its importance in applications, has only been treated in some specific and particular situations (see, for instance, Fitzgibbon [7], Aizicovici [1], Crandall *et al.* [5]) and not always in a systematic way; on the other hand, what is missed in the literature is a general treatment of this problem from a variational point of view (see Artola [2]-[3] for some particular linear and nonlinear partial differential equations with delays). Consequently, in this paper, we shall first establish some existence results by a variational type of argument similar to that one carried out by Lions [13] for a case without delays, but subject to necessary changes to make our scheme go through when $f(t,\cdot) : L^p_H \to H$. Then we will treat the more general case with $f(t,\cdot) : L^p_V \to H$ by using a Galerkin approximation technique.

3 Existence and uniqueness of solutions

Let $A(t,\cdot) : V \to V'$ be a family of (nonlinear) operators defined a.e.t. (for almost every t) and $p \geq 2$. Assume the following hypotheses:

Coercivity: $\exists \alpha > 0$, $\lambda, \nu \in \mathbb{R}$ such that:

$$-2\langle A(t,x), x\rangle + \lambda|x|^2 + \nu \geq \alpha\|x\|^p\,, \ \ \forall x \in V\,, \text{ a.e.}t.; \tag{5}$$

Monotonicity:

$$-2\langle A(t,x) - A(t,y),\, x-y\rangle + \lambda|x-y|^2 \geq 0\,, \ \ \forall x,\, y \in V\,, \text{ a.e.}t.; \tag{6}$$

Boundedness: $\exists\, \gamma > 0$:

$$\|A(t,x)\|_* \leq \gamma\|x\|^{p-1}\,, \ \ \forall x \in V\,, \text{ a.e.}t.; \tag{7}$$

Hemicontinuity:

$$\theta \in \mathbb{R} \to \langle A(t, x+\theta y), z\rangle \in \mathbb{R} \ \text{ is continuous } \ \forall x,\, y,\, z \in V\,, \text{a.e.}t.; \tag{8}$$

Measurability:

$$t \in (0,T) \to A(t,x) \in V' \ \text{ is Lebesgue-measurable } \ \forall x \in V\,, \ \text{ a.e.}t. \tag{9}$$

Let $f(t,\cdot): L^2_H \to H$ be a family of nonlinear operators defined a.e.t., and satisfy the following conditions:

$$\exists c_f \geq 0: \sup_{0\leq t\leq T} |f(t,0)| \leq c_f < +\infty; \tag{10}$$

$$\exists\, k_1 = k_1(h) > 0 \ : |f(t,\eta) - f(t,\xi)| \leq k_1\|\eta - \xi\|_{C_H}, \forall \eta,\ \xi \in C_H, \text{a.e.}t.; \tag{11}$$

$$t \in (0,T) \mapsto f(t,\eta) \in H \quad \text{ is Lebesgue-measurable } \quad \forall \eta \in L^2_H. \tag{12}$$

Given an initial value $\psi \in L^p(-h,0;V)\cap C(-h,0;H)$, the first objective in this Section is, under the conditions described above, to find a unique function $x(\cdot) \in L^p(-h,T;V) \cap C(-h,T;H)$ such that

$$\begin{cases} x(t) = \psi(0) + \int_0^t [A(s,x(s)) + f(s,x_s)]dt, & t \in [0,T], \\ x(t) = \psi(t), & t \in [-h,0], \end{cases} \tag{13}$$

where the first equality is understood in V'. We will refer to this solution as the variational solution to (4).

Remark 1 *(1) First, we notice that, although the results can be proved for $p > 1$, the interesting situations in the applications appear when $p \geq 2$. Because of this, we content ourselves with the analysis of the case $p \geq 2$.*
(2) We have fixed the initial data in the space $L^p(-h,0;V)\cap C(-h,0;H)$ just only to ensure that the solution belongs to $L^p(-h,T;V)\cap C(-h,T;H)$. However, we can of course take $\psi \in L^p_V$ and a value $x_0 \in H$ instead of $\psi(0)$ in the problem. In this case, the argument we will use provides a solution in the space $L^p(-h,T;V) \cap C(0,T;H)$.
(3) Although it is possible to extend the results derived here to more general sytems involving coefficients such that $f(t,x(t),x_t)$, we restrict ourselves to this more simple situation in order to avoid unnecessary technicalities.

Now we shall prove that there exists at most one solution of (13). This result will be deduced mainly from (6) and the energy equality.

Theorem 2 *Assume the preceding hypotheses hold. Then, there exists at most one solution of (13) in $L^p(-h,T;V) \cap C(-h,T;\ H)$.*

Proof. Suppose that $x,\, y \in L^p(-h,T;\ V) \cap C(-h,T;\ H)$ are two solutions of (13). Then, taking into account (6), we obtain

$$\begin{aligned} |x(t) - y(t)|^2 &= 2\int_0^t \langle A(s,x(s)) - A(s,y(s)), x(s) - y(s)\rangle ds \\ &\quad + 2\int_0^t (f(s,x_s) - f(s,y_s), x(s) - y(s))ds \\ &\leq \lambda \int_0^t |x(s) - y(s)|^2 ds \\ &\quad + 2\int_0^t |f(s,x_s) - f(s,y_s)||x(s) - y(s)|ds. \end{aligned}$$

Now, it follows from (11) that for any $t \in [0,T]$

$$\sup_{0\le s\le t} |x(s)-y(s)|^2 \le (|\lambda|+1)\int_0^t |x(s)-y(s)|^2 ds \tag{14}$$

$$+ k_1^2 \int_0^t ||x(s)-y(s)||_{C_H}^2 ds \tag{15}$$

On the other hand, since $x(s)=y(s)$ for $s\le 0$, we easily get

$$\begin{aligned} \textstyle\int_0^t ||x(s)-y(s)||_{C_H}^2 ds &= \textstyle\int_0^t \sup_{-h\le r\le 0} |x_s(r)-y_s(r)|^2 ds \\ &= \textstyle\int_0^t \sup_{-h\le r\le 0} |x(s+r)-y(s+r)|^2 ds \\ &\le \textstyle\int_0^t \sup_{0\le r\le s} |x(r)-y(r)|^2 ds. \end{aligned} \tag{16}$$

Thus, it follows from (14)–(16)

$$\sup_{0\le s\le t} |x(s)-y(s)|^2 \le 2\Big[|\lambda|+1+k_1^2\Big]\int_0^t \sup_{0\le r\le s} |x(r)-y(r)|^2 ds, \quad \forall t\in[0,T],$$

and Gronwall's lemma obviously implies uniqueness. ■

Remark 3 *Observe that if we assume the following monotonicity hypothesis*

For all $\xi, \eta \in L^p(-h,T;V)$ *with* $\xi_0 = \eta_0$ *it holds*

$$\begin{aligned} -2\langle A(t,\xi(t)) + f(t,\xi_t) - A(t,\eta(t)) - f(t,\eta_t), \xi(t)-\eta(t)\rangle \\ + \lambda|\xi(t)-\eta(t)|^2 \ge 0, \quad t\in[0,T], \end{aligned} \tag{17}$$

instead of (6), uniqueness is also easily deduced. Indeed, notice that in this case, (17) implies

$$|x(t)-y(t)|^2 \le \lambda \int_0^t |x(s)-y(s)|^2\, ds \quad \forall t\in[0,T],$$

for arbitrary two solutions x, y *of the problem.*

Now, before proving our first existence result, we shall state a theorem on existence and uniqueness of solutions of evolution equations.

Theorem 4 *Assume (5)–(9) hold with* $\lambda = 0$. *Then, given* $f_1 \in L^{p'}(0,T;V')$ *(with* $\frac{1}{p}+\frac{1}{p'}=1$*) and* $x_0\in H$, *there exists a unique function* $x\in L^p(0,T;V)\cap C(0,T;H)$ *such that*

$$x(t) = x_0 + \int_0^t [A(s,x(s)) + f_1(s)]ds, \quad \textit{for all } t\in[0,T].$$

In addition to this, the following energy equality holds:

$$|x(t)|^2 = |x_0|^2 + 2\int_0^t \langle A(s,x(s)) + f_1(s), x(s)\rangle ds, \quad t\in[0,T].$$

Proof. See Lions [13] (Theorem 1.2, page 162). ■

Theorem 5 *Assume that (5)–(9), (11) and (12) hold. Then, for each initial datum* $\psi \in L^p(-h,0;\,V)\cap C(-h,0;\,H)$*, there exists a unique solution of the problem (13) in* $L^p(-h,T;\,V)\cap C(-h,T;\,H)$.

Proof. Uniqueness follows from Theorem 2. For the existence, we consider the equations

$$\begin{cases} x^1(t) &= \psi(0)+\int_0^t \left[A(s,x^1(s))-\frac{\lambda}{2}x^1(s)\right]ds, \quad t\in[0,T], \\ x^1(t) &= \psi(t), \quad t\in[-h,0], \end{cases} \tag{18}$$

$$\begin{cases} x^{n+1}(t) &= \psi(0)+\int_0^t \left[A(s,x^{n+1}(s))-\frac{\lambda}{2}x^{n+1}(s)\right]ds \\ & \quad +\frac{\lambda}{2}\int_0^t x^n(s)ds+\int_0^t f(s,x_s^n)ds, \quad t\in[0,T], \forall n\geq 1, \\ x^{n+1}(t) &= \psi(t), \quad t\in[-h,0], \quad \forall n\geq 1. \end{cases} \tag{19}$$

By virtue of (5)–(9), the family $A_1(t,.) : V\to V'$ defined as $A_1(t,x)=A(t,x)-(\lambda/2)x$, satisfies assumptions in Theorem 4. Consequently, (18) has a unique solution $x^1\in L^p(-h,T;\,V)\cap C(-h,T;\,H)$. We note that, from (11) it follows that the mapping $t\in(0,T)\mapsto f(t,x_t^1)\in H$ belongs to $L^2(0,T;\,H)$. Consequently, bearing these remarks in mind, we can use Theorem 4 and get that there exists a unique function $x^2\in L^p(-h,T;\,V)\cap C(-h,T;\,H)$, which is the solution of (19) for $n=1$. By recurrence, we obtain a sequence of solutions of (18)–(19), $\{x^n\}_{n\geq1}\subset L^p(-h,T;\,V)\cap C(-h,T;\,H)$. Now, we want to prove that the sequence $\{x^n\}$ converges to a function x in $L^p(-h,T;\,V)\cap C(-h,T;\,H)$, which will be the solution of (13). For this end, we shall first prove the following lemmas.

Lemma 6 $\{x^n\}_{n\geq1}$ *is a Cauchy sequence in* $C(-h,T;\,H)$.

Proof. Indeed, it follows for $n\geq 2$

$$\begin{aligned} |x^{n+1}(t)-x^n(t)|^2 &= 2\int_0^t\langle A(x^{n+1})-A(x^n),x^{n+1}-x^n\rangle ds \\ &\quad -\lambda\int_0^t|x^{n+1}-x^n|^2ds \\ &\quad +\lambda\int_0^t(x^{n+1}-x^n,x^n-x^{n-1})ds \\ &\quad +2\int_0^t(f(x_s^n)-f(x_s^{n-1}),x^{n+1}-x^n)ds, \end{aligned} \tag{20}$$

where, for short we denote, $x^n:=x^n(s)$, $A(x^n):=A(s,x^n(s))$ and $f(x^n):=f(s,x_s^n)$. Now, it is easy to deduce from (6)

$$\begin{aligned} \sup_{0\leq\theta\leq t}|x^{n+1}(\theta)-x^n(\theta)|^2 &\leq |\lambda|\int_0^t|x^{n+1}-x^n||x^n-x^{n-1}|ds \\ &\quad +2\int_0^t|f(x_s^n)-f(x_s^{n-1})||x^{n+1}-x^n|ds. \end{aligned} \tag{21}$$

Firstly,

$$
\begin{aligned}
|\lambda| \int_0^t |x^{n+1} - x^n||x^n - x^{n-1}| ds \;\; &\leq \;\; \tfrac{1}{4} \sup_{0 \leq \theta \leq t} |x^{n+1}(\theta) - x^n(\theta)|^2 \\
&\quad + \lambda^2 T \int_0^t \sup_{0 \leq \theta \leq s} |x^n(\theta) - x^{n-1}(\theta)|^2 ds,
\end{aligned} \tag{22}
$$

and noticing that $x^n(s) = x^{n-1}(s)$ for $-h \leq s \leq 0$,

$$
\begin{aligned}
&2 \int_0^t |f(x_s^n) - f(x_s^{n-1})||x^{n+1} - x^n| ds \\
&\quad \leq \tfrac{1}{4T} \int_0^t |x^{n+1} - x^n|^2 ds + 4k_1^2 T \int_0^t ||x_s^n - x_s^{n-1}||_{C_H}^2 ds \\
&\quad \leq \tfrac{1}{4} \sup_{0 \leq \theta \leq t} |x^{n+1}(\theta) - x^n(\theta)|^2 + 4k_1^2 T \int_0^t \sup_{0 \leq \theta \leq s} |x^{n+1}(\theta) - x^n(\theta)|^2 ds.
\end{aligned} \tag{23}
$$

If we set $\varphi^n(t) = \sup_{0 \leq \theta \leq t} |x^{n+1}(\theta) - x^n(\theta)|^2$, it then follows from (21)–(23) that there exists a positive constant $k > 0$ such that

$$
\varphi^n(t) \leq k \int_0^t \varphi^{n-1}(s) \, ds \, . \tag{24}
$$

By iteration from (24), we get

$$
\varphi^n(t) \leq \frac{k^{n-2} T^{n-1}}{(n-2)!} \varphi^2(T) \, , \;\; \forall n \geq 2 \, , \;\; \forall t \in [0, T] \, . \tag{25}
$$

Therefore,

$$
\sup_{0 \leq \theta \leq T} |x^{n+1}(\theta) - x^n(\theta)|^2 \leq \frac{k^{n-2} T^{n-2}}{(n-2)!} \varphi^2(T) \, , \;\; \forall n \geq 2. \tag{26}
$$

Obviously, since $x^{n+1}(\theta) = x^n(\theta)$ for $\theta \in [-h, 0]$, (26) implies that $\{x^n\}$ is a Cauchy sequence in $C(-h, T; H)$. ■

Lemma 7 *The sequence $\{x^n\}$ is bounded in $L^p(-h, T; V)$.*

Proof. Indeed, for $n \geq 2$ we immediately obtain

$$
\begin{aligned}
|x^n(T)|^2 \;\; &= \;\; 2 \int_0^T \langle A(x^n), x^n \rangle ds - \lambda \int_0^T |x^n|^2 ds \\
&\quad + |\psi(0)|^2 + 2 \int_0^T (f(x^{n-1}), x^n) ds + \lambda \int_0^T (x^{n-1}, x^n) ds.
\end{aligned} \tag{27}
$$

Therefore,

$$
\begin{aligned}
&-2 \int_0^T \langle A(x^n), x^n \rangle ds + \lambda \int_0^T |x^n|^2 ds \\
&\quad \leq |\psi(0)|^2 + 2 \int_0^T |f(x^{n-1})||x^n| ds + |\lambda| \int_0^T |x^{n-1}||x^n| ds.
\end{aligned} \tag{28}
$$

Since $\{x^n\}$ converges in $C(-h, T; H)$, it will be bounded in this space. Now, it is not difficult to check that there exists a positive constant $k' > 0$ such that the right-hand side of (28) is bounded by this constant. For instance, we will estimate one of those terms. Firstly, we observe that

$$\int_0^T ||x_s^{n-1}||_{C_H}^2 \quad \leq \quad \int_0^T \sup_{-h\leq\theta\leq s} |x^{n-1}(\theta)|^2 ds$$

Next, (10) and (11) imply

$$\begin{aligned} 2\int_0^T |f(x^{n-1})||x^n| ds &\leq \int_0^T \left[|f(x^{n-1})|^2 + |x^n|^2\right] ds \\ &\leq T\left(k_1||x^{n-1}||_{C(-h,T;H)} + c_f\right)^2 + T||x^n||^2_{C(0,T;H)}, \end{aligned} \tag{29}$$

which, in addition to (28) and (5), leads to

$$\alpha \int_0^T \|x^n(s)\|^p \, ds \leq -2\int_0^T \langle A(x^n), x^n \rangle \, ds + \lambda \int_0^T |x^n|^2 \, ds + \nu T \leq k',$$

and the lemma is proved. ∎

Now in order to complete the proof of the theorem, we shall prove that the limit of the sequence $\{x^n\}$ is a solution to (13).

Firstly, observe that Lemma 6 implies that there exists $x \in C(-h,T;\, H)$ such that $x^n \to x$ in $C(-h,T;\, H)$. Now, thanks to (11), we have $f(x^n) \to f(x)$ (in $L^\infty(0,T;\, H)$). On the other hand, by virtue of Lemma 7, $\{x^n\}$ has a subsequence which converges weakly in $L^p(-h,T;\, V)$. But, since $x^n \to x$ in $C(-h,T;\, H)$, we can assure that $x^n \to x$ weakly in $L^p(-h,T;\, V)$ (in the sequel, we will denote this by $x^n \rightharpoonup x$ in $L^p(-h,T;\, V)$). Nevertheless, it follows from (7) that $\{A(x^n)\}$ is bounded in $L^{p'}(0,T;\, V')$ (with p' such that $(1/p)+(1/p')=1$), since

$$\int_0^T \|A(t,x^n(t))\|_*^{p/(p-1)} \, dt \leq \gamma \int_0^T \|x^n(t)\|^p dt \leq \gamma k'/\alpha.$$

Therefore, from each subsequence of $\{A(x^n)\}$, we can get another subsequence weakly convergent in $L^{p'}(0,T;\, V')$. Now, as it is easy to see that all the limits of different subsequences coincide, we finally get that $A(x^n) \rightharpoonup v$ in $L^{p'}(0,T;\, V')$. In conclusion, we have proved:

$$x^n \to x \ \text{ in } \ C(0,T;\, H), \tag{30}$$

$$f(x^n) \to f(x) \ \text{ in } \ L^\infty(0,T;\, H), \tag{31}$$

$$x^n \rightharpoonup x \ \text{ in } \ L^p(-h,T;\, V), \tag{32}$$

$$A(x^n) \rightharpoonup v \ \text{ in } \ L^{p'}(0,T;\, V'). \tag{33}$$

Finally, by the virtue of (30)-(33), we can take limits in (19) and obtain

$$x(t) = \psi(0) + \int_0^t v(s)\, ds + \int_0^t f(s,x_s)\, ds. \tag{34}$$

Thus, in order to finish the proof of the theorem, it is sufficient to prove that $A(s,x(s)) = v(s)$ in $L^{p'}(0,T;V')$. To this respect, (27) yields to

$$\begin{aligned} 2\int_0^T \langle A(x^n), x^n\rangle\, ds &= \lambda\int_0^T |x^n|^2 ds + |x^n(T)|^2 - |\psi(0)|^2 \\ &\quad -2\int_0^T (f(x^{n-1}), x^n) ds - \lambda\int_0^T (x^{n-1}, x^n) ds, \end{aligned} \tag{35}$$

which, together with (30)–(31), immediately implies

$$\lim_{n\to\infty} 2\int_0^T \langle A(x^n), x^n\rangle\, ds = |x(T)|^2 - |\psi(0)|^2 - 2\int_0^T (f(x), x) ds. \tag{36}$$

However, (34) and the energy equality yield

$$\lim_{n\to\infty} \int_0^T \langle A(x^n), x^n\rangle\, ds = \int_0^T \langle v, x\rangle\, ds. \tag{37}$$

Thanks to (6),

$$-2\int_0^T \langle A(x^n) - A(z), x^n - z\rangle\, ds + \lambda\int_0^T |x^n - z|^2\, ds \geq 0 \tag{38}$$

for all $z \in L^p(0,T;\,V) \cap L^2(0,T;\,H)$. Nevertheless, (30), (32) and (33) allow us to take limits in (38) and we have

$$-2\int_0^T \langle v - A(z), x - z\rangle\, ds + \lambda\int_0^T |x - z|^2\, ds \geq 0\,. \tag{39}$$

Now, if we set $z = x - \theta z_2$ (for $\theta > 0$, $z_2 \in L^p(0,T;\,V) \cap L^2(0,T;H)$), we obtain

$$-2\int_0^T \langle v - A(x - \theta z_2), \theta z_2\rangle\, ds + \lambda\theta^2\int_0^T |z_2|^2\, ds \geq 0\,. \tag{40}$$

In (40), we divide by θ, take limit as $\theta \to 0$ and then use hemicontinuity (8) to obtain

$$-\int_0^T \langle v - A(x), z_2\rangle\, ds \geq 0\,,\ \ \forall z_2 \in L^p(0,T;\,V) \cap L^2(0,T;H)\,, \tag{41}$$

and therefore $v = A(x)$. Since (34) now holds with $v = A(x)$, the proof of the Theorem is finally complete. ■

Remark 8 *We would like to point out that, in many situations, it is convenient to consider another norm $\|\cdot\|_{L^2_H}$ instead of $\|\cdot\|_{C_H}$ for initial datum spaces in (13). The arguments used in our preceding analysis still carry through when the norm $\|\cdot\|_{C_H}$ is replaced by $\|\cdot\|_{L^2_H}$ in (11).*

In what follows, we shall investigate an existence and uniqueness result for (13) in a more general situation which, in particular, allows f to contain some gradient terms in the applications. Precisely, let us assume hypotheses (5)–(9) for the family of operators $A(t,\cdot)$. Suppose $f(t,\cdot): C_V \to H$ is a family of nonlinear operators defined a.e.t. and satisfying:

$$\text{There exists } c_f \geq 0 \text{ such that } \sup_{t\in[0,T]} |f(t,0)| \leq c_f; \tag{42}$$

$$\exists\, k_1 = k_1(h) > 0 : |f(t,\eta) - f(t,\xi)| \leq k_1 \|\eta - \xi\|_{C_V},\ \forall \eta, \xi \in C_V, \text{ a.e. } t; \tag{43}$$

$$t \in (0,T) \mapsto f(t,\eta) \in H \ \text{ is Lebesgue-measurable}, \forall \eta \in L^2_V. \tag{44}$$

$$\begin{array}{l} \exists C_f > 0 \text{ such that } \forall t \in [0,T], \forall u, v \in C([-h,T];V), \\ \int_0^t |f(s,u_s) - f(s,v_s)|^2\, ds \leq L_f \int_{-h}^t \|u(s) - v(s)\|^2\, ds. \end{array} \tag{45}$$

Observe that (42)-(44) imply that given $u \in C(-h,T;V)$, the function $f_u : t \in [0,T] \mapsto f_u(t) \in H$ defined by $f_u(t) = f(t,u_t)$, for $t \in [0,T]$, is measurable (see Bensoussan et al. [4]) and, in fact, belongs to $L^\infty(0,T;H)$. Then, thanks to (45), the mapping $\mathcal{F} : u \in C(-h,T;V) \to f_u \in L^2(0,T;H)$ has a unique extension to a mapping $\widetilde{\mathcal{F}}$ which is uniformly continuous from $L^2(-h,T;V)$ to $L^2(0,T;H)$. From now on, we will denote $f(t,u_t) = \widetilde{\mathcal{F}}(u)(t)$, for each $u \in L^2(-h,T;V)$, and thus, $\forall t \in [0,T], u, v \in L^2(-h,T;V)$ we will have that (45) holds.

Now we can establish our general existence and uniqueness result. To this end, we will use a Galerkin approximation which requires the existence of a orthonormal basis in H whose elements are in V. More precisely, let us assume that $\{v_1, v_2, ..., v_n, ...\} \subset V$ is an orthonormal basis of H. Let us denote by P_n the orthogonal projector from H onto the vector space generated by $\{v_1, ... v_n\}$, and assume that the sequence $\{\|P_n\|_{\mathcal{L}(H,V)}\}_{n\geq 1}$ is bounded.

Theorem 9 *In addition to (5)–(9) and (42)–(45), suppose the following hypothesis holds:*

(M): For all ξ, $\eta \in L^p(-h,T;V)$ *with* $\xi_0 = \eta_0$ *it holds for a.e.* $t \in [0,T]$,

$$-2\,\langle A(t,\xi(t)) + f(t,\xi_t) - A(t,\eta(t)) - f(t,\eta_t), \xi(t) - \eta(t)\rangle + \lambda |\xi(t) - \eta(t)|^2 \geq 0$$

Then, for each $\psi \in L^p(-h,0;V) \cap C(-h,0;H)$ *there exists a unique solution of the problem (13) in* $L^p(-h,T;V) \cap C(-h,T;H)$.

Proof. Uniqueness follows immediately from Assumption (M) and Gronwall's lemma.

As for the existence, we shall split the proof into the following steps.

STEP 1. ***Finite-dimensional approximation.***

Let $\{v_1, v_2, ..., v_n, ...\}$ be the orthonormal basis of H with $v_i \in V$ for all $i \geq 1$. Let $V_n = H_n = V_n'$ denote the vector space generated by $\{v_1, ... v_n\}$. Let $P_n \in \mathcal{L}(H, H_n)$ be the orthogonal projection from H onto H_n. Then, P_n can be extended to an operator $\tilde{P}_n$ from V' onto V_n' in the following way

$$\tilde{P}_n u = \sum_{i=1}^n \langle u, v_i \rangle v_i,\ u \in V'.$$

Now we consider the problem

$$\begin{cases} \dfrac{d}{dt}(x^n(t), v_i) = \langle A(t, x^n(t)) + f(t, x_t^n), v_i \rangle, \quad 1 \leq i \leq n, \\ x^n(t) = P_n \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{46}$$

This equation can be rewritten in an equivalent way as follows. Let $A^n(t, \cdot)$ denote the family of operators from V_n into V_n' defined as $A^n(t,x) = \tilde{P}_n A(t,x)$, $x \in V_n$ and assume $f^n(t,\cdot) : L^2_{H_n} \to H_n$ is given by $f^n(t,\xi) = P_n f(t,\xi)$ for $\xi \in L^2_{H_n}$ Then, Eq. (46) can be rewritten as

$$\begin{cases} \dfrac{d}{dt}(x^n(t), v_i) = \langle A^n(t, x^n(t)) + f^n(t, x_t^n), v_i \rangle, \\ x^n(t) = \psi^n(t) = P_n \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{47}$$

Although Eq. (47) can be considered as a functional differential equation in $\mathbb{R}^n$, we can not apply the classic results on existence and uniqueness of solutions since, in general, A^n does not satisfy a Lipschitz condition. However, we can apply to this situation the results proved in the preceding section, i.e. Theorem 4 with $\|\cdot\|_{C_H}$ replaced by $\|\cdot\|_{L^2_H}$ in (11). Indeed, it is easy to check that A^n, f^n and ψ^n satisfy the assumptions in Theorem 4 by replacing V, H, V' by V_n, H_n, V_n'. Therefore, for each natural number $n \geq 1$, there exists a unique $x^n \in L^p(-h,T;V_n) \cap C(-h,T;\, H_n)$ which is the solution to (47). Owing to the natural injections, we have that, in fact,

$$x^n \in L^p(-h,T;\, V) \cap C(-h,T;\, H).$$

STEP 2. A priori computations.

As in the proof of Theorem 4, we set $x^n := x^n(s), A^n(x^n) := A^n(s, x^n(s))$ and $f^n(x^n) := f^n(s, x_s^n)$. The energy equality, (42)-(45) and (5) yield that, for any $\varepsilon > 0$ (chosen later),

$$\begin{aligned} |x^n(t)|^2 &= |\psi(0)|^2 + 2\int_0^t \langle A^n(x^n), x^n \rangle \, ds + 2\int_0^t (f^n(x^n), x^n) \, ds \\ &\leq |\psi(0)|^2 + \lambda \int_0^t |x^n|^2 ds + \nu t - \alpha \int_0^t \|x^n\|^p \, ds \\ &\quad + \varepsilon \int_0^t |f^n(x^n)|^2 \, ds + \varepsilon^{-1} \int_0^t |x^n|^2 ds, \end{aligned}$$

and, evaluating

$$
\begin{aligned}
\varepsilon \int_0^t |f^n(x^n)|^2\, ds &\le \varepsilon \int_0^t |f^n(s, x_s^n) - f^n(s,0) + f^n(s,0)|^2\, ds \\
&\le 2\varepsilon \int_0^t |f^n(s, x_s^n) - f^n(s,0)|^2\, ds + 2\varepsilon \int_0^t |f^n(s,0)|^2\, ds \\
&\le 2\varepsilon c_f t + 2\varepsilon L_f \int_{-h}^t \|x^n(s)\|^2\, ds \\
&\le 2\varepsilon c_f t + 2\varepsilon L_f \int_{-h}^t (1 + \|x^n(s)\|^p)\, ds \\
&\le 2\varepsilon c_f t + 2\varepsilon L_f \left[t + h + C \int_{-h}^0 \|\psi(s)\|^p\, ds \right] \\
&+ 2\varepsilon L_f \int_0^t \|x^n(s)\|^p\, ds,
\end{aligned}
$$

so, for ε small enough, namely for ε such that $2\varepsilon L_f < \alpha$, we have

$$
\begin{aligned}
|x^n(t)|^2 + (\alpha - 2\varepsilon L_f) \int_0^t \|x^n\|^p ds \le |\psi(0)|^2 + \nu t + 2\varepsilon c_f t& \\
+ 2\varepsilon L_f \left[t + h + C \int_{-h}^0 \|\psi(s)\|^p\, ds \right]& \\
+ \left(\lambda + \varepsilon^{-1}\right) \int_0^t |x^n|^2 ds.&
\end{aligned}
$$

In particular, this immediately implies

$$
\begin{aligned}
\sup_{0 \le t \le T} |x^n(t)|^2 \le |\psi(0)|^2 + \nu t + 2\varepsilon c_f t& \\
+ 2\varepsilon L_f \left[t + h + C \int_{-h}^0 \|\psi(s)\|^p\, ds \right]& \\
+ \left(|\lambda| + \varepsilon^{-1}\right) \int_0^T |x^n|^2 ds.&
\end{aligned}
$$

So, we finally have that

$$
\begin{aligned}
&\{x^n\}_{n \ge 1} \text{ is bounded in } L^p(-h, T;\, V) \cap C(-h, T;\, H), \\
&\{A(x^n)\}_{n \ge 1} \text{ is bounded in } L^{p'}(0, T;\, V'), \\
&\{f(x^n)\}_{n \ge 1} \text{ is bounded in } L^2(0, T;\, H),
\end{aligned}
$$

where $A(x^n)$ and $f(x^n)$ are defined in the obvious way and p' denotes the conjugate of p.

STEP 3. Taking weak limits

Owing to the last assertions in Step 2, we can claim that there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that

$$\begin{aligned} x^{n_k} &\rightharpoonup x \text{ in } L^p(-h,T;\,V) \text{ and weakly star in } L^\infty(-h,T;\,H),\\ x^{n_k}(T) &\rightharpoonup \xi \text{ in } H,\\ A(x^{n_k}) &\rightharpoonup \chi \text{ in } L^{p'}(0,T;\,V'),\\ f(x^{n_k}) &\rightharpoonup \sigma \text{ in } L^2(0,T;\,H), \end{aligned}$$

Let $\theta : \mathbb{R} \to \mathbb{R}$ be defined as

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

If φ is a function from $[0,T]$ into $\mathbb{R}$, we can define another function $\overline{\varphi} : (-\rho, T+\rho) \to \mathbb{R}$ (where ρ is a positive fixed number) in the following way:

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in [0,T] \\ 0 & \text{otherwise.} \end{cases}$$

This permits us to rewrite Eq. (46) (with $n = n_k$) as follows

$$\begin{aligned} \overline{(x^{n_k}(t), v_i)} &= (\psi(0), v_i)\theta(t) - (x^{n_k}(T), v_i)\theta(t-T) \\ &\quad + \int_0^t \overline{\langle A(x^{n_k}) + f(x^{n_k}), v_i \rangle} ds, \quad i = 1, 2, ..., n_k. \end{aligned} \tag{48}$$

Now, we can take weak limits in (48) and obtain, for all $t \in (-\rho, T+\rho)$,

$$\overline{(x(t), v_i)} = (\psi(0), v_i)\theta(t) - (\xi, v_i)\theta(t-T) + \int_0^t \langle \chi + \sigma, v_i \rangle \, ds, \tag{49}$$

so it follows that

$$\begin{aligned} &\xi = x(T) \\ &dx(t) = (\chi(t) + \sigma(t))dt, \quad t \in [0,T] \\ &x(t) = \psi(t), \quad t \in [-h, 0]. \end{aligned} \tag{50}$$

Therefore, it remains to prove that $\chi + \sigma = A(x) + f(x)$.

STEP 4. Final step.

Now we are going to use the monotonicity method. Consider $v \in L^p(-h,T;\,V) \cap L^2(-h,T;\,H)$ and set

$$\begin{aligned} u^{n_k} &= -2\int_0^T e^{-\lambda s} \langle A(x^{n_k}) + f(x^{n_k}) - A(v) - f(v), x^{n_k} - v \rangle \, ds \\ &\quad + \lambda \int_0^T e^{-\lambda s} |x^{n_k} - v|^2 ds. \end{aligned} \tag{51}$$

Note that $u^{n_k} \geq 0$ due to Assumption (M). On the other hand, we can take limits in the terms of (51) except for the following one

$$y^{n_k} = -2\int_0^T e^{-\lambda s} \langle A(x^{n_k}) + f(x^{n_k}), x^{n_k} \rangle \, ds + \lambda \int_0^T e^{-\lambda s} |x^{n_k}|^2 ds. \tag{52}$$

But, we immediately obtain that

$$|x^{n_k}(t)|^2 = |P_{n_k}\psi(0)|^2 + 2\int_0^t \langle A(x^{n_k}) + f(x^{n_k}), x^{n_k}\rangle\, ds. \tag{53}$$

In particular, (53) proves that the function $t \mapsto |x^{n_k}(t)|^2$ is absolutely continuous and hence

$$d\left[e^{-\lambda t}|x^{n_k}(t)|^2\right] + \lambda e^{-\lambda t}|x^{n_k}(t)|^2 = e^{-\lambda t} d|x^{n_k}(t)|^2. \tag{54}$$

Now, it follows that

$$\begin{aligned} e^{-\lambda T}|x^{n_k}(T)|^2 \quad &\leq \quad |P_{n_k}\psi(0)|^2 - \lambda\int_0^T e^{-\lambda t}|x^{n_k}(t)|^2 dt \\ &\quad +2\int_0^T e^{-\lambda t}\langle A(x^{n_k}) + f(x^{n_k}), x^{n_k}\rangle dt \end{aligned} \tag{55}$$

and, therefore

$$y^{n_k} \leq |\psi(0)|^2 - e^{-\lambda T}|x^{n_k}(T)|^2. \tag{56}$$

As a straightforward consequence, it follows that

$$\limsup_{k\to\infty} y^{n_k} \leq |\psi(0)|^2 - e^{-\lambda T}|x(T)|^2. \tag{57}$$

Applying once again the energy equality for Eq. (50),

$$e^{-\lambda T}|x(T)|^2 = |\psi(0)|^2 - \lambda\int_0^T e^{-\lambda t}|x|^2 dt + 2\int_0^T e^{-\lambda t}\langle \chi + \sigma, x\rangle dt. \tag{58}$$

So

$$\limsup_{k\to\infty} y^{n_k} \leq \int_0^T e^{-\lambda t}\left[2\langle \chi + \sigma, x\rangle + \lambda|x|^2\right] dt, \tag{59}$$

and finally,

$$\begin{aligned} 0 \leq \limsup_{k\to\infty}\, u^{n_k} \quad &\leq \quad -2\int_0^T e^{-\lambda t}\langle \chi + \sigma - A(v) - f(v), x - v\rangle dt \\ &\quad +\lambda\int_0^T e^{-\lambda t}|x - v|^2 dt. \end{aligned} \tag{60}$$

In order to finish the proof, we only need to use hemicontinuity (8). Indeed, we notice that the function f also satisfies a similar property and, in fact, it is easy to deduce from (45) that the map $\theta \in \mathbb{R} \mapsto \int_0^T (f(t, \eta + \theta\xi), x)dt \in \mathbb{R}$ is continuous for all $\eta, \xi \in L^2_V$, $x \in H$ and a.e.$t \in [0, T]$. Now, setting $v = x - \theta u$ for $\theta > 0$ and $u \in L^p(-h, T;\, V) \cap L^2(-h, T;\, H)$ in (60), dividing by θ and letting θ tend to 0, we then get $\forall u \in L^p(-h, T;\, V) \cap L^2(-h, T;\, H)$

$$-2\int_0^T e^{-\lambda t}\langle \chi + \sigma - A(x) - f(x), u\rangle dt \geq 0.$$

Consequently, $\chi + \sigma = A(x) + f(x)$ and the proof of the theorem is complete. ■

4 STABILITY OF SOLUTIONS

In this section we shall show that under suitable conditions exponential stability can be transferred from equations without time lags to those with time lag ones. Since we are mainly interested in exponential stability problems, we will assume there exists $x \in L^2(-h,T;V) \cap C(-h,T;H)$, $\forall T > 0$, which is the variational solution of the following problem:

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t,x(t)) + f(t,x_t) \quad t > 0, \\ x(t) = \psi(t), \quad t \in [-h,0]. \end{cases} \tag{61}$$

In other words, $x(t)$ satisfies the following integral equation (in V'):

$$\begin{cases} x(t) = \psi(0) + \int_0^t [A(s,x(s)) + f(s,x_s)]ds, \quad t \geq 0, \\ x(t) = \psi(t), \ \ t \in [-h,0]. \end{cases} \tag{62}$$

In particular, we suppose in this section that all conditions in Section 3 hold so that there exists a unique solution to the functional differential equation (61).

First of all, we investigate the case without hereditary characteristics. In other words, consider Eq. (61) with $h = 0$ and thus $k_1 > 0$ in (11) does not depend on h. Then Eq. (61) reduces to

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t,x(t)) + f(t,x(t)), \quad t \geq 0, \\ x(0) = x_0. \end{cases} \tag{63}$$

If it is possible to know the existence of a Lyapunov function, we could prove exponential stability of solutions. Indeed, assume there exist $v \in C^2(H;\mathbb{R}^+)$ and positive constants c_i, $1 \leq i \leq 4$, such that $v'(x) \in V$ for all $x \in V$ and

$$c_1|x|^2 \leq v(x) \leq c_2|x|^2, \ \ \mathbf{L}v(x) \leq -c_3 v(x), \ \ |v'(x)| \leq c_4|x|,$$

for all $x \in V$, where $\mathbf{L}$ is the associated Lyapunov operator defined as

$$\mathbf{L}v(x) = \langle A(t,x) + f(t,x), v'(x)\rangle, \quad \forall x \in V.$$

We can get for the function $e^{c_3 t}v(x)$, and $x \in H$

$$\begin{aligned} e^{c_3 t}v(x(t)) \ &= \ v(x(0)) + c_3 \int_0^t e^{c_3 s} v(x(s))ds \\ &\quad + \int_0^t e^{c_3 s}\langle A(s,x(s)) + f(s,x(s)), v'(x(s))\rangle ds. \end{aligned}$$

Observing that $\mathbf{L}v(x) \leq -c_3 v(x)$, we have

$$v(x(t)) \leq e^{-c_3 t} v(x(0)), \quad \forall t \geq 0,$$

and from the assumptions on v, we easily deduce that

$$|x(t)|^2 \leq \frac{c_2}{c_1} e^{-c_3 t}|x(0)|^2, \quad \forall t \geq 0.$$

Consequently, what we have proved is that every solution to (61) converges exponentially to zero, even if zero is not a stationary solution. In particular, if $f(t,0) = 0$ for all $t \geq 0$, our result implies that the trivial solution to (61) is globally exponentially stable.

Although, as we have mentioned before, the construction of Lyapunov functions is not, in general, a trivial problem, there exists a condition which makes $v(x) = |x|^2$ become a natural Lyapunov function. We are refering to the following hypothesis

(H): there exists a positive constant $\gamma > 0$ such that

$$2\langle A(t,x) + f(t,x), x\rangle \leq -\gamma|x|^2, \forall x \in V$$

(observe that on this occasion $\mathbf{L}v(x) = 2\langle A(t,x) + f(t,x), x\rangle \leq -\gamma|x|^2$).

Remark 10 *In what follows, we assume that $f(t,0) = 0$, for all $t \geq 0$ since we are interested in analysing the stability of the trivial solution to our problem. To this respect, it is worth mentioning that in a variety of practical situations, the following assumption (H)$'$ (which seems easier to check) implies (H):*
(H)$'$: There exists a positive constant $\widehat{\alpha} > 0$ such that

$$-2\langle A(t,x), x\rangle \geq \widehat{\alpha}|x|^2, \forall x \in V \text{ and } -\widehat{\alpha} + 2k_1 < 0,$$

where k_1 is the nonnegative constant in (11).

Indeed, note that

$$\begin{aligned} 2\langle A(t,x) + f(t,x), x\rangle &\leq -\widehat{\alpha}|x|^2 + 2(f(t,x), x) \\ &\leq (-\widehat{\alpha} + 2k_1)|x|^2 \end{aligned}$$

and denoting $\gamma = \widehat{\alpha} - 2k_1$, assumption (H) follows.

Now, we shall show that the same hypotheses as above (mainly (11) and (H)$'$) imply exponential stability of the trivial solution to the functional differential equation (61). However, it is particularly worth pointing out that on this occasion the constants k_1 is generally dependent on the time lag constant $h > 0$. This fact simply means that, in order to obtain exponential stability, the time lag must be sufficiently small. However, as will be shown by the examples in the final section (see also Theorem **??**), on some occasions such as the time delay case, the constant k_1 could be independent on $h > 0$ so that the stability holds true for any $h > 0$. For our ends, let us firstly study some stability criteria for the functional differential equation (61) by using a Razumikhin type argument.

Theorem 11 *Assume that operators A and f are continuous with respect to time t, and that there exists a positive constant $\lambda > 0$ such that for all $t \geq 0$,*

$$2\langle A(t,\phi(0)) + f(t,\phi), \phi(0)\rangle < -\lambda|\phi(0)|^2 \tag{64}$$

provided $\phi = \{\phi(s) : -h \leq s \leq 0\} \in C(-h, 0; V)$ *satisfying*

$$\|\phi\|_{C_H}^2 \leq e^{\lambda h}|\phi(0)|^2. \tag{65}$$

Then, there exists a positive constant $K \geq 1$ *such that for all* $\psi \in C(-h, 0; V)$ *for which the solution to (61) corresponding to the initial value* ψ*, denoted by* $x(t, \psi)$*, belongs to* $C(-h, T; V)$ *for all* $T > 0$*, satisfies*

$$|x(t, \psi)|^2 \leq K||\psi||_{C_H}^2 \cdot e^{-\lambda t}, \quad \textit{for all} \;\; t \geq 0. \tag{66}$$

Proof. Suppose the assertion does not hold. Then, for any $K \geq 1$ there exists an initial data ψ such that (66) is not true. Let us fix a $K \geq 1$ and take the corresponding ψ. Thus, we can affirm that there exists a $\rho \geq 0$ such that

$$e^{\lambda t}|x(t; \psi)|^2 \leq e^{\lambda \rho}|x(\rho; \psi)|^2 = K||\psi||_{C_H}^2, \tag{67}$$

for all $0 \leq t \leq \rho$, and there is a sequence $\{t_k\}_{k \geq 1}$ in $\mathbb{R}^+$ such that $t_k \downarrow \rho$, as $k \to \infty$, and

$$e^{\lambda t_k}|x(t_k; \psi)|^2 > e^{\lambda \rho}|x(\rho; \psi)|^2. \tag{68}$$

On the other hand, by virtue of (67) we deduce

$$|x(\rho + \theta; \psi)|^2 \leq e^{\lambda(\rho - t)}|x(\rho; \psi)|^2 \leq e^{\lambda h}|x(\rho; \psi)|^2,$$

for all $-h \leq \theta \leq 0$, which, in view of the assumptions (64)-(65), immediately implies that

$$2 \langle A(\rho, x(\rho)) + f(\rho, x_\rho), x(\rho; \psi) \rangle < -\lambda |x(\rho)|^2.$$

By the continuity of the solution and the functions A and f, we see that for some sufficiently small $h > 0$,

$$2 \langle A(t, x(t)) + f(t, x_t), x(t; \psi) \rangle < -\lambda |x(t)|^2,$$

for all $t \in [\rho, \rho + h]$. Thus, for all sufficiently small $h > 0$,

$$\begin{aligned} & e^{\lambda(\rho+h)}|x(\rho + h; \psi)|^2 - e^{\lambda \rho}|x(\rho; \psi)|^2 \\ & \quad = \textstyle\int_\rho^{\rho+h} e^{\lambda t} \left[\lambda |x(t; \psi)|^2 + 2 \langle A(t, x(t; \psi)) + f(t, x_t), x(t; \psi) \rangle \, dt\right] \\ & \quad \leq 0. \end{aligned}$$

However, this contradicts (68), so the result (66) must be true. ■

Theorem 12 *Suppose the assumptions in Section 3 for existence and uniqueness of solutions to (13) hold. In addition to (11) and (H)′, assume that operators* A *and* f *are continuous with respecto to time* t*. Then, there exist* $\lambda > 0$ *and* $K \geq 1$ *such that for all* $\psi \in C(-h, 0; V)$ *for which the solution to (13) belongs to* $C(-h, T; V)$ *for all* $T > 0$*, i.e.* $x(\cdot; \psi) \in C(-h, T; V)$*, it follows*

$$|x(t; \psi)|^2 \leq K \cdot ||\psi||_{C_H}^2 \cdot e^{-\lambda t}, \quad \textit{for all} \; t \geq 0.$$

Proof. As $0 < \widehat{\alpha} - 2k_1$, we can take a sufficiently small positive λ such that $0 < \widehat{\alpha} - 2k_1 e^{\lambda h}$. Now if $t \geq 0$ and $\phi = \{\phi(s) : -h \leq s \leq 0\} \in C([-h, 0]; V)$ satisfies

$$\|\phi\|_{C_H} \leq e^{\lambda h} |\phi(0)|,$$

then Assumption (11) and condition (H)$'$ imply

$$2\langle A(t, \phi(0)) + f(t, \phi), \phi(0)\rangle \quad \leq \quad \left[-\widehat{\alpha} + 2k_1 e^{\lambda h}\right] |\phi(0)|^2. \tag{69}$$

Therefore, in view of Theorem 11 the proof is complete. ■

Remark 13 *In the finite dimensional case, that is, when $V = H = \mathbb{R}^n$, Theorem 12 guarantees exponential stability for the solutions to (61) if (H)$'$ is fulfilled, where (H)$'$ can be rewritten now as*
(H)$''$ There exists a positive constant $\widehat{\alpha}$ such that

$$-2x^T A(t, x) \geq \widehat{\alpha} |x|^2, \qquad \forall x \in \mathbb{R}^n \quad and \quad -\widehat{\alpha} + 2k_1 < 0,$$

where x^T denotes the transpose of x. Notice that this is the same condition deduced by Hale and Lunel [8], so our assumption becomes a natural extension of the finite-dimensional results. Futhermore, under this condition the results in Mao [14] only ensure exponential stability if the delay function there is $t - h$ with h sufficiently small.

Remark 14 *Observe that the stability result just proved requires some regularity conditions on the problem and some additional continuity assumptions on the operators. Nevertheless, for some particular cases, better results could be obtained. As an example, we shall prove below an exponential stability result for the case of a partial differential equation with variable delay.*

To this respect, let us consider $F : H \to H$ globally Lipschitz with constant $\widehat{k_1}$ and $F(0) = 0$, $\omega : \mathbb{R}^+ \to [0, h]$ continuously differentiable with $\omega'(t) \leq 0$ for all $t \geq 0$, and define $f(t, \phi) = F(\phi(-\omega(t))$, for $\phi \in C(-h, 0; H)$. This situation corresponds to the problem

$$\begin{cases} \frac{dx(t)}{dt} = A(t, x(t)) + F(x(t - \omega(t)), \quad t > 0, \\ x(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{70}$$

Theorem 15 *In the preceding situation, assume that*

$$-2\langle A(t, x), x\rangle \geq \widehat{\alpha} |x|^2, \forall x \in V \;\; with \; -\widehat{\alpha} + 2\widehat{k_1} < 0.$$

Then, there exist $\lambda > 0, K \geq 1$ such that for every $\psi \in C_H$, the solution to (70), denoted again $x(t, \psi)$, satisfies

$$|x(t; \psi)|^2 \leq K \cdot ||\psi||^2_{C_H} \cdot e^{-\lambda t}, \quad for\ all\ t \geq 0.$$

Proof. Let us denote $\rho(t) = t - \omega(t), t \geq 0$. Then, ρ is a continuously differentiable function with $\rho'(t) \geq 1$. This immediately implies that $\rho^{-1}(t) \leq t + \rho^{-1}(0) = t + k$. Now, we can choose $\lambda > 0$ such that $\lambda - \widehat{\alpha} + \widehat{k}_1(1 + e^{\lambda k}) < 0$, and, by setting $x(t) = x(t;\psi)$, it follows that

$$\begin{aligned}
e^{\lambda t} |x(t)|^2 \leq |\psi(0)|^2 &+ \int_0^t e^{\lambda s} |x(s)|^2 \, ds + 2 \int_0^t e^{\lambda s} \langle A(s, x(s)), x(s) \rangle \, ds \\
&+ 2 \int_0^t e^{\lambda s} (F(x(\rho(s)), x(s)) \, ds \\
&\leq ||\psi||^2_{C_H} + (\lambda - \widehat{\alpha}) \int_0^t e^{\lambda s} |x(s)|^2 \, ds \\
&+ 2\widehat{k}_1 \int_0^t e^{\lambda s} |x(\rho(s))| \, |x(s)| \, ds \\
&\leq ||\psi||^2_{C_H} + (\lambda - \widehat{\alpha} + \widehat{k}_1) \int_0^t e^{\lambda s} |x(s)|^2 \, ds \\
&+ \widehat{k}_1 \int_0^t e^{\lambda s} |x(\rho(s))|^2 \, ds,
\end{aligned}$$

and evaluating the last term by using the change of variables $u = \rho(s)$ in the integral, it holds

$$\begin{aligned}
\int_0^t e^{\lambda s} |x(\rho(s))|^2 \, ds &\leq \int_{-h}^t e^{\lambda u + \lambda k} |x(u)|^2 \, du \\
&\leq \int_{-h}^0 e^{\lambda u + \lambda k} |\psi(u)|^2 \, du + \int_0^t e^{\lambda u + \lambda k} |x(u)|^2 \, du \\
&\leq h e^{\lambda k} ||\psi||^2_{C_H} + e^{\lambda k} \int_0^t e^{\lambda u} |x(u)|^2 \, du.
\end{aligned}$$

Thus, we have

$$e^{\lambda t} |x(t)|^2 \leq (1 + \widehat{k}_1 h e^{\lambda k}) ||\psi||^2_{C_H} + (\lambda - \widehat{\alpha} + \widehat{k}_1 + \widehat{k}_1 e^{\lambda k}) \int_0^t e^{\lambda s} |x(s)|^2 \, ds,$$

and therefore the proof is complete. ∎

5 Examples

Now, we are going to apply the results proved in the previous sections to obtain stability of some functional differential equations. First of all, we consider a general situation concerning the semilinear case previously studied by several authors (see Travis and Webb [18],[19], Martin and Smith [15], Wu [23], among others).

Example 1. In our variational setting, consider a linear operator $A \in \mathcal{L}(V, V')$ satisfying the coercivity condition (5) with $\nu = 0$. If we set $D(A) := \{v \in V : Av \in H\}$, then the operator A restricted to this set $D(A)$ becomes a closed and densely defined operator. Moreover, (see

Dautray and Lions [6] for the details) A is the generator of a strongly continuous semigroup of linear operators in H, denoted by $S(t)$, which satisfies:

$$|S(t)| \leq e^{\lambda t/2}, \forall t \geq 0.$$

Conversely, if $S(t)$ is a strongly continuous semigroup satisfying, for some $\omega \in \mathbb{R}$,

$$|S(t)| \leq e^{\omega t}, \forall t \geq 0,$$

then, it is not difficult to prove that its generator A satisfies

$$\langle Ax, x\rangle \leq \omega |x|^2, \forall x \in D(A).$$

Now, consider the semilinear problem

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + f(x_t), t \geq 0, \\ x(t) = \psi(t), t \in [-h, 0]. \end{cases} \tag{71}$$

where we assume that f is continuous, satisfies the Lipschitz condition (11) and $f(0) = 0$. Then, by applying the results in [18], we obtain that there exists a unique mild solution to (71) which, in addition, is a strong solution (whose existence is also ensured by our theory). On the other hand, if $\omega < 0$ and $k_1 < -\omega$, the results in [18] guarantee that the null solution to (71) is exponentially asymptotically stable. As we can easily see, the same result follows from our theory, since on this occasion, (H)$'$ holds with $\widehat{\alpha} = -2\omega$, and therefore, $-\widehat{\alpha} + 2k_1 < 0$ is equivalent to $\omega + k_1 < 0$.
In conclusion, our theory gives a natural extension of the results previously obtained in the semilinear case.

Example 2. Consider the following semilinear heat equation with finite time lags r_1, r_2 ($r > r_1, r_2 \geq 0$) and with $\mu > 0, \alpha_1 \geq 0$

$$\begin{aligned} &\frac{d}{dt}X(t,x) = \mu \frac{\partial^2}{\partial x^2}X(t,x) + \alpha_1 \int_{-r_1}^{0} X(t+u,x)h(u)du + \alpha(X(t))X(t-r_2,x), t \geq 0, \\ &\quad X(t,0) = X(t,\pi) = 0, t \geq 0, X(s,x) = \phi(s,x), \phi(\cdot,x) \in C := C(-r,0;\mathbb{R}), \\ &\quad \phi(s,\cdot) \in L^2(0,\pi), s \in [-r,0], x \in [0,\pi], \|\phi\|_C < \infty, \end{aligned}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$, $h : [-r_1, 0] \to \mathbb{R}$ are two bounded, Lipschitz continuous function with $|\alpha(x)| \leq K$, $|h(u)| \leq M$, $x \in \mathbb{R}$, $u \in [-r_1, 0]$, M, $K > 0$. Define $V = H_0^1[0,\pi]$, $H = L^2[0,\pi]$ with the corresponding boundary conditions above. Let $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$\mathcal{D}(A) = \Big\{u \in L^2(0,\pi),\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2} \in L^2(0,\pi), u(0) = u(\pi) = 0\Big\},$$

so it is easy to deduce

$$2\langle Au, u\rangle \leq -2\mu \|u\|^2, u \in V.$$

On the other hand, it is clear that

$$\left|\alpha_1 \int_{-r_1}^{0} X(t+u,\cdot)h(u)du\right|^2 \leq 2\left[(\alpha_1 r_1 M)^2 + K^2\right] \|X_t\|_C^2.$$

By a straightforward computation and applying Theorems 11, 12 to the above equation, if $\mu > 2\left[(\alpha_1 r_1 M)^2 + K^2\right]$ and $r_2 > 0$ is arbitrary, then the null solution is exponentially stable.

Example 3. Let us now exhibit a final nonlinear example. Let $D = [0,1], 2 < p < +\infty, r > 0$ and consider the following nonlinear partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}u(t,x) = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u(t,x)}{\partial x}\right|^{p-2} \dfrac{\partial u(t,x)}{\partial x}\right) - a(x)u(t,x) + f(t,u_t(x)), t > 0, x \in D, \\ x(t,x) = \psi(t,x), t \in [-r,0], x \in D, \\ x(t,0) = x(t,1) = 0, t > 0, \end{cases} \tag{72}$$

where $a \in L^\infty(D), a(x) \geq \widetilde{a} > 0,$ for all $x \in D$, and $f(t,\phi) = g(\phi(-\omega(t)))$, where $\omega : \mathbb{R} \to [0,r]$ is a measurable function and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant $L > 0$ and satisfying $g(0) = 0$. If we set $H = L^2(D), V = W_0^{1,p}(D)$ and consider the operator A defined as

$$\langle Au, v\rangle = -\int_D \left[\left|\frac{\partial u(x)}{\partial x}\right|^{p-2} \frac{\partial u(x)}{\partial x}\frac{\partial v(x)}{\partial x} + a(x)u(x)v(x)\right] dx, \quad \forall u,v \in V.$$

Then it is not difficult to check that assumptions (5)-(9) hold with $\lambda = -2\widetilde{a}, \alpha = 2$. Consequently, condition (H)$'$ holds provided $-\widetilde{a} + L < 0$. So, we get exponential stability.

6 Conclusions and final remarks

We have proved some results on the existence and uniqueness of solutions to a nonlinear partial functional differential equation with finite delay by using a variational approach. Then, by means of a Razumikhin type argument, we have established a stability result in the case of additional regularity of the problem.

However, nothing has been said in this context concerning the existence of solutions in the case of infinite delays. Also, a study of the multivalued version of our problem could be analyzed and some more stability results should be obtained. Finally, a comparison of these results and the ones previously proved by the evolution operator approach seems to be an interesting task which should be done.

We plan to investigate these questions in some subsequent papers.

Acknowledgments. The author wishes express his thanks to the anonymous referee for helpful suggestions. He also wants to thank Prof. W. Ruess for the useful correpondence on the topic of this paper.
This work has been partially supported by DGICYT Project (Spain) PB98-1134 and Junta de Andalucia FQM733.

References

[1] S. Aizicovici, On a semilinear Volterra integrodifferential equation, *Israel J. Math.* 36 (1980), 273-284.

[2] M. Artola, Sur les perturbations des équations d'evolution, application a des problèmes de retard, *Ann. scient. Ec. Norm. Sup.* 4 série, t. 2 (1969), 137-253.

[3] M. Artola, Equations et inequations variationnelles a retard, *Ann. Sc. Math. Québec* vol. I, no. 2 (1977), 131-152.

[4] A. Bensoussan, G. Da Prato, M.C. Delfour and S.K. Mitter, *Representation and Control of Infinite Dimensional Systems,* Vol. I, Systems and Control: Foundations and Applications, Birkhäuser, Boston (1992).

[5] M.G. Crandall, S.O. Londen and J.A. Nohel, An Abstract Nonlinear Volterra Integrodifferential Equation, *J. Math. Anal. Appl.* 64 (1978), 701-735.

[6] R. Dautray and J.L. Lions, *Analyse Mathèmatique et Calcul Numérique pour les Sciences et les Techniques,* Masson, Paris (1984).

[7] W.E. Fitzgibbon, Asymptotic behavior of solutions to a class of Volterra integrodifferential equations, *J. Math. Anal. Appl.* 146 (1990), 241-253.

[8] J.K. Hale and S.M.V Lunel, *Introduction to Functional Differential Equations,* Springer-Verlag, New York, Inc. (1993).

[9] A.G. Kartsatos, On the construction of methods of lines for functional evolutions in general Banach spaces, *Nonlinear Analysis TMA* 25 (1995), 1321-1331.

[10] A.G. Kartsatos and M.E. Parrott, Existence of solutions and Galerkin approximations for nonlinear functional evolution equations, *Tôhoku Math. J. (2)* 34 (1982), 509-523.

[11] A.G. Kartsatos and M.E. Parrott, Convergence of the Kato approximants for evolution equations involving functional perturbations, *J. Diff. Eqns.* 47 (1983), 358-377.

[12] N.N. Krasovskii, *Stability of motions,* Stanford University Press, Stanford (1963).

[13] J.L. Lions, *Quelque méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier- Villars, Paris (1969).

[14] X. Mao, Exponential stability of nonlinear differential delay equations, *Systems and Control Letters* 28 (1996), 159-165.

[15] R.H. Martin and H.L. Smith, Abstract functional differential equations and reaction-diffusion systems, *Trans. Am. Math. Soc.* 321 (1990), 1-44.

[16] W. Ruess, Existence of solutions to partial functional differential equations with delay, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, (ed. A.G. Kartsatos), Marcel Dekker, New York (1996), 259-288.

[17] W. Ruess, Existence and stability of solutions to partial functional differential equations with delay, *Adv. Diff. Eqns.* 4 (6) (1999), 843-876.

[18] C.C. Travis and G.F. Webb, Existence and stability for partial functional differential equations, *Trans. Am. Math. Soc. 200 (1974), 395-418.*

[19] C.C. Travis and G.F. Webb, Existence, stability, and compactness in the $\alpha-$norm for partial functional differential equations, *Trans. Am. Math. Soc.* 240 (1978), 129-143.

[20] G.F. Webb, Autonomous nonlinear functional differential equations and nonlinear semigroups, *J. Math. Anal. Appl.* 46 (1974), 1-12.

[21] G.F. Webb, Functional differential equations and nonlinear semigroups in L^p-spaces, *J. Diff. Eqn.* 20 (1976), 71-89.

[22] G.F. Webb, Asymptotic stability for abstract nonlinear functional differential equations, *Proc. Am. Math. Soc.* 54 (1976), 225-230.

[23] J. Wu, *Theory and applications of partial functional differential equations,* Springer-Verlag, New York (1996).