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NON-AUTONOMOUS ATTRACTORS FOR INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS

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Abstract. We show that infinite-dimensional integro-differential equations which involve an integral of the solution over the time interval since starting can be formulated as non-autonomous delay differential equations with an infinite delay. Moreover, when conditions guaranteeing uniqueness of solutions do not hold, they generate a non-autonomous (possibly) multi-valued dynamical system (MNDS). The pullback attractors here are defined with respect to a universe of subsets of the state space with sub-exponetial growth, rather than restricted to bounded sets. The theory of non-autonomous pullback attractors is extended to such MNDS in a general setting and then applied to the original integro-differential equations. Examples based on the logistic equations with and without a diffusion term are considered.

1. **Introduction.** The main aim of this paper is to show that a wide class of integro-differential partial differential equations can be analyzed within the framework of non-autonomous dynamical systems, and the long-time behaviour of their solutions can be investigated with the help of the theory of pullback attractors.

This theory is now well established as has been extensively developed over the last one and a half decades. Pullback attractors have proven to be appropriate concepts to describe the long–time behaviour of many dynamical systems arising in science, especially those exhibiting non-autonomity (see, e.g. Caraballo *et al.* [15], Cheban *et al.* [20], Chepyzhov and Vishik [21]), Chueshov [22], Crauel and Flandoli [23], Flandoli and Schmalfuß [25], Kloeden [28], Kloeden and Schmalfuß [29], Robinson [33], Schmalfuß [34], amongst many others).

Integro-differential equations appear in various branches of science (e.g. in modelling the growth of parasite population, in Lotka-Volterra predator-prey systems, in reaction-diffusion models with memory, and their relevance is without doubt. In general, the models containing in their equations some kind of delay terms are now being studied extensively, since it is assumed that in many phenomena from reality,

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the principle of causality does not seem appropriate, and it is assumed that the past history of the phenomena have a decisive influence in the future evolution of the systems.

There are now many papers dealing with the asymptotic behaviour of ordinary or partial differential equations within the framework of the pullback theory for non-autonomous dynamical systems. In principle, as soon as we have an equation with a non-constant delay term, the problem becomes non-autonomous. However, we have not found in the literature any papers concerning its applications to integro-differential equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + F(u) + \int_0^t G(t,s,u(s))ds, \tag{1}$$

where, for instance, A is a linear operator and F, G are nonlinear, in an infinite dimensional Banach state space H. On the other hand, for a finite dimensional state space, we are aware only of some indirectly related papers (see below).

The integral term, essentially a memory term, in equation (1) means that it is in effect a differential equation with unbounded (infinite) delay. Caraballo *et al.* [10, 16] used such a formulation when the space H is finite dimensional for logistic–like equations involving an integral over the entire negative time axis of a function of the solution. The following observation shows that equation (1) can also be formulated as a differential equation with infinite delay. Considering only the last term in the equation and denoting by u_t the segment solution defined for $s \leq 0$ as $u_t(s) = u(t+s)$, a change of variables gives

$$\begin{aligned}\int_0^t G(t,s,u(s))ds &= \int_{-t}^0 G(t,t+s,u(t+s))ds \\ &= \int_{-t}^0 G(t,t+s,u_t(s))ds = \mathbb{G}(t,u_t),\end{aligned}$$

where $\mathbb{G}$ is defined in a suitable phase space C_γ, a Banach subspace of $C(-\infty,0;H)$ satisfying appropriate additional assumptions (e.g. the $\lim_{t\to-\infty} u(t)e^{\gamma t}$ exists for a suitable weight γ). In other words, $\mathbb{G} : \mathbb{R}\times\mathbb{R}\times C_\gamma \to H$ is defined as

$$\mathbb{G}(t,\phi) = \int_{-t}^0 G(t,t+s,\phi(s))ds.$$

Then, equation (1) can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + F(u) + \mathbb{G}(t,u_t)$$

and even the term $F(u)$ can be included in the delay term by setting

$$\mathbb{F}(t,u_t) = F(u_t(0)) + \mathbb{G}(t,u_t)$$

and then our model becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + \mathbb{F}(t,u_t). \tag{2}$$

An analogous situation holds when the Banach space H is infinite dimensional, but leads to an abstract functional partial differential equation.

Although we could carry out our investigation working directly with equation (1), we prefer to develop a general abstract theory for equation (2), and then analyze our motivating model as a particular case, since our results then also apply to many other situations.

We will assume very weak assumptions on the operators in our model so that uniqueness of solutions will be not guaranteed from the very beginning. For this reason we need the theory of multi-valued (or set-valued) non-autonomous dynamical systems.

The structure of the paper is the following. We recall the definition of a non-autonomous set-valued dynamical system in the next section and, then, in section 3 we present the definition and properties of pullback attractors of such systems along with statements of theorems, which will be proved in the appendix, for their existence. The pullback attractors here are defined with respect to a universe of subsets of the state space with sub-exponential growth, rather than restricted to bounded sets. In section 4 we show that general class of infinite-dimensional non-autonomous differential equations with infinite delay generate such a non-autonomous set-valued dynamical system which establish the existence of a pullback attractor under certain structural conditions. Two examples are given in section 5, both with a logistic structure with an integral term, one with and one without an additional diffusion term. An appendix contains proofs of theorems presented earlier in the paper as well as some results that were used earlier.

2. **Non-autonomous set-valued dynamical systems.** First we recall some basic definitions for set–valued non–autonomous dynamical systems and establish a sufficient condition for the existence of a pullback attractor for these systems. For a more general random context the reader is referred to [8]

Let $X=(X,d_X)$ denote a Polish space, let 2^X be the set of all subsets of X and let $\mathcal{P}_c(X)$ be the set of all non-empty closed subsets of the space X. A mapping D $: t\in\mathbb{R}\to D(t)\in 2^X$ is called a multi–function or set-valued mapping. We denote by $\mathcal{C}(X)$ the set of all multi–functions $D: t\in\mathbb{R}\to D(t)\in 2^X$ with closed and non–empty images and use the notation $\widehat{D}=\{D(t): t\in\mathbb{R}\}$ for any element in $\mathcal{C}(X)$.

A multi–valued map $U:\mathbb{R}^2_d\times X\to\mathcal{P}_c(X)$, where $\mathbb{R}^2_d:=\{(t,s)\in\mathbb{R}^2: t\geq s\}$, is called a *multi–valued non–autonomous dynamical system* (MNDS) [12, 13, 14] if

i) $U(s,s,\cdot)=\mathrm{id}_X(\cdot)$ for all $s\in\mathbb{R}$,

ii) $U(t,\tau,x)\subset U(t,s,U(s,\tau,x))$ *for all* $\tau\leq s\leq t,\ x\in X$ *(process property),*

where $U(t,\tau,V):=\cup_{x_0\in V}U(t,\tau,x_0)$ for any non–empty set $V\subset X$.

Moreover, an MNDS is said to be *strict* if

iii) $U(t,\tau,x)=U(t,s,U(s,\tau,x))$ *for all* $\tau\leq s\leq t$ and $x\in X$,

and to be *upper–semicontinuous* at x_0 if

iv) *for every neighborhood* $\mathcal{U}$ *in* X *of the set* $U(t,\tau,x_0)$ *there exists* $\delta>0$ *such that* $U(t,\tau,y)\in\mathcal{U}$ *whenever* $d_X(x_0,y)<\delta$.

Finally, $U(t,\tau,\cdot)$ is said to be upper–semicontinuous, if it is upper–semicontinuous at every x_0 in X.

We note that, if the mapping $U(t,\tau,\cdot)$ is upper–semicontinuous at x_0, then for all $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\mathrm{dist}_X(U(t,\tau,y),U(t,\tau,x_0))\leq\varepsilon,$$

for any y satisfying $d_X(y,x_0)\leq\delta(\varepsilon)$, where dist_X denotes the *Hausdorff semi-distance* which is defined for two non-empty sets A, B as

$$\mathrm{dist}_X(A,B)=\sup_{x\in A}\inf_{y\in B}d_X(x,y).$$

The converse is true when $U(t,\tau,x_0)$ is compact, see Aubin and Cellina [2].

3. **Pullback attractors for MNDS.** We will now establish a sufficient condition ensuring the existence of pullback attractors with respect to a universe of sets (as in [15]). When this universe consists of bounded sets, the results have already been proved in [12].

A multi–valued mapping $\widehat{D} = \{D(t) : t \in \mathbb{R}\}$ is said to be *negatively, strictly, or positively invariant* (resp.) for the MNDS U if

$$D(t) \subset, =, \supset \text{ (resp.) } U(t,\tau,D(\tau)) \quad \text{for } (t,\tau) \in \mathbb{R}^2_d.$$

Let $\mathcal{D}$ be the family of multi–valued mappings with values in $\mathcal{C}(X)$. We say that a family $\widehat{K} \in \mathcal{D}$ is *pullback $\mathcal{D}$-attracting* if for every $\widehat{D} \in \mathcal{D}$

$$\lim_{\tau\to+\infty} \mathrm{dist}_X(U(t,t-\tau,D(t-\tau)),K(t)) = 0, \text{ for all } t \in \mathbb{R}.$$

$\widehat{B} \in \mathcal{D}$ is said to be *pullback $\mathcal{D}$-absorbing* if for every $\widehat{D} \in \mathcal{D}$ and every $t \in \mathbb{R}$, there exists $T := T(t,\widehat{D}) > 0$ such that

$$U(t,t-\tau,D(t-\tau)) \subset B(t) \text{ for all } \tau \geq T. \tag{3}$$

The following definition provides the main objective of this article. For this we need a particular set system called a *universe* (see Schmalfuß [34]): *Let $\mathcal{D}$ be a set of multi–valued mappings in $\mathcal{C}(X)$ satisfying the inclusion closure property: suppose that $\widehat{D} \in \mathcal{D}$ and let $\widehat{D'}$ be a multi–valued mapping in $\mathcal{C}(X)$ such that $D'(t) \subset D(t)$ for $t \in \mathbb{R}$, then $\widehat{D'} \in \mathcal{D}$.*

Definition 3.1. A family $\widehat{A} \in \mathcal{D}$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS U if it satisfies:

i): $A(t)$ is compact for any $t \in \mathbb{R}$;
ii): $\widehat{A}$ is pullback $\mathcal{D}$-attracting;
iii): $\widehat{A}$ is negatively invariant.

$\widehat{A}$ is said to be a strict global pullback $\mathcal{D}$-attractor if the invariance property in the third item is strict.

As usual, the main tool to prove the existence of an attractor is the concept of pullback-omega-limit set. For a multi–valued mappings $\widehat{D}$ we define the pullback-omega-limit set as the t–dependent set $\Lambda(\widehat{D},t)$ given by

$$\Lambda(\widehat{D},t) = \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} U(t,t-s,D(t-s))}.$$

This set is closed, but it may be empty. It can be proved that $y \in \Lambda(\widehat{D},t)$ if and only if there exist $t_n \to +\infty$ and $y_n \in U(t,t-t_n,D(t-t_n))$ such that

$$\lim_{n\to+\infty} y_n = y.$$

We then have the following lemma, which is a generalization of Theorem 6 and Lemma 8 in Caraballo *et al.* [12] to the case in which we consider a general universe $\mathcal{D}$ instead of just the bounded sets of X. The proof is given in the appendix.

Lemma 3.2. *Assume the MNDS* $U(t,\tau,\cdot)$ *is upper–semicontinuous for* $(t,\tau)\in\mathbb{R}^2_d$,. *Let* $\widehat{B}$ *be a multi–valued mapping such that the MNDS is asymptotically compact with respect to* $\widehat{B}$, *i.e., for every sequence* $t_n\to+\infty$, $t\in\mathbb{R}$, *every sequence* $y_n\in U(t,t-t_n,B(t-t_n))$ *is pre–compact.*

Then, for $t\in\mathbb{R}$, *the pullback–omega–limit set* $\Lambda\left(\widehat{B},t\right)$ *is non–empty, compact, and*

$$\lim_{\tau\to+\infty}\operatorname{dist}_X(U(t,t-\tau,B\,(t-\tau)),\Lambda\left(\widehat{B},t\right))=0, \tag{4}$$

$$\Lambda\left(\widehat{B},t\right)\subset U\left(t,\tau,\Lambda\left(\widehat{B},\tau\right)\right),\ \textit{for all}\ (t,\tau)\in\mathbb{R}^2_d. \tag{5}$$

We can now present a sufficient condition ensuring the existence of pullback attractor. The proof of this theorem is also given in the appendix.

Theorem 3.3. *Assume the hypotheses in Lemma 3.2. In addition, suppose that* $\widehat{B}\in\mathcal{D}$ *is pullback* $\mathcal{D}$*–absorbing. Then, the set* $\widehat{A}$ *given by*

$$A\,(t):=\Lambda\left(\widehat{B},t\right)$$

is a pullback $\mathcal{D}$*-attractor. Moreover,* $\widehat{A}$ *is the unique element from* $\mathcal{D}$ *with these properties.*

In addition, if U *is a strict MNDS then* $\widehat{A}$ *is strictly invariant.*

4. **MNDS generated by infinite-delay partial differential equations.** In this section we consider the following evolution equation

$$\frac{dy}{dt}=Ay+f(t,y_t), \tag{6}$$

which includes, in particular, our integro-differential model (1).

Here we suppose that A is the generator of a C_0 contraction semigroup $(\mathrm{e}^{At})_{t\geq0}$ on a separable Banach space $(H,\|\cdot\|)$ such that

$$\|\mathrm{e}^{At}x\|\leq\|x\|e^{-\alpha t},\quad \text{for some } \alpha>0 \quad \text{and every } t\geq0,$$

and assume the operators e^{At} are compact for $t>0$. The non–linear term f depends on t and on a delay term y_t, which is defined as follows:

For a function $y(\cdot):\mathbb{R}\to H$, *and any* $t\in\mathbb{R}$, *we define* $y_t:(-\infty,0]\to H$ *as*

$$y_t(s)=y(t+s),\quad s\in(-\infty,0].$$

When we equip (6) with an initial value in order to have a an initial value problem, we need to set

$$y(t)=\phi(t),\quad \text{for} \quad t\leq0, \tag{7}$$

where $\phi:(-\infty,0]\to H$ is a suitable function. Thus, if $y(\cdot)$ denotes a solution to (6) such that (7) holds, then y_t denotes

$$y_t(s)=\begin{cases} y(t+s) & \text{for } s\in[-t,0] \\ \phi(s+t) & \qquad s<-t \end{cases}$$

where $t\geq0$.

Before describing the assumptions on f, we first introduce the function space

$$C_\gamma=\{u\in C((-\infty,0];H):\ \lim_{\tau\to-\infty}u\,(\tau)\,e^{\gamma\tau}\text{ exists}\},$$

where $\gamma>\alpha$, and set $\|u\|_\gamma:=\sup_{\tau\in(-\infty,0]}e^{\gamma\tau}\|u(\tau)\|<\infty$. This is a separable Banach space [26, p.15].

Our objective now is to show the existence of a *pullback* attractor for the dynamical system generated by (6).

In what follows we assume that there exist two non–negative functions $c_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, such that $t \to c_i(t)$ is integrable with respect to every finite interval (a, b) and sub–exponentially growing for $t \to \pm\infty$. We also suppose that c_2 satisfies

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t c_2(\tau + s)d\tau = \bar{c}_2(s) \in \mathbb{R}, \quad \text{for all} \quad s \in \mathbb{R}.$$

Finally, we suppose that $\phi \in C_\gamma$ and that

$$f : \mathbb{R} \times C_\gamma \to H$$

is a continuous function for which

$$\|f(t, \phi)\| \le c_1(t) + c_2(t)\|\phi\|_\gamma \quad \text{for } t \in \mathbb{R} \quad \text{and } \phi \in C_\gamma. \tag{8}$$

Notice that we do not assume that f is Lipschitz continuous.

We now prove that for every $\phi \in C_\gamma$ (6) possesses at least one mild solution.

Definition 4.1. For a given $\phi \in C_\gamma$, a function $[t_0, T] \ni t \to y_t(\cdot) := y_t(\cdot; t_0, \phi) \in C_\gamma$ is said to be a mild solution of (6) with initial function ϕ at time t_0 $(< T)$, if

$$y_t(s) = \begin{cases} \mathrm{e}^{A(t-t_0+s)}\phi(0) + \int_{t_0}^{s+t} \mathrm{e}^{A(t+s-\tau)} f(\tau, y_\tau)d\tau & : \quad s \in [-(t - t_0), 0] \\ \phi(s + t - t_0) & : \quad s < -(t - t_0), \end{cases} \tag{9}$$

for all $t \in [t_0, T]$.

Note that in the last definition we express that the mild solution has the state space C_γ, not H. Alternatively, we can define a mild solution to (6) with state space H, setting $s = 0$, by

$$y(t) = \begin{cases} \mathrm{e}^{A(t-t_0)}\phi(0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} f(\tau, y_\tau)d\tau & : \quad t \ge t_0 \\ \phi(t - t_0) & : \quad t < t_0. \end{cases}$$

Briefly we can write that $y(\cdot; t_0, \phi)$ is a mild solution to the IVP

$$(\mathrm{IVP})_{t_0,\phi} \qquad \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = Ay + f(t, y_t), \quad \text{for} \quad t \ge t_0, \\ y_{t_0} = \phi. \end{cases}$$

We now introduce the following notation. Let $y \in C([t_0, T]; H)$ with $y(t_0) = \phi(0)$ and $\phi \in C_\gamma$. Then, for $\tau \in [t_0, T]$, we denote by $y \vee_{t_0,\tau} \phi$ the mapping from $\mathbb{R}^-$ to H defined by

$$y \vee_{t_0,\tau} \phi(s) := \begin{cases} y(t_0 + \tau + s) & : \quad s \in (-\tau, 0] \\ \phi(\tau + s) & : \quad s \le -\tau. \end{cases}$$

We observe that for such function y the integral in (9) is well defined.

Theorem 4.2. *Suppose that the above assumptions on* e^{At} *and* f *are satisfied. Then, for every interval* $[t_0, T]$*, and any* $\phi \in C_\gamma$*, the initial value problem* $(IVP)_{t_0,\phi}$ *possesses a mild solution in* C_γ*.*

The global existence follows by the a priori estimates established in Theorem 4.3 below, assuming that one has the local existence of solutions. The proof of local existence, given in the Appendix, follows Pazy [32] Theorem 6.2.1.

Theorem 4.3. *Let y_t be any mild solution of (6) on $[t_0, T)$, $T \in \mathbb{R}^+ \cup \{+\infty\}$ with a initial function $\phi \in C_\gamma$. Then y_t satisfies the inequality*

$$\|y_t\|_\gamma \le e^{-\alpha(t-t_0)+\int_{t_0}^t c_2(\tau)d\tau}\|\phi\|_\gamma + \int_{t_0}^t e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq}c_1(\tau)d\tau. \tag{10}$$

Proof. We have

$$\|y_t\|_\gamma \le \max\Big(\sup_{s\le -(t-t_0)} \|\phi(s+t-t_0)\|e^{\gamma s}, \sup_{s\in[-(t-t_0),0]} \|\mathrm{e}^{A(t-t_0+s)}\phi(0)\|e^{\gamma s}$$
$$+ \sup_{s\in[-(t-t_0),0]} \|\int_{t_0}^{s+t} \mathrm{e}^{A(t+s-\tau)}f(\tau, y_\tau)d\tau\|e^{\gamma s}\Big).$$

The first term on the right hand side of the last inequality is equal to

$$\sup_{s\le 0}\|\phi(s)\|e^{\gamma(s-t+t_0)} = e^{-\gamma(t-t_0)}\|\phi\|_\gamma.$$

For the second term we have the estimate

$$\begin{aligned}\sup_{s\in[-(t-t_0),0]} \|\mathrm{e}^{A(t-t_0+s)}\phi(0)\|e^{\gamma s} &\le \sup_{s\in[-(t-t_0),0]} e^{-\alpha(s+t-t_0)}\|\phi(0)\|e^{\gamma s}\\ &\le e^{-\alpha(t-t_0)} \sup_{s\in[-(t-t_0),0]} e^{(-\alpha+\gamma)s}\|\phi(0)\|\\ &\le e^{-\alpha(t-t_0)}\|\phi(0)\|.\end{aligned}$$

The third term can be estimated as follows

$$\begin{aligned}&\sup_{s\in[-(t-t_0),0]} \|\int_{t_0}^{s+t} \mathrm{e}^{A(t+s-\tau)}f(\tau, y_\tau)d\tau\|e^{\gamma s}\\ &\le \sup_{s\in[-(t-t_0),0]} \int_{t_0}^{s+t} e^{\alpha(-t-s+\tau)}(c_1(\tau)+c_2(\tau)\|y_\tau\|_\gamma)d\tau e^{\gamma s}\\ &\le \int_{t_0}^t e^{-\alpha(t-\tau)}c_1(\tau)d\tau + \int_{t_0}^t e^{-\alpha(t-\tau)}c_2(\tau)\|y_\tau\|_\gamma d\tau.\end{aligned}$$

Collecting all these estimates we have that

$$\begin{aligned}\|y_t\|_\gamma \le& \max\Big(e^{-\gamma(t-t_0)}\|\phi\|_\gamma, e^{-\alpha(t-t_0)}\|\phi(0)\| + \int_{t_0}^t e^{-\alpha(t-\tau)}c_1(\tau)d\tau\\ &+ \int_{t_0}^t e^{-\alpha(t-\tau)}c_2(\tau)\|y_\tau\|_\gamma d\tau\Big)\\ \le& e^{-\alpha(t-t_0)}\|\phi\|_\gamma + \int_{t_0}^t e^{-\alpha(t-\tau)}(c_1(\tau)+c_2(\tau)\|y_\tau\|_\gamma)d\tau.\end{aligned}$$

We obtain the desired inequality by the Gronwall lemma. □

Remark 1. A consequence of this theorem is that, in case of a *finite* maximal interval of existence $[t_0, t_{max})$ of a solution, no explosions are allowed, i.e.

$$\limsup_{t\uparrow t_{max}} \|y_t\|_\gamma < \infty.$$

But the case of such a finite interval carrying a bounded solution can be excluded similar to Pazy [32] Theorem 6.2.2 applying (8). Hence for every $\phi \in C_\gamma$, $t_0 \in \mathbb{R}$ every mild solution of (6) is global.

4.1. **Pullback attractors for the equation with infinite delay.** Throughout this subsection we assume the same conditions on A and f given at the beginning of Section 4.

We define the multi–valued mapping $U(t,\tau,\phi)$ to be the set of mild solutions (9) in the sense of Definition 4.1 at time $t\in\mathbb{R}$, that is, $U(t,\tau,\phi)=\cup\, y_t$, where the union is taken within the set of entire mild solutions $(-\infty,+\infty)\ni t\to y_t(\cdot;\tau,\phi)\in C_\gamma$ such that $y_\tau(\cdot;\tau,\phi)=\phi$. We stress here that we know from above that every local solution can be extended to a global solution.

Lemma 4.4. *The map U is a strict MNDS. In particular, for any fixed $t\in\mathbb{R}$ we have $U(t,\tau,D(\tau))\in\mathcal{C}(C_\gamma)$ if $\hat{D}\in\mathcal{C}(C_\gamma)$.*

Proof. Let us first prove that $U(t,\tau,\phi)\subset U(t,s,U(s,\tau,\phi))$ for all $t\geq s\geq\tau$, and all $\phi\in C_\gamma$. Let $z\in U(t,\tau,\phi)$. Then there exists a solution y of (9) such that $z=y_t\,(=y_t(\cdot;\tau,\phi))$. Denote, for short, $u_t=y_t(\cdot;\tau,\phi)$. Hence $u_s=y_s(\cdot;\tau,\phi)$ (for $\tau\leq s\leq t$), and therefore u solves Eq. (9) with the initial value $y_s(\cdot;\tau,\phi)$ at $t_0=s$.

Indeed, for $\theta\in[-(t-s),0]$ we have

$$\begin{aligned}
u_t\,(\theta) = y_t(\theta;\tau,\phi)) &= e^{A(t-\tau+\theta)}\phi(0)+\int_\tau^{t+\theta} e^{A(t+\theta-\sigma)}f(\sigma,y_\sigma)d\sigma\\
&= e^{A(t-s+\theta)}e^{A(s-\tau)}\phi(0)+\int_\tau^{s} e^{A(t-s+\theta)}e^{A(s-\sigma)}f(\sigma,y_\sigma)d\sigma\\
&\quad+\int_s^{t+\theta} e^{A(t+\theta-\sigma)}f(\sigma,y_\sigma)d\sigma\\
&= e^{A(t-s+\theta)}\left\{e^{A(s-\tau)}\phi(0)+\int_\tau^{s} e^{A(s-\sigma)}f(\sigma,y_\sigma)d\sigma\right\}\\
&\quad+\int_s^{t+\theta} e^{A(t+\theta-\sigma)}f(\sigma,y_\sigma)d\sigma\\
&= e^{A(t-s+\theta)}y_s(0;\tau,\phi)+\int_s^{t+\theta} e^{A(t+\theta-\sigma)}f(\sigma,y_\sigma(\cdot;\tau,\phi))d\sigma,
\end{aligned}$$

and hence

$$u_t(\theta)=y_t(\theta;s,y_s(\cdot;\tau,\phi))\quad\text{ for }\ \theta\in[-(t-s),0].$$

On the other hand, if $\theta\in(-\infty,-(t-s))$, then $u_t(\theta)=y_s(t+\theta-s;\tau,\phi)=y_t(\theta;\tau,\phi)$ and hence $u_t(\cdot)=y_t(\cdot;s,y_s(\cdot;\tau,\phi))$. This implies that

$$z=u_t\in U(t,s,y_s(\cdot;\tau,\phi))\subset U(t,s,U(s,\tau,\phi)),$$

and, consequently, $U(t,\tau,\phi)\subset U\,(t,s,U\,(s,\tau,\phi))$.

As for the other inclusion, let us consider $z\in U(t,s,U(s,\tau,\phi))$. Then, there exist $z^1\in U(s,\tau,\phi)$ (i.e. $z^1=y_s^1(\cdot;\tau,\phi)$) and $z^2\in U(t,s,z^1)=U(t,s,y_s^1(\cdot;\tau,\phi))$ (i.e. $z^2=y_t^2(\cdot;s,y_s^1(\cdot;\tau,\phi))$) such that $z=y_t^2(\cdot;s,y_s^1(\cdot;\tau,\phi))$.

Now, by concatenating these solutions, we construct

$$u_\sigma=\begin{cases} y_\sigma^1(\cdot;\tau,\phi) & \text{if }\ \sigma<s,\\ y_\sigma^2(\cdot;s,y_s^1(\cdot;\tau,\phi)) & \text{if }\ s\leq\sigma.\end{cases}$$

It is not difficult to check that u_t is a mild solution of (6), so $z=u_t=y_t(\cdot;\tau,\phi)$ and, therefore, $z\in U(t,\tau,\phi)$.

We also note that $U(t,\tau,D(\tau))$ belongs to $\mathcal{C}(C_\gamma)$ if $\widehat{D}\in\mathcal{C}(C_\gamma)$ where the proof follows by the continuity of $C_\gamma\ni\phi\to f(\tau,\phi)$, (8) and the Lebesgue domination theorem. □

For our analysis below, we will consider the system $\mathcal{D}$ given by the multi–valued mapping D in $\mathcal{C}(C_\gamma)$ with $D(t) \subset B_{C_\gamma}(0, \varrho(t))$, the closed ball with center zero and radius ϱ, with sub–exponential growth:

$$\lim_{t\to\pm\infty} \frac{\log^+ \varrho(t)}{t} = 0.$$

This universe $\mathcal{D}$ is called the family of sub–exponentially growing multi–functions in $\mathcal{C}(C_\gamma)$.

Of course, the properties of $\mathcal{D}$ given in Definition 3.1 hold.

Lemma 4.5. *In addition to the previous assumptions, suppose that $\overline{c}_2(t) < \alpha$ for all $t \in \mathbb{R}$. Then, the balls $B(t) = B_{C_\gamma}(0, R(t))$ with*

$$R(t) := 2\int_{-\infty}^{0} e^{\alpha\tau + \int_\tau^0 c_2(t+s)ds} c_1(t+\tau)d\tau = 2\int_{-\infty}^{t} e^{\alpha(\tau-t)+\int_\tau^t c_2(s)ds} c_1(\tau)d\tau \tag{11}$$

form a family $\widehat{B} \in \mathcal{D}$. In addition, $\hat{B}$ is pullback $\mathcal{D}$–absorbing in the sense of (3) which is forward invariant, i.e.,

$$U(t, \tau, B(\tau)) \subset B(t)$$

for $(t, \tau) \in \mathbb{R}^2_d$.

Remark 2. We note that $\mathbb{R} \ni t \mapsto R(t)$ is continuous because this function solves the linear ordinary differential equation initial value problem

$$\frac{dr}{dt} = (-\alpha + c_2(t))r + 2c_1(t), \quad r(0) = R(0) = 2\int_{-\infty}^{0} e^{\alpha\tau + \int_\tau^0 c_2(s)ds} c_1(\tau)d\tau.$$

Moreover, by a similar analysis to the one in Caraballo *et al.* [11] for a random situation, it follows that $t \to R(t)$ is sub–exponentially growing.

Proof. The first part of the Lemma holds thanks to the previous remark. Let us now prove that $B(t)$ is pullback $\mathcal{D}$-absorbing. Consider $\widehat{D} \in \mathcal{D}$, and pick $\phi \in \widehat{D}$. Then, if we replace in the formula in Theorem 4.3 the parameter t_0 by $t - s$, we obtain an estimate for $y_t := y_t(\cdot; t-s, \phi) \in U(t, t-s, D(t-s))$,

$$||y_t||_\gamma \le e^{-\alpha s + \int_{t-s}^t c_2(\tau)d\tau} ||\phi||_\gamma + \int_{t-s}^{t} e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq} c_1(\tau)d\tau. \tag{12}$$

We prove now that the right-hand side of Eq. (12) tends to $R(t)$ as s goes to $+\infty$, what means that there exists $T(\widehat{D}, t) > 0$ such that $U(t, t-s, D(t-s)) \subset B(t)$ for all $s > T(\widehat{D}, t)$.

First, since

$$\frac{1}{s}\int_{t-s}^{t} c_2(\tau)d\tau \to \overline{c}_2(t) < \alpha,$$

then there exists $s_0(t) > 0$ such that

$$\alpha - \frac{1}{s}\int_{t-s}^{t} c_2(\tau)d\tau \ge \varepsilon > 0, \quad \text{for all } \ s \ge s_0, \text{ for a certain } \varepsilon > 0,$$

and, hence

$$e^{-\alpha s + \int_{t-s}^t c_2(\tau)d\tau} \to 0 \quad \text{as} \quad s \to +\infty.$$

Therefore, for the first term on the right-hand side of Eq. (12) we have

$$e^{-\alpha s + \int_{t-s}^t c_2(\tau)d\tau} ||\phi||_\gamma \le \frac{1}{2} R(t).$$

On the other hand,

$$\int_{t-s}^{t} e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq} c_1(\tau)d\tau \to \frac{1}{2}R(t) \quad \text{as} \quad s \to +\infty,$$

and because the integrand is positive we have

$$\int_{t-s}^{t} e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq} c_1(\tau)d\tau \le \frac{1}{2}R(t) \quad \text{for all} \quad s \ge 0.$$

Consequently,

$$||y_t||_\gamma \le R(t), \quad \text{for all} \quad s \ge s_0(t, \widehat{D}).$$

Finally, the forward invariance follows by replacing $\|\phi\|_\gamma$ by $R(\tau)$ in (10). Indeed, if we take $\phi \in B(\tau)$, then

$$\begin{aligned}
||y_t(\cdot;\tau,\phi)||_\gamma &\le e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq} R(\tau) + \int_\tau^t e^{-\alpha(t-s)+\int_s^t c_2(q)dq} c_1(s)ds \\
&\le e^{-\alpha(t-\tau)+\int_\tau^t c_2(q)dq} 2\int_{-\infty}^{\tau} e^{\alpha(s-\tau)+\int_s^\tau c_2(q)dq} c_1(s)ds \\
&+ \int_\tau^t e^{-\alpha(t-s)+\int_s^t c_2(q)dq} c_1(s)ds \\
&\le 2\int_{-\infty}^{\tau} e^{-\alpha(t-s)+\int_s^t c_2(q)dq} c_1(s)ds \\
&+ 2\int_\tau^t e^{-\alpha(t-s)+\int_s^t c_2(q)dq} c_1(s)ds \\
&= R(t).
\end{aligned}$$

The proof is therefore complete. □

Let us now prove that U is upper-semicontinuous.

Lemma 4.6. *The mapping $\phi \in C_\gamma \to U(t,\tau,\phi)$ is upper–semicontinuous for fixed $t \ge \tau$.*

Proof. We argue by contradiction. If we suppose that $U(t,\tau,\cdot)$ is not upper–semicontinuous, then there exist a neighborhood $M_{t,\tau}$ of $U(t,\tau,\phi)$, a sequence $\{\phi_n : n \in \mathbb{N}\}$, $\phi_n \to \phi$ in C_γ, and elements $y_t^n := y_t^n(\cdot;\tau,\phi_n) \in U(t,\tau,\phi_n)$ such that $y_t^n \notin M_{t,\tau}$. If we prove that $\lim_{n'\to+\infty} y_t^{n'} =: \phi_0$ for some subsequence (n') in $\mathbb{N}$, which is an element in $U(t,\tau,\phi_0)$, then we will have obtained a contradiction. To prove that y_t^n is relatively compact we apply the Ascoli–Arzelà theorem. By the properties of the sequence ϕ_n (which is pre–compact in C_γ), it is sufficient to show that $y_t^n(s)$, $s \in [-(t-\tau), 0]$ is pre–compact. We note that by Theorem 4.3 the set $\{y_t^n : n \in \mathbb{N}\}$ is bounded in C_γ because $\{\phi_n : n \in \mathbb{N}\}$ is bounded in C_γ. Hence

$$\sup_{n\in\mathbb{N}, s\in[-(t-\tau),0]} \|y_t^n(s)\| < \infty. \tag{13}$$

A similar argument as that in the proof of Theorem 4.2 provides the relative compactness of $Z(s) := \{y_t^n(s) : n \in \mathbb{N}\}$. In particular, $\{e^{A(t+s)}\phi_n(0) : n \in \mathbb{N}\}$ is pre–compact. As for the equicontinuity of $\{y_t^n : n \in \mathbb{N}\}$ at $s \in (-(t-\tau), 0]$ we use the fact that $r \longmapsto e^{Ar}$ is continuous in norm for $r > 0$.

Following the idea of the proof of Theorem 4.2 to see equicontinuity at $s = -(t-\tau)$ we have to study the following equation with $r := s + t - \tau > 0$

$$y_t^n(s) = y^n(r) = e^{Ar}\phi_n(0) + \int_0^r e^{A(r-\sigma)} f(\sigma+\tau, y^n_{\sigma+\tau}) d\sigma.$$

Thanks to applying (8) and (13), the norm of the integral on the right-hand side is small uniformly with respect to n if r is small enough.

To obtain the equicontinuity of the functions formed by the first expression on the right-hand side we have to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $n \in \mathbb{N}$ and $r \le \delta$ we have that $\|e^{Ar}\phi_n(0) - \phi_n(0)\| \le \varepsilon$. If not, there would exist $\varepsilon > 0$, sequences $n \to +\infty, r_n \to 0$ such that $\|e^{Ar_n}\phi_n(0) - \phi_n(0)\| \ge \varepsilon$. Choosing n sufficiently large such that for r in $[0, t-\tau]$ the estimate

$$\|e^{Ar}(\phi_n(0) - \phi(0))\| \le \frac{\varepsilon}{4}$$

holds, and we then have that

$$\begin{aligned} \|e^{Ar_n}\phi_n(0) - \phi_n(0)\| \le & \|e^{Ar_n}(\phi_n(0) - \phi(0))\| + \|e^{Ar_n}\phi(0) - \phi(0)\| \\ & + \|\phi_n(0) - \phi(0)\| \\ \le & \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

for large n. This contradiction finishes the proof. □

Lemma 4.7. *Assume the hypotheses in Lemma 4.5. The multi–valued dynamical system U is pullback $\mathcal{D}$–asymptotically compact with respect to $\widehat{B}$ defined in Lemma 4.5.*

Proof. Let $z_t = z_t(\cdot; s, \phi)$ be the unique (mild) solution of

$$\left\{ \begin{array}{l} \dfrac{dz}{dt} = Az, \qquad t \ge s, \\ z_s = \phi \in C_\gamma \end{array} \right. \tag{14}$$

which is given by

$$z_t(\theta) = \left\{ \begin{array}{lcl} e^{A(t+\theta)}\phi(0) & : & \theta \in [-(t-s), 0], \\ \phi(\theta + t - s) & : & \theta \in (-\infty, -(t-s)), \end{array} \right.$$

so that (since $\gamma > \alpha$)

$$\begin{aligned} \sup_{\theta \le 0} e^{\gamma\theta} ||z_t(\theta)|| \le & \sup_{-(t-s)\le\theta\le 0} e^{\gamma\theta} ||e^{A(t+\theta)}\phi(0)|| + \sup_{\theta \le -(t-s)} e^{\gamma\theta} ||\phi(\theta+t-s)|| \\ \le & \sup_{-(t-s)\le\theta\le 0} e^{\gamma\theta} e^{-\alpha(t+\theta)} ||\phi(0)|| + \sup_{\theta \le -(t-s)} e^{\gamma\theta} ||\phi(\theta+t-s)|| \\ \le & e^{-\alpha t} ||\phi(0)|| + e^{-\gamma(t-s)} ||\phi||_\gamma \\ \le & e^{-\alpha t} ||\phi(0)|| + e^{-\alpha(t-s)} ||\phi||_\gamma, \end{aligned}$$

and

$$\|z_t\|_\gamma \le e^{-\alpha t} ||\phi(0)|| + e^{-\alpha(t-s)} ||\phi||_\gamma. \tag{15}$$

Recall that we have to prove that for every sequence $t_n \to +\infty$, and every sequence $y^n \in U(t, t-t_n, B(t-t_n))$, it follows that $\{y^n\}_{n\ge 1}$ is pre-compact.

Let y_τ denote a solution of (6) with initial value ϕ so that $y_\tau \in U(\tau, t-s, \phi)$ (for $s \geq 0$). Then there exists $u_\tau \in C_\gamma$ such that $y_\tau = z_\tau + u_\tau$, where u_τ is a mild solution of

$$\frac{du}{d\tau} = Au + f(t, y_\tau), \qquad u_{t-s}(\theta) = 0 \quad \text{for } \theta \leq 0. \tag{16}$$

Let $t_n \to \infty$ and $\phi^n \in B(t-t_n)$. The solution of (6) associated to this initial function, with $t - t_n$ as initial time, is denoted by y^n_τ (in other words, $y^n_\tau := y^n_\tau(\cdot; t-t_n, \phi^n)$). Thanks to Lemma 4.5, $y^n_\tau \in B(\tau)$ and, in particular, $y^n_t \in B(t)$, whence $\|y^n_\tau\|_\gamma \leq R(t)$. Let u^n be the solution of (16) with $s = t_n$ which can be written as

$$u^n(t+\theta) = \int_{t-t_n}^{t+\theta} e^{A(t+\theta-\tau)} f(\tau, y^n_\tau) d\tau, \qquad \theta \in [-t_n, 0].$$

In a similar way as we did in the proof of Theorem 4.2 and taking into account the previous calculations, it is not difficult to obtain an estimate of

$$\|u^n(t+s_1) - u^n(t+s_2)\|, \qquad -T \leq s_1 < s_2 \leq 0,$$

for an arbitrary $T > 0$, which gives us the equicontinuity of $\{u^n(t+\cdot) : n \in \mathbb{N}\}$ on $[-T, 0]$. Furthermore, we can prove the pre–compactness of $\{u^n(t+s) : n \in \mathbb{N}\}$ for $s \in [-T, 0]$. Then, by the Ascoli–Arzelà theorem there exist a subsequence $\{n'\}$ and a function $\psi : \mathbb{R}^- \to H$ which is the uniform limit of $u^{n'}(t+\cdot)$ on every interval $[-T, 0]$.

Remark 2, the properties of c_1, c_2, and the continuity of $t \to R(t)$ allow us to obtain an a priori estimate:

$$\begin{aligned}
\|u^n(t+\theta)\| &\leq \int_{t-t_n}^{t+\theta} e^{-\alpha(t+\theta-\tau)} ||f(\tau, y^n_\tau)|| d\tau \\
&= \int_{-t_n}^{\theta} e^{-\alpha(\theta-\sigma)} ||f(\sigma+t, y^n_{\sigma+t})|| d\sigma \\
&\leq \int_{-t_n}^{\theta} e^{-\alpha(\theta-\sigma)} \left(c_1(\sigma+t) + c_2(\sigma+t)||y^n_{\sigma+t}||_\gamma\right) d\sigma \\
&\leq \int_{-\infty}^{0} e^{-\alpha(\theta-\sigma)} \left(c_1(\sigma+t) + c_2(\sigma+t)R(\sigma+t)\right) d\sigma \\
&\leq e^{-\alpha\theta} R(t), \quad \text{for } \theta \leq 0.
\end{aligned} \tag{17}$$

From this inequality we can derive

$$\|u^n(t+\theta)\| e^{\gamma\theta} \leq \int_{-\infty}^{0} e^{\alpha\tau}(2c_1(t+\tau) + c_2(t+\tau)R(t+\tau)) d\tau = R(t), \quad \theta \in \mathbb{R}^-. \tag{18}$$

It follows from (18) that

$$\sup_{\theta \in [-T,0]} \|u^{n'}_t(\theta)\| e^{\gamma\theta} \leq R(t),$$

and then

$$\lim_{n' \to \infty} \sup_{\theta \in [-T,0]} \|u^{n'}(\theta)\| e^{\gamma\theta} = \sup_{\theta \in [-T,0]} \|\psi(\theta)\| e^{\gamma\theta} \leq R(t), \quad \text{for all } T > 0.$$

Hence

$$\sup_{T>0} \sup_{\theta \in [-T,0]} \|\psi(\theta)\| e^{\gamma\theta} \leq R(t)$$

and this not only implies that ψ belongs to C_γ, but also that $\|\psi\|_\gamma \leq R(t)$.

In addition $u_t^{n'}(\cdot)$ converges to ψ in C_γ. Indeed, to prove this statement we have to check that for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\sup_{\theta\le 0} \|u_t^{n'}(\theta) - \psi(\theta)\| e^{\gamma\theta} \le \varepsilon \qquad \text{for all } n' \ge N(\varepsilon). \tag{19}$$

Since we are assuming that $\gamma > \alpha$, then, for every $\varepsilon > 0$ there exists $T_\varepsilon(t) > 0$ such that

$$e^{-(\gamma-\alpha)T_\varepsilon} R(t) \le \frac{\varepsilon}{2}.$$

Since the convergence of $u_t^{n'}(\cdot)$ to ψ holds in compact intervals, in order to prove (19) we only need to check that

$$\sup_{s\le -T_\varepsilon} \|u_t^{n'}(s) - \psi(s)\| e^{\gamma s} \le \varepsilon \qquad \text{for all } n' \ge N(\varepsilon).$$

But, thanks to (17),

$$\|u_t^{n'}(\theta)\| e^{\gamma\theta} \le e^{(\gamma-\alpha)\theta} R(t) \quad \text{for all } \theta \le 0.$$

This fact and the choice of $T_\varepsilon(t)$ implies

$$\sup_{\theta\le -T_\varepsilon} \|u_t^{n'}(\theta)\| e^{\gamma\theta} \le \frac{\varepsilon}{2}.$$

Moreover,

$$\sup_{\theta\in(-T,-T_\varepsilon]} \|\psi(\theta)\| e^{\gamma\theta} \le \lim_{n'\to\infty} \sup_{\theta\in[-T,-T_\varepsilon]} \|u_t^{n'}(\theta)\| e^{\gamma\theta} \le \frac{\varepsilon}{2}$$

for every $T > T_\varepsilon(t)$. Hence the convergence of $\{u_t^{n'}(\cdot)\}$ to ψ takes place in C_γ.

We then have $y_t^n = u_t^n + z_t^n$, where z_t^n is the solution of (14) with initial function ϕ^n. Since $\phi^n \in B(t - t_n)$ it follows from (15)

$$\lim_{t_n\to+\infty} \|z_t^n\|_\gamma = 0,$$

so we can ensure the convergence of $y_t^{n'}$ to ψ in C_γ, which is the conclusion of the lemma. □

According to Theorem 3.3 and taking into account Lemmata 4.5, 4.6, 4.7, we have already proved the following Theorem.

Theorem 4.8. *Under the assumptions in Lemma 4.5, the MNDS generated by (6) has a pullback $\mathcal{D}$–attractor A in $\mathcal{C}(C_\gamma)$.*

Corollary 1. *Suppose that $\gamma' > \alpha$ such that the assumptions on f given at the beginning of Section 4 are satisfied with respect to γ'. Then there exists a pullback $\mathcal{D}_{\gamma'}$–attractor $A_{\gamma'}$ where $\mathcal{D}_{\gamma'}$ consists of the sub–exponentially growing multi–functions in $\mathcal{C}(C_{\gamma'})$. By the embedding*

$$\|u\|_{\gamma'} \le \|u\|_\gamma, \quad \textit{for } u \in C_\gamma,$$

for $\gamma' > \gamma > \alpha$ there exists a $\mathcal{D}_\gamma$–pullback attractor A_γ such that $A_\gamma \subset A_{\gamma'}$.

Remark 3. It is really interesting to stress the relationship that there exists between the uniqueness of pullback attractors A_γ and the systems of attracted sets $\mathcal{D}_\gamma$. Observe that from the Definition 3.1 every pullback $\mathcal{D}_\gamma$-attractor is an invariant set. According to the Corollary 1, the $\mathcal{D}_{\gamma'}$-attractor $A_{\gamma'}$ attracts the infinite number of $\mathcal{D}_\gamma$-attractors A_γ, for $\gamma' > \gamma > \alpha$, since $A_\gamma \in \mathcal{D}_\gamma \subset \mathcal{D}_{\gamma'}$. However, for γ' there exists a unique attractor $A_{\gamma'}$. Indeed, A_γ does not have to attract the elements from $\mathcal{D}_{\gamma'}$.

5. **Absorbing sets for some logistic-type delay equations.** In this section we will analyse some logistic-type equations with an additional integral term representing the accumulative history of the solution since starting. We will prove the existence of absorbing sets for the models and relegate the existence of pullback attractors to a future investigation.

5.1. **An ordinary integro-differential equation.** We consider a logistic like equation with memory term of the form

$$\frac{dx}{dt}(t) = rx(t)\left(1 - K^{-1}x(t) - \gamma\int_0^t w(\tau - t)P(x(\tau))\,d\tau\right). \tag{20}$$

where r, K, $\gamma > 0$ and $w : \mathbb{R}^- \to \mathbb{R}^+$ is continuous and satisfies

$$\int_{-\infty}^0 w(s)e^{-\eta s}\,ds < +\infty$$

for some $\eta > 0$ and $P \in C(\mathbb{R}, \mathbb{R})$ with $P(x) \geq 0$ when $x \geq 0$ with

$$L|x| \leq |P(x)| \leq C_1|x|^m + C_2$$

for certain constants C_1, C_2, $L > 0$ and $m \geq 1$.

Let us denote $C_\gamma^+ := C_\gamma(\mathbb{R}^-, \mathbb{R}^+)$ the non-negative cone. We can rewrite equation (20) as a differential equation with infinite delay

$$\frac{dx}{dt}(t) = f(t, x_t) := rx(t)\left(1 - K^{-1}x(t) - \gamma\int_{-t}^0 w(s)P(x_t(s))\,ds\right), \tag{21}$$

which we can analyze using the methods in the paper [16] for the infinite delay equation

$$\frac{dx}{dt}(t) = rx(t)\left(1 - K^{-1}\int_{-\infty}^0 w(s)P(x(t+s))\,ds\right) \tag{22}$$

with an initial condition being any function $\phi \in C_\gamma^+$, the non-negative cone in C_γ, since only non-negative solutions are relevant and C_γ^+ is positive invariant for an appropriate $\gamma > 0$. An appropriate initial condition for equation (21) is a $\phi \in C_\gamma^+$ with $\phi(0) = x_0 \in \mathbb{R}^+$.

Following [16], we note that the mapping $(t, \phi) \mapsto f(t, \phi) : \mathbb{R}^+ \times C_\gamma \to \mathbb{R}$ is continuous and bounded (i.e. maps bounded sets onto bounded sets) for any $\gamma > 0$. The key step is to show these properties for the mapping

$$M(t, \phi) := \int_{-t}^0 w(s)P(\phi(s))\,ds.$$

Indeed,

$$M(t + \delta t, \psi) - M(t, \phi) = \int_{-t}^0 w(s)\left[P(\psi(s)) - P(\phi(s))\right]\,ds + \int_{-t-\delta t}^{-t} w(s)P(\psi(s))\,ds$$

$$\to 0 \quad \text{as } \delta \to 0 \text{ and } \|\psi - \phi\|_\gamma \to 0,$$

since

$$\int_{-t-\delta t}^{-t} w(s)|P(\psi(s))|\,ds \le \max_{s\in[-t-1,-t]} w(s) \int_{-t-\delta t}^{-t} \left(C_1|\psi(s)|^m + C_2\right)\,ds$$

$$\le \max_{s\in[-t-1,-t]} w(s) \left(C_1 \max_{s\in[-t-1,-t]} |\psi(s)e^{\gamma s}|^m + C_2\right) \times \times e^{m\gamma(t+1)} \cdot \delta t$$

$$\le \max_{s\in[-t-1,-t]} w(s) \left(C_1 \|\psi\|_\gamma^m + C_2\right) e^{m\gamma(t+1)} \cdot \delta t$$

$$\le \max_{s\in[-t-1,-t]} w(s) \left(C_1(\|\phi\|_\gamma + 1)^m + C_2\right) e^{m\gamma(t+1)} \cdot \delta t$$

for all $\psi \in C_\gamma$ with $\|\psi - \phi\|_\gamma \le 1$ and

$$\max_{s\in[-t,0]} |P(\psi(s)) - P(\phi(s))| \to 0 \quad \text{as} \quad \max_{s\in[-t,0]} |\psi(s) - \phi(s)| \to 0,$$

where

$$\max_{s\in[-t,0]} |\psi(s) - \phi(s)| \le \max_{s\in[-t,0]} |(\psi(s) - \phi(s))e^{\gamma s}|e^{\gamma t}$$

$$\le \|\psi - \phi\|_\gamma \, e^{\gamma t} \to 0 \quad \text{as} \quad \|\psi - \phi\|_\gamma \to 0$$

for each fixed $t > 0$.

We can then apply Lemmas 22 and 25 in [16] to equation (21) to conclude that it has at least one positive solution and that the positive cone C_γ^+ is positively invariant under the solution mapping in a weak sense, because what we can ensure is that at least one solution remains therein but, in general, some other solutions can leave it.

We restrict attention to the non-negative cone C_γ^+ and consider the set-valued dynamical system generated by the solutions which intersect the positive cone. To show the existence of an absorbing set we use the properties of P in the differential relationship

$$\frac{d}{dt}x(t)^2 = 2rx(t)^2 \left(1 - K^{-1}x(t) - \gamma \int_{-t}^{0} w(s)P(x_t(s))\,ds\right), \qquad t \ge 0,$$

to obtain the differential inequality

$$\frac{d}{dt}x(t)^2 \le 2rx(t)^2 \left(1 - \frac{1}{K}x(t)\right), \qquad t \ge 0, \tag{23}$$

for $x_t \in C_\gamma^+$. From this we conclude that $x(t) \in [0, K+1]$ after a finite time and, by an argument similar to the one in Lemma 26 from [16], it follows the existence of an absorbing set $\mathcal{B}$.

5.2. **An integro-differential reaction-diffusion equation.** In this subsection we consider a reaction–diffusion version of the integro–differential equation (20), namely

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + ru(x,t) \left(1 - K^{-1}u(x,t) - \gamma \int_0^t w(\tau - t)P(u(x,\tau))\,d\tau\right) \tag{24}$$

on a bounded domain Ω in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and the non-negative initial condition $u(x,0) = u_0(x) \geq 0$.

The crucial step here is to show there exists an absorbing set in $L_2(\Omega)$. For this we use the following estimates for any existing positive solution.

$$\frac{d}{dt}|u(t)|_2^2 = -2|\nabla u(t)| + 2r\int_\Omega u(x,t)^2\,dx - \frac{2r}{K}\int_\Omega u(x,t)^3\,dx$$

$$\leq -2\lambda_1|u(t)|_2^2 + 2r|u(t)|_2^2 - \frac{2r}{K}|u(t)|_3^3$$

$$= -2(\lambda_1 - r)|u(t)|_2^2 - \frac{2r}{K}|u(t)|_3^3$$

by the Poincaré inequality, where $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator on Ω with the Dirichlet boundary condition. Then we use the inequality

$$|u(t)|_2^2 = \int_\Omega u(x,t)^2\,dx \leq \left(\int_\Omega u(x,t)^3\,dx\right)^{2/3}|\Omega|^{1/3} = |u(t)|_3^{2/3}|\Omega|^{1/3}$$

to obtain

$$|\Omega|^{-3/2}|u(t)|_2^9 \leq |u(t)|_3^3.$$

Finally, this leads to the inequality

$$\frac{d}{dt}|u(t)|_2^2 \leq -2(\lambda_1 - r)|u(t)|_2^2 - \frac{2r}{K|\Omega|^{3/2}}|u(t)|_2^9$$

or

$$\frac{d}{dt}|u(t)|_2 \leq -(\lambda_1 - r)|u(t)|_2 - \frac{r}{K|\Omega|^{3/2}}|u(t)|_2^8$$

If $r \leq \lambda_1$ the zero solution is globally asymptotically stable, but if $r > \lambda_1$ then it follows that

$$|u|_2^7 \leq 1 + \frac{K|\Omega|^{3/2}(r - \lambda_1)}{r}.$$

Again, by a similar argument as the one in Lemma 26 from [16], we can obtain the existence of an absorbing set.

6. **Appendix.** *Proof of Lemma 3.2* Consider a sequence $y_n \in U(t, t - t_n, B(t - t_n))$ with $t_n \to +\infty$. As U is pullback–asymptotically compact with respect to $\widehat{B}$, there exists a converging subsequence and its limit y belongs to $\Lambda\left(\widehat{B}, t\right)$, so that $\Lambda\left(\widehat{B}, t\right)$ is non–empty.

We now prove that $\Lambda\left(\widehat{B}, t\right)$ is compact. For any sequence $\{y_n\} \subset \Lambda\left(\widehat{B}, t\right)$ there exist $t_n \to +\infty$ and $z_n \in U(t, t - t_n, B(t - t_n))$, such that $d_X(y_n, z_n) < \frac{1}{n}$. Using again the pullback asymptotic compactness of U the existence of a converging subsequence $z_{n_k} \to z \in \Lambda\left(\widehat{B}, t\right)$ follows. Then, $y_{n_k} \to z$, so that $\Lambda\left(\widehat{B}, t\right)$ is compact.

We prove (4) by contradiction. If (4) does not hold, then there exist $\varepsilon > 0$ and $y_n \in U(t, t - t_n, B(t - t_n))$ with $t_n \to +\infty$, such that

$$\mathrm{dist}_X\left(y_n, \Lambda\left(\widehat{B}, t\right)\right) > \varepsilon.$$

As U is pullback–asymptotically compact with respect to $\widehat{B}$, it follows that there exists a subsequence (relabelled again the same) $y_n \to y \in \Lambda\left(\widehat{B}, t\right)$, which is not possible.

Let us now prove that (5) holds. Fix $(t,\tau) \in \mathbb{R}^2_d$. Then, if $y \in \Lambda\left(\widehat{B}, t\right)$, there exist sequences $y_n \in U\left(t, t-(t_n-\tau), x_n\right)$, $x_n \in B(t-(t_n-\tau))$ with $t_n \to +\infty$, such that $y_n \to y$. For $t_n \geq t$, the process property implies

$$U\left(t, t-t_n+\tau, x_n\right) \subset U\left(t, \tau, U\left(\tau, t-t_n+\tau, x_n\right)\right),$$

and then $y_n \in U\left(t, \tau, z_n\right)$, where $z_n \in U\left(\tau, t-t_n+\tau, x_n\right)$. As before, up to a subsequence, $z_n \to z \in \Lambda\left(\widehat{B}, \tau\right)$. Since $x \mapsto U\left(t, \tau, x\right)$ is upper–semicontinuous with closed values, we have

$$y \in U(t,\tau,z) \subset U(t,\tau,\Lambda\left(\widehat{B},\tau\right)).$$

□

Proof of Theorem 3.3 First we need to prove that

$$\lim_{\tau\to+\infty} \operatorname{dist}_X(U(t,t-\tau,D\left(t-\tau\right)),A(t)) = 0 \quad \text{for every } \widehat{D} \in \mathcal{D}. \tag{25}$$

Indeed, thanks to (4), for every $\varepsilon > 0$ and $t \in \mathbb{R}$, there exists $T(t,\varepsilon)$ such that for $\tau \geq T(t,\varepsilon)$

$$\operatorname{dist}_X(U(t,t-\tau,B\left(t-\tau\right)),A(t) < \varepsilon.$$

But, for every $\widehat{D} \in \mathcal{D}$,

$$U(t-\tau,t-\tau-T(t-\tau,\widehat{D}),D(t-\tau-T(t-\tau,\widehat{D}))) \subset B\left(t-\tau\right)$$

so that

$$\operatorname{dist}_X(U(t,t-\tau,D\left(t-\tau\right)),A(t)) < \varepsilon$$

for τ large.

The third property in Definition 3.1 follows from (5). Since

$$U(t,t-\tau,B(t-\tau)) \subset B(t) \quad \text{for } \tau \geq T(t,\widehat{B}),$$

we have the relation $A(t) \subset B(t)$ for each $t \in \mathbb{R}$, so that $\widehat{A} \in \mathcal{D}$. But this shows that A is unique. Indeed suppose we have another pullback D–attractor $\widehat{A'}$, then as

$$A'(t) \subset U(t,t-\tau,A'(t-\tau))$$

and

$$\lim_{\tau\to+\infty} \operatorname{dist}_X\left(U(t,t-\tau,A'(t-\tau)),A(t)\right) = 0,$$

we have that $A'(t) \subset A(t)$. Exchanging $\widehat{A}$ and $\widehat{A'}$ it follows that $\widehat{A} = \widehat{A'}$.

Finally, assume that U is a strict MNDS. Then,

$$\begin{aligned} U\left(t,r,A\left(r\right)\right) &\subset U\left(t,\tau,U\left(\tau,r-\tau,A(r-\tau)\right)\right) \\ &= U\left(t,r-\tau,A\left(r-\tau\right)\right), \quad \text{for all } \tau \geq 0. \end{aligned}$$

As $\widehat{A}$ pullback attracts itself, it follows that

$$\lim_{\tau\to+\infty} \operatorname{dist}_X(U(t,r-\tau,A(r-\tau)),A(t)) = 0,$$

and, consequently, given $\varepsilon > 0$, there exists $T(\varepsilon,t,r) > 0$ such that, for $\tau \geq T(\varepsilon,t,r)$

$$\operatorname{dist}_X(U(t,r-\tau,A(r-\tau)),A(t)) < \varepsilon,$$

and as $U(t, r, A(r)) \subset U(t, r-\tau, A(r-\tau))$, we have

$$\operatorname{dist}_X(U(t,r,A(r)), A(t)) < \varepsilon, \text{ for all } \varepsilon > 0,$$

so $U(t,r,A(r)) \subset A(t)$, as required. □

Proof of local existence in Theorem 4.2 The proof follows Pazy [32] Theorem 6.2.1. Let us fix some $\phi \in C_\gamma$ and t, $t_0 \in \mathbb{R}$.

Consider

$$B(R) = \{y \in C([t_0,T];H) : y(t_0) = \phi(0), \sup_{s\in[t_0,T]} \|\phi(0) - y(s)\| \le R\}.$$

$B(R)$ is a convex and bounded set in $C([t_0,T];H)$. For any $T > t_0$ (with $T - t_0$ small enough) we define the mapping $\mathcal{T}_T : B(R) \to C([t_0,T];H)$ by

$$\mathcal{T}_T(y)[t] := \mathrm{e}^{A(t-t_0)}\phi(0) + \int_{t_0}^{t} \mathrm{e}^{A(t-t_0-\tau)} f(\tau, y \vee_{t_0,\tau} x_0) d\tau, \quad t \in [t_0,T].$$

We note that $\mathcal{T}_T(y) \in 4C([t_0,T];H)$ because $\tau \to \|f(\tau, y \vee_{t_0,\tau} \phi)\| \in L_1([t_0,T])$. To see that the operator $\mathcal{T}_T$ maps $B(R)$ into itself, for appropriate R and T, we note that

$$\begin{aligned}\|f(r, y \vee_{t_0,r} \phi)\| &\le c_1(r) + c_2(r) \sup_{\varrho\in[0,r]} e^{\gamma(\varrho - r)} \|y(t_0+\varrho)\| \\ &\quad + c_2(r) \sup_{\varrho \le -r} e^{\gamma\varrho}\|\phi(r+\varrho)\| \\ &\le c_1(r) + c_2(r) \sup_{\varrho\in[t_0,T]} \|y(\varrho)\| \\ &\quad + c_2(r) e^{-\gamma r} \sup_{\varrho\le -r} e^{\gamma(\varrho+r)} \|\phi(r+\varrho)\|,\end{aligned}$$

so

$$\|f(r, y \vee_{t_0,r} \phi)\| \le c_1(r) + c_2(r) \sup_{\varrho\in[t_0,T]} \|y(\varrho)\| + c_2(r)\|\phi\|_\gamma. \tag{26}$$

The term $\sup_{\varrho\in[t_0,T]} \|y(\varrho)\|$ is bounded by $\|\phi\|_\gamma + R$. In addition, $\|\mathrm{e}^{A(t-t_0-\tau)}x\| \le e^{-\alpha(t-t_0-\tau)}\|x\|$ so that, for small $T - t_0 > 0$ (depending on r), we have $\mathcal{T}_T(B(R)) \subset B(R)$.

In view of the continuity of $C_\gamma \ni \xi \to f(t,\xi)$ and (8) we obtain by the Lebesgue domination theorem that $\mathcal{T}_T$ is continuous on $B(R)$ with the topology of $C([t_0,T];H)$.

To see that $\mathcal{T}_T$ is compact we first note that the sets

$$Z_t := \{z = \mathcal{T}_T(y)[t], y \in B(R)\}, \quad t \in [t_0,T]$$

are pre–compact. This is trivially true for $t = t_0$. For $t > t_0$ we introduce for sufficiently small $\varepsilon > 0$

$$\begin{aligned}\mathcal{T}_T^\varepsilon(y)[t] &= \mathrm{e}^{A(t-t_0)}\phi(0) + \int_{t_0}^{t-\varepsilon} \mathrm{e}^{A(t-t_0-\tau)} f(\tau, y \vee_{t_0,\tau} \phi) d\tau \\ &= \mathrm{e}^{A(t-t_0)}\phi(0) + \mathrm{e}^{A\varepsilon} \int_{t_0}^{t-\varepsilon} \mathrm{e}^{A(t-t_0-\tau-\varepsilon)} f(\tau, y \vee_{t_0,\tau} \phi) d\tau.\end{aligned}$$

By (26) and the integrability conditions on c_1 and c_2

$$\sup_{y\in B(R)} \| \int_{t_0}^{t-\varepsilon} S(t-t_0-\tau-\varepsilon) f(\tau, y \vee_{t_0,\tau} \phi) d\tau \|$$

is finite for appropriate $\varepsilon > 0$ so that $\mathcal{T}_T^\varepsilon(B(R))[t]$ is pre-compact by the compactness of $e^{A\varepsilon}$. Then for every $\varepsilon' > 0$ we have an $\varepsilon > 0$ such that

$$\|\mathcal{T}_T^\varepsilon(y)[t] - \mathcal{T}_T(y)[t]\| \leq \int_{t-\varepsilon}^t e^{-\alpha(t-t_0-\tau)}\|f(\tau, y \vee_{t_0,\tau} \phi)\|d\tau \leq \varepsilon'$$

uniformly for $y \in B(R)$ so that Z_t is totally bounded, hence pre–compact.

To apply the Arzelà–Ascoli theorem we show that $\mathcal{T}_T(y)$ with $y \in B(R)$, is equicontinuous. Notice that, for $t_2 > t_1 > t_0$,

$$\begin{aligned}\|\mathcal{T}_T(y)[t_2] - \mathcal{T}_T(y)[t_1]\| \leq &\|(\mathrm{e}^{At_2} - \mathrm{e}^{At_1})\phi(0)\| \\ &+ \int_0^{t_1} \|\mathrm{e}^{A(t_2-\tau)} - \mathrm{e}^{A(t_1-\tau)}\|\|f(\tau, y \vee_{t_0,\tau} \phi)\|d\tau \\ &+ \int_{t_1}^{t_2} \|\mathrm{e}^{A(t_2-\tau)}\|\|f(\tau, y \vee_{t_0,\tau} \phi)\|d\tau.\end{aligned}$$

Since $S(t) = e^{At}$ is a compact operator for $t > 0$ we know that the mapping $t \to S(t)$ is norm-continuous for $t > 0$. The Lebesgue domination theorem together with (26) imply the equicontinuity for $t > t_0$. Similar arguments hold for $t_1 = t = t_0$. Indeed, in the above formula the second term on the right hand side disappears for $t_1 = t_0$.

The Schauder theorem gives the existence of a fixed point of $\mathcal{T}_T$ which is a local solution for (6). □

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