LEVITAN/BOHR ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF SECOND-ORDER MONOTONE DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to prove the existence of Levitan/Bohr almost periodic, almost automorphic, recurrent and Poisson stable solutions of the second order differential equation

$$x'' = f(\sigma(t, y), x, x'), \quad (y \in Y) \tag{1}$$

where Y is a complete metric space and $(Y, \mathbb{R}, \sigma)$ is a dynamical system (also called a driving system). When the function f in (1) is increasing with respect to its second variable, the existence of at least one quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent, Levitan almost periodic, almost recurrent, Poisson stable) solution of (1) is proved under the condition that (1) admits at least one solution φ such that φ and φ' are bounded on the real axis.

1. Introduction

The aim of this paper is to analyze the existence of Levitan/Bohr almost periodic, almost automorphic, recurrent and Poisson stable solutions of the second order differential equation

$$x'' = f(\sigma(t, y), x, x'), \quad (y \in Y) \tag{2}$$

where Y is a complete metric space, and $(Y, \mathbb{R}, \sigma)$ is a (driving) dynamical system.

The existence of Bohr almost periodic solutions of equation

$$x'' = f(t, x, x') \tag{3}$$

with Bohr almost periodic right hand-side f with respect to time, uniformly with respect to the variables x, x' on every compact subset in $\mathbb{R}^2$, was studied by C. Corduneanu in [15] (see also [1]), where it was established that, if $\frac{\partial f(t,x,u)}{\partial x} \geq k > 0$ for all $(t, x, u) \in \mathbb{R}^3$, equation (3) admits a unique Bohr almost periodic solution.

When the function $f(t, x, u)$ is only increasing (in the large sense), the same problem was studied by Z. Opial in [21], where the following result was established.

Theorem 1.1. *(Z. Opial [21]) Suppose that the following conditions are fulfilled:*

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(i) *$f \in C(\mathbb{R}^3, \mathbb{R})$ and is increasing in the large sense with respect to the variable x, i.e., the inequality $x_1 \le x_2$ implies $f(t, x_1, u) \le f(t, x_2, u)$ for all $u, t \in \mathbb{R}$;*

(ii) *for all $r > 0$, there exists a number $L(r) > 0$ such that $|f(t, x_1, u_1) - f(t, x_2, u_2)| \le L(r)(|x_1 - x_2| + |u_1 - u_2|)$ for all $|x_i|, |u_i| \le r$ ($i = 1, 2$) and $t \in \mathbb{R}$.*

Then, the following statements hold:

(i) *If equation (3) admits a solution u such that u and its first derivative u' are bounded on $\mathbb{R}$, then this equation admits at least one Bohr almost periodic solution.*

(ii) *If $u(t)$ and $v(t)$ are two Bohr almost periodic solutions of equation (3), then there exists a constant $c \in \mathbb{R}$ such that $u(t) - v(t) = c$ for all $t \in \mathbb{R}$.*

(iii) *If the function f is strictly increasing with respect to the second variable $x \in \mathbb{R}$, then equation (3) admits at most one Bohr almost periodic solution.*

Some generalization of Theorem 1.1 when (3) is a vectorial equation (i.e., $f \in C(Y \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ ($n \ge 2$) are established in [16] and [13].

In [12], P. Cieutat studied the existence of bounded and Bohr almost periodic solutions of the following Liénard equation

$$x'' + f(x)x' + g(x) = p(t), \tag{4}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Bohr almost periodic function, $f(x) \ge 0$ and g is a strictly decreasing function. Namely, it was proved in [12] that every solution, which is bounded on $\mathbb{R}_+$, is asymptotically Bohr almost periodic, and there exists a unique Bohr almost periodic solution of equation (4). A typical model for such equation (4) is

$$x'' + cx' + 1/x^\alpha = p(t), \quad (x \in (0, +\infty)),$$

where $c \ge 0$, $\alpha > 0$ and p is Bohr almost periodic.

Recently, the existence of almost automorphic solutions of equation (4) with almost automorphic forcing term p was studied by Cieutat *et al.* in [14], where they proved the asymptotically almost automorphy of every solution which is bounded on $\mathbb{R}_+$, and the existence of a unique almost automorphic solution of equation (4).

In the periodic case (i.e. when p is periodic), the dynamics of equation (5) was intensively studied by P. Martínez-Amores and P. J. Torres [20] and J. Campos and P. J. Torres [4].

Desheng Li and Jinqiao Duan [19] analyzed the structure of the set of bounded solutions for equation (2). In particular, they proved the existence of a unique periodic (respectively, quasi-periodic, Bohr almost periodic) solution of equation (2) if the point $y \in Y$ is also periodic (respectively, quasi-periodic, Bohr almost periodic), and the function f is strictly increasing with respect to its second variable. Namely, the following theorem was proved in [19].

Theorem 1.2. [19] *Suppose that the following conditions are fulfilled:*

(i) *(H, ρ) is a compact complete metric space and $(H, \mathbb{R}, \theta)$ is a minimal dynamical system on H;*

(ii) *$f : H \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous map, where $I := (a, b) \subseteq \mathbb{R}$;*
(iii) *For any compact subset $V \subset (a, b) \times \mathbb{R}$, there exists an $L > 0$ such that $|f(h, x, p) - f(h, y, q)| \le L(|x - y| + |p - q|)$ for all $(x, p), (y, q) \in V$ and $h \in H$;*
(iv) *$f(h, x, p)$ is strictly increasing in x;*
(v) *For any compact interval $\mathbb{I} \subset I$ there exists $c_0 > 0$ such that $|f(h, x, p)| \le c_0(1 + |p|^2)$ for all $(h, x, p) \in H \times \mathbb{I} \times \mathbb{R}$.*

Then:

(i) *There exists a continuous map $\Gamma : H \mapsto \mathbb{R}$ such that for each $h \in H$, $\gamma_h(t) := \Gamma(\theta(t, h))$ is the unique solution of equation*

$$x'' = f(\theta(t, h), x, x') \tag{5}$$

which is bounded on $\mathbb{R}$;
(ii) *For each $h \in H$, there exists a continuous decreasing function Φ_h defined on a maximal nonempty open interval $D(\Phi_h) \subset I$, such that for any $x \in D(\Phi_h)$, $x(t) := \psi_h(t, x, \Phi_h(x))$ is the unique solution of (5), which is bounded on $\mathbb{R}_+$, that satisfies $x(0) = x$, where $\psi(t, x, x')$ denotes the unique solution of equation (5) passing through the point $(x, x') \in I \times \mathbb{R}$ at the initial moment $t = 0$;*
(iii) *For any compact interval $\mathbb{D} \subset D(\Phi_h)$*

$$\lim_{t \to +\infty} (|\psi_h(t, x, \Phi_h(x)) - \gamma_h(t)| + |\psi_h'(t, x, \Phi_h(x)) - \gamma_h'(t)|) = 0$$

uniformly with respect to $x \in \mathbb{D}$.

We note that, in all of the previously cited works (with the exception of [21]), an assumption of strict monotony is imposed. In the present paper, we consider equation (2) when the function f is increasing with respect to its second variable in the large sense. All of our results will be formulated and proved for this case which includes, of course, the strictly increasing one.

The paper is organized as follows.

In Section 2, we collect some notions (quasi periodicity, Levitan/Bohr almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability) facts and constructions (Bebutov dynamical systems, skew-product dynamical systems, cocycles etc) from the theory of dynamical systems which will be necessary in this paper.

Section 3 is dedicated to the study of a special class of non-autonomous dynamical systems (NDS): the so-called NDS with convergence. The main result in this section is Theorem 3.9 which provides sufficient conditions for the convergence of a NDS.

An application of Theorem 3.9 to study the dynamics of the scalar one-dimensional equation $x' = f(\sigma(t, y), x)$ $(y \in Y)$ with pseudo recurrent base $(Y, \mathbb{R}, \sigma)$ (driving system) is carried out in Section 4. The main result of this section is Theorem 4.2.

Levitan almost periodic and almost automorphic solutions of a second order equation of the form $x'' = f(\sigma(t, y), x, x')$ and with increasing f (in the large sense) are analyzed in Section 5. The main results of this section are Theorem 5.4 and Corollary 5.5.

Section 6 is devoted to the existence of quasi-periodic, Bohr almost periodic and recurrent solutions (in the sense of Birkhoff) of the equation $x'' = f(\sigma(t,y),x,x')$ with increasing f (in the large sense). The main results proved in this section are Theorem 6.1 and Corollary 6.2.

Finally, in Section 7, we discuss some generalizations of our main results (theorems 5.4 and 6.1). One of this type of results is established in Theorem 7.1 (see also corollaries 7.2 and 7.3).

2. Bohr/Levitan Almost Periodic and Almost Automorphic Motions of Dynamical Systems

We recall now some notions, facts and constructions from the theory of dynamical systems.

Although we could refer the readers to other publications for these preliminaries (see, for instance, Caraballo and Cheban [5, 6]), in order to keep our paper as much self-contained as possible, we prefer to include the results here.

2.1. **Recurrent, Bohr Almost Periodic and Almost Automorphic Motions.** Let (X,ρ) be a complete metric space, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$, and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of the additive group $\mathbb{S}$, where $\mathbb{S}_+ := \{s \in \mathbb{S} : \ s \ge 0\}$.

Let $(X,\mathbb{T},\pi)$ be *a dynamical system* on X, i.e., let $\pi : \mathbb{T}\times X \to X$ be a continuous function such that $\pi(0,x) = x$ for all $x \in X$, and $\pi(t_1+t_2,x) = \pi(t_2,\pi(t_1,x))$, for all $x \in X$, and $t_1,t_2 \in \mathbb{T}$.

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$*shift* (respectively, an $\varepsilon-$*almost period*) of x, if $\rho(\pi(\tau,x),x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t,x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n,x)\} \to x \text{ and } \{t_n\} \to \infty\}$ and $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n,x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ is called *Poisson stable in the positive direction* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to +\infty$ as $n \to \infty$.

Let $(X,\mathbb{T},\pi)$ be a two-sided dynamical system (i.e., $\mathbb{T} = \mathbb{S}$). A point $x \in X$ is called *Poisson stable in the negative direction* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to -\infty$ as $n \to \infty$. The point $x \in X$ is called *Poisson stable* if it is Poisson stable in both directions.

A dynamical system $(X,\mathbb{T},\pi)$ is said to be

(i) *transitive*, if there exists a point $x_0 \in X$ such that $H(x_0) = X$, where $H(x_0) := \overline{\{\pi(t,x_0) : \ t \in \mathbb{T}\}}$;
(ii) *pseudo recurrent* if X is compact, the dynamical system $(X,\mathbb{T},\pi)$ is transitive, and every point $x \in X$ is Poisson stable.

A point $x \in X$ is called [26, 28] *pseudo recurrent* if the dynamical system $(H(x),\mathbb{T},\pi)$ is pseudo recurrent.

Remark 2.1. *Every recurrent point is pseudo recurrent, but there exist pseudo recurrent points which are not recurrent* [26, 28].

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}$. Let $(\mathcal{T}^m,\mathbb{T},\sigma)$ be an irrational winding of $\mathcal{T}^m$, i.e., $\sigma(t,\nu) := (\nu_1 t, \nu_2 t, \dots, \nu_m t)$ for all $t \in \mathbb{S}$ and $\nu \in \mathcal{T}^m$.

A point $x \in X$ is called *quasi-periodic* with the frequency $\nu := (\nu_1,\nu_2,\dots,\nu_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ such that $\pi(t,x) := \Phi(\sigma(t,\omega))$ for all $t \in \mathbb{T}$, where $(\mathcal{T}^m,\mathbb{T},\sigma)$ is an irrational winding of the torus $\mathcal{T}^m$ and $\omega \in \mathcal{T}^m$.

A point $x \in X$ of the dynamical system $(X,\mathbb{T},\pi)$ is called *Levitan almost periodic* [18], if there exists a dynamical system $(Y,\mathbb{T},\sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.2. *Let $x_i \in X_i$ $(i = 1,2,\dots,m)$ be a Levitan almost periodic point of the dynamical system $(X_i,\mathbb{T},\pi_i)$. Then the point $x := (x_1,x_2,\dots,x_m) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic in the product dynamical system $(X,\mathbb{T},\pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t,x) := (\pi_1(t,x_1),\pi_2(t,x_2),\ \dots,\ \pi_m(t,x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1,x_2,\dots,x_m) \in X$.*

A point $x \in X$ is called *stable in the sense of Lagrange* (st.L) (respectively, *stable in the sense of Lagrange in the positive direction* (st.L$^+$)), if its trajectory $\{\pi(t,x) \ : \ t \in \mathbb{T}\}$ (respectively, its positive semi-trajectory $\{\pi(t,x) \ : \ t \in \mathbb{T}_+\}$) is relatively compact, where $\mathbb{T}_+; = \{t \in \mathbb{T} : \ t \geq 0\}$.

A point $x \in X$ is called *almost automorphic* [18, 24] in the dynamical system $(X,\mathbb{T},\pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y,\mathbb{T},\sigma)$, a homomorphism h from $(X,\mathbb{T},\pi)$ onto $(Y,\mathbb{T},\sigma)$, and a point $y \in Y$ which is almost periodic, in the sense of Bohr, such that $h^{-1}(y) = \{x\}$.

Remark 2.3. *Notice the following well-known facts.*

1. Every almost automorphic point is Levitan almost periodic.

2. A Levitan almost periodic point is almost automorphic if and only if is stable in the sense of Lagrange.

2.2. **Shift Dynamical Systems, Levitan/Bohr Almost Periodic and Almost Automorphic Functions.** Below we recall a general method of construction of dynamical systems on spaces of continuous functions. In this way, we will obtain many well-known dynamical systems on some functional spaces (see, for example, [2, 23, 26]).

Let $(X,\mathbb{T},\pi)$ be a dynamical system on X, Y a complete pseudo metric space, and $\mathcal{P}$ a family of pseudo metrics on Y. We denote by $C(X,Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d^p_K\}$ $(p \in \mathcal{P},\ K \in \mathcal{C}(X))$, where

$$d^p_K(f,g) := \sup_{x\in K} p(f(x),g(x))$$

and $\mathcal{C}(X)$ denotes the family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X,Y) \to C(X,Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau,x)),\ \ x \in X$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1+\tau_2}$, for all $\tau_1, \tau_2 \in \mathbb{T}$;
c. σ_τ is continuous for all $\tau \in \mathbb{T}$.

Lemma 2.4. [7] *The mapping* $\sigma : \mathbb{T}\times C(X,Y) \to C(X,Y)$, *defined by the equality* $\sigma(\tau,f) := \sigma_\tau f$ $\ (f \in C(X,Y),\ \tau \in \mathbb{T})$, *is continuous, and the triple* $(C(X,Y),\mathbb{T},\sigma)$ *is a dynamical system on* $C(X,Y)$.

Consider now some examples of dynamical systems of the form $(C(X,Y),\mathbb{T},\sigma)$, which are useful in the applications.

Example 2.5. Let $X = \mathbb{T}$, and denote by $(X,\mathbb{T},\pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t,x) := x + t$. The dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$ is called *Bebutov's dynamical system* [2, 23, 26] (dynamical system of translations, or shifts dynamical system).

It is said that the function $\varphi \in C(\mathbb{T},Y)$ *possesses a property* (A), if the motion $\sigma(\cdot,\varphi) : \mathbb{T} \to C(\mathbb{T},Y)$, generated by this function, possesses this property in the Bebutov dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$. As property (A) we can take periodicity, quasi-periodicity, Bohr/Levitan almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability, etc.

Example 2.6. Let $X := \mathbb{T}\times W$, where W is a metric space, and let $(X,\mathbb{T},\pi)$ denote a dynamical system on X defined in the following way: $\pi(t,(s,w)) := (s+t,w)$. Using the general method proposed above, we can define on $C(\mathbb{T}\times W,Y)$ a dynamical system of translations $(C(\mathbb{T}\times W,Y),\mathbb{T},\sigma)$.

The function $f \in C(\mathbb{T}\times W,Y)$ is called *Bohr/Levitan almost periodic (quasi-periodic, recurrent, almost automorphic, etc) with respect to* $t \in \mathbb{T}$, *uniformly in* w *on every compact from* W, if the motion $\sigma(\cdot,f)$ is Bohr/Levitan almost periodic (quasi-periodic, recurrent, almost automorphic, etc.) in the dynamical system $(C(\mathbb{T}\times W,Y),\mathbb{T},\sigma)$.

Remark 2.7. *Recall that for a compact metric space* W, *the topology on* $C(\mathbb{T}\times W,Y)$ *is metrizable. For example, the equality*

$$d(f,g) := \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{d_k(f,g)}{1+d_k(f,g)}$$

defines a complete metric on the space $C(\mathbb{T}\times W, X)$ which is compatible with the compact-open topology on $C(\mathbb{T}\times W, X)$, where $d_k(f,g) := \max_{|t|\le k,\ x\in W} \rho(f(t,x), g(t,x))$.

The space $C(\mathbb{T}\times W, Y)$ is topologically isomorphic to $C(\mathbb{T}, C(W,Y))$ (see [26]*), and also the shifts dynamical systems $(C(\mathbb{T}\times W, Y), \mathbb{T}, \sigma)$ and $(C(\mathbb{T}, C(W,Y)), \mathbb{T}, \sigma)$ are dynamically isomorphic.*

2.3. **Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}_1$).

A triplet $\langle (X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),\ h\rangle$, where h is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$ (i.e., h is continuous and $h(\pi(t,x)) = \sigma(t,h(x))$ for all $t\in\mathbb{T}_1$ and $x\in X$), is called a *non-autonomous dynamical system.*

Let $(Y,\mathbb{T}_2,\sigma)$ be a dynamical system, W a complete metric space, and φ a continuous mapping from $\mathbb{T}_1\times W\times Y$ into W, possessing the following properties:

a. $\varphi(0,u,y) = u$ $(u\in W, y\in Y)$;
b. $\varphi(t+\tau,u,y) = \varphi(\tau,\varphi(t,u,y),\sigma(t,y))$ $(t,\tau\in\mathbb{T}_1,\ u\in W, y\in Y)$.

Then, the triplet $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ (or shortly φ) is called [23] *a cocycle* on $(Y,\mathbb{T}_2,\sigma)$ with fiber W.

Let $X := W\times Y$ and let us define a mapping $\pi : X\times\mathbb{T}_1 \to X$ as follows: $\pi((u,y),t) := (\varphi(t,u,y),\sigma(t,y))$ (i.e., $\pi = (\varphi,\sigma)$). Then, it is easy to see that $(X,\mathbb{T}_1,\pi)$ is a dynamical system on X, which is called a *skew-product dynamical system* [23] and $h = pr_2 : X\to Y$ is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$ and, hence, $\langle (X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ on the dynamical system $(Y,\mathbb{T}_2,\sigma)$ with fiber W, then it generates a non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\sigma), h\rangle$ $(X := W\times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\sigma)\rangle$ on $(Y,\mathbb{T}_2,\sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions, every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of this type.

Example 2.8. Consider the system of differential equations

$$\left\{\begin{array}{ll} u' = & F(y,u)\\ y' = & G(y), \end{array}\right. \tag{6}$$

where $Y\subseteq E^m$ (for example, $Y = \mathcal{T}^m$ is an m–torus), $G\in C(Y,E^n)$ and $F\in C(Y\times E^n, E^n)$. Suppose that, for the system (6), the conditions ensuring existence, uniqueness and extendability of solutions to $\mathbb{R}_+$ are fulfilled. Denote by $(Y,\mathbb{R}_+,\sigma)$ a dynamical system on Y generated by the second equation of the system (6) and by $\varphi(t,u,y)$ we denote the solution of the equation

$$u' = F(\sigma(t,y),u)$$

passing through the point $u\in E^n$ at $t=0$. Then, the mapping $\varphi : \mathbb{R}_+\times E^n\times Y\to E^n$ satisfies conditions a. and b. from the definition of cocycle and, consequently, system (6) generates a non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$ (where $X:=E^n\times Y$, $\pi:=(\varphi,\sigma)$ and $h:=pr_2:X\to Y$).

Example 2.9. Let $(Y,\mathbb{R},\sigma)$ be a dynamical system on the metric space Y. We consider the equation

$$u'=F(\sigma(y,t),u)\quad (y\in Y), \tag{7}$$

where $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$. Suppose again that, for equation (7), the conditions for the existence, uniqueness and extendability of solutions to $\mathbb{R}_+$ are fulfilled. The non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ (respectively, the cocycle $\langle E,\varphi,(Y,\mathbb{R},\sigma)\rangle$), where $X:=\mathbb{R}^n\times Y$, $\pi:=(\varphi,\sigma)$, $\varphi(\cdot,x,y)$ is the solution of (7) passing through the point x at time $t=0$, and $h:=pr_2:X\to Y$ is generated by equation (7).

Example 2.10. We consider the equation

$$u'=f(t,u), \tag{8}$$

where $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Along with equation (8), consider the family of equations

$$u'=g(t,u), \tag{9}$$

where $g\in H(f):=\overline{\{f_\tau:\ \tau\in\mathbb{R}\}}$ and f_τ is the τ-shift of f with respect to the time variable t, i.e., $f_\tau(t,u):=f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$. Suppose that the function f is regular [23], i.e., for all $g\in H(f)$ and $u\in\mathbb{R}^n$ there exists a unique solution $\varphi(t,u,g)$ of equation (9). Denote by $Y=H(f)$ and $(Y,\mathbb{R},\sigma)$ a shift dynamical system on Y induced by the Bebutov dynamical system $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$. Now the family of equations (9) can be written as (7) if we take the mapping $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ defined by $F(g,u):=g(0,u)$, for all $g\in H(f)$ and $u\in\mathbb{R}^n$.

A solution $\varphi(t,u,y)$ of equation (7) is called [26, 28] *compatible (respectively, uniformly compatible) by the character of recurrence* if $\mathfrak{N}_y\subseteq\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_y\subseteq\mathfrak{M}_\varphi$), where $\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) is the set of all sequences $\{t_n\}\subset\mathbb{R}$ such that $\{\varphi(t+t_n,u,y\}$ converges to $\varphi(t,u,y)$ (respectively, $\{\varphi(t+t_n,u,y\}$ converges) in the space $C(\mathbb{T},\mathbb{R}^n)$.

Remark 2.11. *The sequence $\{\varphi(t+t_n,u,y)\}$ converges to the function ψ in the space $C(\mathbb{T},\mathbb{R}^n)$ if and only if $\{\varphi(t_n,u,y)\}$ converges to $\psi(0)$.*

Theorem 2.12. [26, 28] *The following statements hold:*

1. Let $y\in Y$ be a stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable) point. If $\varphi(t,u,y)$ is a compatible solution of equation (7), then so is $\varphi(t,u,y)$.

2. Let $y\in Y$ be a stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) point. If $\varphi(t,u,y)$ is a uniformly compatible solution of equation (7), then so is $\varphi(t,u,y)$.

Example 2.13. Let us consider a second order differential equation

$$x''=f(\sigma(t,y),x,x'),\ (y\in Y) \tag{10}$$

where $f\in C(Y\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, and state a criterion for the existence of Levitan almost periodic and almost automorphic solutions for this equation. Below we will suppose that the function f is *regular*, i.e., for all $y\in Y$ and $x,x'\in\mathbb{R}^n$ the equation

(10) admits a unique solution $\varphi(t,x,x',y)$ defined on $\mathbb{R}_+$ with the initial conditions $\varphi(0,x,x',y)=x$ and $\varphi'(0,x,x',y)=x'$.

As it is well-known, we can reduce equation (10) to the following equivalent system

$$\begin{cases} u'=v \\ v'=f(\sigma(t,y),u,v), \end{cases} \tag{11}$$

$(y\in Y)$ or to the equation

$$z'=F(\sigma(t,y),z)$$

on the product space $\mathbb{R}^n\times\mathbb{R}^n$, where $z:=(u,v)$ and $F\in C(Y\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n\times\mathbb{R}^n)$ is the function defined by the equality $F(y,z):=(v,f(y,u,v))$ for all $y\in Y$ and $z:=(u,v)\in\mathbb{R}^n\times\mathbb{R}^n$.

Theorem 2.14. [26] *Let $\varphi\in C(\mathbb{R},\mathbb{R}^n)$ be a continuously differentiable function. If its derivative $\varphi'\in C(\mathbb{R},\mathbb{R}^n)$ is uniformly continuous on $\mathbb{R}$, then φ' is uniformly comparable by the character of recurrence with φ, i.e., $\mathfrak{M}_\varphi\subseteq\mathfrak{M}_{\varphi'}$.*

We can now prove the following result.

Lemma 2.15. *Suppose that the following conditions hold:*

(i) *Y is compact;*
(ii) *$f\in C(Y\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ is regular;*
(iii) *$\varphi(t,x_0,x_0',y)$ is a solution of equation (10) defined and bounded on $\mathbb{R}$ and such that its derivative $\varphi'(t,x_0,x_0',y)$ is also bounded on $\mathbb{R}$.*

Then, the following two statements are equivalent:

a. *The solution $\varphi(t,x_0,x_0',y)$ of equation (10) is compatible (respectively, uniformly compatible) by the character of recurrence with the right-hand side;*
b. *The solution $(\varphi(t,x_0,x_0',y),\varphi'(t,x_0,x_0',y))$ of equation (11) is compatible (respectively, uniformly compatible) by the character of recurrence with the right-hand side.*

Proof. The implication b.$\Longrightarrow$ a. is evident. Thus, to prove the lemma, it is sufficient to establish the converse implication. Let $\varphi(t,x_0,x_0',y)$ be a solution of equation (10) such that $\varphi(t,x_0,x_0',y)$ and $\varphi'(t,x_0,x_0',y)$ are defined and bounded on $\mathbb{R}$. Then $\mathfrak{N}_y\subseteq\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_y\subseteq\mathfrak{M}_\varphi$). We need to show that the inclusion $\mathfrak{N}_y\subseteq\mathfrak{N}_{\varphi'}$ (respectively, $\mathfrak{M}_y\subseteq\mathfrak{M}_{\varphi'}$) also holds. Indeed, let $\{t_n\}\in\mathfrak{N}_y$ (respectively, $\{t_n\}\in\mathfrak{M}_y$), then the sequence $\{\sigma(t_n,y)\}$ converges to y (respectively, the sequence $\{\sigma(t_n,y)\}$ converges to some point $\tilde{y}\in Y$). Consequently, the functional sequence $\{f(\sigma(t+t_n,y),u,v)\}$ converges to $f(\sigma(t,y),u,v)$ (respectively, to $f(\sigma(t,\tilde{y}),u,v)$) uniformly with respect to t on every compact subset from $\mathbb{R}$ and $u,v\in Q:=\overline{\varphi(\mathbb{R},x_0,x_0',y)}\times\overline{\varphi'(\mathbb{R},x_0,x_0',y)}$. Since $\mathfrak{N}_y\subseteq\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_y\subseteq\mathfrak{M}_\varphi$), the sequence $\{\varphi(t_n,x_0,x_0',y)\}$ converges to x_0 (respectively, to some point $\tilde{x_0}\in\mathbb{R}$). Since the function $f\in C(Y\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ is regular, then the functional sequence $\{\varphi(t+t_n,x_0,x_0',y)\}$ converges to the function $\varphi(t,x_0,x_0',y)$ (respectively, to $\varphi(t,\tilde{x_0},\tilde{x_0}',\tilde{y})$) uniformly with respect to t on every compact subset from $\mathbb{R}$. Note that, under the conditions of Lemma, the second derivative $\varphi''(t,x_0,x_0',y)$ of the function $\varphi(t,x_0,x_0',y)$ is bounded on $\mathbb{R}$ and, consequently, the first derivative $\varphi'(t,x_0,x_0',y)$ is uniformly continuous in $t\in\mathbb{R}$. Thus, according to Theorem 2.14, the first derivative $\varphi'(t, x_0, x_0', y)$ is comparable (respectively, uniformly comparable) by the character of recurrence and, consequently, the sequence $\varphi'(t+t_n, x_0, x_0', y)$ converges to $\varphi'(t, x_0, x_0', y)$ (respectively, $\varphi(t, \tilde{x_0}, \tilde{x_0}', \tilde{y})$), and the lemma is proved. $\square$

Remark 2.16. *1. Notice that, if Y is not compact, then, in general, the boundedness of $\varphi(t, x_0, x_0', y)$ and $\varphi'(t, x_0, x_0', y)$ on $\mathbb{R}$ do not imply the boundedness of the second derivative $\varphi''(t, x_0, x_0', y)$ on $\mathbb{R}$. In this case, the equivalence of statements a. and b. of Lemma 2.15 remains as an open problem.*

2. If $|f(y, x, x')| \le c(1 + |x'|^2)$ for all $(y, x, x') \in Y \times \mathbb{R} \times \mathbb{R}$, then the boundedness of $\varphi(t, x, x', y)$ implies the boundedness of its first derivative $\varphi'(t, x, x', y)$ (see, for example, Lemma 2.1 [19] *and also Lemma 5.1 from* [17, Ch.XII]*).*

3. Non-Autonomous Dynamical Systems with Convergence

Let us now consider a special type of non-autonomous dynamical systems, namely the so-called dynamical systems with convergence. We start by recalling some definitions.

A dynamical system $(X, \mathbb{T}, \pi)$ is called *point dissipative* (respectively, *compact dissipative*), if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t\to+\infty} \rho(\pi(t, x), K) = 0 \tag{12}$$

for all $x \in X$ (respectively, the equality (12) holds uniformly with respect to x on every compact subset M from X).

A compact and invariant set $J \subset X$ is called the *Levinson center* of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$, if, in addition, J attracts every compact subset of X (i.e., (12) holds uniformly with respect to x on every compact subset $M \subset X$) . It is worth noticing that this concept does not coincides in general with that of global attractor (since the latter attracts the bounded subsets of X, i.e., (12) holds uniformly with respect to x on every bounded subset $B \subset X$). For a more detailed analysis on the relationship between these two concepts, see Cheban [7] .

A non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *convergent* if the following conditions hold:

- (i) the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative;
- (ii) the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$ where $X_y := h^{-1}(y) := \{x \in X \mid h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively $(Y, \mathbb{T}_2, \sigma)$).

Remark 3.1. *1. Note that convergent systems are, in some sense, the simplest dissipative dynamical systems. If $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a convergent non-autonomous dynamical system and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$), then J_X and J_Y are dynamically homeomorphic. Although the center of Levinson of a convergent system can be completely described, it may be sufficiently complicated. An example which illustrates this fact can be found in* [10, ChIV].

2. Observe that:

(i) *When Y is compact and invariant, then evidently $(Y, \mathbb{T}_2, \sigma)$ is compact dissipative and its Levinson center J_Y coincides with Y;*
(ii) *If $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a convergent non-autonomous dynamical system, and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) and $J_Y = Y$, then J_X and Y are dynamically homeomorphic. In particular, if the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent), then so is the point $x = h^{-1}(y) \in J_X$.*

Denote by $X \dot{\times} X = \{(x_1, x_2) \in X \times X \mid h(x_1) = h(x_2) \}$. If there exists a function $V : X \dot{\times} X \to \mathbb{R}_+$ with the following properties:

(i) V is continuous;
(ii) V is positive defined, i.e., $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$;
(iii) $V(\pi(t, x_1), \pi(t, x_2)) \le V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$ and $t \in \mathbb{T}_1^+ := \{t \in \mathbb{T}_1 \mid t \ge 0\}$,

then, the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called V - *monotone* (see [7] and [18], [31]).

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be *stable in the sense of Lagrange in the positive direction* (shortly, st. L^+), if for every compact subset K from X its positive semi-trajectory $\sum_K^+ := \bigcup \{\pi(t, x) : \ t \ge 0, \ x \in K\}$ is relatively compact.

Remark 3.2. *1. Every compact dissipative dynamical system is st. L^+.*

2. There are simple examples of st. L^+ dynamical systems which are not compact dissipative.

Let (X, h, Y) be a fiber space, i.e., let X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X onto Y. The subset $M \subseteq X$ is said to be *conditionally relatively compact*, if the pre-image $h^{-1}(Y') \bigcap M$ of every relatively compact subset $Y' \subseteq Y$ is a relatively compact subset of X. In particular, $M_y := h^{-1}(y) \bigcap M$ is relatively compact for every y. The set M is called *conditionally compact* if it is closed and conditionally relatively compact.

Example 3.3. *Let K be a compact space, $X := K \times Y$, $h = pr_2 : X \to Y$, then the triplet (X, h, Y) is a fiber space, the space X is conditionally compact, but not compact.*

Theorem 3.4. [8] *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ be a NDS with the following properties:*

(i) *It admits a conditionally relatively compact invariant set J;*
(ii) *The NDS $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ is positively uniformly stable on J;*
(iii) *Every point $y \in Y$ is two-sided Poisson stable.*

Then,

(i) *All motions on J can be continued uniquely to the left, and define on J a two-sided dynamical system $(J, \mathbb{S}, \pi)$, i.e., the semi-group dynamical system $(X, \mathbb{T}, \pi)$ generates a two-sided dynamical system $(J, \mathbb{S}, \pi)$ on J;*

(ii) *For every $y \in Y$, there are two sequences $\{t_n^1\} \to +\infty$ and $\{t_n^2\} \to -\infty$ such that*

$$\pi(t_n^i, x) \to x \ (i = 1, 2)$$

as $n \to \infty$ for all $x \in J_y$.

Denote by $\mathcal{K} := \{a \in C(\mathbb{R}_+, \mathbb{R}_+) |\ a(0) = 0,\ a \text{ is strictly increasing}\}$.

Theorem 3.5. [8] *(The invariance principle for NDS) Assume the following conditions:*

(i) *$y \in Y$ is Poisson stable;*

(ii) *The NDS $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ admits a conditionally relatively compact invariant set J;*

(iii) *$\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is a V-monotone non-autonomous dynamical system, and there are two functions $a, b \in \mathcal{K}$ such that*
   (a) *$Im(a) = Im(b)$, where $Im(a) := a(\mathbb{R}_+)$ is the image of the values of $a \in \mathcal{K}$;*
   (b) *$a(\rho(x_1, x_2)) \le V(x_1, x_2) \le b(\rho(x_1, x_2))$ for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$.*

Then, $V(\pi(t, x_1), \pi(t, x_2)) = V(x_1, x_2)$ for all $t \in \mathbb{S}$ and $x_1, x_2 \in J_y$.

Recall that the dynamical system $(X, \mathbb{T}_1, \pi)$ is called *asymptotically compact* if for every positively invariant bounded subset $M \subseteq X$ there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t \to +\infty} \beta(\pi(t, M), K) = 0,$$

where $\beta(A, B) := \sup_{a \in A} \rho(a, B)$ and $\rho(a, B) := \inf_{b \in B} \rho(a, b)$.

Denote by ω_x the ω–limit set of the point x, and by $\Omega_X := \overline{\bigcup\{\omega_x |\ x \in X\}}$. Let $M \subseteq X$. Then we set

$$D^+(M) := \bigcap_{\varepsilon > 0} \overline{\bigcup_{t \ge 0} \pi(t, B(M, \varepsilon))},$$

where $B(M, \varepsilon) := \{x \in X |\ \rho(x, M) < \varepsilon\}$.

A subset $M \subseteq X$ is called *orbital stable* if for arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \ge 0$.

Remark 3.6. *1. If the set $M \subseteq X$ is orbital stable, then $D^+(M) = M$ (see* [3]*).*

2. If the space X is locally compact and M is compact, then from the equality $D^+(M) = M$ it follows the orbital stability of M. This fact is known as Theorem of T. Ura [30] *(see also* [3]*).*

Theorem 3.7. [7, ChI] *A point dissipative dynamical system $(X, \mathbb{T}, \pi)$ on the complete metric space X is compact dissipative if and only if $D^+(\Omega_X)$ is compact and orbital stable.*

Corollary 3.8. *Let $(X,\mathbb{T},\pi)$ be point dissipative and Ω_X orbital stable. Then, $(X,\mathbb{T},\pi)$ is compact dissipative and its Levinson center J_X coincides with Ω_X.*

Proof. If Ω_X is orbital stable, then $D^+(\Omega_X)=\Omega_X$ and now to finish the proof it is sufficient to apply Theorem 3.7. □

We can therefore prove our main result in this section.

Theorem 3.9. *Let $\langle(X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

1. *The dynamical system $(Y,\mathbb{S},\sigma)$ is pseudo recurrent;*
2. *The dynamical system $(X,\mathbb{T},\pi)$ is asymptotically compact;*
3. *There exists a point $x_0\in X_{y_0}$ with relatively compact positive semi-trajectory $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$;*
4. *There exists a continuous function $V: X\dot{\times}X\to\mathbb{R}_+$ such that $V(\pi(t,x_1),\pi(t,x_2))<V(x_1,x_2)$ for all $(x_1,x_2)\in X\dot{\times}X\setminus\Delta_X$ and $t>0$ ($t\in\mathbb{T}$), where $\Delta_X:=\{(x,x):\ x\in X\}$;*
5. *There are functions $a,b\in\mathcal{K}$ such that $Im(a)=Im(b)$ and $a(\rho(x_1,x_2)\le V(x_1,x_2)\le b(\rho(x_1,x_2))$ for all $(x_1,x_2)\in X\dot{\times}X$.*

Then, the following statements take place:

(i) *The NDS $\langle(X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ is convergent;*
(ii) *$J_X=\omega_{x_0}$;*
(iii) *$h(J_X)=Y$.*

Proof. Since the point y_0 is Poisson stable and $\omega_{y_0}=H(y_0)=Y$ then, for every $y\in Y$, there exists a sequence $\{t_n\}\subseteq\mathbb{T}_2$ such that $t_n\to+\infty$ as $n\to\infty$ and $\{\sigma(t_n,y_0)\}\to y$. Consider the sequence $\{\pi(t_n,x_0)\}$. Thanks to assumption 3., we can assume that this sequence is convergent. Let p be its limit. Then, it is clear that $p\in\omega_{x_0}\bigcap X_y$. Thus, we have established that $h(\omega_{x_0})=Y$.

First, we notice that ω_{x_0} is compact, invariant and, according to Theorem 3.4, on ω_{x_0} it is defined a two-sided dynamical system $(\omega_{x_0},\mathbb{S},\pi)$ such that $\pi(t,x)=\gamma_x(t)$ for all $x\in\omega_{x_0}$ and $t\in\mathbb{R}_-$, where γ_x is the unique complete trajectory of the dynamical system $(X,\mathbb{T},\pi)$ passing through the point x at the initial moment $t=0$. We will show now that the set $\omega_{x_0}\bigcap X_y$ contains at most one point for all $y\in Y$. Indeed, the set ω_{x_0} is compact, invariant and, according to Theorem 3.5, we have $V(\pi(t,p_1),\pi(t,p_2))=V(p_1,p_2)$ for all $t\in\mathbb{S}$. But the last equality takes place only if $p_1=p_2$.

Let now x be an arbitrary point from X, $y:=h(x)$ and $p\in\omega_{x_0}\bigcap X_y$. According to conditions 4. and 5., we have $a(\rho(\pi(t,x),\pi(t,p)))\le V(\pi(t,x),\pi(t,p))\le V(x,p)\le b(\rho(x,p))$ for all $t\ge 0$ and, consequently, we obtain $\rho(\pi(t,x),\pi(t,p))\le a^{-1}(b(x_1,x_2))$ for all $t\ge 0$. Since $p\in L_X$ (where L_X denotes the set of points $x\in X$ such that there exists a complete relatively compact trajectory γ_x of the dynamical system $(X,\mathbb{T},\pi)$, passing through the point x at the initial moment $t=0$), then from the last inequality we obtain that the set Σ^+_x is bounded. Taking into account that $(X,\mathbb{T},\pi)$ is asymptotically compact, we can conclude that the point x

is stable in the sense of Lagrange in the positive direction. It is easy to show that $\omega_x \bigcap X_y$ contains a single point using the same arguments as we used above for the set ω_{x_0}. We now show that $\omega_x = \omega_{x_0}$. To this end, denote by $M := \omega_{x_0} \bigcup \omega_x$, and repeating the reasoning above for this set we obtain that $M \bigcap X_y$ consists of a single point for all $y \in Y$. Thus, we have $\omega_{x_0} \bigcap X_y = \omega_x \bigcap X_y = M \bigcap X_y$ for all $y \in Y$ and, consequently, $\omega_x = \omega_{x_0}$ for all $x \in X$. This means that the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative and $\Omega_X = M$, where $M := \omega_{x_0}$. Now, we will show that $(X, \mathbb{T}, \pi)$ is compact dissipative. By Theorem 3.7 (see also Corollary 3.8), it is sufficient to establish that the set M is orbitally stable, i.e., for every $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon)$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \geq 0$. If we suppose the opposite, then there are $\varepsilon_0 > 0$, $\delta_n \to 0$ ($\delta_n > 0$) $x_n \in X$ and $t_n \to +\infty$ such that

$$\rho(x_n, M) < \delta_n \quad \text{and} \quad \rho(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{13}$$

Let $m_n \in M$ be a point such that $\rho(x_n, m_n) = \rho(x_n, M)$, and denote by $y_n := h(x_n)$. Since the set M is compact, taking into account (13), we can assume that the sequences $\{x_n\}$, $\{y_n\}$ and $\{m_n\}$ are convergent. Let $\bar{x} := \lim\limits_{n\to\infty} x_n$ and $\bar{m} := \lim\limits_{n\to\infty} m_n$. Then, by (13), we have $\bar{x} = \bar{m}$. Denoting by $m_{y_n} := M \bigcap X_{y_n}$, and taking into consideration the continuity of the mapping $y \mapsto m_y$, we obtain $\lim\limits_{n\to\infty} m_{y_n} = m_{\bar{y}}$, where $\bar{y} := h(\bar{m})$. Note that

$$\bar{m} = m_{\bar{y}} \tag{14}$$

and, consequently,

$$\rho(x_n, m_{y_n}) \leq \rho(x_n, m_n) + \rho(m_n, m_{y_n}). \tag{15}$$

From (14) and (15) it follows that

$$\rho(x_n, m_{y_n}) \to 0$$

as $n \to \infty$. On the other hand,

$$V(\pi(t_n, x_n), \pi(t_n, m_{y_n})) < V(x_n, m_{y_n}) \to 0$$

as $n \to \infty$. It is clear that $\pi(t_n, m_{y_n}) = m_{\sigma(t_n, y_n)}$ and since the space Y is compact, we can assume that the sequence $\{\sigma(t_n, y_n)\}$ is convergent. Let us denote its limit by $\tilde{y}$. Then, $\lim\limits_{n\to\infty} m_{\sigma(t_n, y_n)} = m_{\tilde{y}} \in M$. But, the last equality contradicts inequality (13). This contradiction proves our statement.

Since the set $\Omega_X = M$ is orbitally stable, then, according to Theorem 3.7 and Corollary 3.8, the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and its Levinson center J_X coincides with $\Omega_X = M$. Since we established above that $J_X \bigcap X_y = M \bigcap X_y$ consists of a single point for all $y \in Y$, then the NDS $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ is a system with convergence. The proof is now complete. $\square$

4. First order differential equations

In this section we study a scalar differential equation of the form

$$x' = f(\sigma(t, y), x) \quad (y \in Y), \tag{16}$$

where $f \in C(Y \times \mathbb{R}, \mathbb{R})$, Y is a complete metric space and $(Y, \mathbb{R}, \sigma)$ is a dynamical system.

A function $f \in C(Y\times\mathbb{R},\mathbb{R})$ is said to be *decreasing in the large sense* (respectively, *strictly decreasing*) with respect to the variable $x\in\mathbb{R}$ if for all $x_1,x_2\in\mathbb{R}$ and $y\in Y$ the inequality $x_2>x_1$ implies $f(y,x_2)\le f(y,x_1)$ (respectively, $f(y,x_2)<f(y,x_1)$).

Let us now state a result which collects some properties of the solutions of (16).

Theorem 4.1. [8, 22, 27] *Suppose that the function $f\in C(Y\times\mathbb{R},\mathbb{R})$ is regular and decreasing (in the large sense) with respect to the variable $x\in\mathbb{R}$, and the point $y\in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then, the following statements hold:*

(i) *If (16) admits a solution which is bounded on $\mathbb{R}$, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (16), then $u(t)-v(t)=c$ for all $t\in\mathbb{R}$, where $c\in\mathbb{R}$ is some constant;*
(iii) *If the function f is strictly decreasing with respect to the variable $x\in\mathbb{R}$, then equation (16) admits at most one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Below we will prove some results which improve and generalize the third statement of Theorem 4.1.

Theorem 4.2. *Suppose that the function $f\in C(Y\times\mathbb{R},\mathbb{R})$ is regular and strictly decreasing with respect to the variable $x\in\mathbb{R}$, the dynamical system $(Y,\mathbb{R},\sigma)$ is pseudo recurrent, and $Y=H(y)$. If (16) admits a solution $\varphi(t,u_0,y)$, which is bounded on $\mathbb{R}_+$, then it is convergent, i.e., the non-autonomous dynamical system generated by equation (16) is convergent.*

Proof. Let $\varphi(t,u,y)$ be the unique solution of equation (16) passing through the point $u\in\mathbb{R}$ at the initial moment $t=0$ and $\langle(X,\mathbb{T},\pi),(Y,\mathbb{R},\sigma),h\rangle$ ($X:=\mathbb{R}\times Y$, $\pi:=(\varphi,\sigma)$ and $h=pr_2: X\mapsto Y$) be the non-autonomous dynamical system generated by (16). Consider the mapping $V: X\dot{\times}X\mapsto\mathbb{R}_+$ defined by equality

$$V((u_1,y),(u_2,y)):=\frac{|u_1-u_2|^2}{2}$$

for all $u_1,u_2\in\mathbb{R}$ and $y\in Y$. Note that

$$\frac{dV(\pi(t,(u_1,y)),\pi(t,(u_2,y)))}{dt}\Big|_{t=0}=(u_1-u_2)(f(y,u_1)-f(y,u_2))<0 \tag{17}$$

for all $(u_i,y)\in\mathbb{R}\times Y$ $(i=1,2)$ with $u_1\ne u_2$. From (17) it follows that $V(\pi(t,(u_1,y)),\pi(t,(u_2,y)))<V((u_1,y),(u_2,y))$ for all $t>0$, $u_1,u_2\in\mathbb{R}$ and $y\in Y$. Now, to finish the proof of the Theorem, it is sufficient to apply Theorem 3.9. $\square$

Corollary 4.3. *Suppose that the function $f\in C(Y\times\mathbb{R},\mathbb{R})$ is regular and strictly decreasing with respect to the variable $x\in\mathbb{R}$, and assume that the point y is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent).*

If (16) admits a solution which is bounded on $\mathbb{R}_+$, then it possesses a unique stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) solution which is globally uniformly asymptotically stable.

Proof. This statement follows directly from Theorem 4.2 and Remark 3.1 (item 2 (ii)). □

Remark 4.4. *1. The analog of Theorem 4.2 (as well as Corollary 4.3) holds if we replace the condition "f is strictly decreasing" by "f is strictly increasing". This case can be reduced to the considered one by the time substitution $t \to -t$.*

2. Note that Theorem 4.2 and Corollary 4.3 remain true also for a vectorial equation (i.e., for systems of equations). Indeed, to this end, we assume that $f \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$, and replace the condition "f is strictly decreasing" by the condition

$$\langle f(y,u_1) - f(y,u_2), u_1 - u_2 \rangle < 0$$

for all $y \in Y$ and $u_1, u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$), where $\langle , \rangle$ is the scalar product on the space $\mathbb{R}^n$.

3. If the function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ is continuously differentiable with respect to $x \in \mathbb{R}$ and

$$\frac{\partial f}{\partial x}(y,x) \leq -k < 0 \tag{18}$$

for all $y \in Y$ and $x \in \mathbb{R}$, then Theorem 4.2 and Corollary 4.3 also hold without the requirement that equation (16) admits at least one solution which is bounded on $\mathbb{R}_+$. Owing to condition (18), it follows

$$\langle f(y,u_1) - f(y,u_2), u_1 - u_2 \rangle \leq -k|u_1 - u_2|^2 \tag{19}$$

for all $y \in Y$ and $u_1, u_2 \in \mathbb{R}$. But condition (19) guarantees (see [11]*) that equation (16) is convergent.*

4. We plan to study in more details the multi-dimensional case in one of our next publications.

5. Levitan almost periodic and almost automorphic solutions of second order differential equations

In this section we consider a scalar differential equation of the type (10), i.e., $n = 1$.

In the sequel, we suppose that the function $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and increasing (in the large sense) with respect to the variable x, i.e., if $u_1 \leq u_2$ then $f(y,u_1,v) \leq f(y,u_2,v)$ for all $y \in Y$ and $v \in \mathbb{R}$.

Lemma 5.1. [22] *Let $u(t)$, $v(t)$ be two solutions of equation (10) defined on $\mathbb{R}$. Then, only one of the following three cases is possible:*

- (i) *The function $u(t) - v(t)$ is monotone on the real axis $\mathbb{R}$;*
- (ii) *$u(t) - v(t)$ is positive on $\mathbb{R}$, and there exists a number $t_0 \in \mathbb{R}$ such that this function is non-decreasing on the interval $(t_0, +\infty)$, and non-increasing on $(-\infty, t_0)$;*

(iii) *The function $u(t)-v(t)$ is negative on $\mathbb{R}$, and there exists a number $t_0 \in \mathbb{R}$ such that it is non-increasing on the interval $(t_0, +\infty)$, and non-decreasing on $(-\infty, t_0)$.*

Let $\varphi \in C(\mathbb{R},\mathbb{R})$, and denote by $a_\varphi := \inf\{\varphi(t) |\ t \in \mathbb{R}\}$ and $b_\varphi := \sup\{\varphi(t) |\ t \in \mathbb{R}\}$.

Remark 5.2. *Notice that the following facts take place:*

1. $a_\varphi \leq b_\varphi$ for all $\varphi \in C(\mathbb{R},\mathbb{R})$.

2. The inequalities

$$a_\varphi \leq a_\psi \leq b_\psi \leq b_\varphi \tag{20}$$

hold for all $\psi \in H(\varphi)$.

3. If the function φ is recurrent, then

$$a_\varphi = a_\psi \quad \text{and} \quad b_\psi = b_\varphi$$

for all $\psi \in H(\varphi)$.

Theorem 5.3. [25, 29] *Let $f \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ be Poisson stable with respect to the time variable $t \in \mathbb{R}$. If the equation*

$$x' = f(t, x)$$

admits a solution φ which is bounded on $\mathbb{R}$, then it admits at least a Poisson stable solution $\psi \in H(\varphi)$.

Let us now establish our first main result in this section.

Theorem 5.4. *Suppose that $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and increasing (in the large sense) with respect to the variable $x \in \mathbb{R}$, and assume that the point $y \in Y$ is Poisson stable. Then, the following statements hold:*

(i) *If (10) admits a solution ϕ such that ϕ and ϕ' are bounded on $\mathbb{R}$, then it has at least one compatible (by the character of recurrence with the right-hand side) solution;*
(ii) *If $u(t)$ and $v(t)$ are two compatible solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to the variable $x \in \mathbb{R}$, then equation (10) admits at most one compatible solution which is bounded on $\mathbb{R}$.*

Proof. Let $\phi \in C(\mathbb{R},\mathbb{R})$ be a solution of equation (10) such that ϕ and ϕ' are bounded on $\mathbb{R}$. To prove the first statement, on account of Lemma 2.15, it is sufficient to show that the function ϕ is comparable with y by the character of recurrence, i.e., the functional sequence $\{\phi(t + t_n)\}$ converges to $\phi(t)$ uniformly on every compact subset from $\mathbb{R}$, for every sequence $\{t_n\} \in \mathfrak{N}_y$. Consider the motion $\sigma(t, \phi)$ in the shift dynamical system (Bebutov's system) $(C(\mathbb{R},\mathbb{R}), \mathbb{R}, \sigma)$. According to Theorem 5.3, the set $H(\phi) := \overline{\{\sigma(\tau, \phi) |\ \tau \in \mathbb{R}\}}$ contains at least one Poisson stable solution $\varphi \in H(\phi)$ of equation (10) (in fact, the function φ and the point y are jointly Poisson stable) . We will prove that the solution φ is compatible. To this end, we will show that equation (10) possesses at most one solution from $H(\varphi) \subseteq H(\phi)$. Indeed, if $\psi \in H(\varphi)$ is a solution of equation (10) and $r(t) := \psi(t) - \varphi(t)$ for all $t \in \mathbb{R}$, then, by Lemma 5.1, there exist the limits

$$\lim_{t\to+\infty} r(t) = c_+, \quad \lim_{t\to-\infty} r(t) = c_-$$

and

$$|c_+| + |c_-| > 0.$$

Suppose, for example, that $c_+ > 0$. Then, by the joint Poisson stability of the point y and the solution φ, there exists a sequence $\{t_n\} \in \mathfrak{N}_y \cap \mathfrak{N}\varphi$ such that $t_n \to +\infty$ as $n \to \infty$. Without loss of generality, we can suppose that the sequence $\{\psi(t+t_n)\}$ is convergent in the space $C(\mathbb{R}, \mathbb{R})$. Let $\bar{\psi}$ be its limit, i.e., $\bar{\psi}(t) = \lim_{t\to+\infty} \psi(t+t_n)$. Then,

$$\bar{\psi}(t) = \varphi(t) + c_+ \quad \text{for all} \quad t \in \mathbb{R}. \tag{21}$$

From (20) and the fact that $\bar{\psi} \in H(\psi) \subseteq H(\varphi)$, we have

$$a_\varphi \le a_\psi \le a_{\bar{\psi}} \le b_{\bar{\psi}} \le b_\psi \le b_\varphi. \tag{22}$$

On the other hand, from (21) we have $b_{\bar{\psi}} = b_\varphi + c_+$. From the last equality and (22) we obtain $c_+ \le 0$. This contradiction proves our statement. The other cases can be treated similarly.

Let now $u(t)$ and $v(t)$ be two compatible solutions of equation (10). Then, thanks to Lemma 5.1, there exists a number $t_0 \in \mathbb{R}$ such that the function $r(t) := u(t) - v(t)$ is monotone on one of the two intervals: $(-\infty, t_0)$ or $(t_0, +\infty)$. Consider, for example, the case when $r(t)$ is monotone on the interval $(-\infty, t_0)$. Since the solutions u and v are compatible, and the point y is Poisson stable, the function $r(t)$ is Poisson stable too. In particular, it is Poisson stable in the negative direction. On the other hand, this function is monotone on the interval $(-\infty, t_0)$ and, consequently, is a constant. Thus $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant.

Finally, we prove the third statement of our theorem. Suppose that the function f is strictly increasing with respect to the variable $x \in \mathbb{R}$. If we suppose that equation (10) admits two different solutions u and v which are bounded on $\mathbb{R}$, then the function

$$r(t) := u(t) - v(t) \quad (t \in \mathbb{R}) \tag{23}$$

possesses the limits $c_\pm := \lim_{t\to\pm\infty} r(t)$ and $|c_-| + |c_+| > 0$. Suppose, for example, that $c_+ > 0$. Then, we take a sequence $\{t_n\} \in \mathfrak{N}_y$ such that $t_n \to +\infty$ and the functional sequences $\{u(t+t_n)\}$ and $\{v(t+t_n)\}$ are convergent (since the functions u and v are solutions of (10) which are bounded on $\mathbb{R}$. Denote by $\bar{u}$ (respectively, $\bar{v}$) the limit of the sequence $\{u(t+t_n)\}$ (respectively, $\{v(t+t_n)\}$)). From equality (23) we have

$$\bar{u}(t) := \bar{v}(t) + c_+ \quad \text{for all} t \in \mathbb{R}$$

and, consequently, we obtain $f(\sigma(t,y), \bar{v}(t), \bar{v}'(t)) = f(\sigma(t,y), \bar{v}(t) + c_+, \bar{v}'(t))$ for all $t \in \mathbb{R}$. The last identity contradicts the strict monotony of the function f with respect to the variable x. This contradiction completes the proof. □

As a consequence of Theorem 5.4 and Theorem 2.12, we have the second main result in this section.

Corollary 5.5. *Suppose that $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to the variable $x \in \mathbb{R}$, and the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable). Then, the following statements hold:*

(i) *If (10) admits a solution ϕ such that ϕ and ϕ' are bounded on $\mathbb{R}$, then it has at least one stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to the variable $x \in \mathbb{R}$, then equation (10) admits at most one stationary solution (respectively, τ–periodic, Levitan almost periodic, almost automorphic, almost recurrent, Poisson stable) which is bounded on $\mathbb{R}$.*

6. Quasi-periodic, Bohr almost periodic, almost automorphic and recurrent solutions

In this section we analyze problem (10) in the scalar case, i.e., $n = 1$, and suppose that Y is compact and $(Y, \mathbb{R}, \sigma)$ is a minimal dynamical system, i.e., Y does not contain a proper compact invariant subset.

Our main result below ensures that compatibility is now uniform.

Theorem 6.1. *Suppose that $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to the variable $x \in \mathbb{R}$. Then, the following statements hold:*

(i) *If (10) admits a solution φ such that φ and φ' are bounded on $\mathbb{R}$ then it has at least one uniformly compatible (by the character of recurrence with the right-hand side) solution;*
(ii) *If $u(t)$ and $v(t)$ are two uniformly compatible solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to the variable $x \in \mathbb{R}$, then equation (10) admits at most one uniformly compatible solution.*

Proof. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be a solution such that φ and $var\phi'$ are bounded on $\mathbb{R}$. To prove the first statement, taking into account Lemma 2.15, it is sufficient to show that φ is uniformly comparable with y by the character of recurrence, i.e., the functional sequence $\{\varphi(t + t_n)\}$ is convergent uniformly on every compact subset from $\mathbb{R}$, for every sequence $\{t_n\} \in \mathfrak{M}_y$. Denote by $X := C(\mathbb{R}, \mathbb{R}) \times Y$ and $(X, \mathbb{R}, \pi)$ the product dynamical system, i.e., $\pi(\tau, (\varphi, y)) := (\varphi_\tau, \sigma(\tau, y))$ for all $(\varphi, y) \in C(\mathbb{R}, \mathbb{R}) \times Y$ and $\tau \in \mathbb{R}$, where φ_τ is a τ–shift of the function φ ($\varphi_\tau(t) := \varphi(t + \tau)$ for all $t \in \mathbb{R}$). Consider the motion $\pi(t, (\varphi, y))$ in the product dynamical system $(X, \mathbb{R}, \pi)$. Under the conditions of our theorem, this motion is stable in the sense of Lagrange, i.e., the set $H(\varphi, y) := \overline{\{\pi(\tau, (\varphi, y)) | \ \tau \in \mathbb{R}\}}$ is compact. According to Birkhoff's theorem, the set $H(\varphi, y)$ contains at least one minimal set $\mathcal{M} \subseteq H(\varphi, y)$. Note that the mapping $h := pr_2 \ : \mathcal{M} \mapsto Y$ is an homomorphism of the dynamical system $(H(\varphi, y), \mathbb{R}, \pi)$ onto $(Y, \mathbb{R}, \sigma)$ and, consequently, $\mathcal{M}_y := \{(\psi, y) \ : \ (\psi, y) \in H(\varphi, y)\}$ is a nonempty compact subset of $H(\varphi, y)$. Now we will show that the set $\mathcal{M}_y$ consists of a single point for every $y \in Y$. Indeed, if we assume the opposite, then there exists a point $y_0 \in Y$ such that $\mathcal{M}_{y_0}$ contains at least two different points (v_i, y_0) ($i = 1, 2$ and $v_1 \neq v_2$). According to Theorem 5.4, without loss of generality we may suppose, for example, that v_1 is comparable by the character of recurrence with the point y_0, i.e., $\mathfrak{N}_{y_0} \subseteq \mathfrak{N}_{v_1}$. On the other hand, by Lemma 5.1, there exist the limits $\lim\limits_{t \to \pm\infty} r(t) = c_{\pm}$ and $|c_-| + |c_+| > 0$, where $r(t) := v_2(t) - v_1(t)$ for all $t \in \mathbb{R}$. Suppose, for example, that $c_- > 0$. Then, taking into account the fact that the point (v_1, y_0) is negatively Poisson stable, we have a sequence $\{t_n\} \in \mathfrak{N}_{v_1} \cap \mathfrak{N}_{y_0}$ such that $t_n \to -\infty$ as $n \to \infty$. We can suppose that the sequence $\{v_2(t+t_n)\}$ is convergent. Denote by $\bar{v}_2$ its limit. Then, we have $\bar{v}_2(t) = v_1(t) + c_-$ for all $t \in \mathbb{R}$ and, consequently, we have

$$a_{\bar{v}_2} = a_{v_1} + c_-. \tag{24}$$

But the functions $v_1, \bar{v}_2 \in H(v_1)$, and the function v_1 is recurrent and, consequently, we have

$$a_{\bar{v}_2} = a_{v_1} = a_{v_2}. \tag{25}$$

From (24) and (25) it follows that $c_- = 0$. This contradiction proves our statement. The other cases can be considered in a similar way. Thus, we have established that the set $\mathcal{M}_y$ consists of a single point for all $y \in Y$. Let ϕ be a solution of equation (10) such that $\{(\phi, y)\} = \mathcal{M}_y$. Now, it is easy to show that the solution ϕ is uniformly compatible. Indeed, let $\{t_n\} \in \mathfrak{M}_y$. Then, the sequence $\{\sigma(t_n, y)\}$ converges. Denote by $\tilde{y}$ its limit. We will show that the functional sequence $\{\phi(t + t_n)\}$ is also convergent in the space $C(\mathbb{R}, \mathbb{R})$. If it is not true, then there exist at least two points of accumulation ψ_i ($i = 1, 2$ and $\psi_1 \neq \psi_2$) for this sequence. On the other hand, it is easy to see that $(\psi_i, \tilde{y}) \in \mathcal{M}_{\tilde{y}}$ ($i = 1, 2$). The last inclusion contradicts the fact that every subsets $\mathcal{M}_y \subseteq \mathcal{M}$ consists of a single point for all $y \in Y$. This contradiction proves the first statement of our theorem.

The second and third statements follow from Theorem 5.4. □

Corollary 6.2. *Suppose that $f \in C(Y \times \mathbb{R}^2, \mathbb{R})$ is regular and monotone increasing (in the large sense) with respect to the variable $x \in \mathbb{R}$, and the point $y \in Y$ is stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then, the following statements hold:*

(i) *If (10) admits a solution φ such that φ and φ' are bounded on $\mathbb{R}$, then it has at least one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
(ii) *If $u(t)$ and $v(t)$ are two stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then $u(t) - v(t) = c$ for all $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is some constant;*
(iii) *If the function f is strictly increasing with respect to the variable $x \in \mathbb{R}$, then equation (10) admits at most one stationary (respectively, τ–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. These statements follow from Theorem 6.1 and Theorem 2.12. □

Remark 6.3. *In the particular case in which* Y *is a Bohr almost periodic minimal set, then Corollary 6.2 coincides with the result proved by Opial in* [22].

7. Some generalizations

Let now $I := (a, b)$, where $a, b \in [-\infty, +\infty]$. For example, $I = \mathbb{R}$, $I = (0, +\infty)$, $I = (a, b)$ and $a, b \in \mathbb{R}$, etc. Consider equation (10) when $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$. For example, for the equation

$$x'' + cx' + \frac{1}{x^{\alpha}} = f(\sigma(t, y))$$

we have $f(y, x, x') := -cx' - 1/x^{\alpha} + f(y)$ and $I = (0, +\infty)$, where $\alpha > 0$.

In this context, we will now highlight an extended meaning to the concept of boundedness on $\mathbb{R}$. To this respect, a solution $\varphi \in C(\mathbb{R}, \mathbb{R})$ of equation (10) is said to be *bounded on* $\mathbb{R}$ (respectively, on $\mathbb{R}_+$) if $Q := \overline{\varphi(\mathbb{R})}$ is a compact subset from I, i.e., if there exist two real numbers α and β such that $a < \alpha \leq \varphi(t) \leq \beta < b$ for all $t \in \mathbb{R}$ (respectively, $t \in \mathbb{R}_+$).

All of our results about our second order equation (10) (especially, theorems 5.4, 6.1 and Corollaries 5.5 and 6.2) remain true also when $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$. We will formulate for example the following statements.

Theorem 7.1. *Suppose that* $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ *is regular and increasing (in the large sense) with respect to the variable* $x \in I$. *Then, the following statements hold:*

- (i) *If (10) admits a solution* φ *such that* φ *and* φ' *are bounded on* $\mathbb{R}$, *then it has at least one uniformly compatible (by the character of recurrence with the right-hand side) solution;*
- (ii) *If* $u(t)$ *and* $v(t)$ *are two uniformly compatible solutions of equation (10), then* $u(t) - v(t) = c$ *for all* $t \in \mathbb{R}$, *where* $c \in \mathbb{R}$ *is some constant;*
- (iii) *If the function* f *is strictly increasing with respect to the variable* $x \in I$, *then equation (10) admits at most one uniformly compatible solution.*

Proof. We omit the proof because is completely similar to the proof of Theorem 6.1. □

Corollary 7.2. *Suppose that the function* $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ *is regular and increasing (in the large sense) with respect to the variable* $x \in I$, *and the point* $y \in Y$ *is stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent). Then, the following statements hold:*

- (i) *If (10) admits a solution* φ *such that* φ *and* φ' *are bounded on* $\mathbb{R}$, *then it has at least one stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*
- (ii) *If* $u(t)$ *and* $v(t)$ *are two stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then* $u(t) - v(t) = c$ *for all* $t \in \mathbb{R}$, *where* $c \in \mathbb{R}$ *is some constant;*
- (iii) *If the function* f *is strictly increasing with respect to the variable* $x \in I$, *then equation (10) admits at most one stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. This result follows from Theorem 7.1 and Theorem 2.12. □

Corollary 7.3. *Suppose that the following conditions are fulfilled:*

(i) $f \in C(Y \times I \times \mathbb{R}, \mathbb{R})$ *and there exists a constant* $C > 0$ *such that*

$$|f(y, x, x')| \leq C(1 + |x'|^2) \tag{26}$$

for all $(y, x, x') \in Y \times I \times \mathbb{R}$*;*

(ii) *The function* f *is regular and monotone increasing (in the large sense) with respect to the variable* $x \in \mathbb{R}$*;*

(iii) *The point* $y \in Y$ *is stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent).*

Then, the following statements hold:

(i) *If (10) admits a solution which is bounded on* $\mathbb{R}$*, then it has at least one stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution;*

(ii) *If* $u(t)$ *and* $v(t)$ *are two stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solutions of equation (10), then* $u(t) - v(t) = c$ *for all* $t \in \mathbb{R}$*, where* $c \in \mathbb{R}$ *is some constant;*

(iii) *If the function* f *is strictly increasing with respect to the variable* $x \in I$*, then equation (10) admits at most one stationary (respectively,* τ*–periodic, quasi-periodic, Bohr almost periodic, recurrent) solution.*

Proof. These statements follow from Corollary 7.2. To this end, it is sufficient to note that under condition (26), if $\varphi \in C(\mathbb{R}, \mathbb{R})$ is a solution of equation (10) which is bounded on $\mathbb{R}$, then its derivative φ' is also bounded on $\mathbb{R}$ (see Lemma 2.1 [19] and also Lemma 5.1 from [17, Ch.XII]). □

Remark 7.4. *1. Corollary 7.3 (item (iii)) improves and generalizes some of the results from* [4, 12, 14, 19] *when the function* f *is strictly increasing with respect to the second variable.*

2. We plan to study in more detail this case (f *is strictly increasing with respect to second variable) in one of our future publication.*

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