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INVARIANT MEASURES AND STATISTICAL SOLUTIONS OF THE GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS

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Abstract. We obtain regularity results for solutions of the three dimensional system of globally modified Navier-Stokes equations, and we investigate the relationship between global attractors, invariant measures, time-average measures and statistical solutions of these system in the case of temporally independent forcing.

1. Introduction.

The aim of this paper is to continue with the analysis of the globally modified Navier-Stokes equations, which was initiated recently in the papers [2] and [8]. In fact, we are interested in several aspects related to the statistical analysis of these equations, since statistical solutions have proven to be very useful in the understanding of turbulence in the case of Navier-Stokes equations (see Foias *et al.* [5]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities.
Although there exists an extensive literature on statistical hydrodynamics in fluid mechanics and physics (see, e.g., Kolmogorov [11, 12], Kraichnan [13], Landau and Lifshitz [14], Dubois *et al.* [3], ...), on the mathematical side, we would like to mention the contribution of Hopf [6], the pioneering work of Prodi [18], the book by Vishik and Fursikov [22], and the recent paper by Lukaszewicz [16].

Let us now describe our model.
Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and consider the following system of globally modified Navier-Stokes equations (GMNSE)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N\left(\|u\|\right)\left[(u\cdot\nabla)u\right] + \nabla p = f(t) \quad \text{in} \quad (0,+\infty)\times\Omega, \\ \nabla\cdot u \; = 0 \quad \text{in} \quad (0,+\infty)\times\Omega, \\ u = 0 \;\; \text{on} \; (0,+\infty)\times\Gamma, \\ u(0,x) = u_0(x), \quad x\in\Omega, \end{cases} \tag{1}$$

where $N \in (0,+\infty)$ is given and $F_N : [0,+\infty) \to (0,1]$ is defined by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r\in[0,+\infty).$$

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The GMNSE (1) is indeed a global modification of the Navier-Stokes equations (NSE) on Ω with a homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \quad \text{in} \quad (0,+\infty) \times \Omega, \\ \nabla \cdot u \ = 0 \quad \text{in} \quad (0,+\infty) \times \Omega, \\ u = 0 \ \text{ on } (0,+\infty) \times \Gamma, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, and $f(t)$ a given external force field.

The modifying factor $F_N(\|u\|)$ depends on the norm $\|u\| = \|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It violates the basic laws of mechanics, but mathematically the GMNSE (1) are a well defined system of equations, just like the modified versions of the NSE of Leray and others with other mollifications of the nonlinear term, see the review paper of Constantin [1]. These modifications are local in character, whereas ours is global and essentially reduces estimates of the nonlinear term to those of the two dimensional NSE when the norm of the velocity gradient exceeds a given threshhold. Moreover, unlike in other modifications, the solutions of the GMNSE coincide with those of the NSE as long as this theshold is never exceeded. (We mention in passing that Flandoli and Maslowski [4] used a global cut off function involving the $D(A^{1/4})$ norm for the two dimensional stochastic NSE).

The GMNSE are interesting in themselves, but, more importantly, can be used to obtain useful information about the NSE. In particular, they were recently used as an intermediate step by Kloeden and Valero [10] to prove that the attainability set of the weak solutions of the 3-dim NSE which satisfy an energy constraint is compact and connected set in the weak topology. The present paper is the first in a systematic investigation of statistical solutions of the GMNSE with the long term aim to use their properties to obtain a new understanding of the statistical solutions of the three dimensional NSE.

In this paper we first prove some regularity properties of the solutions of our GMNSE. This ensures that the global attractor for the dynamical system S_N generated by (2) (when $f(t) = f$ does not depend on time t) is a bounded set of the domain of the Stokes operator (sections 3 and 4). Some properties for the invariant measures associated to S_N are proved in Section 5. In particular, we show that any invariant measure is supported by the attractor. Finally, in the last sections we prove the existence of invariant measures and the relationship with the concepts of time-average solutions, statistical solutions and invariant measures. Indeed, we first prove the existence of time-average measures associated to any solution of (2) with initial value in the phase space V (see Section 2 for the definition of V). Then, the existence of invariant measures is obtained from the existence of certain time-average measures. Our analysis in this article is finalized by proving that the invariant probability measures are statistical solutions of our GMNSE. A proof that statistical solutions of the GMNSE are invariant probability measures will be given in [9], since it requires the development of new estimates which are too lengthy to include here. In a future paper we will investigate what information can be obtained about the statistical solutions of the three dimensional NSE on a bounded domain from the results of this paper for the GMNSE. This is not a trivial undertaking in view of the still unresolved problem of uniqueness of strong and weak solutions of the three dimensional NSE, which requires the use of set-valued dynamical systems as in [10].

2. **Preliminaries.** To set our problem in the abstract framework, we consider the following usual abstract spaces (see Lions [15] and Temam [20, 21]):

$$\mathcal{V} = \left\{ u \in \left(C_0^\infty(\Omega)\right)^3 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u,v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad \forall\, u, v, w \in V,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \quad \forall\, u, v, w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v. Evidently we have $b_N(u,v,v) = 0$, for all $u, v \in V$. Moreover, by the properties of b (see [19] or [20]), there exists a constant $C_1 > 0$ only dependent on Ω such that

$$|b(u,v,w)| \le C_1 \, \|u\| \|v\| |w|^{1/4} \|w\|^{3/4}, \quad \forall\, u, v, w \in V, \tag{3}$$

$$|b(u,v,w)| \le C_1 \, |u|^{1/4} \|u\|^{3/4} \|v\| |w|^{1/4} \|w\|^{3/4}, \quad \forall\, u, v, w \in V, \tag{4}$$

$$|b(u,v,w)| \le C_1 \, \|u\| \|v\| \|w\|, \quad \forall\, u, v, w \in V. \tag{5}$$

Thus, if we denote

$$\langle B_N(u,v), w \rangle = b_N(u,v,w), \quad \forall\, u, v, w \in V,$$

we have for example

$$\|B_N(u,v)\|_* \le NC_1 \|u\|, \quad \forall\, u,v \in V. \tag{6}$$

We also consider the operator $A : V \to V'$ defined by $\langle Au, v\rangle = ((u,v))$. Denoting $D(A) = (H^2(\Omega))^3 \cap V$, then $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H).

We recall (see [20] and [19]) that there exists a constant $C_2 > 0$ depending only on Ω such that

$$|b(u,v,w)| \le C_2 |Au|\|v\||w|, \quad \forall\, u \in D(A), v \in V, w \in H, \tag{7}$$

$$|b(u,v,w)| \le C_2 |u|^{1/4}|Au|^{3/4}\|v\||w|, \quad \forall\, u \in D(A), v \in V, w \in H, \tag{8}$$

$$|b(u,v,w)| \le C_2 \|u\|^{1/2}|Au|^{1/2}\|v\||w|, \quad \forall\, u \in D(A), v \in V, w \in H, \tag{9}$$

Definition 1. *Let $u_0 \in H$ and $f \in L^2(0,T;H)$, for all $T > 0$, be given. A weak solution of (1) is any $u \in L^2(0,T;V)$ for all $T > 0$ such that*

$$\begin{cases} u'(t) + \nu Au(t) + B_N(u(t),u(t)) = f(t) \text{ in } \mathcal{D}'(0,+\infty;V'), \\ u(0) = u_0, \end{cases}$$

or equivalently

$$\begin{aligned} &(u(t),w) + \nu \int_0^t ((u(s),w))\, ds + \int_0^t b_N\left(u(s),u(s),w\right)\, ds \\ = \quad &(u_0,w) + \int_0^t (f(s),w)\, ds, \quad \text{for all } t \ge 0 \text{ and all } w \in V. \end{aligned}$$

Remark 2. *Observe that if $u \in L^2(0,T;V)$ for all $T > 0$ and satisfies the equation*

$$u'(t) + \nu Au(t) + B_N(u(t),u(t)) = f(t) \quad \text{in } \mathcal{D}'(0,+\infty;V'),$$

then, as a consequence of (6), $u'(t) \in L^2(0,T;V')$, and consequently (see [21]) $u \in C([0,+\infty);H)$ and satisfies the energy equality

$$|u(t)|^2 - |u(s)|^2 + 2\nu \int_s^t \|u(r)\|^2\, dr = 2\int_s^t (f(r),u(r))\, dr \quad \text{for all } 0 \le s \le t. \tag{10}$$

In [2] we proved that if $u_0 \in V$ and $f \in L^2(0,T;H)$, then there exists a unique solution u of the GMNSE with $u(0) = u_0$, and $u \in L^2(0,T;D(A)) \cap C([0,T];V)$ for all $T > 0$. Consider the Galerkin approximations for the GMNSE, given by

$$u_m' + \nu Au_m + P_m B_N(u_m,u_m) = P_m f, \quad u_m(0) = P_m u_0, \tag{11}$$

where $u_m = \sum_{j=1}^m u_{m,j}\phi_j$, $Au_m = \sum_{j=1}^m \lambda_j u_{m,j}\phi_j$, with λ_j and ϕ_j being the corresponding eigenvalues and orthonormal eigenfunctions of the operator A, and P_m being the projection onto the subspace of H spanned by $\{\phi_1,\dots,\phi_m\}$. From the proof of Theorem 7 in [2] and the uniqueness of u, it follows that if $u_0 \in V$ and $f \in L^2(0,T;H)$, then among other things,

$$\begin{cases} u_m \to u \quad \text{strong in } \ L^2(0,T;V), \\ u_m \rightharpoonup u \quad \text{weak in } \ L^2(0,T;D(A)), \\ u_m' \rightharpoonup u' \quad \text{weak in } \ L^2(0,T;H), \end{cases} \tag{12}$$

for all $T > 0$.

It was also proved in [2] that if $u_0 \in H \setminus V$, and $f \in L^\infty(0,+\infty;H)$, then there exists a solution u of GMNSE with $u(0) = u_0$, but we do not know if it is unique. Nevertheless, in this last case, we know that every solution u of the GMNSE with $u(0) = u_0$ satisfies $u \in L^2(\varepsilon, T; D(A)) \cap C([\varepsilon, T]; V)$ for all $0 < \varepsilon < T$.

3. **Regularity of the solutions. Existence of an absorbing ball in** $D(A)$**.** Let $f \in L^\infty(0,+\infty;H)$, and denote $|f|_\infty = \|f\|_{L^\infty(0,+\infty;H)}$.
Suppose first that $u_0 \in V$, and let $u = u(t)$ be the corresponding solution of the GMNSE.
For the Galerkin approximations u_m we easily have

$$\frac{d}{dt}|u_m(t)|^2 + \nu\lambda_1|u_m(t)|^2 \le \frac{|f(t)|^2}{\nu\lambda_1}, \quad t \ge 0,$$

thus multiplying by $e^{\nu\lambda_1 t}$ and integrating, one obtains

$$|u_m(t)|^2 \le |u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu^2\lambda_1^2} \quad \text{for all } t \ge 0. \tag{13}$$

If we now take the inner product of the Galerkin ODE (11) with $Au_m(t)$ we obtain for all $t \ge 0$

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \nu|Au_m(t)|^2 + b_N(u_m(t), u_m(t), Au_m(t)) = (f(t), Au_m(t)). \tag{14}$$

Evidently,

$$|(f(t), Au_m(t))| \le \frac{\nu}{4}|Au_m(t)|^2 + \frac{|f|_\infty^2}{\nu}.$$

Taking into account that $\lambda_1\|u_m(t)\|^2 \le |Au_m(t)|^2$ and that, by (8),

$$|b_N(u_m(t), u_m(t), Au_m(t))| \le NC_2|u_m(t)|^{1/4}|Au_m(t)|^{7/4},$$

we obtain

$$\frac{d}{dt}\|u_m(t)\|^2 + \nu\lambda_1\|u_m(t)\|^2 \le \frac{2}{\nu}|f|_\infty^2 + C^{(N)}|u_m(t)|^2, \tag{15}$$

with $C^{(N)}$ given by

$$C^{(N)} = \frac{(NC_2)^8 7^7}{2^9\nu^7}. \tag{16}$$

Substituting the bound (13) for $|u_m(t)|^2$ in the differential inequality (15) gives

$$\frac{d}{dt}\|u_m(t)\|^2 + \nu\lambda_1\|u_m(t)\|^2 \le C^{(N)}|u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu}\left(2 + \frac{C^{(N)}}{\nu\lambda_1^2}\right).$$

Integrating this inequality then gives the solution estimate

$$\|u_m(t)\|^2 \le (\|u_0\|^2 + C^{(N)}t|u_0|^2)e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu^2\lambda_1}\left(2 + \frac{C^{(N)}}{\nu\lambda_1^2}\right), \qquad \forall t \ge 0, \tag{17}$$

On the other hand, by (9) and Young's inequality, one obtains

$$|b_N(u_m(t),u_m(t),Au_m(t))| \le NC_2\|u_m(t)\|^{1/2}|Au_m(t)|^{3/2}$$
$$\le \frac{\nu}{4}|Au_m(t)|^2 + C_{(N)}\|u_m(t)\|^2,$$

with

$$C_{(N)} = \frac{27(NC_2)^4}{4\nu^3}. \tag{18}$$

Thus (14) simplifies to

$$\frac{d}{dt}\|u_m(t)\|^2 + \nu|Au_m(t)|^2 \le \frac{2}{\nu}|f|_\infty^2 + 2C_{(N)}\|u_m(t)\|^2 \quad t \ge 0. \tag{19}$$

Let us fix $0 < \varepsilon \le 1$. Integrating (19) between t and $t+\varepsilon$, we obtain in particular

$$\nu\int_t^{t+\varepsilon} |Au_m(s)|^2\,ds \le \frac{2}{\nu}|f|_\infty^2 + 2C_{(N)}\int_t^{t+\varepsilon} \|u_m(s)\|^2\,ds + \|u_m(t)\|^2 \quad \forall\, t \ge 0,$$

and then, by (17), one obtains

$$\begin{aligned} &\int_t^{t+\varepsilon} |Au_m(s)|^2\,ds \\ \le\ & \frac{1+2C_{(N)}}{\nu}(\|u_0\|^2 + C^{(N)}(t+1)|u_0|^2)e^{-\nu\lambda_1 t} \\ &+\frac{|f|_\infty^2}{\nu^2}\left[2 + \frac{1+2C_{(N)}}{\nu\lambda_1}\left(2+\frac{C^{(N)}}{\nu\lambda_1^2}\right)\right] \quad \forall\, t\ge 0 \quad \forall\, m \ge 1. \end{aligned} \tag{20}$$

Suppose now that f', the time derivative of f, also belongs to $L^\infty(0,+\infty;H)$. In [8] it is proved that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \nu\|u_m'(t)\|^2 \\ =\ & -(F_N(\|u_m(t)\|))' b(u_m(t),u_m(t),u_m'(t)) \\ & -b_N(u_m'(t),u_m(t),u_m'(t)) + (f'(t),u_m'(t)) \quad t \ge 0, \end{aligned} \tag{21}$$

where

$$|(F_N(\|u_m(t)\|))'| \le \frac{N\|u_m'(t)\|}{\|u_m(t)\|^2}\chi_{\mathcal{O}}(t) \quad \text{a.e. in } (0,+\infty), \tag{22}$$

with

$$\mathcal{O} = \{t \in (0,+\infty):\ \|u_m(t)\| \ge N\}.$$

From (3), (22) and Young's inequality, we have

$$\begin{aligned} & |2(F_N(\|u_m(t)\|))' b(u_m(t),u_m(t),u_m'(t))| \\ \le\ & 2N\|u_m'(t)\|C_1|u_m'(t)|^{1/4}\|u_m'(t)\|^{3/4} \\ =\ & 2NC_1|u_m'(t)|^{1/4}\|u_m'(t)\|^{7/4} \\ \le\ & \nu\|u_m'(t)\|^2 + \left(\frac{7}{8\nu}\right)^7 2^5(NC_1)^8|u_m'(t)|^2. \end{aligned} \tag{23}$$

By (4) and Young's inequality again

$$\begin{aligned} & |2b_N(u_m'(t), u_m(t), u_m'(t))| \\ \leq\ & 2NC_1|u_m'(t)|^{1/2}\|u_m'(t)\|^{3/2} \\ \leq\ & \nu\|u_m'(t)\|^2 + \frac{27}{16\nu^3}(NC_1)^4|u_m'(t)|^2. \end{aligned} \tag{24}$$

Thus, if we denote

$$L^{(N)} = 1 + \left(\frac{7}{8\nu}\right)^7 2^5(NC_1)^8 + \frac{27}{16\nu^3}(NC_1)^4,$$

from (21), (23) and (24) we easily obtain

$$\frac{d}{dt}|u_m'(t)|^2 \leq L^{(N)}|u_m'(t)|^2 + |f'|_\infty^2 \quad \forall\, t \geq 0 \quad \forall\, m \geq 1. \tag{25}$$

If we integrate this inequality between $s \in [t, t+\varepsilon]$ and $t+\varepsilon$, we have

$$|u_m'(t+\varepsilon)|^2 \leq |u_m'(s)|^2 + L^{(N)}\int_s^{t+\varepsilon} |u_m'(r)|^2\, dr + \varepsilon|f'|_\infty^2 \quad \forall\, 0 \leq t \leq s \leq t+\varepsilon,$$

for all $m \geq 1$. Integrating now this last inequality for s between t and $t+\varepsilon$, we obtain

$$|u_m'(t+\varepsilon)|^2 \leq (\varepsilon^{-1} + L^{(N)})\int_t^{t+\varepsilon} |u_m'(s)|^2\, ds + |f'|_\infty^2 \quad \forall\, t \geq 0, \tag{26}$$

for all $m \geq 1$.

Now, observe that by (11), the definition of F_N and (7),

$$\begin{aligned} |u_m'(t)| &\leq \nu|Au_m(t)| + |B_N(u_m(t), u_m(t))| + |f(t)| \\ &\leq \nu|Au_m(t)| + \frac{N}{\|u_m(t)\|}|b(u_m(t), u_m(t), \cdot)| + |f|_\infty \\ &\leq (\nu + NC_2)|Au_m(t)| + |f|_\infty, \quad t \geq 0, \end{aligned}$$

and therefore

$$\int_t^{t+\varepsilon} |u_m'(s)|^2\, ds \leq 2|f|_\infty^2 + 2(\nu + NC_2)^2 \int_t^{t+\varepsilon} |Au_m(s)|^2\, ds \quad \forall\, t \geq 0, \tag{27}$$

for all $m \geq 1$.

From (20), (26) and (27), it is clear that there exist two positive constants $C_f^{(N)}$ and $D_f^{(N)}$, independent of ε, u_0, t and m, and increasing with $|f|_\infty$ and $|f'|_\infty$, such that

$$|u_m'(t+\varepsilon)|^2 \leq (1+\varepsilon^{-1})\left[C_f^{(N)} + D_f^{(N)}(\|u_0\|^2 + (t+1)|u_0|^2)e^{-\nu\lambda_1 t}\right], \tag{28}$$

for all $t \geq 0$, $m \geq 1$, $\varepsilon \in (0,1]$, $u_0 \in V$.

Again, by (11) and (9),

$$\begin{aligned} \nu|Au_m(t)| &\leq |u_m'(t)| + |B_N(u_m(t), u_m(t))| + |f(t)| \\ &\leq |u_m'(t)| + NC_2\|u_m(t)\|^{1/2}|Au_m(t)|^{1/2} + |f|_\infty \\ &\leq |u_m'(t)| + \frac{\nu}{2}|Au_m(t)| + \frac{N^2C_2^2}{2\nu}\|u_m(t)\| + |f|_\infty, \quad t \geq 0, \end{aligned}$$

and therefore

$$|Au_m(t)|^2 \le \frac{12}{\nu^2}|u'_m(t)|^2 + \frac{3N^4C_2^4}{\nu^4}\|u_m(t)\|^2 + 12|f|_\infty^2, \quad \forall\, t \ge 0, \tag{29}$$

for all $m \ge 1$.
From (17), (28) and (29), one finds that there exist two positive constants $K_f^{(N)}$ and $R_f^{(N)}$, independent of ε, u_0, t and m, and increasing with $|f|_\infty$ and $|f'|_\infty$, such that

$$|Au_m(t)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + K_f^{(N)}(1+t)\|u_0\|^2 e^{-\nu\lambda_1 t}\right] \quad \forall\, t \ge \varepsilon, \tag{30}$$

for all $m \ge 1$, $\varepsilon \in (0,1]$, $u_0 \in V$.
Let $t \ge \varepsilon$ be fixed. By (30) we obtain

$$|Au_m(s)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + K_f^{(N)}(2+t)\|u_0\|^2 e^{-\nu\lambda_1 t}\right] \quad \forall\, s \in [t, t+1], \tag{31}$$

for all $m \ge 1$.
Now, we will make use of the following result (see [19] for a proof).

Lemma 3. *Let $X \subset Y$ be Banach spaces such that X is reflexive and the injection of X in Y is compact. Suppose that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0,T;X)$ such that $u_n \rightharpoonup u$ weakly in $L^p(t_0,T;X)$ for some $p \in [1,+\infty)$ and $u \in C^0([t_0,T];Y)$. Then, $u(t) \in X$ for all $t \in [t_0,T]$ and*

$$\|u(t)\|_X \le \sup_{n\ge 1}\|u_n\|_{L^\infty(t_0,T;X)}, \quad \forall\, t \in [t_0,T]. \tag{32}$$

■

From this lemma, inequality (31) and convergences in (12), we have

$$u(t) \in D(A),\ \ |Au(t)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + K_f^{(N)}(2+t)\|u_0\|^2 e^{-\nu\lambda_1 t}\right] \ \ \forall\, t \ge \varepsilon, \tag{33}$$

where the inequality is valid for all $u_0 \in V$ and all $\varepsilon \in (0,1]$.

Suppose now that $u_0 \in H$ and $u(t)$ is a solution of the GMNSE with initial datum u_0. We know that $u(t) \in V$ for all $t > 0$.
Let $\varepsilon \in (0,1]$ be fixed and let $v(t)$ be the unique solution of the GMNSE with initial datum $u(\varepsilon)$ and forcing term $\widehat{f}(t) = f(t+\varepsilon)$. By (33),

$$v(t) \in D(A) \ \text{ and } \ |Av(t)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + K_f^{(N)}(2+t)\|u(\varepsilon)\|^2 e^{-\nu\lambda_1 t}\right] \ \ \forall\, t \ge \varepsilon.$$

But, by uniqueness, $v(t) = u(t+\varepsilon)$ for all $t \ge 0$, and thus, from the above inequality we have

$$u(t) \in D(A) \quad \forall\, t \ge 2\varepsilon, \tag{34}$$

$$|Au(t)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + K_f^{(N)}(2+t)\|u(\varepsilon)\|^2 e^{-\nu\lambda_1 (t-1)}\right] \quad \forall\, t \ge 2\varepsilon. \tag{35}$$

Now let $w(t)$ be the unique solution of the GMNSE with initial datum $u(\varepsilon/2)$ and forcing term $\widetilde{f}(t) = f(t+\varepsilon/2)$. By uniqueness we know that $w(t) = u(t+\varepsilon/2)$ for all $t \ge 0$.

From estimate (39) in Proposition 15 in [8] we have

$$\varepsilon/2\|w(1/2)\|^2 \le K_N e^{K_N}\left(|u(\varepsilon/2)|^2 + \int_0^{1/2} |\widetilde{f}(s)|^2 + \int_0^{1/2} |\widetilde{f}'(s)|^2\,ds\right),$$

where $K_N > 0$, is a constant depending only on C_1, N, ν and λ_1. Consequently,

$$\|u(\varepsilon)\|^2 \le 2\varepsilon^{-1} K_N e^{K_N}\left(|u(\varepsilon/2)|^2 + |f|_\infty^2 + |f'|_\infty^2\right). \tag{36}$$

Finally, the estimate

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \frac{|f(t)|^2}{\nu\lambda_1} \quad t \ge 0,$$

is well known and, in particular, implies that

$$|u(\varepsilon/2)|^2 \le |u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}. \tag{37}$$

From (36), (37), (33) y (35), we obtain the following result.

Proposition 4. *Suppose that $f \in W^{1,\infty}(0,+\infty;H)$, and let $u = u(t)$ be a solution of GMNSE. Then*

$$u(t) \in D(A) \quad \forall\, t > 0, \tag{38}$$

and there exist two positive constants $K_f^{(N)}$ and $M_f^{(N)}$, independent of ε, u_0 and t, and increasing with $|f|_\infty$ and $|f'|_\infty$, such that

a) *if $u(0) \in V$, then*

$$|Au(t)|^2 \le (1+\varepsilon^{-1})\left[R_f^{(N)} + M_f^{(N)}(1+t)\|u_0\|^2 e^{-\nu\lambda_1 t}\right] \quad \forall\, t \ge \varepsilon, \tag{39}$$

for all $\varepsilon \in (0,1]$;

b) *in general, if $u(0) \in H$, then*

$$|Au(t)|^2 \le (1+\varepsilon^{-1})R_f^{(N)} + \varepsilon^{-1}(1+\varepsilon^{-1})M_f^{(N)}(1+t)(1+|u_0|^2)e^{-\nu\lambda_1 t}, \tag{40}$$

for all $t \ge 2\varepsilon$, $0 < \varepsilon \le 1$. In particular, there exists a $T_0 = T_0(|u_0|)$ depending only on $|u_0|$, $|f|_\infty$, $|f'|_\infty$, C_1, C_2, N, ν and λ_1 such that

$$|Au(t)|^2 \le 2R_f^{(N)} \quad \forall\, t \ge T_0(|u_0|). \tag{41}$$

■

Remark 5. *Observe that (40) implies that if $f \in W^{1,\infty}(0,+\infty;H)$, then every solution of the GMNSE belongs to $L^\infty(\varepsilon,+\infty;D(A))$ for all $\varepsilon > 0$. If, moreover, the initial datum $u_0 \in D(A)$, then it can be proved that the corresponding solution $u = u(t)$ of the GMNSE belongs to $L^\infty(0,+\infty;D(A))$, and, more exactly,*

$$\sup_{t\ge 0}|Au(t)| < +\infty.$$

4. **An estimate for $|Au|^\beta$.** We now derive an a priori estimate for the solutions of the GMNSE, which is like the a priori estimate (A.43) obtained for NSE in [5] in the case $\beta = 2/3$.

Lemma 6. *Suppose that $f \in L^\infty(0,+\infty;H)$, and $u_0 \in H$. Let $u = u(t)$ be a solution of the GMNSE with $u(0) = u_0$. Then, for each $0 < \beta < 1$, there exists a constant $C_\beta > 0$, independent of T, such that*

$$\int_0^T |Au(t)|^\beta \, dt \le C_\beta \,(1+T) \quad \forall\, T \ge 0. \tag{42}$$

Proof.- We know that $u \in L^2(\varepsilon, T; D(A)) \cap C([\varepsilon, T]; V)$ for all $0 < \varepsilon < T$. Let us fix $\varepsilon > 0$. By the energy equality, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 + b_N(u(t), u(t), Au(t)) = (f(t), Au(t)) \quad t \ge \varepsilon. \tag{43}$$

Evidently,

$$|(f(t), Au(t))| \le \frac{\nu}{4}|Au(t)|^2 + \frac{|f|_\infty^2}{\nu}$$

In addition, by (9) and Young's inequality, one obtains

$$\begin{aligned} |b_N(u(t), u(t), Au(t))| &\le NC_2\|u(t)\|^{1/2}|Au(t)|^{3/2} \\ &\le \frac{\nu}{4}|Au(t)|^2 + C_{(N)}\|u(t)\|^2, \end{aligned}$$

with $C_{(N)}$ given by (18).

Thus (43) simplifies to

$$\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 \le \frac{2}{\nu}|f|_\infty^2 + 2C_{(N)}\|u(t)\|^2 \quad t \ge \varepsilon. \tag{44}$$

Let us fix $0 < \beta < 1$, and denote $\delta = (2-2\beta)/\beta > 0$.
Evidently, taking $C'_N = \max\left(\frac{2}{\nu}|f|_\infty^2, 2C_{(N)}\right) > 0$, we have a constant independent of ε such that

$$\begin{aligned} \frac{2}{\nu}|f|_\infty^2 + 2C_{(N)}\|u(t)\|^2 &\le C'_N(1+\|u(t)\|^2) \\ &\le C'_N(1+\|u(t)\|^2)^{1+\delta} \quad t \ge \varepsilon. \end{aligned}$$

From this inequality and (44) we have

$$\frac{1}{(1+\|u(t)\|^2)^{1+\delta}}\frac{d}{dt}\|u(t)\|^2 + \frac{\nu|Au(t)|^2}{(1+\|u(t)\|^2)^{1+\delta}} \le C'_N \quad t \ge \varepsilon.$$

Integrating between ε and T we obtain

$$\begin{aligned} &\frac{1}{\delta(1+\|u(\varepsilon)\|^2)^\delta} + \int_\varepsilon^T \frac{\nu|Au(t)|^2}{(1+\|u(t)\|^2)^{1+\delta}}\, dt \\ \le\ &\frac{1}{\delta(1+\|u(T)\|^2)^\delta} + C'_N T \\ \le\ &\frac{1}{\delta} + C'_N T, \end{aligned}$$

and in particular,

$$\int_\varepsilon^T \frac{\nu|Au(t)|^2}{(1+\|u(t)\|^2)^{1+\delta}}\,dt \le \frac{1}{\delta} + C'_N T,$$

for all $0<\varepsilon<T$. Thus, making $\varepsilon \to 0$ in the last inequality, we obtain

$$\int_0^T \frac{\nu|Au(t)|^2}{(1+\|u(t)\|^2)^{1+\delta}}\,dt \le \frac{1}{\delta} + C'_N T \quad \forall T \ge 0. \tag{45}$$

But

$$\int_0^T |Au(t)|^\beta\,dt = \int_0^T \frac{|Au(t)|^\beta}{(1+\|u(t)\|^2)^{\beta(1+\delta)/2}}(1+\|u(t)\|^2)^{\beta(1+\delta)/2}\,dt,$$

and taking $p = 2/\beta$ in the Hölder inequality, by (45) and the choice of δ, we obtain

$$\begin{aligned}
&\int_0^T |Au(t)|^\beta\,dt \qquad (46)\\
\le\ & \left(\int_0^T \frac{|Au(t)|^2}{(1+\|u(t)\|^2)^{(1+\delta)}}\,dt\right)^{\beta/2}\left(\int_0^T (1+\|u(t)\|^2)^{\beta(1+\delta)/(2-\beta)}\,dt\right)^{(2-\beta)/2}\\
\le\ & \left(\frac{1}{\nu\delta} + \frac{C'_N T}{\nu}\right)^{\beta/2}\left(\int_0^T (1+\|u(t)\|^2)\,dt\right)^{(2-\beta)/2}
\end{aligned}$$

for all $T \ge 0$.

On the other hand, the estimate

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \frac{|f(t)|^2}{\nu\lambda_1} \quad t \ge 0,$$

is well known and implies that

$$\nu\int_0^T \|u(t)\|^2\,dt \le |u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}T \quad \forall T \ge 0. \tag{47}$$

Now, (42) follows easily from (46) and (47). ■

5. **The attractor in the autonomous case is a bounded subset of $D(A)$.** We now assume that the forcing term f does not depend on time. Then, for each $N>0$ and $u_0 \in V$, we denote $S_N(t)u_0 = u(t,u_0)$, where $u(t,u_0)$ is the unique strong solution of (1) with initial datum u_0.

It is known (see [2]) that, $\{S_N(t)\}_{t\ge 0}$ is a C^0 semigroup in V, i.e., $S_N(t)$ maps V into V for each $t \ge 0$, and

(a) $S_N(0) = I$, the identity map on V,
(b) $S_N(t+s) = S_N(t)S_N(s)$ for all s, $t \ge 0$,
(c) The function $(t,u_0) \in [0,+\infty)\times V \mapsto S_N(t)u_0 \in V$ is continuous.

It was also shown in [2] that the semigroup in V here is asymptotically compact and hence that the GMNSE has a unique global attractor $\mathcal{A}_N$ in V for each $N>0$, i.e., a compact subset $\mathcal{A}_N \subset V$ such that

$$S_N(t)\mathcal{A}_N = \mathcal{A}_N \quad \text{for all } t \geq 0, \tag{48}$$

and

$$\lim_{t\to+\infty} \mathrm{dist}_V(S_N(t)D, \mathcal{A}_N) = 0 \quad \text{for all bounded subset } D \subset V, \tag{49}$$

where $\mathrm{dist}_V(D_1, D_2)$ is the Hausdorff semidistance in V between D_1 and D_2, i.e.

$$\mathrm{dist}_V(D_1, D_2) = \sup_{v\in D_1} \inf_{w\in D_2} \|v-w\| \quad \text{for } D_1, D_2 \subset V.$$

From the results in [8] we know that the fractal dimension of $\mathcal{A}_N$ is finite, and also that it attracts all solutions of the GMNSE starting in H. More exactly, for each $u_0 \in H$, let us denote $\widetilde{S}_N(t)u_0$ the set of values $u(t)$ at time $t \geq 0$ of all weak solutions $u(\cdot)$ of the (1) with initial datum u_0, then

$$\lim_{t\to+\infty} \mathrm{dist}_V\left(\bigcup_{u_0\in D} \widetilde{S}_N(t)u_0, \mathcal{A}_N\right) = 0 \quad \text{for all bounded subset } D \text{ of } H. \tag{50}$$

Observe that (41) implies that the closed ball $\mathcal{B}_N$ in $D(A)$ defined by

$$\mathcal{B}_N := \{v \in D(A) : \ |Av|^2 \leq 2R_f^{(N)}\}, \tag{51}$$

is absorbing for the semigroup $\{S_N(t)\}_{t\geq 0}$. Moreover, by the compactness of the injection of $D(A)$ into V, as $\mathcal{B}_N$ is bounded in $D(A)$, it is relatively compact in V. Suppose that $v_n \in \mathcal{B}_N$ is a sequence such that $v_n \to v$ in V. As $\mathcal{B}_N$ is a bounded subset of $D(A)$, we can extract a subsequence v_μ of v_n, weakly convergent in $D(A)$, i.e., $v_\mu \rightharpoonup w$ weakly in $D(A)$, but then $v_\mu \rightharpoonup w$ weakly in V, and so $w = v$. Consequently, $v \in D(A)$ and $|Av|^2 \leq \liminf_{\mu\to+\infty} |Av_\mu|^2 \leq R_f^{(N)}$, and so $v \in \mathcal{B}_N$. Hence, we have proved that the set $\mathcal{B}_N$ is a compact subset of V.
Thus, in fact, the existence of $\mathcal{A}_N$ is a consequence of Proposition 4. Moreover, from (41) and the invariance property (48), a new regularity property for the global attractor follows.

Corollary 7. *The global attractor $\mathcal{A}_N$ is a bounded subset of $D(A)$.*

6. **S_N-invariant measures in the autonomous case.** Henceforth we will suppose that $f \in H$ is independent of time t. By a probability measure on H we will understand a probability measure on the σ-algebra of Borel subsets of H.
We recall that a Borel set in V is a Borel set in H and that a set $E \subset V$ is a Borel subset of V if and only if there exists a Borel subset F of H such that $E = F \cap V$. The same properties follow with $D(A)$ instead of V (see [5]).

Definition 8. *Let μ be a probability measure on H. We will say that μ is S_N-invariant if*

$$\mu(V) = 1 \quad \text{and} \quad \mu(E) = \mu(S_N(t)^{-1}E) \quad \forall\, t \geq 0, \tag{52}$$

for every Borel subset E of V.

An interesting property of S_N-invariant probability measures is the following.

Lemma 9. *Let μ be an S_N-invariant probability measure on H and let E be a measurable subset of V such that for all $r > 0$ exists a $t_r > 0$ such that*

$$S_N(t_r)u_0 \in E \quad \text{for all } u_0 \in V \text{ such that } \|u_0\| \le r.$$

Then,

$$\mu(E) = 1.$$

Proof.- Let us denote $\mathcal{B}_V(r) = \{v \in V : \ \|v\| \le r\}$. We know that for each $r > 0$ there exists $t_r > 0$ such that $S_N(t_r)\mathcal{B}_V(r) \subset E$. Hence $\mathcal{B}_V(r) \subset S_N(t_r)^{-1}(E)$, and so

$$\mu(\mathcal{B}_V(r)) \le \mu(S_N(t_r)^{-1}(E)) = \mu(E) \le 1.$$

Now observe that, $V = \bigcup_{r>0} \mathcal{B}_V(r)$, and that $\mathcal{B}_V(r)$ is increasing with r, so $\mu(\mathcal{B}_V(r)) \to \mu(V) = 1$. Hence $\mu(E) = 1$. ■

Now we will prove that the support of an S_N-invariant measure on H is included in the global attractor $\mathcal{A}_N$.
We recall that $\mathcal{B}_N$ is a compact set of V which is absorbing for the semigroup $\{S_N(t)\}_{t\ge 0}$. Thus

$$\mathcal{A}_N = \bigcap_{t\ge 0} \overline{\bigcup_{s\ge t} S_N(s)\mathcal{B}_N},$$

where the closure is taken in V. In [5] it is proved that

$$\mathcal{A}_N = \bigcap_{k\ge 1} S_N(t_k)\mathcal{B}_N, \tag{53}$$

where t_k is any sequence of positive numbers such that $t_k \to +\infty$ as $k \to +\infty$.

Theorem 10. *The support of any S_N-invariant measure on H is included in the global attractor $\mathcal{A}_N$.*

Proof.- Let $T_0 > 0$ be such that $S_N(t)\mathcal{B}_N \subset \mathcal{B}_N$ for all $t \ge T_0$. Hence, the sequence of sets $S_N(kT_0)\mathcal{B}_N$ is decreasing, and thus, by (53),

$$\mu_N(\mathcal{A}_N) = \lim_{k\to\infty} \mu(S_N(kT_0)\mathcal{B}_N). \tag{54}$$

Since $\mathcal{B}_N$ is S_N-absorbing, the set $S_N(kT_0)\mathcal{B}_N$ is also S_N-absorbing and is, moreover, a subset of $D(A)$, so, by Lemma 9, $\mu(S_N(kT_0)\mathcal{B}_N) = 1$. Consequently, $\mu(\mathcal{A}_N) = 1$ by (54). Since $\mathcal{A}_N$ is compact in V, it is closed in H and thus includes the support of μ, which is the smallest closed subset of H of μ-measure equal to 1. ■

7. **Time-averages solutions in the autonomous case.** Let us denote $\mathcal{B}([0,\infty))$ the space of all bounded real-valued functions on $[0,\infty)$, and $\mathcal{B}_+([0,\infty))$ the space of all functions $g \in \mathcal{B}([0,\infty))$ such that $g(s) \ge 0$ for all $s \ge 0$.

Definition 11. *A generalized limit is any linear functional, denoted* $\mathrm{LIM}_{T\to\infty}$*, on* $\mathcal{B}([0,\infty))$ *such that*

a) $\mathrm{LIM}_{T\to\infty} g(T) \ge 0 \qquad \forall\, g \in \mathcal{B}_+([0,\infty))$;
b) $\mathrm{LIM}_{T\to\infty} g(T) = \lim_{T\to\infty} g(T)$ *for all* $g \in \mathcal{B}([0,\infty))$ *for which the usual limit* $\lim_{T\to\infty} g(T)$ *exists.*

For the existence of generalized limits see [5], where the following properties are also proved

c) $\liminf_{T\to\infty} g(T) \le \mathrm{LIM}_{T\to\infty} g(T) \le \limsup_{T\to\infty} g(T), \quad \forall\, g \in \mathcal{B}([0,\infty))$;

d) $|\mathrm{LIM}_{T\to\infty} g(T)| \le \limsup_{T\to\infty} |g(T)| \le \sup_{T\ge 0} |g(T)|, \quad \forall\, g \in \mathcal{B}([0,\infty))$;

e) $\mathrm{LIM}_{T\to\infty} \dfrac{1}{T+\tau}\displaystyle\int_0^{T+\tau} f(t)\,dt = \mathrm{LIM}_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T f(t)\,dt, \quad \forall\,\tau \ge 0, \quad \forall\, f \in L^\infty(0,\infty)$.

From now on we suppose we have fixed a generalized limit, denoted $\mathrm{LIM}_{T\to\infty}$.

Let $u(t)$ be a solution of the autonomous GMNSE. We remember that, by Proposition 4, there exists a $T_0 = T_0(|u(0)|) > 0$ such that $u(t) \in \mathcal{B}_N$ for all $t \ge T_0$. Then, as $u \in C([0,+\infty); H)$, if $\varphi \in C(H)$, the function $\varphi(u(t))$ is a continuous and bounded real function on $[0,+\infty)$, and consequently the function $T \in [0,+\infty) \mapsto \dfrac{1}{T}\displaystyle\int_0^T \varphi(u(t))\,dt \in \mathbb{R}$, with the convention $0/0 = 0$, is well defined and bounded, and therefore the generalized limit $\mathrm{LIM}_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T \varphi(u(t))\,dt$ is defined.
Observe that, if $u(0) \in V$, then $u \in C([0,+\infty); V)$ and by the same argument as above the generalized limit $\mathrm{LIM}_{T\to\infty}\dfrac{1}{T}\displaystyle\int_0^T \varphi(u(t))\,dt$ is also defined for all $\varphi \in C(V)$.

Definition 12. *Let $u(t)$ be a solution of the autonomous GMNSE. A time-average measure of the solution $u(t)$ is any probability measure μ on H such that $C(H) \subset L^1(H,\mu)$ and*

$$\mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u(t))\,dt = \int_H \varphi(v)\,d\mu(v) \quad \forall\,\varphi \in C(H). \tag{55}$$

Before proving the existence of time-average measures, we prove the following result.

Proposition 13. *Any time-average measure μ of a solution $u(t)$ of the autonomous GMNSE is carried by $D(A)$, i.e.,*

$$\mu(D(A)) = 1. \tag{56}$$

Proof.- Let $u(t)$ be a solution of the autonomous GMNSE, and consider a time-average measure μ of a $u(t)$.
Let us consider the projections P_m onto the subspace of H spanned by $\{\phi_1,\dots,\phi_m\}$, ϕ_j being the orthonormal eigenfunctions of the operator A.
Let us fix $0 < \beta < 1$. It is clear that for any $m \ge 1$ the function $|AP_m v|^\beta$ is continuous on H and consequently

$$\int_H |AP_m v|^\beta\,d\mu(v) = \mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T |AP_m u(t)|^\beta\,dt. \tag{57}$$

On the other hand, $|AP_m v|^\beta \nearrow |Av|^\beta$ as $m \to +\infty$ for all $v \in H$, with the convention that $|Av| = +\infty$ for all $v \in H \setminus D(A)$. Thus, by the monotone convergence theorem,

$$\int_H |Av|^\beta\,d\mu(v) = \lim_{m\to\infty}\int_H |AP_m v|^\beta\,d\mu(v). \tag{58}$$

If we prove that

$$\mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T |AP_m u(t)|^\beta\, dt$$

remains bounded when $m\to+\infty$, by (57) and (58) we will obtain that

$$\int_H |Av|^\beta\, d\mu(v) < +\infty,$$

and consequently $\mu(H\setminus D(A)) = 0$.
Recall that by Lemma 6 there exists a constant $C_\beta > 0$, independent of T, such that

$$\int_0^T |Au(t)|^\beta\, dt \le C_\beta\,(1+T) \quad \forall\, T\ge 0.$$

Then, since $u(t)\in D(A)$ for all $t>0$, and consequently $|AP_m u(t)| = |P_m Au(t)| \le |Au(t)|$, we obtain

$$\frac{1}{T}\int_0^T |AP_m u(t)|^\beta\, dt \le 2C_\beta \quad \forall\, T\ge 1.$$

Therefore,

$$\limsup_{T\to+\infty}\frac{1}{T}\int_0^T |AP_m u(t)|^\beta\, dt \le 2C_\beta,$$

for all $m\ge 1$, and by the properties of generalized limits,

$$\mathrm{LIM}_{T\to+\infty}\frac{1}{T}\int_0^T |AP_m u(t)|^\beta\, dt \le 2C_\beta,$$

for all $m\ge 1$, as desired. ■

In the case of an initial datum $u_0\in V$, we can obtain existence of more regular time-average measures.

Proposition 14. *For any solution $u(t)$ of the autonomous GMNSE such that $u(0)\in V$ there exists a time-average measure μ of this solution such that moreover $C(V)\subset L^1(H,\mu)$ and*

$$\mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u(t))\, dt = \int_H \varphi(v)\, d\mu(v) \quad \forall\, \varphi\in C(V). \tag{59}$$

Proof.- Let $u(t)$ be a solution of the autonomous GMNSE with $u(0)\in V$. We have seen that there exists a $T_0 = T_0(|u(0)|) > 0$ such that $u(t)\in\mathcal{B}_N$ for all $t\ge T_0$, $\mathcal{B}_N$ is a compact subset of V, and

$$L(\varphi) := \mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u(t))\, dt$$

is well defined as a real number for all $\varphi\in C(H)\cup C(V)$.
Moreover, for any $\varphi\in C(H)\cup C(V)$, the value $L(\varphi)$ depends only on the restriction of φ to $\mathcal{B}_N$. Indeed, if $\widetilde{\varphi}\in C(H)\cup C(V)$ is another function such that $\varphi(v)=\widetilde{\varphi}(v)$ for all $v\in\mathcal{B}_N$, then $\varphi(u(t))=\widetilde{\varphi}(u(t))$ for all $t\ge T_0$, and therefore, by the linearity of $\mathrm{LIM}_{T\to\infty}$,

$$\begin{aligned} L(\varphi) - L(\widetilde{\varphi}) &= \mathrm{LIM}_{T\to\infty} \frac{1}{T} \int_0^T (\varphi(u(t)) - \widetilde{\varphi}(u(t)))\, dt \\ &= \mathrm{LIM}_{T\to\infty} \frac{1}{T} \int_0^{T_0} (\varphi(u(t)) - \widetilde{\varphi}(u(t)))\, dt \\ &= 0. \end{aligned}$$

Thus $L(\varphi) = L(\widetilde{\varphi})$.
Let now $\psi \in C(\mathcal{B}_N)$, where $\mathcal{B}_N$ is considered as a metric subspace of H. As $\mathcal{B}_N$ is a closed subset of H and ψ is continuous and bounded, we can extend ψ to a continuous function $\varphi \in C(H)$. By the considerations above, the value $L(\varphi)$ is the same for any $\varphi \in C(H) \cup C(V)$ such that $\varphi_{|\mathcal{B}_N} = \psi$. Therefore, we can define a functional l on $C(\mathcal{B}_N)$ by $l(\psi) = L(\varphi)$, where $\varphi \in C(H) \cup C(V)$ is any continuous extension of ψ.
It is evident that l is a positive linear functional on $C(\mathcal{B}_N)$, and because $\mathcal{B}_N$ is compact, it follows from Kakutani-Riesz representation theorem (see [5]) that there exists a positive measure μ on $\mathcal{B}_N$ such that

$$l(\psi) = \int_{\mathcal{B}_N} \psi(v)\, d\mu(v) \quad \forall\, \psi \in C(\mathcal{B}_N).$$

The measure μ can be extended to a measure on H by setting $\mu(F) = \mu(F \cap \mathcal{B}_N)$ for all Borel measurable subset F of H. It is clear that $\mu(H \setminus \mathcal{B}_N) = 0$, and observe that if $\varphi \in C(V)$, then $\varphi_{|\mathcal{B}_N} \in C(\mathcal{B}_N)$ (if $v_n \to v_0$ in $\mathcal{B}_N$, then, as $\mathcal{B}_N$ is a compact subset of V, $v_n \to v_0$ in V, and therefore $\varphi(v_n) \to \varphi(v_0)$).
Consequently for any $\varphi \in C(H) \cup C(V)$ we have

$$\begin{aligned} \mathrm{LIM}_{T\to\infty} \frac{1}{T} \int_0^T \varphi(u(t))\, dt &= L(\varphi) = l(\varphi_{|\mathcal{B}_N}) \\ &= \int_{\mathcal{B}_N} \varphi_{|\mathcal{B}_N}(v)\, d\mu(v) \\ &= \int_H \varphi(v)\, d\mu(v). \end{aligned}$$

Finally, note that taking $\varphi \equiv 1$, we deduce that $\mu(H) = \mathrm{LIM}_{T\to\infty} 1 = 1$, so that μ is a probability measure on H. ■

Remark 15. *With an almost identical proof to that of the preceding theorem, one can prove that there exists a time-average measure of any solution of the autonomous GMNSE.*

Now, we can obtain existence of S_N-invariant measures.

Proposition 16. *Let $u(t) = S_N(t)u_0$ be the solution of the autonomous GMNSE corresponding to $u_0 \in V$, and let μ be a time-average measure of $u(t)$ such that $C(V) \subset L^1(H, \mu)$ and (59) is satisfied for all $\varphi \in C(V)$. Then μ is an S_N-invariant measure.*

Proof.- Let $\psi \in C(H)$ and $\tau > 0$. The function $\psi \circ S_N(\tau) : v \mapsto \psi(S_N(\tau)v)$ is also continuous in V, and by (59) with φ replaced by $\psi \circ S_N(\tau)$, we have

$$\begin{aligned}\int_H \psi(S_N(\tau)v)\,d\mu(v) &= \mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \psi(S_N(t+\tau)u_0)\,dt \\ &= \mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_\tau^{T+\tau} \psi(S_N(t)u_0)\,dt \\ &= \mathrm{LIM}_{T\to\infty}\left[\frac{1}{T}\int_0^T \psi(S_N(t)u_0)\,dt\right. \\ &\quad \left. +\frac{1}{T}\int_T^{T+\tau} \psi(S_N(t)u_0)\,dt - \frac{1}{T}\int_0^\tau \psi(S_N(t)u_0)\,dt\right].\end{aligned}$$

But, observe that $S_N(t)u_0$ belongs to a compact set of V, and hence also of H, for all $t \geq 0$. Therefore $\psi(S_N(t)u_0)$ remains bounded for all $t \geq 0$, so

$$\mathrm{LIM}_{T\to\infty}\left[\frac{1}{T}\int_T^{T+\tau} \psi(S_N(t)u_0)\,dt - \frac{1}{T}\int_0^\tau \psi(S_N(t)u_0)\,dt\right] = 0.$$

Thus,

$$\begin{aligned}\int_H \psi(S_N(\tau)v)\,d\mu(v) &= \mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \psi(S_N(t)u_0)\,dt \\ &= \int_H \psi(v)\,d\mu(v),\end{aligned}$$

for all $\tau > 0$ and any $\psi \in C(H)$. By density, we then obtain

$$\int_H \phi(S_N(\tau)v)\,d\mu(v) = \int_H \phi(v)\,d\mu(v) \quad \forall\, \phi \in L^1(H,\mu).$$

In particular, taking the characteristic function of any measurable subset E of V, we then have

$$\mu(E) = \mu(S_N(\tau)^{-1}E) \quad \forall\, \tau > 0,$$

and the S_N-invariance of μ follows. ∎

8. **Stationary Statistical Solutions of the GMNSE in the autonomous case.**

Definition 17. *We define $\mathcal{T}$ as the set of real valued functionals $\Phi = \Phi(v)$ on H such that*

(i) $c_r := \sup_{|v|\leq r} |\Phi(v)| < +\infty$ *for all* $r > 0$*;*

(ii) *for any* $v \in V$ *there exists* $\Phi'(v) \in V$ *such that*

$$\frac{|\Phi(v+w) - \Phi(v) - (\Phi'(v), w)|}{|w|} \to 0 \quad \text{as } |w| \to 0 \text{ with } w \in V; \tag{60}$$

(iii) *the mapping* $v \mapsto \Phi'(v)$ *is continuous and bounded as function from* V *into* V*.*

Let us denote $\|v\| = +\infty$ if $v \in H \setminus V$. With this convention, if μ is a probability measure on H and $\int_H \|v\|^2 \, d\mu(v) < +\infty$, then $\mu(H \setminus V) = 0$.
We define

$$G_N(v) = -\nu A v - B_N(v,v) + f \quad \forall\, v \in V. \tag{61}$$

Taking into account (5) and that

$$r|F_N(r) - F_N(s)| \leq |r - s| \quad \forall\, r, s \geq 0,$$

it is easy to obtain that

$$\|B_N(v,v) - B_N(u,u)\|_* \leq NC_1(2\|v\| + \|u\|)\|v - u\| \quad \forall\, u, v \in V,$$

and therefore the mapping $G_N : V \to V'$ is continuous. Also, by (6),

$$\|G_N(v)\|_* \leq (\nu + NC_1)\|v\| + \lambda_1^{-1/2}|f| \quad \forall\, v \in V. \tag{62}$$

Thus, if $\Phi \in \mathcal{T}$,

$$|\langle G_N(v), \Phi'(v)\rangle| \leq [(\nu + NC_1)\|v\| + \lambda_1 - 1/2|f|] \sup_{w \in V} \|\Phi'(w)\| \quad \forall\, v \in V,$$

and consequently, if μ is a probability measure on H with $\int_H \|v\| \, d\mu(v) < +\infty$, then the integral $\int_H \langle G_N(v), \Phi'(v)\rangle \, d\mu(v)$ is finite.

Definition 18. *A stationary statistical solution of the GMNSE is a probability measure μ on H such that*

(i) $\displaystyle\int_H \|v\|^2 \, d\mu(v) < +\infty;$

(ii) $\displaystyle\int_H \langle G_N(v), \Phi'(v)\rangle \, d\mu(v) = 0$ *for any* $\Phi \in \mathcal{T}$;

(iii) $\displaystyle\int_{\{a \leq |v|^2 < b\}} \{\nu\|v\|^2 - (f,v)\} \, d\mu(v) \leq 0$ *for any* $0 \leq a < b \leq +\infty$.

We have the following result

Theorem 19. *Any S_N-invariant probability measure on H is a stationary statistical solution of the GMNSE.*

Proof.- Let μ be a S_N-invariant probability measure on H. We know by Proposition 4 and Lemma 9 that $\mu(H \setminus \mathcal{B}_N) = 0$. The set $\mathcal{B}_N$ is a compact subset of V and hence the function $\|v\|$ is bounded on $\mathcal{B}_N$. Thus, for any $\beta > 0$,

$$\int_H \|v\|^\beta \, d\mu(v) = \int_{\mathcal{B}_N} \|v\|^\beta \, d\mu(v) < +\infty,$$

and in particular condition (i) in Definition 18 holds.
Let us fix $0 \leq a < b \leq +\infty$, an let us denote

$$E = \{v \in V : \ a \leq |v|^2 < b\}, \qquad F = \{v \in H : \ a \leq |v|^2 < b\}.$$

Since μ is S_N-invariant, and by (i) the function $v \mapsto \nu\|v\|^2 - (f,v)$ is μ-integrable, we have

$$\begin{aligned} & \int_F [\nu\|v\|^2 - (f,v)] \, d\mu(v) \qquad (63) \\ = \ & \int_E [\nu\|v\|^2 - (f,v)] \, d\mu(v) \\ = \ & \int_E [\nu\|S_N(t)v\|^2 - (f, S_N(t)v)] \, d\mu(v) \quad \forall\, t \geq 0. \end{aligned}$$

Now, observe that reasoning as in the proof of Lemma 6 one can obtain that

$$\|S_N(t)v\|^2 \le \left(\frac{2}{\nu}|f|^2 + \|v\|^2\right) e^{2C_{(N)}t} \quad \forall\, t \ge 0 \tag{64}$$

for any $v \in V$. Consequently, taking into account condition (i), we can integrate in (63) and apply Fubini's theorem, to obtain

$$\begin{aligned} &\int_F [\nu\|v\|^2 - (f,v)]\, d\mu(v) \\ = \;& \frac{1}{T}\int_0^T \int_E [\nu\|S_N(t)v\|^2 - (f, S_N(t)v)]\, d\mu(v)\, dt \\ = \;& \frac{1}{T}\int_E \int_0^T [\nu\|S_N(t)v\|^2 - (f, S_N(t)v)]\, dt\, d\mu(v) \end{aligned} \tag{65}$$

for all $T > 0$.
But we know that for all $v \in V$ and all $T > 0$,

$$|S_N(T)v|^2 - |v|^2 + 2\nu \int_0^T \|S_N(t)v\|^2\, dt = 2\int_0^T (f, S_N(t)v)\, dt,$$

and hence, by (65),

$$\int_F [\nu\|v\|^2 - (f,v)]\, d\mu(v) = \frac{1}{2T}\int_E (|v|^2 - |S_N(T)v|^2)\, d\mu(v) \quad \forall\, T > 0. \tag{66}$$

Now observe that

$$|S_N(t)v|^2 \le |v|^2 e^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu^2\lambda_1^2} \quad \forall\, t \ge 0 \quad \forall\, v \in V. \tag{67}$$

Suppose first that $b < +\infty$. Then, from (66) and (67), and making $T \to +\infty$, we find that

$$\int_F [\nu\|v\|^2 - (f,v)]\, d\mu(v) = 0.$$

If $b = +\infty$, it is enough to consider a sequence $b_n \nearrow +\infty$. Thus we have proved that μ satisfies condition (iii) in Definition 18.
Finally, we must prove that μ satisfies condition (ii) in Definition 18.
Let $\Phi \in \mathcal{T}$ be given. For each integer $m \ge 1$, denote

$$\Phi_m(v) = \Phi(P_m v) \quad \forall\, v \in H.$$

It is easy to see that $\Phi_m \in C^1(H)$, with $\Phi'_m(v) = P_m\Phi'(P_m v)$ for all $v \in H$.
Evidently,

$$\sup_{|v|\le r} |\Phi_m(v)| \le \sup_{|w|\le r} |\Phi(w)| = c_r < +\infty,$$

and the mapping $v \mapsto \Phi'_m(v)$ is continuous and bounded as a function from V into V. Thus (see for example [17] Theorem 4.2, page 65) for any $m \ge 1$ and all $v \in V$ we have

$$\Phi_m(S_N(T)v) - \Phi_m(v) = \int_0^T \langle G_N(S_N(t)v), \Phi'_m(S_N(t)v)\rangle\, dt \quad \forall\, T > 0. \tag{68}$$

Since μ is S_N-invariant,

$$\int_H \langle G_N(v), \Phi'_m(v)\rangle\, d\mu(v) = \int_V \langle G_N(S_N(t)v), \Phi'_m(S_N(t)v)\rangle\, d\mu(v)$$

for all $t \geq 0$. Now we integrate, and taking into account (62), (64) and condition (i), we can apply Fubini theorem, and we obtain

$$\begin{aligned}\int_H \langle G_N(v), \Phi'_m(v)\rangle \, d\mu(v) &= \frac{1}{T}\int_0^T \int_V \langle G_N(S_N(t)v), \Phi'_m(S_N(t)v)\rangle \, d\mu(v)\, dt \\ &= \frac{1}{T}\int_V \int_0^T \langle G_N(S_N(t)v), \Phi'_m(S_N(t)v)\rangle \, dt\, d\mu(v),\end{aligned}$$

and thus, by (68),

$$\int_H \langle G_N(v), \Phi'_m(v)\rangle \, d\mu(v) = \frac{1}{T}\int_V [\Phi_m(S_N(T)v) - \Phi_m(v)]\, d\mu(v) \quad \forall T > 0.$$

Taking $T \to +\infty$ in the last equality, and using the mean value theorem, the boundedness of Φ'_m on V, and the inequality (67), we obtain

$$\int_H \langle G_N(v), \Phi'_m(v)\rangle \, d\mu(v) = 0. \tag{69}$$

Now, observe that

$$\begin{aligned}\|\Phi'_m(v) - \Phi'(v)\| &= \|P_m\Phi'(P_m v) - \Phi'(v)\| \\ &\leq \|P_m\Phi'(P_m v) - P_m\Phi'(v)\| + \|P_m\Phi'(v) - \Phi'(v)\| \\ &\leq \|\Phi'(P_m v) - \Phi'(v)\| + \|P_m\Phi'(v) - \Phi'(v)\|.\end{aligned}$$

Therefore, by the continuity of Φ' on V, we obtain

$$\|\Phi'_m(v) - \Phi'(v)\| \to 0 \quad \text{as } m \to +\infty \text{ for all } v \in V. \tag{70}$$

Finally, by (62), the boundedness of Φ' and Φ'_m on V, and (70), from (69) we have that

$$\int_H \langle G_N(v), \Phi'(v)\rangle \, d\mu(v) = 0.$$

■

As a direct consequence of Proposition 16 and Theorem 19, we have

Corollary 20. *Let μ be a time-average measure of a solution $u(t)$ of the GMNSE such that $C(V) \subset L^1(H,\mu)$ and (59) is satisfied for all $\varphi \in C(V)$. Then μ is a stationary statistical solution of the GMNSE.*

Remark 21. *Let $u_0 \in V$. Let μ_N be for each $N > 0$ a time-average measure of the solution $S_N(t)u_0$ of GMNSE. We know that there exists a subsequence of solutions $S_{N'}(t)u_0$ that converges in an adequate sense to a solution $u(t)$ of NSE (see page 432 in* [2]*). To our knowledge, the question remains open if the $\mu_{N'}$ converge in some sense to a measure related to $u(t)$.*

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