H^2-boundedness of the pullback attractor for a non-autonomous reaction-diffusion equation

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Abstract

We prove some regularity results for the pullback attractor of a reaction-diffusion model. First we establish a general result about H^2-boundedness of invariant sets for an evolution process. Then, as a consequence, we deduce that the pullback attractor of a non-autonomous reaction-diffusion equation is bounded not only in $L^2(\Omega) \cap H_0^1(\Omega)$ but in $H^2(\Omega)$.

Key words: reaction-diffusion equations, non-autonomous (pullback) attractors, invariant sets, H^2-regularity.
Mathematics Subject Classifications (2000): 35B41, 35Q35

1 Introduction and setting of the problem

Let us consider the following problem for a non-autonomous reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty)\,, \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $f \in C^1(\mathbb{R})$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. We assume that there exist positive constants α_1, α_2, k, l, and $p > 2$ such that

$$-k - \alpha_1 |s|^p \leq f(s)s \leq k - \alpha_2 |s|^p, \quad \forall s \in \mathbb{R}, \tag{2}$$

$$f'(s) \leq l, \quad \forall s \in \mathbb{R}. \tag{3}$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r)dr.$$

Then, there exist positive constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{k}$ such that

$$-\widetilde{k} - \widetilde{\alpha}_1 |s|^p \leq \mathcal{F}(s) \leq \widetilde{k} - \widetilde{\alpha}_2 |s|^p, \qquad \forall s \in \mathbb{R}. \tag{4}$$

It is well-known (see, e.g. [5] or [8]) that under the conditions above, for any initial condition $u_\tau \in L^2(\Omega)$, there exists a unique solution $u(\cdot) = u(\cdot; \tau, u_\tau)$ of (1), i.e., a unique function $u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C^0([\tau, T]; L^2(\Omega))$ for all $T > \tau$, such that

$$u(t) - \int_\tau^t \Delta u(s)\, ds = u_\tau + \int_\tau^t (f(u(s)) + h(s))\, ds \quad \forall\, t \geq \tau,$$

where the equality must be understood in the sense of the dual of $H_0^1(\Omega) \cap L^p(\Omega)$.

Therefore, we can define a process $U = \{U(t,\tau), \ \tau \leq t\}$ in $L^2(\Omega)$ as

$$U(t,\tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega), \quad \forall \tau \leq t. \tag{5}$$

A pullback attractor for the process U defined by (5) (cf. [1], [2], [3]) is a family $\mathcal{A} = \{\mathcal{A}(t) : \ t \in \mathbb{R}\}$ of compact subsets of $L^2(\Omega)$ such that

a) (invariance) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $\tau \leq t$,
b) (pullback attraction) $\lim\limits_{\tau \to -\infty} \sup\limits_{u_\tau \in B} \inf\limits_{v \in \mathcal{A}(t)} |U(t,\tau)u_\tau - v| = 0$, for all $t \in \mathbb{R}$, for any bounded subset $B \subset L^2(\Omega)$,

where $|\cdot|$ denotes the norm in $L^2(\Omega)$.

It can be proved that, under the above conditions, if in addition f satisfies

$$\int_{-\infty}^t e^{\lambda_1 r} |h(r)|^2\, dr < +\infty \qquad \forall t \in \mathbb{R},$$

where λ_1 denotes the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in Ω, then there exists a pullback attractor for the process U defined by (5).

Several studies on this model have already been published (see [4], [6], [7], [9]). However, as for as we know, no one refers to the H^2-regularity we will consider in this paper.

In the next section we prove some results which, in particular, imply that, under suitable assumptions, any pullback attractor $\mathcal{A}$ for U satisfies that $\mathcal{A}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$, for every $t \in \mathbb{R}$.

2 H^2-boundedness of invariants sets

In this section we prove that, under suitable assumptions, every family of bounded subsets of $L^2(\Omega)$ which is invariant for the process U, is in fact bounded in $H^2(\Omega)$.

First, we recall a lemma (see [5]) which is necessary for the proof of our result.

Lemma 2.1 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$.*

Then, $u(t) \in X$ for all $t \in [t_0, T]$ and

$$\|u(t)\|_X \leq \sup_{n \geq 1} \|u_n\|_{L^\infty(t_0,T;X)} \qquad \forall t \in [t_0, T].$$

We will denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$, by $\|\cdot\| = |\nabla \cdot|$ the norm in $H_0^1(\Omega)$, and by $\langle \cdot, \cdot \rangle$ the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (1), which is given by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j, \tag{6}$$

and is the solution of

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) = \langle \Delta u_n(t), w_j \rangle + (f(u_n(t)), w_j) + (h(t), w_j), \\ (u_n(\tau), w_j) = (u_\tau, w_j) \qquad j = 1, .., n, \end{cases} \tag{7}$$

where $\{w_j : \ j \geq 1\}$ is the Hilbert basis of $L^2(\Omega)$ formed by the eigenfunctions associated to $-\Delta$ in $H_0^1(\Omega)$.

We first prove the following result.

Proposition 2.2 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2) and (3). Suppose moreover that $\Omega \subset \mathbb{R}^N$ is a bounded C^s domain, with $s \geq \max(2, N(p-2)/2p)$, and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. Then, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > \tau + \varepsilon$, the set $\{u_n(r; \tau, u_\tau) : \ r \in [\tau + \varepsilon, t], \ u_\tau \in B, \ n \geq 1\}$, is a bounded subset of $H^1_0(\Omega) \cap L^p(\Omega)$.*

Proof. Observe that by the regularity of Ω, all the eigenfunctions w_j associated to $-\Delta$ in $H^1_0(\Omega)$ belong to $H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega)$.

Let us fix a bounded set $B \subset L^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_\tau \in B$.

Multiplying by γ_{nj} in (7), and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dr}|u_n(r)|^2 + \|u_n(r)\|^2 = (f(u_n(r)), u_n(r)) + (h(r), u_n(r)). \tag{8}$$

Using (2),

$$\begin{aligned}(f(u_n(r)), u_n(r)) &\leq \int_\Omega (k - \alpha_2 |u_n(x,r)|^p)\, dx \\ &= k|\Omega| - \alpha_2 \|u_n(r)\|^p_{L^p(\Omega)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(h(r), u_n(r)) &\leq \frac{1}{2\lambda_1}|h(r)|^2 + \frac{\lambda_1}{2}|u_n(r)|^2 \\ &\leq \frac{1}{2\lambda_1}|h(r)|^2 + \frac{1}{2}\|u_n(r)\|^2.\end{aligned}$$

Thus, from (8) we deduce

$$\frac{d}{dr}|u_n(r)|^2 + \|u_n(r)\|^2 + 2\alpha_2\|u_n(r)\|^p_{L^p(\Omega)} \leq \frac{1}{\lambda_1}|h(r)|^2 + 2k|\Omega|,$$

and integrating between τ and r

$$\begin{aligned}&|u_n(r)|^2 + \int_\tau^r \|u_n(s)\|^2\, ds + 2\alpha_2 \int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)}\, ds \\ &\quad \leq |u_\tau|^2 + \frac{1}{\lambda_1}\int_\tau^t |h(s)|^2\, ds + 2k|\Omega|(t-\tau), \ \ \forall r \in [\tau, t], \ \ \forall n \geq 1.\end{aligned} \tag{9}$$

Now, multiplying by the derivative γ'_{nj} in (7), and summing from $j = 1$ to n,

$$\begin{aligned}|u'_n(r)|^2 + \frac{1}{2}\frac{d}{dr}\|u_n(r)\|^2 &= (f(u_n(r)), u'_n(r)) + (h(r), u'_n(r)) \\ &\leq \frac{1}{2}|h(r)|^2 + \frac{1}{2}|u'_n(r)|^2 + \frac{d}{dr}\int_\Omega \mathcal{F}(u_n(x,r))\, dx.\end{aligned}$$

Integrating now between $s \in [\tau, r]$ and $r \le t$, we obtain

$$\int_s^r |u_n'(\theta)|^2 \, d\theta + \|u_n(r)\|^2 \le \|u_n(s)\|^2 + \int_\tau^t |h(\theta)|^2 \, d\theta \\ + 2\int_\Omega \mathcal{F}(u_n(x,r))\, dx - 2\int_\Omega \mathcal{F}(u_n(x,s))\, dx,$$

which, jointly with (4), yields that

$$\int_s^r |u_n'(\theta)|^2 \, d\theta + \|u_n(r)\|^2 + 2\widetilde{\alpha}_2 \|u_n(r)\|^p_{L^p(\Omega)} \\ \le \|u_n(s)\|^2 + \int_\tau^t |h(\theta)|^2 \, d\theta + 4\widetilde{k}\,|\Omega| + 2\widetilde{\alpha}_1 \|u_n(s)\|^p_{L^p(\Omega)}, \tag{10}$$

for all $s \in [\tau, r]$, and any $r \in [\tau, t]$.

Integrating in this last inequality with respect to s from τ to r, we in particular obtain

$$(r-\tau)\left(\|u_n(r)\|^2 + 2\widetilde{\alpha}_2 \|u_n(r)\|^p_{L^p(\Omega)}\right) \le \int_\tau^t \|u_n(s)\|^2 \, ds + (t-\tau)\int_\tau^t |h(s)|^2 \, ds \\ + 4\widetilde{k}\,|\Omega|\,(t-\tau) + 2\widetilde{\alpha}_1 \int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)} \, ds,$$

for all $r \in [\tau, t]$, and for any $n \ge 1$. From this inequality and (9), our result holds. ■

Corollary 2.3 *Under the assumptions in Proposition 2.2, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the set $\bigcup_{r \in [\tau+\varepsilon, t]} U(r,\tau)B$ is a bounded subset of $H_0^1(\Omega) \cap L^p(\Omega)$.*

Proof. This is a straightforward consequence of Lemma 2.1, Proposition 2.2, and the well known fact that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;H_0^1(\Omega)) \cap L^p(\tau,t;L^p(\Omega))$. ■

Proposition 2.4 *In addition to the assumptions in Proposition 2.2, assume that $h \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega))$. Then, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau + \varepsilon$, the set $\{u_n(r;\tau,u_\tau) : \ r \in [\tau+\varepsilon, t], \ u_\tau \in B, \ n \ge 1\}$ is a bounded subset of $H^2(\Omega)$.*

Proof. Let us fix a bounded set $B \subset L^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_\tau \in B$.

As we are assuming that $h \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega))$, we can differentiate with respect to time in (7), and then, multiplying by γ_{nj}', and summing from $j = 1$ to n, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dr}\left|u_n'(r)\right|^2+\left\|u_n'(r)\right\|^2 &= (f'(u_n(r))u_n'(r),u_n'(r))+(h'(r),u_n'(r))\\ &\le l\left|u_n'(r)\right|^2+\frac{1}{2}\left|u_n'(r)\right|^2+\frac{1}{2}\left|h'(r)\right|^2.\end{aligned}$$

In particular, integrating in the last inequality,

$$\left|u_n'(r)\right|^2\le\left|u_n'(s)\right|^2+(2l+1)\int_{\tau+\varepsilon/2}^{t}\left|u_n'(\theta)\right|^2d\theta+\int_{\tau+\varepsilon/2}^{t}\left|h'(\theta)\right|^2d\theta,$$

for all $\tau+\varepsilon/2\le s\le r\le t$. Now, integrating with respect to s between $\tau+\varepsilon/2$ and r,

$$\begin{aligned}(r-\tau-\varepsilon/2)\left|u_n'(r)\right|^2 &\le [(2l+1)(t-\tau-\varepsilon/2)+1]\int_{\tau+\varepsilon/2}^{t}\left|u_n'(\theta)\right|^2d\theta\\ &\quad+(r-\tau-\varepsilon/2)\int_{\tau+\varepsilon/2}^{t}\left|h'(\theta)\right|^2d\theta,\end{aligned}$$

for all $\tau+\varepsilon/2\le r\le t$, and, in particular,

$$\begin{aligned}\left|u_n'(r)\right|^2 &\le 2\varepsilon^{-1}[(2l+1)(t-\tau-\varepsilon/2)+1]\int_{\tau+\varepsilon/2}^{t}\left|u_n'(\theta)\right|^2d\theta\\ &\quad+\int_{\tau+\varepsilon/2}^{t}\left|h'(\theta)\right|^2d\theta,\end{aligned}\tag{11}$$

for all $r\in[\tau+\varepsilon,t]$.

On the other hand, multiplying in (7) by $\lambda_j\gamma_{nj}$, where λ_j is the eigenvalue associated to the eigenfunction w_j, and summing once more from $j=1$ to n, we obtain

$$(u_n'(r),\Delta u_n(r))=\left|\Delta u_n(r)\right|^2+(f(u_n(r)),\Delta u_n(r))+(h(r),\Delta u_n(r)).\tag{12}$$

But, it follows from (3) that

$$\begin{aligned}-(f(u_n(r)),\Delta u_n(r)) &= -\int_\Omega (f(u_n(x,r))-f(0))\,\Delta u_n(x,r)dx\\ &\quad-f(0)\int_\Omega \Delta u_n(x,r)dx\\ &\le l\left\|u_n(r)\right\|^2+\frac{1}{4}\left|\Delta u_n(r)\right|^2+(f(0))^2\left|\Omega\right|\\ &= l\,(u_n(r),-\Delta u_n(r))+\frac{1}{4}\left|\Delta u_n(r)\right|^2+(f(0))^2\left|\Omega\right|\\ &\le l^2\left|u_n(r)\right|^2+\frac{1}{2}\left|\Delta u_n(r)\right|^2+(f(0))^2\left|\Omega\right|,\end{aligned}$$

and thus, from (12) we obtain

$$|\Delta u_n(r)|^2 \leq 8\,|u_n'(r)|^2 + 8\,|h(r)|^2 + 4l^2\,|u_n(r)|^2 + 4\,(f(0))^2\,|\Omega|\,, \tag{13}$$

for all $r \geq \tau$.

Finally, observe that by (10)

$$\int_{\tau+\varepsilon/2}^{t} |u_n'(\theta)|^2\,d\theta \leq \|u_n(\tau+\varepsilon/2)\|^2 + \int_{\tau}^{t} |h(\theta)|^2\,d\theta + 4\widetilde{k}\,|\Omega| \tag{14}$$
$$+ 2\widetilde{\alpha}_1\,\|u_n(\tau+\varepsilon/2)\|_{L^p(\Omega)}^p\,.$$

Taking into account that, in particular, $h \in C^0([\tau,t];L^2(\Omega))$, the result is a direct consequence of Proposition 2.2 and estimates (11), (13) and (14). ■

Corollary 2.5 *Under the assumptions of Proposition 2.4, for any bounded set $B \subset L^2(\Omega)$, any $\tau \in \mathbb{R}$, any $\varepsilon > 0$, and any $t > \tau+\varepsilon$, the set $\bigcup_{r\in[\tau+\varepsilon,t]} U(r,\tau)B$ is a bounded subset of $H^2(\Omega)$.*

Proof. This follows from Lemma 2.1, propositions 2.2 and 2.4, and the well known facts that $u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;H_0^1(\Omega))$, and $u(\cdot;\tau,u_\tau) \in C^0([\tau+\varepsilon,t];H_0^1(\Omega))$. ■

As a direct consequence of the above results, we can now establish our main results.

Theorem 2.6 *Under the assumptions in Proposition 2.4, if $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a family of bounded subsets of $L^2(\Omega)$, such that $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for any $\tau \leq t$, then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.*

In particular, we have the following result for pullback attractors.

Corollary 2.7 *Under the assumptions in Proposition 2.4, if $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is a pullback attractor for the process defined by (5), then for any $T_1 < T_2$, the set $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ is a bounded subset of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^p(\Omega)$.*

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