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Gradient Infinite-Dimensional Random Dynamical Systems*

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Abstract. In this paper we introduce the concept of a gradient random dynamical system as a random semiflow possessing a continuous random Lyapunov function which describes the asymptotic regime of the system. Thus, we are able to analyze the dynamical properties on a random attractor described by its Morse decomposition for infinite-dimensional random dynamical systems. In particular, if a random attractor is characterized by a family of invariant random compact sets, we show the equivalence among the asymptotic stability of this family, the Morse decomposition of the random attractor, and the existence of a random Lyapunov function.

Key words. Morse decomposition, attractor, repeller, Morse set, Lyapunov function, random dynamical systems

AMS subject classifications. 37L55, 60H15, 37B25, 37B35, 37B55

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1. Introduction.

One important aspect of the qualitative analysis of differential equations and dynamical systems is the study of asymptotic, long-term behavior of solutions. To this aim, the analysis of dynamical systems generally involves the study of the existence and structure of invariant sets and their stability properties.

When an autonomous infinite-dimensional dynamical system, i.e., related to semiflows in an infinite-dimensional phase space, with a global attractor is shown to possess a Lyapunov function, the system is said to be gradient (see, for instance, Hale [19]), and most of the important asymptotic regime of solutions can be deduced from the existence of this function. In particular, alpha- and omega-limit sets of solutions converge to equilibria, and there are no cycles between them. The existence of a finite family of invariant sets in the global attractor describing the forward and backward behavior of solutions with no cycles between them is defined in Carvalho and Langa [9] as a gradient-like dynamical system. Very recently, it has been shown that this gradient-like dynamical description of a system, a consequence of the existence of a Lyapunov map, is also a sufficient condition for the existence of such a function (see Aragao-Costa et al. [1]); i.e., a system is gradient if and only if it is gradient-like in the sense of Carvalho and Langa [9]. This fact allows us to describe a gradient system from asymptotical dynamical properties of global solutions instead of the existence of an abstract

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Lyapunov function, for which no methods are known to obtain its existence. Moreover, as gradient-like systems are robust under perturbation, in fact what is proved in [1] is that gradient systems are persistent under (autonomous or nonautonomous) perturbations. The argument in this result goes through the proof of the equivalence between a gradient-like structure and the existence of a Morse decomposition on the global attractor.

On the other hand, when a semiflow in a phase space X is allowed to have random influences, a description of the asymptotic behavior of the associated infinite-dimensional random dynamical system is usually analyzed from the study of random attractors and their characterization. A random attractor (see Crauel and Flandoli [17]) is an invariant random compact set attracting in the pullback sense (see Definition 2.9). We prove that a random attractor is an invariant compact set for which there exists a continuous (in the space variable) random Lyapunov function describing a decreasing energy level on the evolutions of entire orbits.

Recently, Liu has introduced a random version of Morse decomposition theory in Conley [13] adapted to random invariant compact sets for flows or even semiflows (see Liu [27, 28, 29] and Liu, Ji, and Su [30]). In particular, given a random attractor, it is first possible to define a random attractor-repeller pair associated to a random dynamical system, from which to describe a finite family $\{M_i(\omega), i = 1, \ldots, n\}$ of random compact invariant sets named as random Morse decomposition of the random attractor (see Definition 4.14). In these last papers some dynamical properties of the Morse sets are proved. In this work, and in the framework of infinite-dimensional dynamical systems, we prove the equivalence between a gradient-like dynamics on a finite family of invariant random compact sets (see Definition 4.17) and the existence of a Morse decomposition on the random attractor.

On the other hand, in Liu [29] it is shown that any random Morse decomposition implies the existence of a measurable random Lyapunov function on the phase space. In this paper we prove that this function is in fact continuous in the phase space X and, conversely, its existence gives rise to a Morse decomposition on the random attractor, which, as a consequence, implies the equivalence with gradient-like dynamics on the associated finite family of invariant random compact sets. Note that, in applications, the determination of a concrete Lyapunov function is always a difficult problem, even in the deterministic case. Thus, our results allow us to conclude the existence of such a Lyapunov function of a system from a detailed analysis of the structure and asymptotic dynamics on the random attractor.

These results, as in the deterministic case (see Aragao-Costa et al. [1]), allow us to define a concept of a gradient random dynamical system from two different but equivalent approaches: an abstract one, by proving the existence of a random Lyapunov function, and a dynamical one, by the description of the internal asymptotic behavior of entire orbits on the random attractors with respect to the family $M_i(\omega)$.

Other concepts of attraction and, consequently, attractor-repeller pairs and Morse decomposition have been introduced in the framework of random dynamical systems. Among them, the one on weak attractors, related to convergence in probability, has been used to prove the existence of Lyapunov functions on the random attractor (see Arnold and Schmalfuss [3]) or the existence of weak random Morse decomposition, as in Ochs [32]; see also [16]. We have adopted convergence $\mathbb{P}$-a.s., in the pullback sense of a local attractor and in the pullback-backwards sense in the case of a repeller. It is remarkable that this kind of convergence implies forward attraction in probability to local attractors and backwards attraction in probability to associated repellers, as in the previous referenced papers, which is the same as we observe in the autonomous deterministic case (see Conley [13], Rybakowski [33], or Aragao-Costa et al. [1]).

2. Random dynamical systems and attractors.

In this section, we will recall some definitions and propositions for later use. First, we establish the definition of continuous random dynamical systems (cf. Arnold [2]).

Definition 2.1. *Let (X, d) be a Polish metric space. Denote by $\mathbb{T}$ a subset of real numbers $\mathbb{R}$ which satisfies either $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and let $\mathbb{T}^+$ be defined by $\mathbb{T}^+ = \mathbb{T} \cap \mathbb{R}^+$. A continuous* random dynamical system (RDS)*, denoted by φ, consists of two ingredients:*

(i) *A model of the noise, namely, a metric dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t\omega$ is a measurable flow which leaves $\mathbb{P}$ invariant, i.e., $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{T}$.*

(ii) *A model of the system driven by noise, namely, a cocycle φ over θ, i.e., a measurable mapping $\varphi : \mathbb{T}^+ \times \Omega \times X \to X, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, such that $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and the family $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : X \to X$ of random self-mappings of X satisfies the cocycle property:*

$$\varphi(0, \omega) = \mathrm{id}_X, \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \quad \text{for all } t, s \in \mathbb{T}^+, \omega \in \Omega, \tag{2.1}$$

where $\circ$ means composition.

Remark 2.2. The time for the base flow (θ_t) is always assumed to be two-sided, even if φ is defined for nonnegative time only. Furthermore, the maps $\varphi(t, \omega) : X \to X$ are not assumed to be invertible a priori. If the cocycle property (2.1) holds for two-sided time $\mathbb{T}$ instead of $\mathbb{T}^+$, then $\varphi(t, \omega)$ is automatically invertible for every $t \in \mathbb{T}$. In fact, in this case $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega)$ for every $t \in \mathbb{T}$.

We now establish the definition of random set, which is a basic concept for an RDS.

Definition 2.3. *Let X be a metric space with a metric d. A set-valued map $\omega \mapsto D(\omega)$ taking values in the closed/compact subsets of X is said to be a* random closed/compact set *if the mapping $\omega \mapsto d(x, D(\omega))$ is measurable for any $x \in X$, where $d(x, B) := \inf_{y\in B} d(x, y)$. A set-valued map $\omega \mapsto U(\omega)$ taking values in the open subsets of X is said to be a* random open set *if $\omega \mapsto U^c(\omega)$ is a random closed set, where U^c denotes the complement of U, i.e., $U^c := X \setminus U$.*

When we say a "random set" in what follows but do not specify that the set is open, closed, or compact, then either it is clear from the context or it can be any one of these three types, which, in our eyes, will not confuse the reader.

Definition 2.4. *A random set D is said to be* forward invariant *under the RDS φ if $\varphi(t, \omega)D(\omega) \subset D(\theta_t\omega)$ for all $t \geq 0$ a.s. It is said to be* invariant *if $\varphi(t, \omega)D(\omega) = D(\theta_t\omega)$ for all $t \geq 0$ a.s.*

Now we enumerate some basic results about random sets in the following proposition. For more details the reader is referred to Arnold [2], Castaing and Valadier [10], Crauel [14], Hu and Papageorgiou [21], and Arnold and Schmalfuss [3].

Proposition 2.5. *Let X be a Polish space. Then the following assertions hold:*

(i) *D is a random closed set if and only if the set $\{\omega \in \Omega \mid D(\omega) \bigcap U \neq \emptyset\}$ is measurable for any open set $U \subset X$.*

(ii) *If D is a random closed set, then so is the closure of D^c.*

(iii) *If D is a random open set, then the closure $\overline{D}$ of D is a random closed set; if D is a random closed set, then* $\mathrm{int}D$, *the interior of D, is a random open set.*

(iv) *D is a random compact set in X if and only if $D(\omega)$ is compact for every $\omega \in \Omega$ and the set $\{\omega \in \Omega \mid D(\omega) \bigcap C \neq \emptyset\}$ is measurable for any closed set $C \subset X$.*

(v) *If $\{D_n, n \in \mathbb{N}\}$ is a sequence of random closed sets with nonvoid intersection, and there exists $n_0 \in \mathbb{N}$ such that D_{n_0} is a random compact set, then $\bigcap_{n\in\mathbb{N}} D_n$ is a random compact set in X.*

(vi) *If $f : \Omega \times X \to X$ is a function such that $f(\omega, \cdot)$ is continuous for all ω and $f(\cdot, x)$ is measurable for all x, then $\omega \mapsto f(\omega, D(\omega))$ is a random compact set, provided that D is a random compact set.*

(vii) *If D is a random closed set, then* $\mathrm{graph}(D) := \{(\omega, x) \in \Omega \times X \mid x \in D(\omega)\}$ *is a measurable subset of $\mathcal{F} \times \mathcal{B}(X)$; conversely, given $D : \Omega \to 2^X$, taking values in the closed subsets of X, if* $\mathrm{graph}(D) \in \mathcal{F} \times \mathcal{B}(X)$, *then D is an $\mathcal{F}^u$-measurable (in particular, $\mathcal{F}^{\mathbb{P}}$-measurable, with $\mathcal{F}^{\mathbb{P}}$ being the completion of the σ-algebra $\mathcal{F}$ with respect to the measure $\mathbb{P}$) random closed set; i.e., the mapping $\omega \in \Omega \mapsto d(x, D(\omega))$ is $\mathcal{F}^u$-measurable (universally measurable) for any $x \in X$.*

(viii) *If D is an $\mathcal{F}^{\mathbb{P}}$-measurable random closed set, then there exists an $\mathcal{F}$-measurable random closed set $\tilde{D}$ such that $D = \tilde{D}$ a.s.*

(ix) (Measurable selection theorem.) *Let a multifunction $\omega \mapsto D(\omega)$ take values in the subspace of closed nonvoid subsets of X. Then D is a random closed set if and only if there exists a sequence $\{v_n : n \in \mathbb{N}\}$ of measurable maps $v_n : \Omega \to X$ such that*

$$v_n(\omega) \in D(\omega) \quad \textit{and} \quad D(\omega) = \overline{\{v_n(\omega) \in X \mid \; n \in \mathbb{N}\}} \quad \textit{for all} \quad \omega \in \Omega.$$

In particular if D is a random closed set, then there exists a measurable selection, i.e., a measurable map $v : \Omega \to X$ such that $v(\omega) \in D(\omega)$ for all $\omega \in \Omega$.

(x) (Projection theorem.) *Let X be a Polish space and let $M \subset \Omega \times X$ be a set which is measurable with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}(X)$. Then the set*

$$\Pi_\Omega M = \{\omega \in \Omega | (\omega, x) \in M \;\; \textit{for some } x \in X\}$$

is universally measurable, i.e., belongs to $\mathcal{F}^u$, where Π_Ω stands for the canonical projection of $\Omega \times X$ to Ω. In particular, it is measurable with respect to the $\mathbb{P}$-completion $\bar{\mathcal{F}}^{\mathbb{P}}$ of $\mathcal{F}$.

Remark 2.6. By (vii) of the previous proposition, the intersection of a finite or countable number of random closed sets is an $\mathcal{F}^u$-measurable random closed set; and by (viii), we can assume that it is just a random closed set.

Definition 2.7. *For any $D : \Omega \to 2^X$, the* omega-limit set of D, *denoted by Ω_D, is defined by*

$$\Omega_D(\omega) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \varphi(s, \theta_{-s}\omega) D(\theta_{-s}\omega)}$$

for each $\omega \in \Omega$.

Definition 2.8. *Given two random sets D and A, we say that A* (pullback) attracts *D if*

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega) | A(\omega)) = 0$$

holds a.s., where $d(A|B)$ *stands for the Hausdorff semimetric between two sets* A, B*, i.e.,* $d(A|B) := \sup_{x\in A} d(x, B)$*; and we say* A attracts D in probability *or* weakly attracts D *if*

$$\mathbb{P} - \lim_{t\to\infty} d(\varphi(t,\omega)D(\omega)|A(\theta_t\omega)) = 0;$$

i.e., given $\epsilon > 0$*, there exists* $t(\epsilon)$ *such that*

$$\mathbb{P}(\{\omega \in \Omega \mid d(\varphi(t,\omega)D(\omega), A(\theta_t\omega)) > \epsilon\}) \le \epsilon \quad \text{for all } t \ge t(\epsilon).$$

By the measure preserving property of θ_t, it is clear that pullback attraction implies weak attraction.

Global random attractors were introduced by Crauel and Flandoli [17] and Schmalfuss [35] and were studied for many SDEs; see [5, 6, 8, 15, 26, 34, 36], among others. First, let us recall the definition of a global random attractor. Here we adopt the point of view from [36], also considered in [2, 34] and others. This more flexible version allows us to consider some local properties.

Definition 2.9 (see [2, 34, 36]). *Assume that* φ *is a random semiflow on a Polish space* X*. A* universe $\mathcal{D}$ *is a collection of families* $(D(\omega))_{\omega\in\Omega}$ *of nonempty subsets of* X *which is closed with respect to set inclusion; i.e., if* $D_1 \in \mathcal{D}$ *and* $D_2(\omega) \subset D_1(\omega)$ *for all* ω*, then* $D_2 \in \mathcal{D}$*. A random compact set* $S \in \mathcal{D}$ *is called a* global random attractor of φ in $\mathcal{D}$ *if*

- S *is invariant, i.e.,*

$$\varphi(t,\omega)S(\omega) = S(\theta_t\omega) \quad \text{for all } t \ge 0 \tag{2.2}$$

  for almost all $\omega \in \Omega$*;*
- S *pullback attracts in* $\mathcal{D}$*; i.e., for any* $D \in \mathcal{D}$*, we have*

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)|S(\omega)) = 0 \tag{2.3}$$

  a.s.;
- *there exists a neighborhood* $U \in \mathcal{D}$ *of* S*; i.e.,* $S(\omega) \subset \mathrm{int}U(\omega)$ *for almost all* $\omega \in \Omega$*.*

Note that not every element of the universe $\mathcal{D}$ is a random set. Throughout the paper, we assume that S is the global attractor of φ in the universe $\mathcal{D}$. In specific theorems or results, we will point out what elements $\mathcal{D}$ need to contain.

Remark 2.10. (i) It is immediate to check that the global random attractor defined above for the RDS φ is the *minimal* random closed set in $\mathcal{D}$ which attracts all the elements in $\mathcal{D}$, and it is the *largest* random compact set which is invariant in the sense of (2.2).

(ii) Note that the definition of global random attractor is stronger than that of [17] by requesting that the attractor itself be an element of the universe and there be a random neighborhood of it which belongs to the universe, but these are satisfied, for instance, when the universe consists of all the random tempered sets.

(iii) If there exists a random compact set $C \in \mathcal{D}$ which pullback attracts in $\mathcal{D}$, then there exists a unique global random attractor that coincides with the omega-limit set of C. For details, the reader is referred to [25, Theorem 2.2].

We list the following two results from [28, 27] for later use.

Lemma 2.11 (see [28, Lemma 3.1]). *Assume that U is a random open set and that an invariant random compact set $A \subset U$ satisfies that $\Omega_U = A$ a.s. Then there exists a forward invariant random open set $\tilde{U}$ with the same properties as U, i.e., $A \subset \tilde{U}$ and $\Omega_{\tilde{U}} = A$ a.s.*

Lemma 2.12 (see [27, Lemma 3.5]). *Let U be a random open set, and x a random variable. Define*

$$t(\omega) := \inf\{t \in \mathbb{R}^+ |\ \varphi(t,\omega)x(\omega) \in \overline{U(\theta_t\omega)}\}, \tag{2.4}$$

i.e., the first entrance time of x into U under the cocycle φ. Then $\omega \mapsto t(\omega)$ is a random variable, which is measurable with respect to the universal σ-algebra $\mathcal{F}^u$.

Remark 2.13. By Proposition 2.5 (viii), the random entrance time t in Lemma 2.12 can be assumed to be measurable with respect to $\mathcal{F}$. Furthermore, by the measurable selection theorem, Lemma 2.12 also holds when the random variable x is replaced by a random closed set and U is forward invariant.

3. Random attractors and associated Lyapunov functions.

The following "backward orbit" and "entire orbit" were introduced in [29] for random semiflows.

Definition 3.1. (i) *For fixed ω and x, a mapping $\sigma_\cdot(\omega) : \mathbb{R}^- \to X$ is called a* backward orbit of φ through x driven by ω *if it satisfies the cocycle property:*

$$\sigma_0(\omega) = x,\ \ \sigma_{t+s}(\omega) = \varphi(s,\theta_t\omega)\sigma_t(\omega) \ \ \textit{for all } t \le 0, s \ge 0, t+s \le 0.$$

(ii) *Let $\mathcal{M}$ denote the set of all X-valued random variables and let $x \in \mathcal{M}$. A mapping $\sigma : \mathbb{R}^- \to \mathcal{M}$ is called a* backward orbit of φ through x *if for all $\omega \in \Omega$ the following cocycle property holds:*

$$\sigma_0(\omega) = x(\omega),\ \ \sigma_{t+s}(\omega) = \varphi(s,\theta_t\omega)\sigma_t(\omega) \ \ \textit{for all } t \le 0, s \ge 0, t+s \le 0.$$

Definition 3.2. (i) *For fixed ω and x, a mapping $\sigma_\cdot(\omega) : \mathbb{R} \to X$ is called an* entire orbit of φ through x driven by ω *if it satisfies the cocycle property:*

$$\sigma_0(\omega) = x,\ \ \sigma_{t+s}(\omega) = \varphi(s,\theta_t\omega)\sigma_t(\omega) \ \ \textit{for all } t \in \mathbb{R}, s \ge 0.$$

(ii) *Let $x \in \mathcal{M}$. A mapping $\sigma : \mathbb{R} \to \mathcal{M}$ is called an* entire orbit of φ through x *if for all $\omega \in \Omega$ the following cocycle property holds:*

$$\sigma_0(\omega) = x(\omega),\ \ \sigma_{t+s}(\omega) = \varphi(s,\theta_t\omega)\sigma_t(\omega) \ \ \textit{for all } t \in \mathbb{R}, s \ge 0.$$

Remark 3.3. Note that by the definition of entire orbit, for $s \ge 0, t \in \mathbb{R}$,

$$\sigma_{t+s}(\omega) = \varphi(s,\theta_t\omega)\sigma_t(\omega),$$

but usually we do not have

$$\varphi(s,\theta_t\omega)\sigma_t(\omega) = \sigma_0(\theta_{t+s}\omega).$$

That is,

$$\sigma_{t+s}(\omega) = \sigma_0(\theta_{t+s}\omega)$$

does not hold usually. Only when $\sigma_0 = x$ is a random fixed point, i.e., $\varphi(s,\omega)x(\omega) = x(\theta_s\omega)$ for $s \geq 0$ and $\omega \in \Omega$, does the above relation hold.

Remark 3.4. (i) Note that when φ is restricted to an entire orbit σ through x driven by ω, which will be denoted by φ^σ, it can be extended to be defined for all $t \in \mathbb{R}$ along the entire orbit σ. Indeed, let

$$\varphi^\sigma(t,\omega)(x) := \begin{cases} \varphi(t,\omega)(x) & \text{for } t \geq 0, \\ \sigma_t(\omega) & \text{for } t < 0, \end{cases}$$

or simply $\varphi^\sigma(t,\omega)(x) = \sigma_t(\omega)$ for $t \in \mathbb{R}$, taking into account that $\varphi(t,\omega)(x) = \sigma_t(\omega)$ for $t \geq 0$, by the definition of entire orbit. A similar fact holds for an entire orbit through a random variable $x \in \mathcal{M}$, i.e., for all $\omega \in \Omega$,

$$\varphi^\sigma(t,\omega)(x(\omega)) := \begin{cases} \varphi(t,\omega)(x(\omega)) & \text{for } t \geq 0, \\ \sigma_t(\omega) & \text{for } t < 0. \end{cases}$$

(ii) In the case that σ is an entire orbit of φ through $x \in X$ driven by $\omega \in \Omega$, φ^σ is a mapping from $\mathbb{R} \times \{\omega\} \times \{\sigma_0\}$ to X defined through $\varphi^\sigma(t,\omega)\sigma_0 := \sigma_t$ for all $t \in \mathbb{R}$. In the case that σ is an entire orbit of φ through a random variable $x \in \mathcal{M}$, φ^σ is a mapping from $\{(t,\omega,\sigma_0(\omega)) \in \mathbb{R} \times \Omega \times X\}$ to X defined through $\varphi^\sigma(t,\omega)\sigma_0(\omega) := \sigma_t(\omega)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$.

(iii) Note that for any fixed $t \geq 0$ and $\omega \in \Omega$, $\varphi(t,\omega) : X \to X$ is a continuous mapping on X, but not necessarily a homeomorphism. Generally, we cannot extend the definition of φ from $\mathbb{R}^+$ to $\mathbb{R}$ compatibly, i.e., extend φ from a random semiflow to a random flow, which is just like saying that we cannot extend a semiflow to a flow in the deterministic case without additional assumptions; see [37, section 2 of Part II] for details. So, generally, φ^σ is not a mapping from $\mathbb{R} \times \Omega \times X$ to X. But for any point or random variable in an invariant random compact set, there is always an entire orbit through it; see Remark 3.5 and Lemma 3.6 for details. Note also that the backward orbit through the point or the random variable is not unique in general, which is also the main reason we cannot extend the definition of φ from $\mathbb{R}^+$ to $\mathbb{R}$ compatibly.

Remark 3.5. A random set D is forward invariant if and only if $D = D_\varphi^+$ a.s., where

$$D_\varphi^+(\omega) := \{x \in X \mid \varphi(t,\omega)x \in D(\theta_t\omega) \text{ for all } t \geq 0\}.$$

A random set D is invariant if and only if $D = D_\varphi$ a.s., where, for all $\omega \in \Omega$,

$$D_\varphi(\omega) := \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists an entire orbit } \sigma : \mathbb{R} \to X \\ \text{of } \varphi \text{ through } x \text{ driven by } \omega \text{ which} \\ \text{satisfies } \sigma_t(\omega) \in D(\theta_t\omega) \text{ for all } t \in \mathbb{R} \end{array} \right\}.$$

The following result from [29] will be used later.

Lemma 3.6 (see [29, Lemma 4.2 and Corollary 4.2]). *Assume that D is a forward invariant random compact set; then for any random variable on Ω_D there exists a backward orbit lying on Ω_D through this random variable. In particular, if D is an invariant random compact set, then for any random variable on D, there exists a backward orbit lying on D through it.*

Next, we prove a simple result which confirms that, like in the deterministic case, the global random attractor consists of entire orbits. For a given entire orbit σ through a random variable, denote by $\mathrm{Tr}\sigma$ the *trace of* σ, i.e., $\mathrm{Tr}\sigma(\omega) := \{\sigma_t(\omega)|t \in \mathbb{R}\}$ for each $\omega \in \Omega$; denote by $\tilde{\mathcal{M}}$ the subset of $\mathcal{M}$ that consists of all $x \in \mathcal{M}$ satisfying that there exists an entire orbit σ through x such that S attracts $\mathrm{Tr}\sigma$. Then we have the following.

Proposition 3.7. *The global random attractor S satisfies*

$$S(\omega) = \{x(\omega) \in X \mid x \in \tilde{\mathcal{M}}\} \tag{3.1}$$

for almost all $\omega \in \Omega$.

Proof. By Lemma 3.6, the global random attractor S is a subset of the right-hand side of (3.1), so we need to show the converse inclusion. For a given random variable x belonging to the right-hand side of (3.1), let σ be an entire orbit through x with trace being attracted by S. By the definition of entire orbit, for all $t \in \mathbb{R}$, σ is also an entire orbit through the random variable $\sigma_t \in \mathrm{Tr}\sigma$, that is, $\mathrm{Tr}\sigma$ is an invariant set. Since S attracts $\mathrm{Tr}\sigma$ and S is compact, it follows that the omega-limit set $\Omega_{\mathrm{Tr}\sigma}$ of $\mathrm{Tr}\sigma$ is nonempty and $\Omega_{\mathrm{Tr}\sigma} \subset S$ a.s.; see [15, Theorem 2.1]. Note that $\Omega_{\mathrm{Tr}\sigma} = \overline{\mathrm{Tr}\sigma}$ since $\mathrm{Tr}\sigma$ is invariant. Therefore, $\mathrm{Tr}\sigma \subset S$ and, in particular, $x(\omega) \in S(\omega)$ a.s. The proof is complete. ■

Lemma 3.8. *Assume that S is the global random attractor with U being a closed forward invariant neighborhood of S such that $\Omega_U(\omega) = S(\omega)$ a.s. Then, for any $D \in \mathcal{D}$, there exists a random variable $T_D \geq 0$ such that, for almost all $\omega \in \Omega$,*

$$\varphi(t, \omega)D(\omega) \subset U(\theta_t\omega) \quad \textit{for all } t \geq T_D(\omega). \tag{3.2}$$

Proof. Since $\Omega_D \subset S$ a.s., there exists a random $T \geq 0$ such that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset U(\omega) \quad \text{for all } t \geq T(\omega).$$

Note that, since U is forward invariant, if for some $t_0 \geq 0$ we have $\varphi(t_0, \omega)D(\omega) \subset U(\theta_{t_0}\omega)$, then the same holds for any $t \geq t_0$. Therefore, if the result is not true, then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) > 0$ such that

$$\varphi(t, \omega)D(\omega) \not\subset U(\theta_t\omega) \quad \text{for all } t \geq 0, \omega \in \Omega_1.$$

That is,

$$d(\varphi(t, \omega)D(\omega)|U(\theta_t\omega)) > 0 \quad \text{for all } t \geq 0, \omega \in \Omega_1.$$

On the other hand, since U is a neighborhood of A, for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}\{\omega|d(U(\omega)|S(\omega)) \geq \delta\} > 1 - \epsilon.$$

In particular, if we choose $\epsilon \leq \frac{1}{2}\mathbb{P}(\Omega_1)$, then it follows that

$$\mathbb{P}\{\omega|d(\varphi(t, \omega)D(\omega)|S(\theta_t\omega)) \geq \delta\} > 1 - \frac{1}{2}\mathbb{P}(\Omega_1) \geq \frac{1}{2}\mathbb{P}(\Omega_1) \quad \text{for all } t \geq 0.$$

This is a contradiction of the fact that S attracts D in probability. So, if we let

$$T_D(\omega) := \inf\{t \geq 0|d(\varphi(t, \omega)D(\omega)|U(\theta_t\omega)) = 0\},$$

then T_D is the desired first entrance time. By Lemma 2.12 and Remark 2.13, we obtain that T_D is a random variable. The proof is complete. ■

Remark 3.9. In the previous lemma, we assume that there is a closed forward invariant neighborhood U of S such that $\Omega_U = S$ a.s. The forward invariance and closedness of U are not restrictive assumptions, i.e., such a neighborhood does exist. Actually, by Definition 2.9, there exists a neighborhood U (not necessarily closed and forward invariant) of S satisfying $\Omega_U = S$ a.s. Then it is clear that int$U$ is an open random neighborhood of S with $\Omega_{\mathrm{int}U} = \Omega_U = S$ a.s. By Lemma 2.11, there exists an open forward invariant neighborhood U_1 of S with $\Omega_{U_1} = S$ a.s. Thus the closure of U_1 is the required closed forward invariant neighborhood of S.

Remark 3.10. We may call the property (3.2) *forward absorption*, which appears in [3, Proposition 4.4] for random flows. In contrast, there is also a concept of pullback absorption; see [17, Definition 3.5] for details.

Lemma 3.11. *Assume that U is a forward invariant random closed set. Then, for any nonrandom constant $T \geq 0$, $U_T(\omega) := \overline{\varphi(T, \theta_{-T}\omega)U(\theta_{-T}\omega)}$ is still a forward invariant random closed set. Furthermore, for any $t > s \geq 0$, $U_t(\omega) \subset U_s(\omega)$ for all $\omega \in \Omega$. In particular,*

$$\Omega_U(\omega) = \bigcap_{T \geq 0} U_T(\omega) = \bigcap_{n \in \mathbb{N}} U_n(\omega) \qquad \textit{for all } \omega \in \Omega.$$

Proof. Note that, for any $t \geq 0$,

$$\begin{aligned}
\varphi(t,\omega)U_T(\omega) &= \varphi(t,\omega)\overline{\varphi(T,\theta_{-T}\omega)U(\theta_{-T}\omega)} \\
&\subset \overline{\varphi(t,\omega) \circ \varphi(T,\theta_{-T}\omega)U(\theta_{-T}\omega)} \\
&= \overline{\varphi(t+T,\theta_{-T}\omega)U(\theta_{-T}\omega)} \\
&= \overline{\varphi(T,\theta_{-T} \circ \theta_t\omega) \circ \varphi(t,\theta_{-T}\omega)U(\theta_{-T}\omega)} \\
&= \overline{\varphi(T,\theta_{-T} \circ \theta_t\omega)U(\theta_{-T} \circ \theta_t\omega)} \\
&= U_T(\theta_t\omega),
\end{aligned}$$

where the inclusion holds since $f(\overline{A}) \subset \overline{f(A)}$ for any continuous f, the second through fourth equalities hold by the cocycle property, and the last equality holds by the definition of U_T. To see the second claim, note that

$$\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) = \varphi(s,\theta_{-s}\omega) \circ \varphi(t-s,\theta_{-t}\omega)U(\theta_{-t}\omega) \subset \varphi(s,\theta_{-s}\omega)U(\theta_{-s}\omega),$$

where the inclusion holds thanks to the forward invariance of U. The proof is complete. ■

Theorem 3.12. *Assume that $\mathcal{D}$ is a universe which contains all the singleton sets consisting of a single deterministic point in X. Assume further that S is the global random attractor of φ in $\mathcal{D}$. Then there exists a Lyapunov function $L : \Omega \times X \to [0,1]$ satisfying the following:*

(i) *$x \mapsto L(\omega, x)$ is continuous for each $\omega \in \Omega$ and $\omega \mapsto L(\omega, x)$ is measurable for each $x \in X$.*

(ii) *$L(\omega, x) = 0$ when $x \in S(\omega)$ and $L(\omega, x) > 0$ when $x \in X \setminus S(\omega)$.*

(iii) *L is strictly decreasing along the orbits outside S, i.e., $L(\theta_t\omega, \varphi(t,\omega)x) < L(\omega, x)$ for $t > 0$ when $x \in X \setminus S(\omega)$.*

Proof. Assume that U is a forward invariant random closed neighborhood of S in X such that $\Omega_U = S$ a.s. Define

$$U_n(\omega) := \overline{\varphi(n, \theta_{-n}\omega)U(\theta_{-n}\omega)}, \quad n \in \mathbb{N}.$$

Then, by Lemma 3.11, U_n is also a forward invariant random closed set. Furthermore, $U_{n+1} \subset U_n$ and $\Omega_{U_n} = S$ a.s. Let

$$\bar{l}_n(\omega, x) := d(x, U_n(\omega))$$

and

$$l_n(\omega, x) := \sup_{t\geq 0} \bar{l}_n(\theta_t\omega, \varphi(t,\omega)x) = \sup_{t\geq 0} d(\varphi(t,\omega)x, U_n(\theta_t\omega)).$$

Then $l_n(\omega, x) > 0$ for $x \in X \setminus U_n(\omega)$, and $l_n(\omega, x) = 0$ for $x \in U_n(\omega)$ by the forward invariance of U_n. Furthermore, l_n is decreasing along orbits of φ. Actually, by the definition of $l_n(\omega, x)$, for $s \geq 0$,

$$\begin{aligned} l_n(\theta_s\omega, \varphi(s,\omega)x) &= \sup_{t\geq 0} \bar{l}_n(\theta_t \circ \theta_s\omega, \varphi(t, \theta_s\omega) \circ \varphi(s,\omega)x) \\ &= \sup_{t\geq 0} \bar{l}_n(\theta_{t+s}\omega, \varphi(t+s,\omega)x) \\ &= \sup_{t\geq s} \bar{l}_n(\theta_t\omega, \varphi(t,\omega)x) \\ &\leq \sup_{t\geq 0} \bar{l}_n(\theta_t\omega, \varphi(t,\omega)x) = l_n(\omega, x). \end{aligned} \tag{3.3}$$

Note that, by the forward invariance of U_n, for $0 \leq t \leq s$, we have $\varphi(s-t, \theta_t\omega)U_n(\theta_t\omega) \subset U_n(\theta_s\omega)$, so

$$d(\varphi(s,\omega)x, U_n(\theta_s\omega)) \leq d(\varphi(s,\omega)x, \varphi(s-t, \theta_t\omega)U_n(\theta_t\omega)). \tag{3.4}$$

On the other hand, by the continuity of the mapping $t \mapsto \varphi(t, \omega, x)$ for fixed (ω, x), we have

$$\lim_{s\searrow t} d(\varphi(s,\omega)x, \varphi(s-t, \theta_t\omega)U_n(\theta_t\omega)) = d(\varphi(t,\omega)x, U_n(\theta_t\omega)). \tag{3.5}$$

Thus, (3.4) and (3.5) imply that

$$l_n(\omega, x) := \sup_{t\in\mathbb{R}^+\cap\mathbb{Q}} d(\varphi(t,\omega)x, U_n(\theta_t\omega)), \tag{3.6}$$

so l_n is measurable with respect to $(\omega, x) \in \Omega \times X$.

For fixed ω and x we have from Lemma 3.8 that $\varphi(t,\omega)x \in \mathrm{int}U_n(\theta_t\omega)$ for some $t \geq 0$. By the continuity of φ with respect to x, there exists a neighborhood N_x of x such that $\varphi(t,\omega)N_x \subset \mathrm{int}U_n(\theta_t\omega)$. By the forward invariance of $\mathrm{int}U_n$ (note that since U_n is forward invariant, $\mathrm{int}U_n$ is forward invariant),

$$\varphi(s,\omega)N_x \subset \mathrm{int}U_n(\theta_s\omega) \qquad \text{for all } s \geq t.$$

It follows that, for any $y \in N_x$,

$$l_n(\omega, y) = \sup_{0\leq s\leq t} d(\varphi(s,\omega)y, U_n(\theta_s\omega)).$$

Therefore, for any $y \in N_x$, by the triangle inequality,

$$|l_n(\omega, x) - l_n(\omega, y)| \leq \sup_{0 \leq s \leq t} |\varphi(s, \omega)x - \varphi(s, \omega)y|,$$

which implies that $l_n(\omega, \cdot)$ is continuous at x.

Let

$$\tilde{l}_n(\omega, x) := \frac{l_n(\omega, x)}{l_n(\omega, x) + 1}.$$

Note that $\tilde{l}_n(\omega, x) = 0$ when $x \in U_n(\omega)$ and $l_n(\omega, x) > 0$ when $x \notin U_n(\omega)$. Furthermore, since $l_n \geq 0$ and the derivative of the function $x/(1+x)$ is positive, we have $\tilde{l}_n(\theta_t \omega, \varphi(t, \omega)x) \leq \tilde{l}_n(\omega, x)$ since l_n satisfies this property. Let

$$\hat{l}(\omega, x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \tilde{l}_n(\omega, x).$$

Since the sum is uniformly convergent, the mapping $x \mapsto \hat{l}(\omega, x)$ is continuous for fixed $\omega \in \Omega$; $\hat{l}(\theta_t \omega, \varphi(t, \omega)x) \leq \hat{l}(\omega, x)$ because each $\tilde{l}_n$ satisfies this property. Furthermore, $\hat{l}(\omega, x) = 0$ if and only if $l_n(\omega, x) = 0$ for each n, that is, $x \in \cap_{n=1}^{\infty} U_n(\omega) = S(\omega)$; hence $\hat{l}(\omega, x) > 0$ for $x \notin S(\omega)$. Now $\hat{l}$ satisfies all the properties needed except that it is decreasing but not necessarily strictly decreasing along orbits outside S. To this end, and similar to the arguments in [3, 29], let

$$L(\omega, x) := \frac{1}{2} \left[\hat{l}(\omega, x) + \int_0^{\infty} \mathrm{e}^{-t} \hat{l}(\theta_t \omega, \varphi(t, \omega)x) \mathrm{d}t \right].$$

Then it is not hard to check that L is continuous with respect to x, measurable with respect to ω, and $L(\omega, x) = 0$ for $x \in S(\omega)$, $L(\omega, x) > 0$ for $x \notin S(\omega)$ and $L(\theta_t \omega, \varphi(t, \omega)x) \leq L(\omega, x)$ for $t \geq 0$. We only need to check that L is strictly decreasing along the orbits outside S. If for some (ω, x) and $t_0 > 0$ we have that $L(\theta_{t_0} \omega, \varphi(t_0, \omega)x) = L(\omega, x)$, then by the monotonicity of $\hat{l}$ along the orbits of φ,

$$\hat{l}(\theta_s \omega, \varphi(s, \omega)x) = \hat{l}(\omega, x) > 0 \qquad \text{for all } 0 \leq s \leq t_0 \tag{3.7}$$

and

$$\hat{l}(\theta_{s+t_0} \omega, \varphi(s + t_0, \omega)x) = \hat{l}(\theta_s \omega, \varphi(s, \omega)x) \qquad \text{for Lebesgue almost all } s \geq 0.$$

Hence

$$\hat{l}(\theta_{nt_0+s} \omega, \varphi(nt_0 + s, \omega)x) = \tilde{l}(\theta_s \omega, \varphi(s, \omega)x) \tag{3.8}$$

for all $n \in \mathbb{N}$ and for Lebesgue almost all $s \geq 0$. There exists a $\tau \geq 0$ such that (3.7) and (3.8) hold, i.e.,

$$\hat{l}(\theta_{nt_0+\tau} \omega, \varphi(nt_0 + \tau, \omega)x) = \hat{l}(\omega, x) > 0 \qquad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

By Lemma 3.8, for each $k \in \mathbb{N}$, $\varphi(nt_0 + \tau, \omega)x \in \mathrm{int} U_k(\theta_{nt_0+\tau} \omega)$ when n is large enough. By the standard diagonal method, there exists a subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\varphi(n_k t_0 + \tau, \omega)x) \in \mathrm{int} U_k(\theta_{n_k t_0+\tau} \omega)$ for each $k \in \mathbb{N}$, so

$$\lim_{k \to \infty} \hat{l}(\theta_{n_k t_0+\tau} \omega, \varphi(n_k t_0 + \tau, \omega)x) = 0,$$

a contradiction to (3.9). The proof is complete. ■

4. Morse decomposition for random dynamical systems.

Recall that we use S to denote the global random attractor of the given random dynamical system φ. By Remark 3.5 and Lemma 3.6, for any point (random variable) in S, there exists a backward orbit lying in S through this point (random variable). Afterward, when we say backward orbits, we refer to those lying in S unless otherwise stated (since there may be backward orbits not lying in S but lying in the entire phase space X).

Definition 4.1. *An invariant random compact set $A \subset S$ is called a* (local) attractor *if there exists a random closed neighborhood U of A in X such that $\Omega_U(\omega) = A(\omega)$. Here A attracts in the universe given by taking all the subsets of U. The* basin of attraction of A *is defined by*

$$B(A)(\omega) := \{x \in X | \varphi(t, \omega)x \in \mathrm{int}U(\theta_t\omega) \text{ for some } t \geq 0\}$$

and the dual repeller R of A *is defined by*

$$R(\omega) := S(\omega) \backslash B(A)(\omega).$$

(A, R) is called an attractor-repeller pair in S. *We will denote $B(A; S) := B(A) \cap S$ in what follows.*

Remark 4.2. (i) Note that by Lemma 2.11 and Remark 3.9, without loss of generality, we can assume that U in Definition 4.1 is forward invariant.

(ii) The basin of attraction $B(A)$ of A is independent of U, and this is why we use the notation $B(A)$ instead of $B(A, U)$ in Definition 4.1. Indeed, by [27, Lemma 3.2], the basin of attraction is independent of U when the entire state space X is compact; when A is compact and attracting, we can show that the basin of attraction of A is still independent of U even if X loses compactness; see the forthcoming Lemma 4.8 for details. We also remark that when X is not compact and A is not compact or attracting, the basin may depend on the neighborhood U; see [22, 23] for details.

Remark 4.3. (i) By the definition of local attractor, it is clear that the universe, in which the local attractor attracts, is not unique since different U may determine the same local attractor. But a local attractor has a maximal universe which contains all the subsets of $B(A)$ that are attracted by the local attractor; see the forthcoming Lemma 4.8. In what follows, if we do not write explicitly the universe of a local attractor, then the maximal universe is assumed. Furthermore, by Lemma 4.6 below, the maximal universe of a local attractor contains all the random compact sets in $B(A)$.

(ii) Although it may seem that the definition of local attractor in Definition 4.1 depends on the global attractor S, this is not the case. Indeed, an invariant random compact set A is a local attractor if it is the omega-limit set of one of its neighborhoods. But in this section we are mainly concerned with Morse decomposition of the global random attractor, so we assume the existence of global random attractor S from the beginning of this section. Note that S is the largest invariant random compact set (see Remark 2.10 (i)), so any local attractor is contained in S. That is why in Definition 4.1 we write $A \subset S$. Note also that a local attractor can be regarded as the global attractor in its maximal universe; conversely, by Definition 2.9, the global attractor S can be regarded as a local attractor since it pullback attracts a neighborhood U of itself.

Remark 4.4. Note that the above definition of attractor-repeller pair is slightly different from that in [29]: here the attractor A attracts a random neighborhood of itself in X; there the attractor A attracts a random neighborhood in S, like the definition in [13] and [33] for the deterministic case. A definition similar to ours is also adopted in [1], where the authors show that both definitions actually coincide for deterministic dynamical systems. But we do not know whether or not the two definitions coincide in the random case.

The following lemmas will be used in what follows, so we list them for the convenience of the reader.

Lemma 4.5 (see [29, Lemma 4.3]). *Assume that (A, R) is an attractor-repeller pair in S. Then A, $B(A; S)$, and R are invariant random sets.*

Lemma 4.6 (see [30, Lemma 5.2]). *Assume that A_1 and A_2 are two random attractors with basins of attraction $B(A_1)$ and $B(A_2)$, respectively. Assume that D is a random compact set satisfying $D \subset B(A_1) \cup B(A_2)$ a.s. Then $A_1 \cup A_2$ pullback attracts D.*

Remark 4.7. Denote $B^*(R; S)(\omega) := S(\omega) \setminus A(\omega)$ for each ω.

(i) Similar to the proof of [29, Lemma 4.3 (ii)], we obtain that if $D \subset S$ is forward invariant, then $S \setminus D$ is backward invariant. Furthermore, $S \setminus D$ is *strongly backward invariant* in the sense that *any* backward orbit through the point (or the random variable) on S lies on $S \setminus D$.

(ii) Observe that, in contrast to the random flow case, the complement of a backward invariant set need not be forward invariant. Particularly, $B^*(R; S)$ is not necessarily forward invariant since the forward orbit through a point in $B^*(R; S)$ may enter A.

(iii) Since A is forward invariant, $B^*(R; S)$ is strongly backward invariant. Similarly, the random set $S \setminus (A \cup R)$ is strongly backward invariant, but not necessarily forward invariant. Note that the forward orbit through the point in $S \setminus (A \cup R)$ can enter A, but never enter R.

(iv) Note that if a random set $D \subset S$ is strongly backward invariant in the above sense, then $S \setminus D$ is forward invariant. That is, the reason that the complement of a backward invariant set is not necessarily forward invariant lies in that the set is not strongly backward invariant.

Lemma 4.8. *Assume that A is an invariant random compact set in X and U is a closed forward invariant random neighborhood of A such that $\Omega_U = A$ a.s. Then the basin of attraction $B(A)$ of A, defined in Definition 4.1, is independent of U.*

Proof. Assume that $\tilde{U}$ is also a closed forward invariant random neighborhood of A with $\Omega_{\tilde{U}} = A$. First, since $\tilde{U}$ is attracted by A and U is a closed forward invariant neighborhood of A, by Lemma 3.8, there exists a random variable $t_1 \geq 0$ such that

$$\varphi(t, \omega)\tilde{U}(\omega) \subset U(\theta_t\omega) \quad \text{for all } t \geq t_1(\omega).$$

We use $B(A, U)$ and $B(A, \tilde{U})$ to denote the basins of attraction of A with respect to U and $\tilde{U}$, respectively. That is,

$$B(A, U)(\omega) := \{x \in X \,|\, \varphi(t, \omega)x \in \mathrm{int}U(\theta_t\omega) \text{ for some } t \geq 0\}$$

and

$$B(A, \tilde{U})(\omega) := \{x \in X \,|\, \varphi(t, \omega)x \in \mathrm{int}\tilde{U}(\theta_t\omega) \text{ for some } t \geq 0\}.$$

For arbitrary $x \in B(A, \tilde{U})(\omega)$, by the definition of $B(A, \tilde{U})$, there exists $t_0 \geq 0$ such that $\varphi(t_0, \omega)x \in \tilde{U}(\theta_{t_0}\omega)$. Then, when $s \geq t_1(\theta_{t_0}\omega)$, we have

$$\varphi(s + t_0, \omega)x = \varphi(s, \theta_{t_0}\omega)\varphi(t_0, \omega)x \subset \varphi(s, \theta_{t_0}\omega)\tilde{U}(\theta_{t_0}\omega) \subset U(\theta_{s+t_0}\omega),$$

i.e., $x \in B(A,U)(\omega)$. Hence $B(A,\tilde{U})(\omega) \subset B(A,U)(\omega)$. In the same way we obtain $B(A,U)(\omega) \subset B(A,\tilde{U})(\omega)$. This completes the proof. ■

Definition 4.9. *Assume that x is a random variable in S, and σ is an entire orbit through x. Then the* omega-limit set Ω_x of x *and the* alpha-limit set $\Omega_x^{*,\sigma}$ of x along the entire orbit σ *are defined by*

$$\Omega_x(\omega) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega)}$$

and

$$\Omega_x^{*,\sigma}(\omega) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi^\sigma(-t,\theta_t\omega)x(\theta_t\omega)},$$

respectively.

Remark 4.10. (i) Clearly Ω_x is actually the omega-limit set of the random set $\{x\}$. By definition, a point $y \in \Omega_x(\omega)$ (respectively, $y \in \Omega_x^{*,\sigma}(\omega)$) if and only if there exist sequences $t_n \to +\infty$ (respectively, $t_n \to -\infty$) and $y_n = \varphi^\sigma(t_n,\theta_{-t_n}\omega)x(\theta_{-t_n}\omega)$ such that $y_n \to y$ as $n \to +\infty$.

(ii) The above definition is the same as that in [29, Definition 4.3], but the notation there is a little confusing. So we write it more precisely here.

For later use, we recall the following result from [29].

Lemma 4.11 (see [29, Lemma 4.5]). *Assume that $x \in S$ is a random variable with σ being an entire orbit through x, and (A,R) is a random attractor-repeller pair on S. Then the following statements hold:*

(i) *If $x \in R$ a.s., then $\Omega_x \subset R$ and $\Omega_x^{*,\sigma} \subset R$ a.s.*

(ii) *If $x \in B(A;S)\backslash A$ a.s., then $\Omega_x \subset A$ and $\Omega_x^{*,\sigma} \subset R$ a.s.*

(iii) *If $x \in A$ a.s., then $\Omega_x \subset A$ a.s.; if $\Omega_x^{*,\sigma} \subset A$ a.s., then σ lies in A a.s.; i.e., for arbitrary $t \in \mathbb{R}$, we have $\sigma_t \subset A$ a.s.*

(iv) *If $x \in B(A;S)$ a.s., then $\Omega_x \subset A$ a.s.; if $x \in B^*(R;S)$ a.s., then $\Omega_x^{*,\sigma} \subset R$ a.s.*

Lemma 4.12. *Assume that $\sigma : \mathbb{R} \to \mathcal{M}$ is an entire orbit lying in S through the random variable x. Then, for all $\omega \in \Omega$, we have*

$$\Omega_{\sigma_t}^{*,\sigma}(\omega) = \Omega_{\sigma_\tau}^{*,\sigma}(\theta_{t-\tau}\omega) \quad \text{and} \quad \Omega_{\sigma_t}(\omega) = \Omega_{\sigma_\tau}(\theta_{t-\tau}\omega) \qquad \text{for all } t,\tau \in \mathbb{R}.$$

In particular, $\Omega_{\sigma_t}^{*,\sigma}(\omega) = \Omega_x^{*,\sigma}(\theta_t\omega)$ and $\Omega_{\sigma_t}(\omega) = \Omega_x(\theta_t\omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$.

Proof. Note that, for all $\omega \in \Omega$, we have $\sigma_0(\omega) = x(\omega)$, $\sigma_t(\omega) = \varphi(t-\tau,\theta_\tau\omega)\sigma_\tau(\omega)$ for $t \geq \tau$, and $\sigma_t(\omega) = \varphi^\sigma(t-\tau,\theta_\tau\omega)\sigma_\tau(\omega)$ for $t \leq \tau$. For notational simplicity, we just assume that $\tau = 0$ and $t \geq \tau$, and the general case can be proved similarly. Therefore,

$$\begin{aligned}
\Omega_{\sigma_t}^{*,\sigma}(\omega) &= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi^\sigma(-s,\theta_s\omega)\sigma_t(\theta_s\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi^\sigma(-s,\theta_s\omega)\varphi(t,\theta_s\omega)x(\theta_s\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi^\sigma(-(s-t),\theta_s\omega)x(\theta_s\omega)} \\
&= \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \varphi^\sigma(-(s-t),\theta_{s-t}\circ\theta_t\omega)x(\theta_{s-t}\circ\theta_t\omega)}
\end{aligned}$$

$$= \Omega^{*,\sigma}_{\sigma_0}(\theta_t\omega),$$

where the first and the last equalities hold by the definition of the alpha-limit, and the third one by the cocycle property. The corresponding result for the omega-limit is proved similarly, so we omit the details. ■

Lemma 4.13. *Assume that S is the global random attractor in universe $\mathcal{D}$ and that (A, R) is an attractor-repeller pair in S. Then, for any random variable $x \in X \setminus (A \cup R)$ a.s. and the associated singleton random set $\{x\} \in \mathcal{D}$, we have*

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega), A(\omega) \cup R(\omega)) = 0 \tag{4.1}$$

a.s. In particular,

$$\lim_{t\to\infty} d(\varphi(t, \omega)x(\omega), A(\theta_t\omega) \cup R(\theta_t\omega)) = 0$$

in probability.

Proof. Assume that U is a random closed neighborhood of A in X, disjoint from R, such that $\Omega_U = A$ a.s. By Lemma 2.11, we may assume that U is forward invariant. Note that

$$B(A)(\omega) = \{x \in X | \varphi(t, \omega)x \in \mathrm{int}U(\theta_t\omega) \text{ for some } t \geq 0\},$$

and by the definition of attractor-repeller, we have

$$B(A) \cap S = S \setminus R \qquad \text{a.s.}$$

By Lemma 4.6, for any random compact set $D \subset B(A)$, A pullback attracts D. In particular, for any random variable $x \in B(A)$, A pullback attracts x.

For any random variable $y \in X \setminus B(A)$, by the definition of $B(A)$, we obtain that the forward orbit of y never enters U. That is, for any $t \geq 0$, we have

$$\begin{aligned} &\mathbb{P}\{\omega \in \Omega \,|\, d(\varphi(t, \theta_{-t}\omega)y(\theta_{-t}\omega), U(\omega)) = 0\} \\ &= \mathbb{P}\{\omega \in \Omega \,|\, d(\varphi(t, \omega)y(\omega), U(\theta_t\omega)) = 0\} = 0 \end{aligned}$$

by the measure preserving property of θ_t. Noting that $A = \Omega_U \subset U$, we have

$$\Omega_y \cap \Omega_U = \emptyset \qquad \text{a.s.}$$

On the other hand, note that the random variable y is attracted by the global attractor S, so Ω_y is an invariant random compact set, and $\Omega_y \subset S$ a.s. Since A pullback attracts any random compact set in $B(A)$, this enforces that $\Omega_y \subset R$ a.s. As y is attracted by Ω_y, y is attracted by R.

Now for any random variable $x \in X \setminus (A \cup R)$ with $\{x\} \in \mathcal{D}$, choose random variables $x_1 \in B(A)$ a.s. and $x_2 \in X \setminus B(A)$ a.s. such that x_2 is attracted by R and that

$$x(\omega) = x_1(\omega) \text{ for } \omega \in \Omega_1 \quad \text{and} \quad x(\omega) = x_2(\omega) \text{ for } \omega \in \Omega_2,$$

where $\Omega_1 := \{\omega | x(\omega) \in B(A)(\omega)\}$ and $\Omega_2 := \{\omega | x(\omega) \in X \setminus B(A)(\omega)\}$. Then, by Lemma 4.6, it follows that

$$\Omega_x \subset \Omega_{x_1 \cup x_2} \subset A \cup R.$$

That is, (4.1) holds. The proof is complete. ■

Definition 4.14. *Assume that* (A_i, R_i), $i = 1, \dots, n$, *are attractor-repeller pairs of* φ *on* S *with*

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = S \quad \text{and} \quad S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset.$$

Then the family $D = \{M_i\}_{i=1}^n$ *of invariant random compact sets, defined by*

$$M_i = A_i \cap R_{i-1}, \ 1 \le i \le n,$$

is called a random Morse decomposition *of* S, *and each* M_i *is called a* Morse set. *If* D *is a Morse decomposition,* M_D *is defined to be* $\bigcup_{i=1}^n M_i$.

The following important result describes the internal asymptotic dynamics between the invariant sets in a Morse decomposition of a random attractor S.

Theorem 4.15. *Assume that* $D = \{M_i\}_{i=1}^n$ *is a Morse decomposition of the global attractor* S, *determined by attractor-repeller pairs* (A_i, R_i), $i = 1, \dots, n$. *Then* M_D *determines the limiting behavior of* φ *on* S. *More precisely, we have the following:*

(i) *For any random variable* x *in* S, *there is an entire orbit* σ *through* x *such that* $\Omega_x \subset M_D$ *and* $\Omega_x^{*,\sigma} \subset M_D$ *a.s.*

(ii) *If* σ *is an entire orbit through the random variable* x *satisfying that* $\Omega_x \subset M_p$ *a.s. and* $\Omega_x^{*,\sigma} \subset M_q$ *a.s. for some* $1 \le p, q \le n$, *then* $p \le q$. *Moreover,* $p = q$ *if and only if* σ *lies on* M_p.

(iii) *For each* $1 \le k \le n$, *there exists a neighborhood* U_k *of* $\cup_{i=1}^k M_i$ *in* X *and a neighborhood* V_k *of an invariant random compact set* A_k^* *in* X, *disjoint from* A_k, *such that* $U_k \cap V_k = \emptyset$ *and* $U_k \cup V_k$ *is a random neighborhood of* S *in* X. *Furthermore,* $\Omega_x \subset \cup_{i=1}^k M_i$ *for any random variable* x *in* U_k, $\Omega_x \subset A_k^*$ *for any random variable* x *in* $\overline{V_k} \setminus S$, *and* $\Omega_x^{*,\sigma} \subset A_k^*$ *for any random variable* x *in* $\overline{V_k} \cap S$ *with* σ *being any entire orbit through it.*

(iv) *The attractors* $A_1, \dots, A_n$ *are uniquely determined by*

$$A_k(\omega) = \{x(\omega) \in X \mid x \in \mathcal{M}_k\}, \qquad k = 1, \dots, n. \tag{4.2}$$

for almost all $\omega \in \Omega$, *where*

$$\mathcal{M}_k := \left\{ x \in \mathcal{M} \,\middle|\, \begin{array}{l} x \in S \text{ a.s. and there exists an entire orbit } \sigma \\ \text{through } x \text{ such that } \Omega_x^{*,\sigma} \subset \cup_{i=1}^k M_i \end{array} \right\}. \tag{4.3}$$

(v) *If* $\sigma_1, \dots, \sigma_l$ *are* l *entire orbits through the random variables* $x_1, \dots, x_l$, *respectively, such that for some* $1 \le j_0, \dots, j_l \le n$, $\Omega_{x_k} \subset M_{j_{k-1}}$ *and* $\Omega_{x_k}^{*,\sigma_k} \subset M_{j_k}$ *for* $k = 1, \dots, l$, *then* $j_0 \le j_l$. *Moreover,* $j_0 < j_l$ *if and only if* σ_k *does not lie on* M_D *with positive probability for some* k, *and* $j_0 = \cdots = j_l$ *otherwise.*

Proof. Note that our definition of attractor-repeller pairs is slightly stronger than that in [29], so (i), (ii), and (v) have been proved in [29, Theorem 5]; we need to verify (iii) and (iv). First choose a neighborhood U of S in X with $\Omega_U = S$ and a neighborhood $U_k \subset U$ of A_k in X with $\Omega_{U_k} = A_k$. Let $V_k := (S \setminus U_k) \cup (U \setminus B(A))$, $A_k^* := R_k$. Then $U_k \cap V_k = \emptyset$ and $U_k \cup V_k = U$. Noting that $\cup_{i=1}^k M_i \subset A_k$, then by (i) and the proofs of Lemmas 4.13 and 4.11 (iv), we obtain that (iii) holds.

Note again that the definitions of attractor-repeller pair and Morse decomposition are slightly stronger than that in [29]. Even in that case, we can prove (iv). Actually, for fixed

k, $A_1, \ldots, A_k$ are random local attractors in $\mathcal{A}_k$ with dual repellers given by $R_1 \cap \mathcal{A}_k, \ldots, R_k \cap \mathcal{A}_k$, and it follows that the associated Morse decomposition of $\mathcal{A}_k$ induced by the filtration of attractors $A_1, \ldots, A_k$ is given by $M_i = A_i \cap (R_{i-1} \cap \mathcal{A}_k) = M_i$ for $i = 1, \ldots, k$. That is, $\{M_1, \ldots, M_k\}$ is a Morse decomposition of $\mathcal{A}_k$. For any random variable $x \in \mathcal{A}_k$, by the invariance of $\mathcal{A}_k$ there exists an entire orbit σ through x on $\mathcal{A}_k$. By (i), $\Omega_x^{*,\sigma} \subset M_1 \cup \cdots \cup M_k$; i.e., $\mathcal{A}_k$ is a subset of the right-hand side of (4.2). Since A_k is an attractor in S, for any random variable $x \in S \backslash A_k$ a.s., we have $\Omega_x^{*,\sigma} \subset R_k$ by Lemma 4.11 (iv), hence $\Omega_x^{*,\sigma} \cap (M_1 \cup \cdots \cup M_k) = \emptyset$ a.s. So the right-hand side of (4.2) is a subset of A_k, and (iv) is proved. ■

Remark 4.16. The random Morse decomposition defined in Definition 4.14 is the random version of the original definition of Morse decomposition due to Conley [13]. In [18], Franzosa proposed an alternative definition of Morse decomposition like Theorem 4.15 (ii), which is adopted by many authors; see [31] for details. Indeed, Conley [13, page 40] had shown that both definitions are equivalent. But for random Morse decomposition, we do not know whether or not the two definitions are equivalent.

A natural question that comes to mind is what conditions can characterize a Morse decomposition for RDSs. The following theorem shows that conditions (i)–(iv) in Theorem 4.15 are actually sufficient for that end, so that we introduce the following concept.

Definition 4.17. *Assume that S is the random global attractor of φ in universe $\mathcal{D}$ and that $D = \{M_i\}_{i=1}^n$ is a family of invariant random compact sets in S. Then the semiflow φ is said to be* dynamically gradient *(with respect to D) if the following conditions hold:*

(g1) *For any random variable x in S, there is an entire orbit σ through x such that $\Omega_x \subset M_D$ and $\Omega_x^{*,\sigma} \subset M_D$ a.s.*

(g2) *If σ is an entire orbit through the random variable x satisfying that $\Omega_x \subset M_p$ a.s. and $\Omega_x^{*,\sigma} \subset M_q$ a.s. for some $1 \le p, q \le n$, then $p \le q$. Moreover, $p = q$ if and only if σ lies on M_p.*

(g3) *Let*

$$A_k(\omega) := \{x(\omega) \in X \mid x \in \mathcal{M}_k\}, \qquad k = 1, \ldots, n, \tag{4.4}$$

recalling that $\mathcal{M}_k$ is defined in (4.3). Then A_k is a random compact set for each $k = 1, 2, \ldots, n$.

(g4) *For each $1 \le k \le n$, there exists a neighborhood U_k of $\cup_{i=1}^k M_i$ in X and a neighborhood V_k of an invariant random compact set A_k^* in X, disjoint from A_k, such that $U_k \cap V_k = \emptyset$, $U_k \cup V_k$ is a random neighborhood of S in X, and $\overline{U_k \cup V_k} \in \mathcal{D}$. Furthermore, $\Omega_x \subset \cup_{i=1}^k M_i$ for any random variable x in U_k, $\Omega_x \subset A_k^*$ for any random variable x in $\overline{V_k} \backslash S$, and $\Omega_x^{*,\sigma} \subset A_k^*$ for any random variable x in $\overline{V_k} \cap S$ with σ being any entire orbit through it.*

Theorem 4.18. *Assume that $M_1, \ldots, M_n$ are disjoint invariant random compact sets in S and the RDS φ is dynamically gradient with respect to $M_1, \ldots, M_n$. Then $\{M_1, \ldots, M_n\}$ is a Morse decomposition for S with A_k being the associated increasing family of local attractors.*

Proof. It suffices to verify that A_k given by (4.4), for $k = 1, \ldots, n$, is actually an attractor in X with dual repeller R_k and $M_k = A_k \cap R_{k-1}$. First we show that A_k defined by (4.4) is invariant. For an arbitrary random variable $x \in A_k$, by the definition of A_k, there exists an entire orbit σ through x such that $\Omega_x^{*,\sigma} \subset M_1 \cup \cdots \cup M_k$ a.s. Note that, for any given $t \in \mathbb{R}$, σ is an entire orbit passing through the random variable σ_t at time t. On the other hand, by Lemma 4.12, $\Omega_{\sigma_t}^{*,\sigma}(\cdot) = \Omega_{\sigma_0}^{*,\sigma}(\theta_t \cdot)$ and hence is a subset of $(M_1 \cup \cdots \cup M_k)(\theta_t \cdot)$ a.s. By the definition (4.4) of A_k, $\sigma_t(\cdot) \in A_k(\theta_t \cdot)$, so A_k is an invariant set.

To show that A_k is an attractor in X, we need to show that A_k attracts a neighborhood of itself in X. First, for a given random variable $y_0 \in A_k$ a.s., there exists an entire orbit σ through it with $\Omega^{*,\sigma}_{y_0} \subset \cup_{i=1}^k M_i$. By (g1) and (g2), it follows that $\Omega_{y_0} \subset \cup_{i=1}^k M_i$ a.s. On the other hand, for any random variable $y \in \overline{V_k} \cap S$, we have $\Omega^{*,\sigma}_y \subset A_k^*$ for any entire orbit σ through y and $\Omega_x \subset A_k^*$ for any random variable $x \in \overline{V_k} \setminus S$. This implies that $y_0 \in U_k$ a.s. That is, $A_k \subset U_k$ a.s.

U_k is a neighborhood of A_k in X. Actually, if U_k is not a neighborhood of A_k a.s., then there exists a random variable $x \in A_k$ a.s. and meantime $x \in \overline{V_k}$ with positive probability. Since A_k is an invariant random compact set, by Lemma 3.6, there is an entire orbit σ through x lying in A_k. By the measure preserving property of θ_t, we have

$$\lim_{t\to\infty} d(\varphi^\sigma(-t,\omega)x(\omega), A_k(\theta_{-t}\omega)) = 0$$

in probability. Similarly, if $x \in \overline{V_k} \cap S$ with positive probability, then, by the property of V_k and the measure preserving property of θ_t, we have

$$\lim_{t\to\infty} d(\varphi^\sigma(-t,\omega)x(\omega), A_k^*(\theta_{-t}\omega)) = 0$$

with positive probability. This is a contradiction since $A_k \cap A_k^* = \emptyset$ a.s., recalling that $U_k \cap V_k = \emptyset$ and V_k is a neighborhood of A_k^*. If $x \in \overline{V_k} \setminus S$ with positive probability, then by the property of V_k we have

$$\lim_{t\to\infty} d(\varphi(t,\omega)x(\omega), A_k^*(\theta_t\omega)) = 0$$

with positive probability. This is a contradiction because A_k attracts x in probability, and $A_k \cap A_k^* = \emptyset$ a.s. Therefore, U_k is a neighborhood of A_k a.s. in X.

Furthermore, U_k is pullback attracted by A_k, so A_k is an attractor in X. If not, then $\Omega_{U_k} \setminus A_k \neq \emptyset$ a.s. (Note that since Ω_{U_k} and A are invariant sets, if $\mathbb{P}$ is ergodic under θ_t, this naturally holds; if $\mathbb{P}$ is not ergodic under θ_t, then $\Omega_{U_k} \not\subset A$ holds on at least one ergodic component, and we may consider the problem on the ergodic component.) Taking $y_0 \in \Omega_{U_k}(\omega_0) \setminus A_k(\omega_0)$, by the definition of omega-limit set, there exist sequences $t_m \to \infty$ as $m \to \infty$ and $y_m \in U_k(\theta_{-m}\omega_0)$ such that $\varphi(t_m, \theta_{-t_m}\omega_0)y_m \to y_0$ as $m \to \infty$. Choose a random variable $\tilde{y} \in U_k$ a.s. such that $\tilde{y}(\theta_{-t_m}\omega_0) = y_m$. Note that $\tilde{y} \in U_k$ a.s. implies that $\Omega_{\tilde{y}} \subset \cup_{i=1}^k M_i$ a.s. by (g4); then, by the definition of omega-limit sets, $y_0 \in \Omega_{\tilde{y}}(\omega_0) \subset \cup_{i=1}^k M_i(\omega_0) \subset A_k(\omega_0)$, which is a contradiction.

Next we show that $M_{k+1} = A_{k+1} \cap R_k$ a.s. with R_k being the dual repeller of A_k, hence completing the proof. Since (A_k, R_k) is an attractor-repeller pair in S, R_k is the maximal invariant random compact set in S disjoint from A_k. That is, $M_{k+1} \subset R_k$ a.s. Therefore, $M_{k+1} \subset A_{k+1} \cap R_k$ a.s.

For an arbitrary random variable $x \in A_{k+1} \cap R_k$, there exists an entire orbit σ such that $\Omega^{*,\sigma}_x \subset M_1 \cup \cdots \cup M_{k+1}$. Since $x \in R_k$ we have $\Omega_x \subset R_k$ by Lemma 4.11 (i). Note also that $\Omega_x \cap (M_1 \cup \cdots \cup M_k) = \emptyset$ since $M_1 \cup \cdots \cup M_k \subset A_k$. By (g1),

$$\Omega_x \subset M_{k+1} \cup \cdots \cup M_n \qquad \text{a.s.} \tag{4.5}$$

Note that $\Omega_x^{*,\sigma} \subset R_k$ a.s. by Lemma 4.11 (i), so

$$\Omega_x^{*,\sigma} \subset (M_1 \cup \cdots \cup M_{k+1}) \cap R_k = M_{k+1} \cap R_k = M_{k+1} \subset A_{k+1} \qquad \text{a.s.} \tag{4.6}$$

By Lemma 4.11 (iii), σ lies on A_{k+1} a.s. In particular, $\Omega_x \subset A_{k+1}$ a.s. Hence, by (4.5),

$$\Omega_x \subset (M_{k+1} \cup \cdots \cup M_n) \cap A_{k+1} = M_{k+1} \qquad \text{a.s.},$$

noting that A_{k+1} is disjoint from $M_{k+2} \cup \cdots \cup M_n$ by the definition (4.4) of A_k. This fact, together with (4.6) and (g2), implies that σ lies on M_{k+1} a.s. In particular, $x \in M_{k+1}$ a.s. That is, $A_{k+1} \cap R_k \subset M_{k+1}$ a.s. Therefore, $M_{k+1} = A_{k+1} \cap R_k$ a.s. The proof is complete. ■

Remark 4.19. Note that property (i) in Theorem 4.15 is much weaker than its deterministic counterpart. Note that, for any entire orbit σ, we have $\Omega_x \subset M_i$ and $\Omega_x^{*,\sigma} \subset M_j$ for some i, j in the deterministic case. This property in the deterministic case produces a partial order $\preccurlyeq$ among the invariant sets $M_i, i = 1, \ldots, n$: $M_i \preccurlyeq M_j$ if $\Omega_x \subset M_i$ and $\Omega_x^{*,\sigma} \subset M_j$ for some entire orbit σ. However, for property (i) in the random case, it cannot produce any partial order among $M_i, i = 1, \ldots, n$. The property (ii) in Theorem 4.15 is similar: it is also much weaker than that in the deterministic case and cannot determine any order among $M_i, i = 1, \ldots, n$, if the entire orbit satisfying the condition (ii) is not known a priori. In the deterministic case, the property (ii) always holds for any entire orbits, so it is simpler.

Remark 4.20. Again a natural question arises: can properties (i) and (ii) of a Morse decomposition in Theorem 4.15 be improved like in the deterministic case pointed out in Remark 4.19? Unfortunately, the answer is no. This can be seen from a very simple observation. Assume that σ_1 and σ_2 are entire orbits through the random variables x and y, respectively, with $\Omega_x \subset M_i$ and $\Omega_y \subset M_j$ a.s. for different i and j. Construct a new random variable $z(\omega) = x(\omega)$ for $\omega \in \Omega_1$ and $z(\omega) = y(\omega)$ for $\omega \in \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2$; then Ω_z can be contained by neither M_i nor M_j a.s., nor by other M_k's. This holds similarly for the alpha-limit even if we consider only the random flow case instead of the random semiflow case.

Remark 4.21. It seems a little artificial that, to characterize a Morse decomposition of an invariant random compact set, we need the condition (g4). Actually this condition is necessary. We know well that to characterize a Morse decomposition, we need to determine a partial order among the given disjoint invariant sets $M_i, i = 1, \ldots, n$. But note that conditions (g1) and (g2) are not enough; see Remark 4.19. Now, condition (g4) induces a partial order to obtain the Morse decomposition.

5. Lyapunov functions for Morse decompositions. In this section, we consider the relation between Lyapunov functions and Morse decompositions. First, let us prepare some lemmas for later use.

Lemma 5.1. *Assume that S is the global random attractor of φ in universe $\mathcal{D}$ and that (A, R) is an attractor-repeller pair in S. Then, for any random neighborhood V of R in $X \setminus B(A)$ with $V \in \mathcal{D}$, we have $\Omega_V = R$ a.s.*

Proof. Take a random neighborhood $V \subset X \setminus B(A)$ of R in $X \setminus B(A)$ with $V \in \mathcal{D}$. Then $\Omega_V \subset S$ a.s. since S is the global attractor in $\mathcal{D}$. For arbitrary $x \in X \setminus B(A)(\omega)$, by the definition of $B(A)$, we have $\varphi(t, \omega)x \in X \setminus B(A)(\theta_t\omega)$ for any $t \geq 0$. That is, $X \setminus B(A)$ is a forward invariant random closed set, noting that $B(A)$ is a random open set since $B(A)(\omega) =$

$\cup_{t\geq 0}\varphi(t,\omega)^{-1}\mathrm{int}U(\theta_t\omega)$, where $\varphi(t,\omega)^{-1}\mathrm{int}U(\theta_t\omega)$ denotes the preimage of int$U$ under φ. The measurability of $B(A)$ follows from the same proof as that in [12, Proposition 1.5.1].

By the definition of omega-limit sets, it follows that $\Omega_V \subset \Omega_{X\setminus B(A)} \subset X\setminus B(A)$. Therefore, $\Omega_V \subset S\cap(X\setminus B(A)) = R$ a.s. The other inclusion is easy to check. The proof is complete. ■

Remark 5.2. By Lemma 2.11, the neighborhood V in the above lemma can be chosen forward invariant.

By Lemmas 5.1 and 3.8, we have the following lemma.

Lemma 5.3. *Assume that V is a forward invariant neighborhood of R in $X \setminus B(A)$ with $\Omega_V = R$ a.s. Then, for any $D \in \mathcal{D}$ with $D \subset X \setminus B(A)$ a.s., there exists a random variable $T_D \geq 0$ such that, for almost all $\omega \in \Omega$,*

$$\varphi(t,\omega)D(\omega) \subset V(\theta_t\omega) \quad \textit{for all } t \geq T_D(\omega).$$

To construct continuous Lyapunov functions for attractor-repeller pairs, we need the following assumption.

(H) Given (A,R) being an attractor-repeller pair on the global attractor S, assume that there are a forward invariant random closed neighborhood U of A and a forward invariant random closed neighborhood V of R in $X \setminus B(A)$ such that

$$\mathrm{dist}_{\min}(U(\omega),V(\omega)) \geq \frac{1}{2}\mathrm{dist}_{\min}(A(\omega),R(\omega)) \quad \text{for all } \omega\in\Omega, \tag{5.1}$$

where $\mathrm{dist}_{\min}(A,B) := \inf_{x\in A}\inf_{y\in B} d(x,y)$.

Remark 5.4. Note that since A is an attractor, there is a forward invariant neighborhood U of A, disjoint from R such that $\Omega_U = A$ a.s. By Lemma 3.11, we have $\lim_{n\to\infty} d(U_n(\omega)|A(\omega)) = 0$ a.s. with each $U_n(\omega) = \overline{\varphi(n,\theta_{-n}\omega)U(\theta_{-n}\omega)}$ being a forward invariant random closed set containing A. By Lemma 5.1, a similar result holds for a forward invariant neighborhood V of R in $X \setminus B(A)$ with V_n defined similarly.

Note that $\mathrm{dist}_{\min}(A(\omega),R(\omega)) > 0$ for all $\omega\in\Omega$ since A and R are compact. It follows that for any $\epsilon > 0$ there is N such that

$$\mathbb{P}\left\{\omega\in\Omega \mid \mathrm{dist}_{\min}(U_n(\omega),V_n(\omega)) \geq \frac{1}{2}\mathrm{dist}_{\min}(A(\omega),R(\omega))\right\} > 1-\epsilon \quad \text{for } n\geq N.$$

Proposition 5.5. *Assume that (A,R) is an attractor-repeller pair in S, and that hypothesis* (H) *holds. Then there exists a Lyapunov function $L:\Omega\times X\to[0,1]$ satisfying that*

(i) *$x\mapsto L(\omega,x)$ is continuous for each $\omega\in\Omega$ and $\omega\mapsto L(\omega,x)$ is measurable for each $x\in X$;*

(ii) *$L(\omega,x)=0$ when $x\in A(\omega)$ and $L(\omega,x)=1$ when $x\in R(\omega)$;*

(iii) *L is decreasing along all the orbits and is strictly decreasing along the orbits on $S\setminus(A\cup R)$, i.e., $0\leq L(\theta_t\omega,\varphi(t,\omega)x) < L(\omega,x) < 1$ for $t>0$ when $x\in S(\omega)\setminus(A(\omega)\cup R(\omega))$.*

Proof. Since A is an attractor, i.e., there exists a forward invariant random closed neighborhood U of A in X, disjoint from R, such that $\Omega_U = A$ a.s., we denote

$$U_n(\omega) := \overline{\varphi(n,\theta_{-n}\omega)U(\theta_{-n}\omega)}.$$

Then, by Lemma 3.11, U_n is also a forward invariant random closed set. Furthermore, $U_{n+1} \subset U_n$ a.s. Similarly, by Lemma 5.1, choose a forward invariant random closed neighborhood V of R in $X \setminus B(A)$ with $\Omega_V = R$ a.s., and denote

$$V_n(\omega) := \overline{\varphi(n, \theta_{-n}\omega)V(\theta_{-n}\omega)},$$

with $V_{n+1} \subset V_n$ a.s. being forward invariant. Let

$$l_1^n(\omega, x) := \frac{d(x, U_n(\omega))}{d(x, U_n(\omega)) + d(x, V_n(\omega))}$$

and

$$l_2^n(\omega, x) := \sup_{t \geq 0} l_1^n(\theta_t\omega, \varphi(t, \omega)x).$$

Analogously to (3.3) in the proof of Theorem 3.12, for each n, l_2^n is decreasing along the orbits, i.e.,

$$l_2^n(\theta_t\omega, \varphi(t, \omega)x) \leq l_2^n(\omega, x) \qquad \text{for any } t \geq 0.$$

By the forward invariance of U_n and V_n, we have

$$l_2^n(\omega, x) = \begin{cases} 0, & x \in U_n(\omega), \\ 1, & x \in V_n(\omega). \end{cases}$$

Similar to the proof of (3.6), we have

$$l_2^n(\omega, x) := \sup_{t \in \mathbb{R}^+ \cap \mathbb{Q}} l_1^n(\theta_t\omega, \varphi(t, \omega)x),$$

so l_2^n is measurable with respect to $(\omega, x) \in \Omega \times X$.

We now show that, for fixed $\omega \in \Omega$, the mapping $l_2^n(\omega, \cdot) : X \to [0, 1]$ is continuous. Note that

$$B(A)(\omega) = \{x \in X | \varphi(t, \omega)x \in \text{int}U_n(\theta_t\omega) \text{ for some } t \geq 0\}.$$

For any $x \in B(A)(\omega)$, there exists $t_0 \geq 0$ such that $\varphi(t, \omega)x \in \text{int}U_n(\theta_t\omega)$ for $t \geq t_0$ by the forward invariance of U_n, and hence $\text{int}U_n$. In particular, there exists a neighborhood N_x of x with $N_x \subset B(A)(\omega)$ such that $\varphi(t_0, \omega)N_x \subset \text{int}U_n(\theta_{t_0}\omega)$ and hence $\varphi(t, \omega)N_x \subset \text{int}U_n(\theta_t\omega)$ for $t \geq t_0$ by the forward invariance of $\text{int}U_n$. That is,

$$l_2^n(\theta_t\omega, \varphi(t, \omega)x) = 0 \qquad \text{for all } t \geq t_0.$$

Therefore,

$$l_2^n(\omega, x) = \sup_{0 \leq t \leq t_0} \frac{d(\varphi(t, \omega)x, U_n(\theta_t\omega))}{d(\varphi(t, \omega)x, U_n(\theta_t\omega)) + d(\varphi(t, \omega)x, V_n(\theta_t\omega))}.$$

For any $y \in N_x$, we have

$$\begin{aligned} |l_2^n(\omega, x) - l_2^n(\omega, y)| &\leq \sup_{0 \leq t \leq t_0} \frac{2d(\varphi(t, \omega)x, \varphi(t, \omega)y)}{\text{dist}_{\min}(U_n(\theta_t\omega), V_n(\theta_t\omega))} \\ &\leq \sup_{0 \leq t \leq t_0} \frac{4d(\varphi(t, \omega)x, \varphi(t, \omega)y)}{\text{dist}_{\min}(A(\theta_t\omega), R(\theta_t\omega))} \\ &\leq \alpha \sup_{0 \leq t \leq t_0} d(\varphi(t, \omega)x, \varphi(t, \omega)y), \end{aligned}$$

where the first inequality follows from [1, Proposition 3.3]; the second inequality follows from the assumption (H); the third inequality holds for some constant α since the mappings $t \mapsto A(\theta_t\omega)$ and $t \mapsto R(\theta_t\omega)$ are continuous by the invariance of A and R, which implies that

$$\inf_{0\le t\le t_0} \mathrm{dist}_{\min}(A(\theta_t\omega), R(\theta_t\omega)) \ge c$$

for some constant $c > 0$ since A and R are compact. So, we have obtained the continuity of the mapping $x \mapsto l_2^n(\omega, x)$ at $x \in B(A)(\omega)$ for fixed $\omega \in \Omega$.

We next show that, for fixed ω, the mapping $x \mapsto l_2^n(\omega, x)$ is continuous at $x \in R(\omega)$. Note that for any $x_0 \in R(\omega)$ and $x \in X$, we have

$$|l_2^n(\omega, x) - l_2^n(\omega, x_0)| \le 1 - l_2^n(\omega, x) \le 1 - l_1^n(\omega, x).$$

Note that for arbitrary $\epsilon > 0$, when x is close to $R(\omega)$, we have $1 - l_1^n(\omega, x) < \epsilon$.

Observe also that $X = B(A)(\omega) \cup B(R)(\omega)$ for each $\omega \in \Omega$, where

$$B(R)(\omega) := X \setminus B(A)(\omega).$$

Now to show the continuity of the mapping $l_2^n(\omega, \cdot)$ on X, we only need to show that it is continuous in $B(R)(\omega)$. For $x \in B(R)(\omega)$, by Lemma 5.3, there exists $t \ge 0$ such that $\varphi(t, \omega)x \in V_n(\theta_t\omega)$, that is, $l_2^n(\omega, x) = 1$. Therefore, $l_2^n(\omega,)$ is continuous in $B(R)(\omega)$.

So far, the continuity of the mapping $l_2^n(\omega, \cdot) : X \to X$ has been proved.

Similar to the proof of Theorem 3.12, let

$$l_3^n(\omega, x) := \frac{1}{2}\left[l_2^n(\omega, x) + \int_0^\infty \mathrm{e}^{-t} l_2^n(\theta_t\omega, \varphi(t, \omega)x)\mathrm{d}t\right].$$

Then l_3^n satisfies all the properties that l_2^n possesses. By a similar argument to that in the proof of Theorem 3.12, l_3^n is strictly decreasing along orbits on $S\setminus(U_n \cup V_n)$, but not necessarily strictly decreasing along orbits outside S.

Let

$$L(\omega, x) := \sum_{n=1}^{\infty} \frac{1}{2^n} l_3^n(\omega, x).$$

Then $\omega \mapsto L(\omega, x)$ is measurable for fixed x since each $\omega \mapsto l_3^n(\omega, x)$ is also measurable; $x \mapsto L(\omega, x)$ is continuous for fixed ω since each $x \mapsto l_3^n(\omega, x)$ is continuous, and the series is uniformly convergent. Noting that $A = \cap_{n=1}^{\infty} U_n$, we have $L(\omega, x) = 0$ when $x \in A(\omega)$. Furthermore, $L(\omega, x) = 1$ when $x \in \cap_{n=1}^{\infty} V_n(\omega) = R(\omega)$ and L is strictly decreasing along orbits on $S \setminus (A \cup R)$. ■

The previous proposition can be partly improved, as can be seen in the following corollary.

Corollary 5.6. *Assume that the hypotheses of Theorem 3.12 hold, that (A, R) is an attractor-repeller pair in S, and that hypothesis* (H) *holds. Then there exists a Lyapunov function $L : \Omega \times X \to [0, 2]$ satisfying that*

(i) *$x \mapsto L(\omega, x)$ is continuous for each $\omega \in \Omega$ and $\omega \mapsto L(\omega, x)$ is measurable for each $x \in X$;*

(ii) *$L(\omega, x) = 0$ when $x \in A(\omega)$ and $L(\omega, x) = 1$ when $x \in R(\omega)$;*

(iii) *L is decreasing along all the orbits and is strictly decreasing along the orbits on $X \setminus (A \cup R)$, i.e., $0 \le L(\theta_t\omega, \varphi(t,\omega)x) < L(\omega, x) < 2$ for $t > 0$ when $x \in X \setminus (A(\omega) \cup R(\omega))$.*

Proof. The result follows by setting $\tilde{L} = L_1 + L_2$, with L_1 being the Lyapunov function in Theorem 3.12 and L_2 being the Lyapunov function in Proposition 5.5. ∎

In what follows, we need the following lemmas on omega-limit sets.

Lemma 5.7. *Let S be the global attractor for φ. Then, for any forward invariant random closed set $D \in \mathcal{D}$, Ω_D is the maximal invariant random compact set in D.*

Proof. First, by the definition of omega-limit set and the forward invariance of D, we have $\Omega_D \subset D$. Since S is the global attractor, by [17, Proposition 3.6], $\Omega_D \subset S$ and Ω_D is invariant. If $E \subset D$ is another invariant random compact set and $E \setminus \Omega_D \neq \emptyset$ a.s., then by the definition of omega-limit set, we have

$$E = \Omega_E \subset \Omega_D$$

since $E \subset D$ implies $\Omega_E \subset \Omega_D$, a contradiction. ∎

Lemma 5.8. *Let S be the global attractor of φ in the universe $\mathcal{D}$. Assume that $D \in \mathcal{D}$ is a forward invariant random closed set. Then*

$$\Omega_{D \cap S} = \Omega_D \qquad a.s.$$

Proof. By the definition of omega-limit set, $\Omega_{D\cap S} \subset \Omega_D$, so we only need to show that the converse inclusion holds a.s. Note that $\Omega_D \subset S$ and Ω_D is invariant since S is the global attractor. Since $D \cap S$ is forward invariant, by Lemma 5.7, $\Omega_{D\cap S}$ is the maximal invariant random compact set in $D \cap S$. On the other hand, $\Omega_D \subset D$ and $\Omega_S \subset S$, so $\Omega_D \cap \Omega_S (= \Omega_D)$ is an invariant random compact set in $D \cap S$. Therefore, $\Omega_D \subset \Omega_{D\cap S}$. The proof is complete. ∎

The following result ensures the existence of an attractor-repeller pair from the existence of a Lyapunov function.

Proposition 5.9. *Assume that A and R are two disjoint invariant random compact sets and L is a continuous Lyapunov function for (A, R) satisfying the properties in Proposition 5.5. Then (A, R) is an attractor-repeller pair of φ on the global attractor S.*

Proof. The proof is a modification of [30, Lemma 4.6] for random flows on compact spaces.

Note that $A \cup R \subset S$ a.s. since S is the maximal invariant random compact set. Define a random set M by

$$M(\omega) := \{x \in X \mid \ L(\omega, x) < 1\} \cap S(\omega).$$

Then it is easy to see that $R = S \setminus M$. On the one hand, $L(\omega, x) < 1$ implies $L(\theta_t\omega, \varphi(t,\omega)x) < 1$ for $t \ge 0$, so M is forward invariant. On the other hand, M is the complement of R in S, an invariant set, so M is backward invariant (see Remark 4.7 (i)). That is, M is an invariant random open set in S. For given $0 < \alpha < 1$, define the random sets $\tilde{M}_\alpha$ and M_α by

$$\tilde{M}_\alpha(\omega) := \{x \in X \mid \ L(\omega, x) \le \alpha\}$$

and

$$M_\alpha(\omega) := \{x \in X \mid L(\omega, x) \le \alpha\} \cap S(\omega),$$

respectively. That is, $M_\alpha = \tilde{M}_\alpha \cap S$. Since, for any $(x, \omega) \in X \times \Omega$, we have

$$L(\omega, x) \ge L(\theta_t\omega, \varphi(t,\omega)x), \quad t \ge 0,$$

hence $x \in \tilde{M}_\alpha(\omega)$ implies $\varphi(t,\omega,x) \in \tilde{M}_\alpha(\theta_t\omega)$; i.e., $\tilde{M}_\alpha$ is a forward invariant random closed set (thus, M_α is also forward invariant) and it is a random neighborhood of A in X. By Lemma 5.8, we have

$$\Omega_{\tilde{M}_\alpha} = \Omega_{M_\alpha}.$$

Let A_α be the omega-limit set of M_α, i.e.,

$$A_\alpha(\omega) := \Omega_{M_\alpha}(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t,\theta_{-t}\omega)M_\alpha(\theta_{-t}\omega)}.$$

Then, by the forward invariance of M_α, we have

$$A_\alpha(\omega) = \bigcap_{t\geq 0} \varphi(t,\theta_{-t}\omega)M_\alpha(\theta_{-t}\omega).$$

On the one hand, for all $\omega \in \Omega$, we have

$$A(\omega) = \bigcap_{t\geq 0} \varphi(t,\theta_{-t}\omega)A(\theta_{-t}\omega) \subset \bigcap_{t\geq 0} \varphi(t,\theta_{-t}\omega)M_\alpha(\theta_{-t}\omega) = A_\alpha(\omega).$$

Note that M_α is attracted by the global attractor S and S is compact, so A_α is an invariant random compact set. Consider

$$\tilde{L}(\omega) := \sup_{x\in A_\alpha(\omega)} L(\omega,x).$$

On the other hand we have $A_\alpha \subset A$ $\mathbb{P}$-a.s. If the assertion is false, similarly to the argument of Proposition 6.2 in [3], we then have $\tilde{L}(\cdot) > 0$ with positive probability and hence

$$\tilde{L}(\cdot) > \tilde{L}(\theta_t\cdot) \quad \text{for all } t > 0 \tag{5.2}$$

with positive probability. Note that $\tilde{L}(\omega) \leq \alpha$ for all ω, so $\tilde{L}$ is integrable. Then by the invariance of $\mathbb{P}$ under θ, we have

$$\int_\Omega (\tilde{L}(\omega) - \tilde{L}(\theta_t\omega))\mathrm{d}\mathbb{P}(\omega) = 0,$$

a contradiction to (5.2). Hence, we have obtained $A = A_\alpha$ $\mathbb{P}$-a.s. Therefore, $A = \Omega_{M_\alpha}$ and, consequently, $A = \Omega_{\tilde{M}_\alpha}$ $\mathbb{P}$-a.s., i.e., A is an attractor. Now, we only need to show that M is in fact the basin of attraction of A on S, i.e., $B(A;S) = M$, recalling that $B(A;S)$ is defined in Definition 4.1.

For any random compact set $D \subset M$, by the strict decreasing property of the Lyapunov function L on $S \setminus (A \cup R)$ and the compactness of D, for $\mathbb{P}$-almost all $\omega \in \Omega$, we have, for some $\alpha < 1$,

$$\varphi(t,\omega)D(\omega) \subset M_\alpha(\theta_t\omega) \quad \text{for all } t \geq T_D(\omega).$$

Analogously to the proof of [30, Lemma 4.3], we can conclude that A pullback attracts D. Since A and R are two disjoint invariant random compact sets, A can never pullback attract

R. On the other hand, $R = S \setminus M$, so M is the basin of attraction of A on S, and (A, R) is an attractor-repeller pair on S. The proof is complete. ∎

It seems that the following result has dynamical meaning, although the proof we provide is entirely algebraic.

Lemma 5.10. *Assume that $D = \{M_i\}_{i=1}^n$ is a Morse decomposition of S determined by attractors $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = S$. Then we have*

$$\bigcup_{i=1}^n M_i = \bigcap_{i=1}^n (A_i \cup R_i).$$

Proof. The proof amounts to a verification of

$$\bigcup_{i=1}^n (A_i \cap R_{i-1}) = \bigcap_{i=1}^n (A_i \cup R_i).$$

Suppose $x \in \bigcap_{i=1}^n (A_i(\omega) \cup R_i(\omega))$. Let $k := \min\{i | x \in A_i(\omega)\}$. Then $x \notin A_{k-1}(\omega)$, so $x \in R_{k-1}(\omega)$. That is, $x \in A_k(\omega) \cap R_{k-1}(\omega) = M_k(\omega) \subset M_D(\omega)$. On the other hand, if $x \in M_D(\omega)$, then $x \in M_k(\omega) = A_k(\omega) \cap R_{k-1}(\omega)$ for some $1 \leq k \leq n$. Hence $x \in A_k(\omega) \subset A_{k+1}(\omega) \subset \cdots \subset A_n(\omega)$ and $x \in R_{k-1}(\omega) \subset R_{k-2}(\omega) \subset \cdots \subset R_1(\omega)$. It follows that

$$\begin{aligned} x &\in \left(\bigcap_{i=k}^n A_i(\omega)\right) \cap \left(\bigcap_{i=1}^{k-1} R_i(\omega)\right) \subset \left(\bigcap_{i=k}^n (A_i(\omega) \cup R_i(\omega))\right) \cap \left(\bigcap_{i=1}^{k-1} (A_i(\omega) \cup R_i(\omega))\right) \\ &= \bigcap_{i=1}^n (A_i(\omega) \cup R_i(\omega)). \end{aligned}$$

The proof is complete. ∎

We can now conclude with the following important result.

Theorem 5.11. *Assume that $D = \{M_1, M_2, \ldots, M_n\}$ is a Morse decomposition of the global random attractor S and that hypothesis* (H) *holds for each of the attractor-repeller pairs which induce the Morse decomposition. Then there exists a Lyapunov function $L : \Omega \times X \to [0, 1]$ such that the following hold:*

(i) *The mapping $x \mapsto L(\omega, x)$ is continuous for fixed ω and the mapping $\omega \mapsto L(\omega, x)$ is measurable for fixed x.*

(ii) *L is constant on each M_i, i.e., for all $x, y \in M_i(\omega)$, $L(\omega, x) = L(\omega, y) = \alpha_i$, and α_i is independent of ω, $i = 1, \ldots, n$.*

(iii) *$0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n$, i.e., $L(\cdot, M_1(\cdot)) < L(\cdot, M_2(\cdot)) < \cdots < L(\cdot, M_n(\cdot))$.*

(iv) *For any $x \in X$ and $t > 0$, $L(\omega, x) \geq L(\theta_t\omega, \varphi(t, \omega)x)$; for $x \in S(\omega) \setminus (\bigcup_{i=1}^n M_i(\omega))$ and $t > 0$, $L(\omega, x) > L(\theta_t\omega, \varphi(t, \omega)x)$.*

Proof. Assume that the Morse decomposition $D = \{M_i\}_{i=1}^n$ is determined by attractor-repeller pairs (A_i, R_i), $i = 0, 1, \ldots, n$, and assume that l_i is the Lyapunov function constructed in Proposition 5.5 for the attractor-repeller pair (A_i, R_i). Let

$$L(\omega, x) := \sum_{i=1}^n \frac{2 l_i(\omega, x)}{3^{i+1}}. \tag{5.3}$$

Then L is the desired Lyapunov function. Clearly (i) holds. For the Morse set M_i, $1 \leq i \leq n$, we have

$$M_i \subset A_j,\ j \geq i, \qquad \text{and} \qquad M_i \subset R_j,\ j \leq i-1.$$

Hence by the definition of l_i, we have $L(\cdot, M_i(\cdot)) = \sum_{j=1}^{i-1} \dfrac{2}{3^{j+1}}$, which verifies (ii)–(iii). For $x \in X \backslash (\bigcup_{i=1}^{n} M_i(\omega))$, by Lemma 5.10 we know that there exists an i for $1 \leq i \leq n$ such that $x \notin A_i(\omega) \cup R_i(\omega)$. Therefore, we have $l_i(\omega, x) > l_i(\theta_t\omega, \varphi(t,\omega)x)$ for all $t > 0$, which, together with the fact $l_j(\omega, x) \geq l_j(\theta_t\omega, \varphi(t,\omega)x)$ for each $1 \leq j \leq n$, imply (iv). ∎

Corollary 5.12. *Assume that $D = \{M_1, M_2, \ldots, M_n\}$ is a Morse decomposition of the global random attractor S. Then there is a Lyapunov function $\tilde{L} : \Omega \times X \to [0,1]$ satisfying* (i)–(iii) *in Theorem* 5.11*; furthermore $\tilde{L}$ is strictly decreasing on $X \setminus (\bigcup_{i=1}^{n} M_i)$; i.e., for any $t > 0$ and $x \in X \setminus (\bigcup_{i=1}^{n} M_i(\omega))$, $\tilde{L}(\omega, x) > \tilde{L}(\theta_t\omega, \varphi(t,\omega)x)$.*

Proof. Let $\tilde{L}(\omega, x) := \sum_{i=1}^{n} \frac{2l_i(\omega,x)}{3^{i+1}}$ with each l_i being the Lyapunov function for the attractor-repeller pair (A_i, R_i) given in Corollary 5.6. Then $\tilde{L}$ satisfies the desired property. ∎

The following result shows the importance of the existence of a Lyapunov function in order to provide a Morse decomposition on a random attractor which, by Theorem 4.15, implies the RDS to be dynamically gradient as in Definition 4.17.

Theorem 5.13. *Let $D = \{M_1, M_2, \ldots, M_n\}$ be a finite collection of mutually disjoint invariant random compact sets, and assume that there exists a continuous Lyapunov function for D satisfying* (i)–(iv) *in Theorem* 5.11*. Then D is a Morse decomposition of the global attractor S.*

Proof. The proof is a modification of [30, Lemma 5.4] for the random flow case on compact spaces.

Note that S is the maximal invariant random compact set, so $M_1, \ldots, M_n$ are subsets of S. Assume that L is a Lyapunov function for D. Without loss of generality, let $L(\cdot, M_i(\cdot)) = \alpha_i$. By Theorem 5.11 (ii)–(iii), α_i are nonrandom constants and $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Let $A_1 := M_1$. For arbitrary $\alpha_{1,2}$ with $\alpha_1 < \alpha_{1,2} < \alpha_2$, define the random sets $\tilde{N}_{1,2}$ and $N_{1,2}$ by

$$\tilde{N}_{1,2}(\omega) := \{x \in X |\ \alpha_1 \leq L(\omega, x) \leq \alpha_{1,2}\} \quad \text{and} \quad N_{1,2}(\omega) = \tilde{N}_{1,2}(\omega) \cap S(\omega),$$

respectively. Note that $\tilde{N}_{1,2}$ is a forward invariant neighborhood of A_1 in X. Then, completely identical to the proof of Proposition 5.9, we know that $A_1 (= M_1)$ is an attractor with $\Omega_{\tilde{N}_{1,2}} = A_1$, and the corresponding basin of attraction $B(A_1; S)$ on S is

$$B(A_1; S)(\omega) = \{x \in X |\ \ \alpha_1 \leq L(\omega, x) < \alpha_2\} \cap S(\omega).$$

Therefore, the repeller R_1 corresponding to A_1 on S is

$$R_1(\omega) = \{x \in X |\ \ L(\omega, x) \geq \alpha_2\} \cap S(\omega).$$

Hence $M_2, \ldots, M_n \subset R_1$.

For each $\alpha_{2,3} \in (\alpha_2, \alpha_3)$, define the random sets $\tilde{N}_{2,3}$ and $N_{2,3}$ by

$$\tilde{N}_{2,3}(\omega) := \{x \in X |\ \alpha_1 \leq L(\omega, x) \leq \alpha_{2,3}\} \quad \text{and} \quad N_{2,3}(\omega) = \tilde{N}_{2,3}(\omega) \cap S(\omega),$$

respectively. Then $M_1 \cup M_2 \subset N_{2,3}$ and $\tilde{N}_{2,3}$ is a forward invariant random neighborhood of A_2 in X. Assuming that A_2 is the attractor inside $\tilde{N}_{2,3}$, by Lemma 5.8, for $\mathbb{P}$-almost all $\omega \in \Omega$, we obtain

$$A_2(\omega) = \bigcap_{t \geq 0} \varphi(t, \theta_{-t}\omega) N_{2,3}(\theta_{-t}\omega). \tag{5.4}$$

Hence, we have $M_1 \cup M_2 \subset A_2$ $\mathbb{P}$-a.s. Therefore, we have obtained $A_2 \cap R_1 \supset M_2$ $\mathbb{P}$-a.s. Next, we show that $A_2 \cap R_1 \subset M_2$ $\mathbb{P}$-a.s. For any $x \in N_{2,3}(\omega) \backslash (M_1(\omega) \cup M_2(\omega))$ and for all $t > 0$, we have

$$L(\theta_t \omega, \varphi(t, \omega)x) < L(\omega, x).$$

Therefore, by the proof of Proposition 5.9, for every $\alpha \in (\alpha_2, \alpha_3)$, the forward invariant random compact set N_α, given by

$$N_\alpha(\omega) := \{x \in X \mid \ \alpha_1 \leq L(\omega, x) \leq \alpha\} \cap S(\omega),$$

is always a forward invariant neighborhood of A_2 in S and $\Omega_{N_\alpha} = A_2$. Hence, we have

$$A_2 \subset \bigcap_{n \in \mathbb{N}} N_{\alpha_2 + \frac{1}{n}} \quad \mathbb{P}\text{-a.s.},$$

and, similarly, we also have

$$R_1 \subset \bigcap_{n \in \mathbb{N}} \hat{N}_{\alpha_2 - \frac{1}{n}} \quad \mathbb{P}\text{-a.s.},$$

where

$$N_{\alpha_2 + \frac{1}{n}}(\omega) := \left\{ x \in X \mid \ \alpha_1 \leq L(\omega, x) \leq \alpha_2 + \frac{1}{n} \right\} \cap S(\omega)$$

and

$$\hat{N}_{\alpha_2 - \frac{1}{n}}(\omega) := \left\{ x \in X \mid \ L(\omega, x) \geq \alpha_2 - \frac{1}{n} \right\} \cap S(\omega).$$

Thus, for $\mathbb{P}$-almost all ω,

$$\begin{aligned} A_2(\omega) \cap R_1(\omega) &\subset \left(\bigcap_{n \in \mathbb{N}} N_{\alpha_2 + \frac{1}{n}}(\omega) \right) \cap \left(\bigcap_{n \in \mathbb{N}} \hat{N}_{\alpha_2 - \frac{1}{n}}(\omega) \right) \\ &\subset \bigcap_{n \in \mathbb{N}} (N_{\alpha_2 + \frac{1}{n}}(\omega) \cap \hat{N}_{\alpha_2 - \frac{1}{n}}(\omega)) \\ &= \{x \in X \mid \ L(\omega, x) = \alpha_2\} = M_2(\omega); \end{aligned}$$

i.e., we have obtained $A_2 \cap R_1 = M_2$ $\mathbb{P}$-a.s. Then we can obtain R_2 from A_2, i.e.,

$$R_2(\omega) = \{x \in X \mid L(\omega, x) \geq \alpha_3\} \cap S(\omega).$$

Similar to the above arguments, for $\alpha_{3,4} \in (\alpha_3, \alpha_4)$, define the random sets $\tilde{N}_{3,4}$ and $N_{3,4}$ by

$$\tilde{N}_{3,4}(\omega) = \{x \in X \mid \alpha_1 \leq L(\omega, x) \leq \alpha_{3,4}\} \quad \text{and} \quad N_{3,4}(\omega) = \tilde{N}_{3,4}(\omega) \cap S(\omega),$$

and we immediately obtain A_3 similar to (5.4). Hence we at once obtain the repeller R_3 corresponding to A_3. Similarly, we can obtain A_4, R_4, …, A_{n-1}, R_{n-1} in the same way. Let $A_0 = R_n = \emptyset, A_n = R_0 = S$. Therefore we have obtained

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = S \ \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad S = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset \ \ \mathbb{P}\text{-a.s.}$$

from $M_i, i = 1, \ldots, n$, satisfying

$$M_i = A_i \cap R_{i-1}, \ 1 \leq i \leq n.$$

This shows that D is a Morse decomposition of S and hence completes the proof of the theorem. ■

6. Applications. For infinite-dimensional dynamical systems, the structure and characterization of global attractors is a difficult task. Indeed, there is only a small set of examples in which the description of the geometrical structure of attractors has been satisfactorily carried out (see, for instance, Hale [19]). The same problem appears in the random case. In the deterministic case, one of these canonical models is the Chafee–Infante equation, for which the attractor consists of an odd number of stationary points (which bifurcate from the origin) and the unstable manifolds joining them (see Hale [19], Henry [20], and Chafee and Infante [11]). The following example is a random version of this model, and we show, from the study of dynamical properties on the random attractor, the existence of a gradient infinite-dimensional dynamical system.

Suppose there exists a single multiplicative *Stratonovich* term on the Chafee–Infante equation on the interval $D = (0, \pi)$,

$$\mathrm{d}u = [\Delta u + \beta u - u^3]\,\mathrm{d}t + \sigma u \circ \mathrm{d}W_t, \quad u(0,t) = u(\pi,t) = 0 \tag{6.1}$$

(W_t is a two-sided one-dimensional Brownian motion), using the framework of RDSs (see [7] for more details). The equation can be rewritten in the form of an evolution equation on $X = L^2(D)$,

$$\mathrm{d}u = [-Au + \beta u - u^3]\,\mathrm{d}t + \sigma u \circ \mathrm{d}W_t, \tag{6.2}$$

where $A = -\Delta$ on D with Dirichlet boundary condition. For the details of the finite-dimensional Stratonovich integral, the reader is referred to [24, pages 100–102]. There is no essential difference to rewrite the definition and properties for the Hilbert space–valued Stratonovich integral, which is sufficient for our purpose here. We also remark that the mild solution to (6.2) satisfies a variant of constants formula, i.e.,

$$u(t) = T(t)u(0) + \int_0^t T(t-s)(\beta u(s) - u^3(s))\mathrm{d}s + \sigma \int_0^t T(t-s)u(s) \circ \mathrm{d}W_s,$$

where $T(t)_{t\geq 0}$ is the strongly continuous semigroup generated by $-A$.

Nevertheless, the procedure to prove that (6.2) generates an RDS (see [4, 7]) does not make use of this formulation, as it is carried out by performing a change of variables which transforms (6.2) into a problem for a random partial differential equation, i.e., a partial differential equation whose coefficients depend on the random parameter ω, and which can be analyzed for every fixed $\omega \in \Omega$.

The study in Caraballo et al. [4, 7] shows that (6.2) generates an RDS φ in the space X, and with respect to a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$, which possesses a positive $\xi(\omega)$, and, respectively, a negative $-\xi(\omega)$, random fixed point; i.e., $\xi(\cdot)$ is a random variable such that $\varphi(t,\omega)\xi(\omega) = \xi(\theta_t\omega)$ in the interior of the positive and, respectively, negative, invariant cones

$$\mathcal{K}^+ = \{u \in X : u(x) \geq 0 \text{ a.e.}\}$$

and

$$\mathcal{K}^- = \{u \in X : u(x) \leq 0 \text{ a.e.}\}.$$

Note that $\{0\}$ is also a fixed point of the equation in $\mathcal{K}^+ \cup \mathcal{K}^-$. Then there exist random attractors $S^+(\omega)$ and $S^-(\omega)$ in $\mathcal{K}^+$ and $\mathcal{K}^-$, respectively. Let λ_i denote the eigenvalues of the operator A. It is also proved in [7] that if $\beta \in (\lambda_1, \lambda_2)$, 0 is locally unstable, and it is conjectured that $\xi(\omega)$ and $-\xi(\omega)$ are pullback attracting random compact sets inside $\mathcal{K}^+$ and $\mathcal{K}^-$. For this concrete model, Liu [29] describes the Morse sets for the attractor $S(\omega) = S^+(\omega) \cup S^-(\omega)$ in the phase space $\mathcal{K}^+ \cup \mathcal{K}^-$. Indeed, to the local attractors

$$A_0 = \emptyset, \quad A_1(\omega) = \{\xi(\omega)\}, \quad A_2(\omega) = \{-\xi(\omega), \xi(\omega)\}, \quad A_3(\omega) = S(\omega)$$

correspond the associated repellers

$$R_0 = S(\omega), \quad R_1(\omega) = [-\xi(\omega), 0], \quad R_2(\omega) = \{0\}, \quad R_3(\omega) = \emptyset,$$

so that the Morse sets are given by

$$M_1(\omega) = \{\xi(\omega)\}, \quad M_2(\omega) = \{-\xi(\omega)\}, \quad M_3(\omega) = \{0\}.$$

That is, $\{M_1, M_2, M_3\}$ is a Morse decomposition of the attractor S. By the results of this paper, we can say more: we conclude from Theorem 5.11 that there exists a continuous random Lyapunov function associated to this Morse decomposition, so that (6.1) is a gradient RDS.

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