

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR DELAY STOCHASTIC EVOLUTION EQUATIONS

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Abstract

Some results on the existence and uniqueness of solutions for stochastic evolution equations containing some hereditary characteristics are proved. In fact, our theory is developed from a variational point of view and in a general functional setting which permit us to deal with several kinds of delay terms in a unified formulation.

1. Introduction and statement of the problem

When one wants to model some evolution phenomena arising in physics, biology, engineering, etc., some hereditary characteristics such as after-effect, time-lag, time-delay can appear in the variables. Typical examples can be found in the researches of materials with thermal memory, biochemical reactions, population models, etc. (see, for instance, Ruess [10], Wu [11] and references cited therein). This enables us to think that the problem could be better modeled by considering a functional differential equation which takes into account the history of the system. However, in most cases, some kind of randomness can appear in the problem, so that the system should be modeled by a stochastic form of the functional equation. Motivated by these facts, our main purpose in this paper is to analyse the existence and uniqueness of solutions for a class of nonlinear stochastic PDEs with time delays in a variational context, which, in particular, extend and complete the results in Caraballo [2] and Caraballo et al. [4].

First of all, we would like to mention that, in the deterministic framework, there exists a wide literature on the existence of different kind of solutions (strong, mild, integral, etc.) to functional differential equations even in the more general context of differential inclusions. It is well worth reading the work by Ruess [10] where one can find a description of the different techniques used to handle this question, in addition to a large list of references concerning these methods (e.g. method of lines, Galerkin approximations, Kato approximants, etc.). However, from a variational point of view, only a few works have been published (Artola [1] for linear and semilinear retarded equations, Caraballo [3] for a more general nonlinear monotone situation in the functional framework, among others).

As for the stochastic problem in the variational setting, even much less has been done. As far as we know, only the works by Real [8,9] (in the linear case), Caraballo [2] (nonlinear problem with variable delay) and Caraballo et al. [4] have appeared up to date. But, as the assumptions in these works are rather restrictive so that the operators involved in the equations cannot be general enough, we are now interested in developing a theory which, in particular, contains the previous works and which permits us to prove existence and uniqueness of solution for a wider class of systems.

To start off, let us state the abstract framework in which our analysis will be carried out. Let V and H be two real separable Hilbert spaces such that

$$V \subset H \equiv H^* \subset V^*,$$

where the injections are continuous and dense.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V , H and V^* respectively; by $((\cdot, \cdot))$ and (\cdot, \cdot) the scalar products in V and H respectively; and by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V .

Assume that $\{\Omega, \mathcal{F}, P\}$ is a complete probability space, equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, i.e., \mathcal{F}_0 contains all $A \in \mathcal{F}$ such that $P(A) = 0$ and $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$, $\forall t \geq 0$. Denote $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$.

We suppose also given $\{W(t)\}_{t \geq 0}$, a real valued $\{\mathcal{F}_t\}$ -Wiener process.

Given real numbers $a < b$, and a separable Hilbert space \mathcal{H} we will denote by $I^2(a, b; \mathcal{H})$ the space of all processes $X \in L^2(\Omega \times (a, b), \mathcal{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$ (where $\mathcal{B}((a, b))$ denotes the Borel σ -algebra on (a, b)) such that $X(t)$ is \mathcal{F}_t -measurable a.e. $t \in (a, b)$. The space $I^2(a, b; \mathcal{H})$ is a closed subspace of $L^2(\Omega \times (a, b), \mathcal{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$.

We will denote by $C(a, b; \mathcal{H})$ the Banach space of all continuous functions from $[a, b]$ into \mathcal{H} equipped with *sup* norm. We will write $L^2(\Omega; C(a, b; \mathcal{H}))$ instead of $L^2(\Omega, \mathcal{F}, dP; C(a, b; \mathcal{H}))$.

Let us also consider two fixed real numbers $T > 0$ and $h > 0$. If we consider a function $x \in C(-h, T; \mathcal{H})$, for each $t \in [0, T]$ we will denote by $x_t \in C(-h, 0; \mathcal{H})$ the function defined by $x_t(s) = x(t + s) \forall s \in [-h, 0]$. Moreover, if $y \in L^2(-h, T; \mathcal{H})$, we will also denote by $y_t \in L^2(-h, 0; \mathcal{H})$, for a.e. $t \in (0, T)$, the function defined by $y_t(s) = y(t + s)$ a.e. $s \in (-h, 0)$.

Let $A(t, \cdot) : V \rightarrow V^*$ be a family of nonlinear operators defined a.e. $t \in (0, T)$ and satisfying:

(A.1) (Measurability) $\forall v \in V$, the map $t \in (0, T) \rightarrow A(t, v) \in V^*$ is Lebesgue measurable.

(A.2) (Hemicontinuity) the map

$$\theta \in \mathbf{R} \rightarrow \langle A(t, u + \theta v), w \rangle \in \mathbf{R}$$

is continuous $\forall u, v, w \in V$, and a.e. $t \in (0, T)$.

(A.3) (Boundedness) there exists $c > 0$ such that $\|A(t, v)\|_* \leq c\|v\| \quad \forall v \in V$, a.e. $t \in (0, T)$.

(A.4) (Monotonicity and Coercivity): there exist $\alpha > 0$ and $\lambda \in \mathbf{R}$ such that

$$-2 \langle A(t, u) - A(t, v), u - v \rangle + \lambda \|u - v\|^2 \geq \alpha \|u - v\|^2, \quad \forall u, v \in V, \text{ a.e. } t \in (0, T).$$

Let $F_1 : (0, T) \times C(-h, 0; H) \rightarrow V^*$ and $F_2 : (0, T) \times C(-h, 0; V) \rightarrow V^*$ be two families of nonlinear operators defined a.e. $t \in (0, T)$ such that:

(F_{1.1}) $\forall \xi \in C(-h, 0; H)$, the map $t \in (0, T) \mapsto F_1(t, \xi) \in V^*$ is Lebesgue measurable,

(F_{1.2}) $F_1(t, 0) = 0$, a.e. $t \in (0, T)$,

(F_{1.3}) there exists $C_{F_1} > 0$ such that

$$\|F_1(t, \xi) - F_1(t, \eta)\|_*^2 \leq C_{F_1} |\xi - \eta|_{C(-h, 0; H)}^2, \quad \forall \xi, \eta \in C(-h, 0; H), \text{ a.e. } t \in (0, T),$$

(F_{2.1}) $\forall \xi \in C(-h, 0; V)$, the map $t \in (0, T) \mapsto F_2(t, \xi) \in V^*$ is Lebesgue measurable,

(F_{2.2}) $F_2(t, 0) = 0$, a.e. $t \in (0, T)$,

(F_{2.3}) there exists $C_{F_2} > 0$ such that

$$\|F_2(t, \xi) - F_2(t, \eta)\|_*^2 \leq C_{F_2} \|\xi - \eta\|_{C(-h, 0; V)}^2, \quad \forall \xi, \eta \in C(-h, 0; V), \text{ a.e. } t \in (0, T),$$

(F_{2.4}) there exists $K_{F_2} > 0$ such that $\forall x, y \in C(-h, T; V)$, and $\forall t \in [0, T]$,

$$\int_0^t \|F_2(s, x_s) - F_2(s, y_s)\|_*^2 ds \leq K_{F_2} \int_{-h}^t \|x(s) - y(s)\|^2 ds.$$

Let also $G_0 : (0, T) \times C(-h, 0; H) \rightarrow H$ and $G_1 : (0, T) \times C(-h, 0; V) \rightarrow H$ be another two families of nonlinear operators defined a.e. $t \in (0, T)$ such that:

(G_{0.1}) $\forall \xi \in C(-h, 0; H)$, the map $t \in (0, T) \mapsto G_0(t, \xi) \in H$ is Lebesgue measurable,

(G_{0.2}) $G_0(t, 0) = 0$, a.e. $t \in (0, T)$,

(G_{0.3}) there exists $C_{G_0} > 0$ such that

$$|G_0(t, \xi) - G_0(t, \eta)|^2 \leq C_{G_0} |\xi - \eta|_{C(-h, 0; H)}^2, \quad \forall \xi, \eta \in C(-h, 0; H), \text{ a.e. } t \in (0, T),$$

(G_{1.1}) $\forall \xi \in C(-h, 0; V)$, the map $t \in (0, T) \mapsto G_1(t, \xi) \in H$ is Lebesgue measurable,

(G_{1.2}) $G_1(t, 0) = 0$, a.e. $t \in (0, T)$,

(G_{1.3}) there exists $C_{G_1} > 0$ such that

$$|G_1(t, \xi) - G_1(t, \eta)|^2 \leq C_{G_1} \|\xi - \eta\|_{C(-h, 0; V)}^2, \quad \forall \xi, \eta \in C(-h, 0; V), \text{ a.e. } t \in (0, T),$$

(G_{1.4}) there exists $K_{G_1} > 0$ such that $\forall x, y \in C(-h, T; V)$, and $\forall t \in [0, T]$,

$$\int_0^t |G_1(s, x_s) - G_1(s, y_s)|^2 ds \leq K_{G_1} \int_{-h}^t \|x(s) - y(s)\|^2 ds.$$

We consider the problem

$$\begin{cases} u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\ u(t) = \psi(0) + \int_0^t A(s, u(s)) ds + \int_0^t (F_1(s, u_s) + F_2(s, u_s) + f(s)) ds \\ \quad + \int_0^t (G_0(s, u_s) + G_1(s, u_s) + g(s)) dW(s), \quad t \in [0, T], \\ u(t) = \psi(t), \quad t \in [-h, 0], \end{cases} \quad (P)$$

where $f \in I^2(0, T; V^*)$, $g \in I^2(0, T; H)$ and $\psi \in I^2(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$ are given.

Remark 1.1. It is not difficult to deduce from (F1.1)-(F1.3) that if $u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, the process $F_1(t, u_t)$ belongs to $I^2(0, T; V^*)$. Also, by means of (G0.1)-(G0.3), the process $G_0(t, u_t)$ belongs to $I^2(0, T; H)$.

Remark 1.2. Observe that by (F2.1)-(F2.3), for a given $x \in C(-h, T; V)$, the function $F_2^x : (0, T) \rightarrow V^*$ defined by $F_2^x(t) = F_2(t, x_t)$ a.e. $t \in (0, T)$, belongs to $L^2(0, T; V^*)$. Then, thanks to (F2.4), the mapping

$$\Xi : x \in C(-h, T; V) \mapsto F_2^x \in L^2(0, T; V^*)$$

has a unique extension to a mapping $\tilde{\Xi}$ which is uniformly continuous from $L^2(-h, T; V)$ into $L^2(0, T; V^*)$. From now on, we will also write $F_2(t, x_t) = \tilde{\Xi}(x)(t)$ for each $x \in L^2(-h, T; V)$, and for every $x, y \in L^2(-h, T; V)$ it holds

$$\int_0^t \|F_2(s, x_s) - F_2(s, y_s)\|_*^2 ds \leq K_{F_2} \int_{-h}^t \|x(s) - y(s)\|^2 ds \quad \forall t \in [0, T]. \quad (1.1)$$

By a similar argument, we can define $G_1(t, x_t) \in L^2(0, T; H)$ for each $x \in L^2(-h, T; V)$, and $\forall x, y \in L^2(-h, T; V)$ it follows

$$\int_0^t |G_1(s, x_s) - G_1(s, y_s)|^2 ds \leq K_{G_1} \int_{-h}^t \|x(s) - y(s)\|^2 ds \quad \forall t \in [0, T]. \quad (1.2)$$

Thus, if $u \in I^2(-h, T; V)$ is given, the process $F_2(t, u_t)$ belongs to $I^2(0, T; V^*)$, the process $G_1(t, u_t)$ belongs to $I^2(0, T; H)$, and, consequently, $\forall u, v \in I^2(-h, T; V)$ we obtain

$$\int_0^t \|F_2(s, u_s) - F_2(s, v_s)\|_*^2 ds \leq K_{F_2} \int_{-h}^t \|u(s) - v(s)\|^2 ds \quad \forall t \in [0, T], \quad P - a.s., \quad (1.3)$$

and

$$\int_0^t |G_1(s, u_s) - G_1(s, v_s)|^2 ds \leq K_{G_1} \int_{-h}^t \|u(s) - v(s)\|^2 ds \quad \forall t \in [0, T], \quad P - a.s.. \quad (1.4)$$

As a consequence of the preceding remarks, the terms appearing in problem (P) make sense. Now, we are interested in establishing some results on the existence and uniqueness of solution to (P) under some additional assumptions. To this respect, it is worth mentioning that in the absence of hereditary characteristics (i.e. when $h = 0$), our problem has been solved by Pardoux [6] (see also Da Prato and Zabczyk [5] for a different approach); in the linear case

containing variable delays, it has also been treated by Real [9]; Caraballo [2] considered the nonlinear monotone situation with variable delay but for bounded operators F_i and G_i , and finally, Caraballo et al. [4] provided an answer to our problem in the particular situations in which $F_2 \equiv 0$, $F_1(t, \cdot)$ is a family of operators from V into H and, what is more important, under stronger assumptions on the family of operators which do not allow us to cover a wide class of applications (e.g. in the case of unbounded operators, essentially the ones containing distributed delays satisfy the assumptions in [4]). Thus, on the one hand, the results we shall obtain can be considered as extensions to the nonlinear case of those obtained in Real [9]. On the other hand, the presence of the term F_1 and the hypotheses that we shall impose on F_2 and G_1 , permit us, as we have already mentioned, to treat examples which cannot be handled with the results in Caraballo et al. [4].

The paper is organized as follows. In Section 2, we prove a first result on the existence and uniqueness of solution for the problem (P) in the particular case $F_2 \equiv G_1 \equiv 0$. Then, in Section 3, we establish an existence and uniqueness result for the complete problem. Finally, an example is considered in the last Section to illustrate our results.

2. A first existence and uniqueness result

In this section, we shall consider the problem

$$\begin{cases} u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\ u(t) = \psi(0) + \int_0^t A(s, u(s)) ds + \int_0^t (F_1(s, u_s) + f(s)) ds \\ \quad + \int_0^t (G_0(s, u_s) + g(s)) dW(s), \quad t \in [0, T], \\ u(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \quad (P')$$

We can now prove the following result:

Theorem 2.1 *Assume that hypotheses (A.1)-(A.4), (F_{1.1})-(F_{1.3}) and (G_{0.1})-(G_{0.3}) hold. Then, for every $\psi \in I^2(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$, $f \in I^2(0, T; V^*)$ and $g \in I^2(0, T; H)$, there exists a unique solution u to the problem (P') .*

Proof.

Uniqueness of solutions. Assume that $u, v \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$ are two solutions of (P') . Then, Itô's formula and condition (A.4) imply that for all $t \in [0, T]$

$$\begin{aligned} & |u(t) - v(t)|^2 + \alpha \int_0^t \|u(s) - v(s)\|^2 ds \\ & \leq \lambda \int_0^t |u(s) - v(s)|^2 ds \\ & \quad + 2 \int_0^t \langle F_1(s, u_s) - F_1(s, v_s), u(s) - v(s) \rangle ds \\ & \quad + 2 \int_0^t \langle G_0(s, u_s) - G_0(s, v_s), u(s) - v(s) \rangle dW(s) \\ & \quad + \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds. \end{aligned}$$

Therefore,

$$\begin{aligned}
& E \left[\sup_{0 \leq s \leq t} |u(s) - v(s)|^2 \right] + \alpha E \int_0^t \|u(s) - v(s)\|^2 ds \\
& \leq 2|\lambda| E \int_0^t |u(s) - v(s)|^2 ds + 2E \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds \\
& \quad + 4E \int_0^t \|F_1(s, u_s) - F_1(s, v_s)\|_* \|u(s) - v(s)\| ds \\
& \quad + 4E \left[\sup_{0 \leq s \leq t} \int_0^s (G_0(\theta, u_\theta) - G_0(\theta, v_\theta), u(\theta) - v(\theta)) dW(\theta) \right] \tag{2.1}
\end{aligned}$$

for all $t \in [0, T]$. Now, we can estimate the terms on the right-hand side of (2.1). On the one hand,

$$\begin{aligned}
& 4E \int_0^t \|F_1(s, u_s) - F_1(s, v_s)\|_* \|u(s) - v(s)\| ds \\
& \leq E \int_0^t \left[\frac{8}{\alpha} \|F_1(s, u_s) - F_1(s, v_s)\|_*^2 + \frac{\alpha}{2} \|u(s) - v(s)\|^2 \right] ds \\
& \leq \frac{8}{\alpha} C_{F_1} E \int_0^t |u_s - v_s|_{C(-h, 0; H)}^2 ds + \frac{\alpha}{2} E \int_0^t \|u(s) - v(s)\|^2 ds \\
& \leq \frac{8}{\alpha} C_{F_1} E \int_0^t \sup_{0 \leq r \leq s} |u(r) - v(r)|^2 ds + \frac{\alpha}{2} E \int_0^t \|u(s) - v(s)\|^2 ds. \tag{2.2}
\end{aligned}$$

On the other hand, Burkholder-Davis-Gundy's inequality yields that

$$\begin{aligned}
& 4E \left[\sup_{0 \leq s \leq t} \int_0^s (G_0(\theta, u_\theta) - G_0(\theta, v_\theta), u(\theta) - v(\theta)) dW(\theta) \right] \\
& \leq 12E \left\{ \sup_{0 \leq s \leq t} |u(s) - v(s)| \left[\int_0^t |G_0(\theta, u_\theta) - G_0(\theta, v_\theta)|^2 d\theta \right]^{\frac{1}{2}} \right\} \\
& \leq \frac{1}{2} E \left(\sup_{0 \leq s \leq t} |u(s) - v(s)|^2 \right) + 72E \int_0^t |G_0(\theta, u_\theta) - G_0(\theta, v_\theta)|^2 d\theta \\
& \leq \frac{1}{2} E \left(\sup_{0 \leq s \leq t} |u(s) - v(s)|^2 \right) + 72C_{G_0} E \int_0^t |u_\theta - v_\theta|_{C(-h, 0; H)}^2 d\theta \\
& \leq \frac{1}{2} E \left(\sup_{0 \leq s \leq t} |u(s) - v(s)|^2 \right) + 72C_{G_0} E \int_0^t \sup_{0 \leq r \leq \theta} |u(r) - v(r)|^2 d\theta. \tag{2.3}
\end{aligned}$$

Thus, (2.1)-(2.3) imply that for all $t \in [0, T]$

$$\begin{aligned}
& \frac{1}{2} E \left[\sup_{0 \leq s \leq t} |u(s) - v(s)|^2 \right] + \frac{\alpha}{2} E \int_0^t \|u(s) - v(s)\|^2 ds \\
& \leq \left[2|\lambda| + \frac{8}{\alpha} C_{F_1} + 74C_{G_0} \right] E \int_0^t \sup_{0 \leq r \leq \theta} |u(r) - v(r)|^2 d\theta.
\end{aligned}$$

Now, uniqueness follows immediately from Gronwall's lemma.

Existence of solutions: We denote $u^0 \equiv 0$, and define by recurrence a sequence $\{u^n\}_{n \geq 1}$ of processes as solutions to the problem

$$\begin{cases} u^n \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\ u^n(t) = \psi(0) + \int_0^t (A(s, u^n(s)) - \frac{\lambda}{2} u^n(s)) ds + \frac{\lambda}{2} \int_0^t u^{n-1}(s) ds \\ + \int_0^t (F_1(s, u_s^{n-1}) + f(s)) ds + \int_0^t (G_0(s, u_s^{n-1}) + g(s)) dW(s), t \in [0, T], \\ u^n(t) = \psi(t), t \in [-h, 0]. \end{cases} \quad (P'_n)$$

Observe that $u^0 \equiv 0 \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, and by Remark 1.1, if $u^{n-1} \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, it follows that $F_1(t, u_t^{n-1}) \in I^2(0, T; V^*)$, and $G_0(t, u_t^{n-1}) \in I^2(0, T; H)$, and consequently, from the results in Pardoux [6], there exists a unique $u^n \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$ which is a solution of (P'_n) .

Now, we want to prove that $\{u^n\}_{n \geq 1}$ converges in $I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$ to a process u which will be the solution of problem (P') .

Applying Itô's formula to the process $u^{n+1}(t) - u^n(t)$, $n \geq 1$, and using condition (A.4), we obtain

$$\begin{aligned} & |u^{n+1}(t) - u^n(t)|^2 + \alpha \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\ & \leq \lambda \int_0^t (u^{n+1}(s) - u^n(s), u^n(s) - u^{n-1}(s)) ds \\ & \quad + 2 \int_0^t \langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle ds \\ & \quad + 2 \int_0^t (G_0(s, u_s^n) - G_0(s, u_s^{n-1}), u^{n+1}(s) - u^n(s)) dW(s) \\ & \quad + \int_0^t |G_0(s, u_s^n) - G_0(s, u_s^{n-1})|^2 ds \end{aligned} \quad (2.4)$$

for all $t \in [0, T]$.

Consequently, (2.4) yields

$$\begin{aligned} & E \left[\sup_{0 \leq s \leq t} |u^{n+1}(s) - u^n(s)|^2 \right] + \alpha E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\ & \leq 2|\lambda| E \int_0^t |u^{n+1}(s) - u^n(s)| |u^n(s) - u^{n-1}(s)| ds \\ & \quad + 4E \int_0^t |\langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle| ds \\ & \quad + 4E \left[\sup_{0 \leq s \leq t} \int_0^s (G_0(\theta, u_\theta^n) - G_0(\theta, u_\theta^{n-1}), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right] \\ & \quad + 2E \int_0^t |G_0(s, u_s^n) - G_0(s, u_s^{n-1})|^2 ds. \end{aligned} \quad (2.5)$$

Now, observe that

$$\begin{aligned}
& 2|\lambda|E \int_0^t |u^{n+1}(s) - u^n(s)| |u^n(s) - u^{n-1}(s)| ds \\
& \leq 2\beta|\lambda|E \int_0^t \|u^{n+1}(s) - u^n(s)\| |u^n(s) - u^{n-1}(s)| ds \\
& \leq \frac{\alpha}{3}E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds + \frac{3\lambda^2\beta^2}{\alpha}E \int_0^t \sup_{0 \leq \theta \leq s} |u^n(\theta) - u^{n-1}(\theta)|^2 ds, \quad (2.6)
\end{aligned}$$

where $\beta > 0$ is a constant such that $|v| \leq \beta \|v\|, \forall v \in V$.

On the other hand, thanks to condition $(F_1.3)$, we can obtain

$$\begin{aligned}
& 4E \int_0^t |\langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle| ds \\
& \leq 4E \int_0^t \|F_1(s, u_s^n) - F_1(s, u_s^{n-1})\|_* \|u^{n+1}(s) - u^n(s)\| ds \\
& \leq E \int_0^t \left[\frac{12}{\alpha} \|F_1(s, u_s^n) - F_1(s, u_s^{n-1})\|_*^2 + \frac{\alpha}{3} \|u^{n+1}(s) - u^n(s)\|^2 \right] ds \\
& \leq \frac{12}{\alpha} C_{F_1} E \int_0^t |u_s^n - u_s^{n-1}|_{C(-h, 0; H)}^2 ds + \frac{\alpha}{3} E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\
& \leq \frac{12}{\alpha} C_{F_1} E \int_0^t \sup_{0 \leq r \leq s} |u^n(r) - u^{n-1}(r)|^2 ds + \frac{\alpha}{3} E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds. \quad (2.7)
\end{aligned}$$

In a similar manner as for uniqueness, we can obtain from Burkholder-Davis-Gundy's inequality that

$$\begin{aligned}
& 4E \left[\sup_{0 \leq s \leq t} \int_0^s (G_0(\theta, u_\theta^n) - G_0(\theta, u_\theta^{n-1}), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right] \\
& \leq \frac{1}{2}E \left(\sup_{0 \leq s \leq t} |u^{n+1}(s) - u^n(s)|^2 \right) + 72C_{G_0}E \int_0^t \sup_{0 \leq r \leq \theta} |u^n(r) - u^{n-1}(r)|^2 d\theta. \quad (2.8)
\end{aligned}$$

Then, we can get from (2.5)-(2.8) and $(G_0.3)$, that there exists a positive constant k such that for all $n \geq 1$ and all $t \in [0, T]$

$$\begin{aligned}
& \frac{1}{2}E \left[\sup_{0 \leq s \leq t} |u^{n+1}(s) - u^n(s)|^2 \right] + \frac{\alpha}{3}E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\
& \leq \frac{k}{2}E \int_0^t \sup_{0 \leq r \leq \theta} |u^n(r) - u^{n-1}(r)|^2 d\theta. \quad (2.9)
\end{aligned}$$

Now, we define

$$\rho^n(t) = \frac{1}{2}E \left[\sup_{0 \leq s \leq t} |u^{n+1}(s) - u^n(s)|^2 \right] + \frac{\alpha}{3}E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds, \quad \forall n \geq 1, \forall t \in [0, T].$$

Then, (2.9) immediately implies that

$$\rho^n(t) \leq k \int_0^t \rho^{n-1}(s) ds, \quad \forall n \geq 1, \forall t \in [0, T],$$

and, consequently, by iterating the preceding inequality, we obtain

$$\rho^n(t) \leq \frac{k^{n-1}T^{n-1}}{(n-1)!} \rho^1(T), \quad \forall n \geq 1, \quad \forall t \in [0, T]. \quad (2.10)$$

Since $u^{n+1}(t) = u^n(t)$, $\forall t \in [-h, 0]$, (2.10) implies that $\{u^n\}_{n \geq 1}$ is a Cauchy sequence in $I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$. Thus, there exists $u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$ such that

$$u^n \rightarrow u \text{ in } I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)).$$

Thanks to conditions (F_{1.3}) and (G_{0.3}), we have in particular that

$$F_1(t, u_t^n) \rightarrow F_1(t, u_t) \text{ in } I^2(0, T; V^*),$$

and

$$G_0(t, u_t^n) \rightarrow G_0(t, u_t) \text{ in } I^2(0, T; H).$$

Moreover, by (A.3), the sequence $\{A(t, u^n(t))\}_{n \geq 1}$ is bounded in $I^2(0, T; V^*)$. Thus, there exist a subsequence $\{A(t, u^{n_k}(t))\}_{n_k \geq 1} \subset \{A(t, u^n(t))\}_{n \geq 1}$ and $\xi \in I^2(0, T; V^*)$, such that

$$A(t, u^{n_k}(t)) \rightharpoonup \xi \text{ in } I^2(0, T; V^*),$$

where \rightharpoonup denotes weak convergence. Thus, we can take limits in (P'_{n_k}), and obtain that u is solution of

$$\begin{cases} u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\ u(t) = \psi(0) + \int_0^t \xi(s) ds + \int_0^t (F_1(s, u_s) + f(s)) ds \\ \quad + \int_0^t (G_0(s, u_s) + g(s)) dW(s), \quad t \in [0, T], \\ u(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \quad (P'')$$

To simplify the notation, observe that ξ is uniquely determined by u , and thus, the whole sequence $\{A(t, u^n(t))\}_{n \geq 1}$ converges weakly to ξ in $I^2(0, T; V^*)$.

In order to prove that u is in fact a solution of problem (P'), we only need to prove that $\xi(t) = A(t, u(t))$ in $(0, T)$.

First of all, applying Itô's formula to $|u^n(t)|^2$ and to $|u(t)|^2$ on the interval $[0, T]$, we obtain

$$\begin{aligned} E|u^n(T)|^2 &= E|\psi(0)|^2 + 2E \int_0^T \langle A(s, u^n(s)), u^n(s) \rangle ds \\ &\quad + \lambda E \int_0^T (u^n(s), u^{n-1}(s)) ds \\ &\quad - \lambda E \int_0^T |u^n(s)|^2 ds + 2E \int_0^T \langle F_1(s, u_s^{n-1}) + f(s), u^n(s) \rangle ds \\ &\quad + E \int_0^T |G_0(s, u_s^{n-1}) + g(s)|^2 ds \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} E|u(T)|^2 &= E|\psi(0)|^2 + 2E \int_0^T \langle \xi(s), u(s) \rangle ds + 2E \int_0^T \langle F_1(s, u_s) + f(s), u(s) \rangle ds \\ &\quad + E \int_0^T |G_0(s, u_s) + g(s)|^2 ds. \end{aligned} \quad (2.13)$$

Taking limits in (2.12) as $n \rightarrow \infty$, and comparing with (2.13), we deduce:

$$\lim_{n \rightarrow \infty} E \int_0^T \langle A(s, u^n(s)), u^n(s) \rangle ds = E \int_0^T \langle \xi(s), u(s) \rangle ds. \quad (2.14)$$

In particular, thanks to (A.4),

$$-2E \int_0^T \langle A(s, u^n(s)) - A(s, X(s)), u^n(s) - X(s) \rangle ds + \lambda E \int_0^T |u^n(s) - X(s)|^2 ds \geq 0, \quad (2.15)$$

for all $X \in I^2(0, T; V)$ and all $n \geq 1$. Taking limits in (2.15), we get

$$-2E \int_0^T \langle \xi(s) - A(s, X(s)), u(s) - X(s) \rangle ds + \lambda E \int_0^T |u(s) - X(s)|^2 ds \geq 0,$$

for all $X \in I^2(0, T; V)$. Now, if we set $X = u - \delta Z$, with $Z \in I^2(0, T; V)$ and $\delta > 0$, we have

$$-2E \int_0^T \langle \xi(s) - A(s, u(s) - \delta Z(s)), \delta Z(s) \rangle ds + \lambda \delta^2 E \int_0^T |Z(s)|^2 ds \geq 0, \quad (2.16)$$

for all $Z \in I^2(0, T; V)$ and all $\delta > 0$.

If we divide by δ in (2.16) and take limits as $\delta \rightarrow 0$, we obtain from (A.2)

$$E \int_0^T \langle \xi(s) - A(s, u(s)), Z(s) \rangle ds \leq 0, \quad \forall Z \in I^2(0, T; V),$$

and, therefore, $A(s, u(s)) = \xi(s)$ in $(0, T)$. ■

Remark 2.1. We want to point out once again that in [4], a similar result to the preceding theorem is obtained but under the assumption of being $F_1(t, \cdot)$ a family of operators taking values in H instead of V^* .

3. Existence and uniqueness of solution of problem (P)

Now, under suitable additional assumptions, we can show existence and uniqueness of solution of problem (P). In fact, we will prove the following theorem:

Theorem 3.1. *Assume that the injection $V \subset H$ is compact, and hypotheses (A.1)-(A.4), (F_{1.1})-(F_{1.3}), (F_{2.1})-(F_{2.4}), (G_{0.1})-(G_{0.3}) and (G_{1.1})-(G_{1.4}) hold. Suppose also the three following hypotheses:*

(A.5) *There exists $\widehat{\lambda} > 0$ such that for all $x, y \in L^2(-h, T; V)$ and for all $t \in [0, T]$ it holds*

$$\begin{aligned} & \alpha \int_0^t \|x(s) - y(s)\|^2 ds + \int_0^t |G_1(s, x_s) - G_1(s, y_s)|^2 ds \\ & \leq -2 \int_0^t \langle A(s, x(s)) - A(s, y(s)), x(s) - y(s) \rangle ds + \lambda \int_0^t |x(s) - y(s)|^2 ds \\ & \quad + \widehat{\lambda} \int_{-h}^0 \|x(s) - y(s)\|^2 ds - 2 \int_0^t \langle F_2(s, x_s) - F_2(s, y_s), x(s) - y(s) \rangle ds. \end{aligned}$$

($F_1.4$) There exists $C^{F_1} > 0$ such that for all $X, Y \in L^2(\Omega; C(-h, T; H))$ such that $X \equiv Y$ on $[-h, 0]$, and all $t \in [0, T]$,

$$E \int_0^t \|F_1(s, X_s) - F_1(s, Y_s)\|_*^2 ds \leq C^{F_1} \int_0^t \sup_{0 \leq \theta \leq s} E |X(\theta) - Y(\theta)|^2 ds.$$

($G_0.4$) There exists $C^{G_0} > 0$ such that for all $X, Y \in L^2(\Omega; C(-h, T; H))$ such that $X \equiv Y$ on $[-h, 0]$, and all $t \in [0, T]$,

$$E \int_0^t |G_0(s, X_s) - G_0(s, Y_s)|^2 ds \leq C^{G_0} \int_0^t \sup_{0 \leq \theta \leq s} E |X(\theta) - Y(\theta)|^2 ds.$$

Then, for every $\psi \in I^2(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$, $f \in I^2(0, T; V^*)$ and $g \in I^2(0, T; H)$, there exists a unique solution u of the problem (P).

Proof.

Uniqueness of solutions. Assume that $u, v \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$ are two solutions to problem (P). Then, applying Itô's formula to $|u(t) - v(t)|^2$, and using (A.5), we obtain for each $t \in [0, T]$

$$\begin{aligned} & |u(t) - v(t)|^2 + \alpha \int_0^t \|u(s) - v(s)\|^2 ds \\ & \leq \lambda \int_0^t |u(s) - v(s)|^2 ds + 2 \int_0^t \langle F_1(s, u_s) - F_1(s, v_s), u(s) - v(s) \rangle ds \\ & \quad + 2 \int_0^t (G_0(s, u_s) - G_0(s, v_s), G_1(s, u_s) - G_1(s, v_s)) ds \\ & \quad + 2 \int_0^t (G_0(s, u_s) - G_0(s, v_s), u(s) - v(s)) dW(s) \\ & \quad + 2 \int_0^t (G_1(s, u_s) - G_1(s, v_s), u(s) - v(s)) dW(s) \\ & \quad + \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & E |u(t) - v(t)|^2 + \alpha E \int_0^t \|u(s) - v(s)\|^2 ds \\ & \leq \lambda E \int_0^t |u(s) - v(s)|^2 ds + E \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds \\ & \quad + 2E \int_0^t (G_0(s, u_s) - G_0(s, v_s), G_1(s, u_s) - G_1(s, v_s)) ds \\ & \quad + 2E \int_0^t \langle F_1(s, u_s) - F_1(s, v_s), u(s) - v(s) \rangle ds, \end{aligned} \tag{3.1}$$

for all $t \in [0, T]$. We can now deduce from ($F_1.4$) that

$$\begin{aligned} 2E \int_0^t \langle F_1(s, u_s) - F_1(s, v_s), u(s) - v(s) \rangle ds & \leq \frac{3}{\alpha} C^{F_1} \int_0^t \sup_{0 \leq \theta \leq s} E |u(\theta) - v(\theta)|^2 ds \\ & \quad + \frac{\alpha}{3} \int_0^t E \|u(s) - v(s)\|^2 ds. \end{aligned} \tag{3.2}$$

From (G_{1.4}) it follows that

$$\begin{aligned}
& 2E \int_0^t (G_0(s, u_s) - G_0(s, v_s), G_1(s, u_s) - G_1(s, v_s)) ds \\
& \leq 2E \int_0^t |G_0(s, u_s) - G_0(s, v_s)| |G_1(s, u_s) - G_1(s, v_s)| ds \\
& \leq \frac{3K_{G_1}}{\alpha} E \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds + \frac{\alpha}{3} E \int_0^t \|u(s) - v(s)\|^2 ds. \tag{3.3}
\end{aligned}$$

Thanks to condition (G_{0.4}), we have

$$\begin{aligned}
& \left(\frac{3K_{G_1}}{\alpha} + 1 \right) E \int_0^t |G_0(s, u_s) - G_0(s, v_s)|^2 ds \\
& \leq \left(\frac{3K_{G_1}}{\alpha} + 1 \right) C^{G_0} \int_0^t \sup_{0 \leq \theta \leq s} E |u(\theta) - v(\theta)|^2 ds. \tag{3.4}
\end{aligned}$$

From (3.1)-(3.4), it follows that for all $t \in [0, T]$

$$\begin{aligned}
& \sup_{0 \leq s \leq t} E |u(s) - v(s)|^2 + \frac{\alpha}{3} E \int_0^t \|u(s) - v(s)\|^2 ds \\
& \leq 2 \left(|\lambda| + \left(\frac{3K_{G_1}}{\alpha} + 1 \right) C^{G_0} + \frac{3}{\alpha} C^{F_1} \right) \int_0^t \sup_{0 \leq \theta \leq s} E |u(\theta) - v(\theta)|^2 ds,
\end{aligned}$$

and Gronwall's lemma implies now uniqueness.

Existence of solutions. We will proceed in two steps.

Step 1. Firstly, we consider that $F_1 \equiv G_0 \equiv 0$ and $\lambda = 0$. We have to prove existence of solution to

$$\begin{cases}
u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\
u(t) = \psi(0) + \int_0^t A(s, u(s)) ds + \int_0^t (F_2(s, u_s) + f(s)) ds \\
\quad + \int_0^t (G_1(s, u_s) + g(s)) dW(s), \quad t \in [0, T], \\
u(t) = \psi(t), \quad t \in [-h, 0].
\end{cases} \tag{\widehat{P}}$$

We will use a Galerkin scheme. As injection $V \subset H$ is compact, there exists a Hilbert basis of H , $\{v_i\}_{i \geq 1} \subset V$ and a nondecreasing sequence $\{\mu_i\}_{i \geq 1}$ of positive numbers, with $\lim_{i \rightarrow \infty} \mu_i = +\infty$, such that $((w, v_i) = \mu_i(w, v_i) \forall i \geq 1, \forall w \in V$. The subspace of V spanned by v_1, \dots, v_m will be denoted by V_m . Consider the projector $P_m : H \rightarrow V_m$ given by $P_m u = \sum_{i=1}^m (u, v_i) v_i$, and define $u^m(t) = \sum_{i=1}^m \gamma^{mi}(t) v_i$, where

$$\begin{cases}
u^m \in I^2(-h, T; V_m) \cap L^2(\Omega; C(-h, T; V_m)), \\
(u^m(t), v) = (P_m \psi(0), v) + \int_0^t \langle A(s, u^m(s)) + F_2(s, u_s^m) + f(s), v \rangle ds \\
\quad + \int_0^t (G_1(s, u_s^m) + g(s), v) dW(s), \quad \forall t \in [0, T], \quad \forall v \in V_m, \\
u^m(t) = P_m \psi(t), \quad t \in [-h, 0].
\end{cases} \tag{\widehat{P}^m}$$

The existence and uniqueness of solution to problem (\widehat{P}^m) is therefore guaranteed by Theorem 2.1 (notice that in this case, $V = H = V^* = V_m$).

Observe also that $u^m = P_m \psi$ in $[-h, 0]$, and, by the particular choice of the basis $\{v_i\}$, for each $m \geq 1$, $\|u^m(t)\| \leq \|\psi(t)\|$ for all $t \in [-h, 0]$, and the sequence u^m converges to ψ in $I^2(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$.

Applying Itô's formula to $|u^m(t)|^2$, and using $A(t, 0) = 0$ a.e. $t \in (0, T)$, (A.5), (F₂.2) and (G₁.2), we have (bearing in mind that we are assuming $\lambda = 0$):

$$\begin{aligned} |u^m(t)|^2 + \alpha \int_0^t \|u^m(s)\|^2 ds &\leq |P_m \psi(0)|^2 + 2 \int_0^t \langle f(s), u^m(s) \rangle ds + \int_0^t |g(s)|^2 ds \\ &\quad + 2 \int_0^t (G_1(s, u_s^m), g(s)) ds + \widehat{\lambda} \int_{-h}^0 \|P_m \psi(s)\|^2 ds \\ &\quad + 2 \int_0^t (G_1(s, u_s^m) + g(s), u^m(s)) dW(s), \end{aligned}$$

for all $t \in [0, T]$, and therefore,

$$\begin{aligned} E |u^m(t)|^2 + \alpha E \int_0^t \|u^m(s)\|^2 ds &\leq E |\psi(0)|^2 + 2E \int_0^t \langle f(s), u^m(s) \rangle ds + E \int_0^t |g(s)|^2 ds \\ &\quad + \widehat{\lambda} E \int_{-h}^0 \|\psi(s)\|^2 ds + 2E \int_0^t (G_1(s, u_s^m), g(s)) ds, \quad (3.5) \end{aligned}$$

for all $t \in [0, T]$. But

$$2E \int_0^t \langle f(s), u^m(s) \rangle ds \leq \frac{3}{\alpha} E \int_0^t \|f(s)\|_*^2 ds + \frac{\alpha}{3} E \int_0^t \|u^m(s)\|^2 ds,$$

and, by (G₁.2) and (G₁.4),

$$2E \int_0^t (G_1(s, u_s^m), g(s)) ds \leq \frac{3K_{G_1}}{\alpha} E \int_0^t |g(s)|^2 ds + \frac{\alpha}{3} E \int_{-h}^t \|u^m(s)\|^2 ds.$$

Then, from (3.5) we can obtain

$$\begin{aligned} E |u^m(t)|^2 + \frac{\alpha}{3} E \int_0^t \|u^m(s)\|^2 ds &\leq E |\psi(0)|^2 + \frac{3}{\alpha} E \int_0^t \|f(s)\|_*^2 ds \\ &\quad + \left(\widehat{\lambda} + \frac{\alpha}{3} \right) E \int_{-h}^0 \|\psi(s)\|^2 ds \\ &\quad + \left(1 + \frac{3K_{G_1}}{\alpha} \right) E \int_0^t |g(s)|^2 ds, \end{aligned}$$

for all $t \in [0, T]$, and, consequently, the sequence $\{u^m\}_{m \geq 1}$ is bounded in $I^2(-h, T; V)$ (notice that u^m converges to ψ in $I^2(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$), and the sequence $\{u^m(T)\}_{m \geq 1}$ is bounded in $L^2(\Omega; H)$. In fact, from (A.3), (F₂.2), (F₂.4), (G₁.2) and (G₁.4), we can affirm that

$$\begin{aligned} \{u^m(\cdot)\}_{m \geq 1} &\text{ is bounded in } I^2(-h, T; V), \\ \{u^m(T)\}_{m \geq 1} &\text{ is bounded in } L^2(\Omega; H), \end{aligned}$$

$\{A(\cdot, u^m(\cdot))\}_{m \geq 1}$ is bounded in $I^2(0, T; V^*)$,

$\{F_2(\cdot, u^m)\}_{m \geq 1}$ is bounded in $I^2(0, T; V^*)$,

$\{G_1(\cdot, u^m)\}_{m \geq 1}$ is bounded in $I^2(0, T; H)$.

Thus, we can ensure that there exists a subsequence $\{u^{m_k}\}$ of $\{u^m\}$, a random variable $\xi \in L^2(\Omega; H)$ and four processes $u \in I^2(-h, T; V)$, $\eta \in I^2(0, T; V^*)$, $\sigma \in I^2(0, T; V^*)$, and $\zeta \in I^2(0, T; H)$ such that

$$\begin{aligned} u^{m_k}(T) &\rightharpoonup \xi \text{ in } L^2(\Omega; H), \\ u^{m_k}(\cdot) &\rightharpoonup u \text{ in } I^2(-h, T; V), \\ A(\cdot, u^{m_k}(\cdot)) &\rightharpoonup \eta \text{ in } I^2(0, T; V^*), \\ F_2(\cdot, u^{m_k}) &\rightharpoonup \sigma \text{ in } I^2(0, T; V^*), \\ G_1(\cdot, u^{m_k}) &\rightharpoonup \zeta \text{ in } I^2(0, T; H). \end{aligned}$$

Now, let χ be an absolutely continuous real function on $[0, T]$ such that $\chi' \in L^2(0, T)$ and $\chi(T) = 0$. Fix m_j and $v \in V_{m_j}$. Applying Itô's formula to $(u^{m_k}(t), v)\chi(t)$ with $1 \leq m_j \leq m_k$, we get

$$\begin{aligned} 0 &= (P_{m_k} \psi(0), v)\chi(0) + \int_0^T \langle A(s, u^{m_k}(s)) + F_2(s, u_s^{m_k}) + f(s), v \rangle \chi(s) ds \\ &\quad + \int_0^T (G_1(s, u_s^{m_k}) + g(s), v)\chi(s) dW(s) + \int_0^T (u^{m_k}(s), v)\chi'(s) ds. \end{aligned} \quad (3.6)$$

We can take limits in (3.6) as $m_k \rightarrow \infty$, and observing that m_j is arbitrary and that $\cup_{m \geq 1} V_m$ is dense in V , we can ensure that

$$\begin{aligned} 0 &= (\psi(0), v)\chi(0) + \int_0^T \langle \eta(s) + \sigma(s) + f(s), v \rangle \chi(s) ds \\ &\quad + \int_0^T (\zeta(s) + g(s), v)\chi(s) dW(s) + \int_0^T (u(s), v)\chi'(s) ds, \forall v \in V. \end{aligned} \quad (3.7)$$

Consequently, if we fix $t \in (0, T)$, and for each integer $n \geq 1$ such that $t + 1/2n \leq T$ we denote by χ^n the function

$$\chi^n(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq t - 1/2n, \\ \frac{1}{2} + n(t - s) & \text{if } t - 1/2n \leq s \leq t + 1/2n, \\ 0 & \text{if } t + 1/2n \leq s \leq T, \end{cases}$$

we deduce from (3.7),

$$\begin{aligned} n \int_{t-1/2n}^{t+1/2n} (u(s), v) ds &= (\psi(0), v) + \int_0^T \langle \eta(s) + \sigma(s) + f(s), v \rangle \chi^n(s) ds \\ &\quad + \int_0^T (\zeta(s) + g(s), v)\chi^n(s) dW(s), \forall v \in V. \end{aligned} \quad (3.8)$$

We can take limits in (3.8) a.e. $t \in (0, T)$, and obtain

$$(u(t), v) = (\psi(0), v) + \int_0^t \langle \eta(s) + \sigma(s) + f(s), v \rangle ds + \int_0^t (\zeta(s) + g(s), v) dW(s), \quad (3.9)$$

a.e. $t \in (0, T)$, for all $v \in V$. Taking into account the separability of V , (3.9) implies

$$u(t) = \psi(0) + \int_0^t (\eta(s) + \sigma(s) + f(s)) ds + \int_0^t (\zeta(s) + g(s)) dW(s) \text{ in } V^*, \quad (3.10)$$

a.e. $t \in (0, T)$.

Thus, u is a.e. in $[0, T]$ equal to a process in $L^2(\Omega; C(0, T; H))$ which satisfies (3.10) for all $t \in [0, T]$. We denote again by u such a process, and thus u satisfies

$$\begin{cases} u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)) \\ u(t) = \psi(0) + \int_0^t (\eta(s) + \sigma(s) + f(s)) ds + \int_0^t (\zeta(s) + g(s)) dW(s), \quad t \in [0, T], \\ u(T) = \xi, \\ u(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \quad (3.11)$$

Now, to finish the proof, it is enough to prove that $\eta(s) + \sigma(s) = A(s, u(s)) + F_2(s, u_s)$ and $\zeta(s) = G_1(s, u_s)$.

Consider $X \in I^2(-h, T; V)$ such that $X = \psi$ in $(-h, 0)$, and denote

$$\begin{aligned} x^{m_k} &= 2E \int_0^T \langle A(t, u^{m_k}(t)) - A(t, X(t)) + F_2(t, u_t^{m_k}) - F_2(t, X_t), u^{m_k}(t) - X(t) \rangle dt \\ &\quad + \alpha E \int_0^T \|u^{m_k}(t) - X(t)\|^2 dt - \widehat{\lambda} E \int_{-h}^0 \|\psi(t) - P_{m_k} \psi(t)\|^2 ds \\ &\quad + E \int_0^T |G_1(t, u_t^{m_k}) - G_1(t, X_t)|^2 dt, \end{aligned}$$

$$y^{m_k} = 2E \int_0^T \langle A(t, u^{m_k}(t)) + F_2(t, u_t^{m_k}), u^{m_k}(t) \rangle dt + E \int_0^T |G_1(t, u_t^{m_k})|^2 dt + \alpha E \int_0^T \|u^{m_k}(t)\|^2 dt.$$

Then, thanks to condition (A.5) (with $\lambda = 0$), $x^{m_k} \leq 0$.

Observe also that

$$\begin{aligned} x^{m_k} - y^{m_k} &= 2E \int_0^T \langle -A(t, X(t)) - F_2(t, X_t), u^{m_k}(t) \rangle dt \\ &\quad + 2E \int_0^T \langle A(t, u^{m_k}(t)) - A(t, X(t)) + F_2(t, u_t^{m_k}) - F_2(t, X_t), -X(t) \rangle dt \\ &\quad + E \int_0^T |G_1(t, X_t)|^2 dt - 2E \int_0^T (G_1(t, u_t^{m_k}), G_1(t, X_t)) dt + \alpha E \int_0^T \|X(t)\|^2 dt \\ &\quad - 2\alpha E \int_0^T ((u^{m_k}(t), X(t))) dt - \widehat{\lambda} E \int_{-h}^0 \|\psi(t) - P_{m_k} \psi(t)\|^2 ds, \end{aligned}$$

and consequently,

$$\begin{aligned}
\lim_{k \rightarrow \infty} (x^{m_k} - y^{m_k}) &= 2E \int_0^T \langle -A(t, X(t)) - F_2(t, X_t), u(t) \rangle dt \\
&\quad + 2E \int_0^T \langle \eta(t) - A(t, X(t)) + \sigma(t) - F_2(t, X_t), -X(t) \rangle dt \\
&\quad + E \int_0^T |G_1(t, X_t)|^2 dt - 2E \int_0^T (\zeta(t), G_1(t, X_t)) dt \\
&\quad - 2\alpha E \int_0^T ((u(t), X(t))) dt + \alpha E \int_0^T \|X(t)\|^2 dt. \tag{3.12}
\end{aligned}$$

Applying Itô's formula to $|u^{m_k}(t)|^2$ on the interval $[0, T]$,

$$\begin{aligned}
E |u^{m_k}(T)|^2 &\leq E |\psi(0)|^2 + E \int_0^T |G_1(t, u_t^{m_k}) + g(t)|^2 dt \\
&\quad + 2E \int_0^T \langle A(t, u^{m_k}(t)) + F_2(t, u_t^{m_k}) + f(t), u^{m_k}(t) \rangle dt,
\end{aligned}$$

and, thus

$$\begin{aligned}
y^{m_k} &\geq E |u^{m_k}(T)|^2 - E |\psi(0)|^2 - E \int_0^T |g(t)|^2 dt + \alpha E \int_0^T \|u^{m_k}(t)\|^2 dt \\
&\quad - 2E \int_0^T (G_1(t, u_t^{m_k}), g(t)) dt - 2E \int_0^T \langle f(t), u^{m_k}(t) \rangle dt.
\end{aligned}$$

Letting $k \rightarrow \infty$,

$$\begin{aligned}
\liminf_{k \rightarrow \infty} y^{m_k} &\geq E |u(T)|^2 - E |\psi(0)|^2 - E \int_0^T |g(t)|^2 dt + \alpha E \int_0^T \|u(t)\|^2 dt \\
&\quad - 2E \int_0^T (\zeta(t), g(t)) dt - 2E \int_0^T \langle f(t), u(t) \rangle dt. \tag{3.13}
\end{aligned}$$

Applying once again Itô's formula to $|u(t)|^2$ on $[0, T]$,

$$E |u(T)|^2 = E |\psi(0)|^2 + E \int_0^T |\zeta(t) + g(t)|^2 dt + 2E \int_0^T \langle \eta(t) + \sigma(t) + f(t), u(t) \rangle dt,$$

and so, from (3.13)

$$\liminf_{k \rightarrow \infty} y^{m_k} \geq 2E \int_0^T \langle \eta(t) + \sigma(t), u(t) \rangle dt + E \int_0^T |\zeta(t)|^2 dt + \alpha E \int_0^T \|u(t)\|^2 dt. \tag{3.14}$$

From (3.12) and (3.14) we have

$$\begin{aligned}
0 &\geq \liminf_{k \rightarrow \infty} x^{m_k} \geq 2E \int_0^T \langle \eta(t) - A(t, X(t)) + \sigma(t) - F_2(t, X_t), u(t) - X(t) \rangle dt \\
&\quad + E \int_0^T |\zeta(t) - G_1(t, X_t)|^2 dt + \alpha E \int_0^T \|u(t) - X(t)\|^2 dt. \tag{3.15}
\end{aligned}$$

If we take $X(t) = u(t)$ in (3.15), it follows that $\zeta(t) = G_1(t, u_t)$, $t \in [0, T]$. Now, we will set $X(t) = u(t) - \delta Z(t)$, where $\delta > 0$ and $Z \in I^2(-h, T; V)$ is such that $Z = 0$ in $(-h, 0)$. Then, by (3.15),

$$\begin{aligned} 0 \geq 2E \int_0^T \langle \eta(t) - A(t, u(t) - \delta Z(t)) + \sigma(t) - F_2(t, u_t - \delta Z_t), \delta Z(t) \rangle dt \\ + E \int_0^T |G_1(t, u_t) - G_1(t, u_t - \delta Z_t)|^2 dt. \end{aligned} \quad (3.16)$$

Dividing by δ in (3.16), and letting $\delta \rightarrow 0$, we get by (A.2), (F2.2) and (F2.4),

$$2E \int_0^T \langle \eta(t) - A(t, u(t)) + \sigma(t) - F_2(t, u_t), Z(t) \rangle dt \leq 0,$$

and since $Z \in I^2(0, T; V)$ is arbitrary, clearly $\eta(t) + \sigma(t) = A(t, u(t)) + F_2(t, u_t)$ in $[0, T]$.

Step 2. Now, we consider problem (P) under the conditions in the theorem. We denote $u^0 \equiv 0$, and define by recurrence a sequence $\{u^n\}_{n \geq 1}$ of processes by

$$\left\{ \begin{array}{l} u^n \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \\ u^n(t) = \psi(0) + \int_0^t (A(s, u^n(s)) - \frac{\lambda}{2} u^n(s)) ds + \frac{\lambda}{2} \int_0^t u^{n-1}(s) ds \\ \quad + \int_0^t (F_1(s, u_s^{n-1}) + F_2(s, u_s^n) + f(s)) ds \\ \quad + \int_0^t (G_0(s, u_s^{n-1}) + G_1(s, u_s^n) + g(s)) dW(s), \quad t \in [0, T], \\ u^n(t) = \psi(t), \quad t \in [-h, 0]. \end{array} \right. \quad (P^n)$$

Observe that if $u^{n-1} \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, then $F_1(t, u_t^{n-1}) \in I^2(0, T; V^*)$, and $G_0(t, u_t^{n-1}) \in I^2(0, T; H)$. Moreover, the family of operators defined by $\tilde{A}(t, v) = A(t, v) - \frac{\lambda}{2} v \forall v \in V$, a.e. $t \in (0, T)$, satisfies conditions (A.1) – (A.5) with $\lambda = 0$. Consequently, we can use Step 1 to ensure that problem (P^n) has a unique solution.

Now, arguing as in the proof of Theorem 2.1, we can prove that $\{u^n\}_{n \geq 1}$ is a Cauchy sequence in $I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, and thus, it converges to a process $u \in I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, which will be the solution to (P).

In order to obtain our objective, we first apply Itô's formula to the process $u^{n+1}(t) - u^n(t)$, $t \geq 0$, $n \geq 1$, and using (A.5) we have

$$\begin{aligned} & |u^{n+1}(t) - u^n(t)|^2 + \alpha \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\ & \leq \lambda \int_0^t (u^{n+1}(s) - u^n(s), u^n(s) - u^{n-1}(s)) ds \\ & \quad + 2 \int_0^t \langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle ds \\ & \quad + 2 \int_0^t (G_0(s, u_s^n) - G_0(s, u_s^{n-1}), G_1(s, u_s^{n+1}) - G_1(s, u_s^n)) ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t |G_0(s, u_s^n) - G_0(s, u_s^{n-1})|^2 ds \\
& + 2 \int_0^t (G_0(s, u_s^n) - G_0(s, u_s^{n-1}), u^{n+1}(s) - u^n(s)) dW(s) \\
& + 2 \int_0^t (G_1(s, u_s^{n+1}) - G_1(s, u_s^n), u^{n+1}(s) - u^n(s)) dW(s), \tag{3.17}
\end{aligned}$$

which, together with conditions (F_{1.4}), (G_{0.4}) and (G_{1.4}) imply

$$\begin{aligned}
& E |u^{n+1}(t) - u^n(t)|^2 + \alpha E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\
& \leq \frac{3\alpha}{4} E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds + \frac{\lambda^2 \beta^2}{\alpha} \int_0^t \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds \\
& + \frac{4}{\alpha} C^{F_1} \int_0^t \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds \\
& + \left(\frac{3K_{G_1}}{\alpha} + 1 \right) C^{G_0} \int_0^t \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds, \tag{3.18}
\end{aligned}$$

where $\beta > 0$ is the constant such that $|v| \leq \beta \|v\| \forall v \in V$. Consequently, (3.18) yields

$$\begin{aligned}
& \sup_{0 \leq s \leq t} E |u^{n+1}(s) - u^n(s)|^2 + \frac{\alpha}{4} E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds \\
& \leq k \int_0^t \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds, \tag{3.19}
\end{aligned}$$

for all $t \in [0, T]$ and all $n \geq 1$, where $k = \frac{2\lambda^2 \beta^2}{\alpha} + \frac{8}{\alpha} C^{F_1} + 2 \left(\frac{3K_{G_1}}{\alpha} + 1 \right) C^{G_0}$. Now, if we denote

$$\rho^n(t) = \sup_{0 \leq s \leq t} E |u^{n+1}(s) - u^n(s)|^2 + \frac{\alpha}{4} E \int_0^t \|u^{n+1}(s) - u^n(s)\|^2 ds, \quad \forall n \geq 1, \forall t \in [0, T],$$

we can deduce from (3.19) that

$$\rho^n(t) \leq \frac{(kT)^{n-1}}{(n-1)!} \rho^1(T), \quad \forall n \geq 1, \forall t \in [0, T],$$

and thus, $\forall n \geq 1$,

$$\sup_{0 \leq s \leq T} E |u^{n+1}(s) - u^n(s)|^2 + \frac{\alpha}{4} E \int_0^T \|u^{n+1}(s) - u^n(s)\|^2 ds \leq \frac{(kT)^{n-1}}{(n-1)!} \rho^1(T), \tag{3.20}$$

and, in particular, $\{u^n\}_{n \geq 1}$ is a Cauchy sequence in $I^2(-h, T; V)$.

Now, in order to prove that $\{u^n\}_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega; C(-h, T; H))$, we consider again (3.17), take $\sup_{0 \leq s \leq T}$ and, finally, expectation, so that we obtain

$$E \left(\sup_{0 \leq s \leq T} |u^{n+1}(s) - u^n(s)|^2 \right)$$

$$\begin{aligned}
&\leq |\lambda| E \int_0^T |(u^{n+1}(s) - u^n(s), u^n(s) - u^{n-1}(s))| ds \\
&\quad + 2E \int_0^T |\langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle| ds \\
&\quad + 2E \int_0^T |(G_0(s, u_s^n) - G_0(s, u_s^{n-1}), G_1(s, u_s^{n+1}) - G_1(s, u_s^n)| ds \\
&\quad + 2E \left(\sup_{0 \leq s \leq T} \int_0^s (G_0(\theta, u_\theta^n) - G_0(\theta, u_\theta^{n-1}), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right) \\
&\quad + 2E \left(\sup_{0 \leq s \leq T} \int_0^s (G_1(\theta, u_\theta^{n+1}) - G_1(\theta, u_\theta^n), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right) \\
&\quad + E \int_0^T |G_0(s, u_s^n) - G_0(s, u_s^{n-1})|^2 ds \tag{3.21}
\end{aligned}$$

for all $t \in [0, T]$ and all $n \geq 1$.

Using (F_{1.4}), we obtain

$$\begin{aligned}
&2E \int_0^T |\langle F_1(s, u_s^n) - F_1(s, u_s^{n-1}), u^{n+1}(s) - u^n(s) \rangle| ds \\
&\leq C^{F_1} \int_0^T \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds + E \int_0^T \|u^{n+1}(s) - u^n(s)\|^2 ds. \tag{3.22}
\end{aligned}$$

From Burkholder-Davis-Gundy's inequality, (G_{0.4}) and (G_{1.4}), we have

$$\begin{aligned}
&2E \left(\sup_{0 \leq s \leq T} \int_0^s (G_0(\theta, u_\theta^n) - G_0(\theta, u_\theta^{n-1}), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right) \\
&\leq \frac{1}{3} E \left(\sup_{0 \leq s \leq T} |u^{n+1}(s) - u^n(s)|^2 \right) + 27C^{G_0} \int_0^T \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds, \tag{3.23}
\end{aligned}$$

$$E \int_0^T |G_0(s, u_s^n) - G_0(s, u_s^{n-1})|^2 ds \leq C^{G_0} \int_0^T \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds, \tag{3.24}$$

$$\begin{aligned}
&2E \int_0^T |(G_0(s, u_s^n) - G_0(s, u_s^{n-1}), G_1(s, u_s^{n+1}) - G_1(s, u_s^n)| ds \\
&\leq C^{G_0} \int_0^T \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds + K_{G_1} E \int_0^T \|u^{n+1}(s) - u^n(s)\|^2 ds, \tag{3.25}
\end{aligned}$$

and

$$\begin{aligned}
&2E \left(\sup_{0 \leq s \leq T} \int_0^s (G_1(\theta, u_\theta^{n+1}) - G_1(\theta, u_\theta^n), u^{n+1}(\theta) - u^n(\theta)) dW(\theta) \right) \\
&\leq \frac{1}{3} E \left(\sup_{0 \leq s \leq T} |u^{n+1}(s) - u^n(s)|^2 \right) + 27K_{G_1} E \int_0^T \|u^{n+1}(s) - u^n(s)\|^2 ds. \tag{3.26}
\end{aligned}$$

Also,

$$\begin{aligned}
&|\lambda| E \int_0^T (u^{n+1}(s) - u^n(s), u^n(s) - u^{n-1}(s)) ds \\
&\leq \frac{1}{2} E \int_0^T \|u^{n+1}(s) - u^n(s)\|^2 ds + \frac{\lambda^2 \beta^2}{2} \int_0^T \sup_{0 \leq \theta \leq s} E |u^n(\theta) - u^{n-1}(\theta)|^2 ds. \tag{3.27}
\end{aligned}$$

From (3.20)-(3.27), we deduce that $\{u^n\}_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega; C(-h, T; H))$. Thus, there exists u such that $u^n \rightarrow u$ in $I^2(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$. Now, by a similar argument to the one in the proof of theorem 2.1, we can deduce that u is the solution of problem (P).■

Remark 3.1. The hypothesis concerning the compactness of the injection $V \subset H$ can be omitted if, for example, $\psi \equiv 0$.

Remark 3.2. Theorems 2.1. and 3.1. can be extended to the case in which $W(t)$ is an \mathbb{R}^n -valued (or Hilbert valued) Wiener process.

4. An example

To illustrate our theory, mainly Theorem 3.1, we shall consider the following situation, which cannot be handled with the results in Caraballo [2] or Caraballo et al. [4].

Assume $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set. Let us set $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$ and $V^* = H^{-1}(\mathcal{O})$.

Let $\phi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous map such that there exists $c_\phi > 0$ such that $|\phi(t, x)|_{\mathbb{R}^n} \leq c_\phi |x|_{\mathbb{R}^n}$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$, and suppose that

$$(\phi(t, x) - \phi(t, y)) \cdot (x - y) \leq 0 \quad \forall t \in [0, T], \quad \forall x, y \in \mathbb{R}^n, \quad (4.1)$$

where we denote by \cdot the escalar product in \mathbb{R}^n . It is easy to see that the family of operators $A(t, \cdot)$ defined by

$$\langle A(t, u), v \rangle = - \int_{\mathcal{O}} \nabla u(x) \cdot \nabla v(x) dx - \int_{\mathcal{O}} \phi(t, \nabla u(x)) \cdot \nabla v(x) dx \quad \forall t \in [0, T], \quad \forall u, v \in V, \quad (4.2)$$

satisfies hypotheses (A.1)-(A.4), with $\lambda = 0$ and $\alpha \leq 2$.

Let us consider now a measurable map $k_1 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^n$ and a measurable function $\omega_1 : [0, T] \rightarrow \mathbb{R}$ such that $0 \leq \omega_1(t) \leq h$ for all $t \in [0, T]$. Suppose that $k_1(t, 0) = 0$, $\forall t \in [0, T]$, and that there exists $L_{k_1} > 0$ such that

$$|k_1(t, a) - k_1(t, b)|_{\mathbb{R}^n} \leq L_{k_1} |a - b|, \quad \forall t \in [0, T], \quad \forall a, b \in \mathbb{R}. \quad (4.3)$$

Denote by $F_1(t, \cdot)$ the family of operators defined by

$$\langle F_1(t, \xi), v \rangle = - \int_{\mathcal{O}} k_1(t, \xi(-\omega_1(t)))(x) \cdot \nabla v(x) dx, \quad \forall \xi \in C(-h, 0; H), \quad \forall v \in V, \quad (4.4)$$

for each $t \in [0, T]$.

Then, the family $F_1(t, \cdot)$ satisfies assumptions (F₁.1)-(F₁.4), with $C_{F_1} = L_{k_1}^2$ and $C^{F_1} = L_{k_1}^2$.

Consider also $k_2 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, measurable, and $\omega_2 \in C^1([0, T])$ such that $0 \leq \omega_2(t) \leq h$ for all $t \in [0, T]$, and $\omega_2^* = \max_{t \in [0, T]} \omega_2'(t) < 1$. Suppose that $k_2(t, 0) = 0$, $\forall t \in [0, T]$, and that there exists $L_{k_2} > 0$ such that

$$|k_2(t, x) - k_2(t, y)|_{\mathbb{R}^n} \leq L_{k_2} |x - y|_{\mathbb{R}^n}, \quad \forall t \in [0, T], \quad \forall x, y \in \mathbb{R}^n. \quad (4.5)$$

Denote by $F_2(t, \cdot)$ the family of operators defined by

$$\langle F_2(t, \xi), v \rangle = - \int_{\mathcal{O}} k_2(t, \nabla \xi(-\omega_2(t))(x)) \cdot \nabla v(x) dx, \quad \forall \xi \in C(-h, 0; V), \quad \forall v \in V, \quad (4.6)$$

for each $t \in [0, T]$. Then, the family $F_2(t, \cdot)$ satisfies hypotheses (F_{2.1})-(F_{2.4}), with $C_{F_2} = L_{k_2}^2$ and $K_{F_2} = \frac{L_{k_2}^2}{1 - \omega_2^*}$.

Finally, let $l_0 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $l_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be two measurable functions, such that $l_0(t, 0) = l_1(t, 0) = 0$ for all $t \in [0, T]$, and there exist $L_{l_0} > 0$ and $L_{l_1} > 0$ such that

$$|l_0(t, a) - l_0(t, b)| \leq L_{l_0} |a - b|, \quad \forall t \in [0, T], \quad \forall a, b \in \mathbb{R}, \quad (4.6)$$

and

$$|l_1(t, x) - l_1(t, y)| \leq L_{l_1} |x - y|_{\mathbb{R}^n}, \quad \forall t \in [0, T], \quad \forall x, y \in \mathbb{R}^n. \quad (4.7)$$

Let us also fix $\rho_i : [0, T] \rightarrow \mathbb{R}$, $i = 0, 1$, two measurable functions such that $0 \leq \rho_i(t) \leq h$ for all $t \in [0, T]$ and $i = 0, 1$, $\rho_1 \in C^1([0, T])$, and $\rho_1^* = \max_{t \in [0, T]} \rho_1'(t) < 1$.

Then, if we define

$$G_0(t, \xi)(x) = l_0(t, \xi(-\rho_0(t))(x)), \quad \forall t \in [0, T], \quad \forall \xi \in C(-h, 0; H), \quad \text{a.e. } x \in \mathcal{O}, \quad (4.8)$$

and

$$G_1(t, \xi)(x) = l_1(t, \nabla \xi(-\rho_1(t))(x)), \quad \forall t \in [0, T], \quad \forall \xi \in C(-h, 0; V), \quad \text{a.e. } x \in \mathcal{O}, \quad (4.9)$$

it is easy to check that G_0 satisfies (G_{0.1})-(G_{0.4}), and that G_1 satisfies hypotheses (G_{1.1})-(G_{1.4}), with $C_{G_0} = C^{G_0} = L_{l_0}^2$, $C_{G_1} = L_{l_1}^2$, and $K_{G_1} = \frac{L_{l_1}^2}{1 - \rho_1^*}$.

As for hypothesis (A.5), it is fulfilled with $\hat{\lambda}$ large enough provided

$$\frac{2L_{k_2}}{\sqrt{1 - \omega_2^*}} + \frac{L_{l_1}^2}{1 - \rho_1^*} < 2. \quad (4.10)$$

Consequently, under all the hypotheses above, we can ensure that given $\psi \in I^2(-h, 0; H_0^1(\mathcal{O})) \cap L^2(\Omega; C(-h, 0; L^2(\mathcal{O})))$, $f \in I^2(0, T; H^{-1}(\mathcal{O}))$, and $g \in I^2(0, T; L^2(\mathcal{O}))$, there exists a unique solution $u \in I^2(-h, T; H_0^1(\mathcal{O})) \cap L^2(\Omega; C(-h, T; L^2(\mathcal{O})))$ to the corresponding problem (P). Such a solution, satisfies, in a generalized sense, the problem

$$\begin{cases} \frac{\partial u(t)}{\partial t} = \Delta u(t) + \nabla \cdot (\phi(t, \nabla u(t))) + \nabla \cdot (k_2(t, \nabla u(t - \omega_2(t)))) + \nabla \cdot (k_1(t, u(t - \omega_1(t)))) \\ + f(t) + (l_1(t, \nabla u(t - \rho_1(t))) + l_0(t, u(t - \rho_0(t))) + g(t)) \frac{\partial W(t)}{\partial t} \quad \text{in } \mathcal{O} \times (0, T), \\ u(0) = 0 \quad \text{on } \partial \mathcal{O} \times (0, T), \\ u(t) = \psi(t) \quad \text{in } \mathcal{O} \times [-h, 0], \end{cases}$$

where, for a vector function $\vec{v} = (v_1, \dots, v_n)$ defined on \mathcal{O} , we denote by $\nabla \cdot \vec{v}$ the divergence of \vec{v} defined by $\nabla \cdot \vec{v} = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$.

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