Autonomous and non-autonomous attractors for differential equations with delays

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Dedicated to Professor George R. Sell on the occasion of his 65th birthday

Abstract

The asymptotic behaviour of some types of retarded differential equations, with both variable and distributed delays, is analyzed. In fact, the existence of global attractors is established for different situations: with and without uniqueness, and for both autonomous and non-autonomous cases, using the classical notion of attractor and the recently new concept of pullback one respectively.

Key words: autonomous and non-autonomous (pullback) attractors, delay differential equations, integro-differential equations, non-uniqueness of solutions, multi-valued semiflows, multi-valued processes. *Mathematics Subject Classifications (2000):* 34D45, 34K20, 34K25, 37L30

1 Introduction

Physical reasons, non instant transmission phenomena, memory processes, and specially biological motivations (e.g. [20], [31], [37]) like species' growth or incubating time on disease models among many others, make retarded differential equations an important area of applied mathematics.

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Moreover, the asymptotic behaviour of such models has meaningful interpretations like permanence of species on a given domain, with or without competition, their possible extinction, instability and sometimes chaotic developments, being therefore of obvious interest. However, most studies use to deal with stability concepts concerning fixed points. The study of global attractors and the equations for which the existence of an attractor (and so both stable and unstable regions) can be ensured is therefore an interesting subject.

The theory of global attractors for autonomous systems as developed by Hale in [23] owes much to examples arising in the study of (finite and infinite) retarded functional differential equations [26] (for slightly different approaches see Babin and Vishik [3], Ladyzhenskaya [32], or Temam [41]). Although the classical theory can be extended in a relatively straightforward manner to deal with time-periodic equations, general non-autonomous equations such as

$$x'(t) = F(t, x(t), x(t - \rho(t))), \tag{1}$$

with variable delay, or

$$x'(t) = \int_{-h}^{0} b(t, s, x(t + s))\, \mathrm{d}s, \tag{2}$$

for distributed delay terms, including the possibility of being $h = +\infty$, fall outside its scope.

Recently, a theory of 'pullback attractors' has been developed for stochastic and non-autonomous systems in which the trajectories can be unbounded when time increases to infinity, allowing many of the ideas for the autonomous theory to be extended to deal with such examples. In this case, the global attractor is defined as a parameterized family of sets $A(t)$ depending on the final time, such that attracts solutions of the system 'from $-\infty$', i.e. initial time goes to $-\infty$ while the final time remains fixed.

Moreover, in [9] this theory has been successfully extended to deal with variable delay equations, and some sufficient conditions have been proved to guarantee the existence of pullback attractor for equation (1) (see also Cheban [12] and Cheban & Schmalfuss [15]).

However, as far as we know, there exists a wide variety of situations of great interest from the point of view of applications that still has not been analyzed. For instance, delay differential equations without uniqueness (in both the autonomous and non-autonomous framework), differential inclusions, integro-differential equations in a non-autonomous context with or without uniqueness, all the previous situations but considering infinite delays, etc...

Consequently, we are mainly interested in providing some results on two of the previous situations: autonomous functional equations without uniqueness, and non-autonomous functional and/or integro-differential equations with and without uniqueness with finite delay.

The content of the paper is as follows. In Section 2 we include some preliminaries on the existence of solutions to functional differential equations and their properties. The construction of the semiflows and processes associated to our delay models is carried out in Section 3. Some results ensuring the existence of autonomous and non-autonomous attractors are collected in Section 4. Finally, in Section 5, which is the main one, our theory is applied to some interesting and general situations arising in applications and several examples are exhibited.

2 Preliminaries

First, let us introduce some notation.

Let $h > 0$ be a given positive number (the delay time) and denote by $\mathcal{C}$ the Banach space $C([-h,0];\mathbb{R}^n)$ endowed with the norm $\|\psi\| = \sup_{\sigma\in[-h,0]} |\psi(\sigma)|$, which is the *usual* phase space when we deal with delay differential equations. However, it is sometimes useful to consider the solutions as mappings from $\mathbb{R}$ into $\mathbb{R}^n$ (we will consider in $\mathbb{R}^n$ its usual Euclidean topology and denote by $\langle\cdot,\cdot\rangle$, $|\cdot|$ its scalar product and norm, respectively). Let us point out that the case of infinite delay needs a more careful choice of the phase space (cf. [1], [25]), but we will not get into those details here. By x_t we will denote the element in $\mathcal{C}$ given by $x_t(s) = x(t+s)$ for all $s \in [-h,0]$. Also, it will be useful to denote $\mathbb{R}_d = \{(t,s) \in \mathbb{R}^2,\ t \geq s\}$.

We will now recall some well known results for a general functional differential equation with finite delay (cf. [24, Ch.2]):

$$x'(t) = f(t, x_t), \qquad x_{t_0} = \psi \in \mathcal{C}, \tag{3}$$

Theorem 1 (Existence of solutions) *Suppose Ω is an open subset in $\mathbb{R}\times\mathcal{C}$ and $f \in C(\Omega;\mathbb{R}^n)$. If $(t_0,\psi) \in \Omega$, then there is a solution of (3), i.e. a function $x : [t_0-h, t_0+\alpha) \to \mathbb{R}^n$ with $\alpha > 0$, which satisfies (3) in a classical sense.*

Remark 2 *As in the non-delay case, uniqueness results hold if, for instance, f satisfies a locally Lipschitz condition on compact sets with respect to its second variable (cf. [24, Ch.2, Th.2.3]).*
However, we will be concerned with both situations, i.e. with and without uniqueness, establishing a more general theory.

The existence of global solutions in time of (3) is obviously essential for our purpose. We have the following result from [24, Ch.2]:

Theorem 3 (Non-continuable solutions) *Suppose Ω is an open set in $\mathbb{R}\times\mathcal{C}$ and $f \in C(\Omega;\mathbb{R}^n)$. If x is a non-continuable solution of equation (3) on $[t_0-h,b)$, then, for any compact set W in Ω, there is a t_W such that $(t,x_t)\notin W$ for $t_W \le t < b$.*

As a straightforward consequence of this result, we have an analogous result to the non-delay case with non-explosion a priori estimates.

Suppose E is a metric space and denote by $P(E)$, $C(E)$, $\mathcal{B}(E)$ and $K(E)$ the sets of nonempty, nonempty and closed, nonempty and bounded, and nonempty and compact subsets of E.

Definition 4 *Given two metric spaces X and Y, a single (or multi-valued resp.) mapping $\Upsilon : X \to Y$ ($P(Y)$ resp.) is said to be bounded if for every $B \in \mathcal{B}(X)$, $\Upsilon(B) \in \mathcal{B}(Y)$.*

Remark 5 *Observe that, if the map f is only bounded, we cannot in general ensure that the solutions of (3) are defined in the future, even in the case without delays, as the simple example $x' = x^2$ shows.*

Corollary 6 *Let $f \in C(\mathbb{R}\times\mathcal{C};\mathbb{R}^n)$ be a bounded map, and assume that the equation in (3) satisfies the property that* possible solutions *x corresponding to an initial datum ψ remain in a bounded set of $\mathcal{C}$, in other words,*

$$\begin{aligned} &\forall\,(t,t_0)\in\mathbb{R}_d, \quad \forall\psi\in\mathcal{C}, \quad \exists\; D = D(t,t_0,\psi)\in\mathcal{B}(\mathcal{C}) \;\; \textit{such that} \\ &\forall \textit{ solution } x(\cdot) \textit{ of (3) defined in } [t_0-h,t) \textit{ it holds } \; x_{t'}\in D \;\; \forall t'\in[t_0,t). \end{aligned} \tag{4}$$

Then, all solutions are defined globally in time.

Proof. By a contradiction argument, consider a non-continuable solution x of (3) defined in $[t_0-h,t)$, with initial datum $\psi\in\mathcal{C}$. Then, from (4) we deduce the existence of a bounded set $D = D(t,t_0,\psi)\in\mathcal{B}(\mathcal{C})$ such that $x_{t'}\in D$ $\forall t'\in[t_0,t)$. Define now the set

$$\omega = \{\varphi\in C^1([-h,0];\mathbb{R}^n) : \|\varphi\| \le \|D\|, \|\varphi'\| \le \sup_{(r,\eta)\in[t_0-h,t]\times D} |f(r,\eta)| = M\},$$

which is compact thanks to the Ascoli-Arzelà Theorem. Thus, the set $W = [t_0,t]\times\omega$ is compact and we can apply Theorem 3 to obtain the existence of t_W such that $(t',x_{t'})\notin W$ for $t_W \le t' < t$. In particular, for t_W it holds that either $\|x_{t_W}\| > \|D\|$ or $\left\|x'_{t_W}\right\| > M$. The first possibility obviously contradicts

(4). For the second, observe that

$$\begin{aligned}\left\|x'_{t_W}\right\| &= \sup_{\theta\in[-h,0]}\left|x'_{t_W}(\theta)\right| \\ &= \sup_{\theta\in[-h,0]}\left|x'(t_W+\theta)\right| \\ &= \sup_{\theta\in[-h,0]}\left|f(t_W+\theta, x_{t_W+\theta})\right| \\ &\le \sup_{(r,\eta)\in[t_0-h,t]\times D}\left|f(r,\eta)\right| = M,\end{aligned}$$

and the proof is complete. ■

3 Semiflows and processes for retarded differential equations

In this section we aim to establish the definitions of (multi-valued) semiflows and processes associated to our two cases under study (autonomous functional equations and non-autonomous integro-differential equations with or without uniqueness) and some useful properties about them.

In order to avoid unnecessary repetitions, we shall first state the results for the non-autonomous case and will particularize later on for the autonomous framework.

Hereafter Ω will denote the set $\mathbb{R}\times\mathcal{C}$ unless otherwise is specified. We also suppose that the assumptions in Corollary 6 hold, what jointly with Theorem 1, guarantees the existence of global solutions to (3).

Since in [9] only the case containing variable delays was considered, we will develop here most of the time our theory and applications for a mixed case of both retarded terms, with the following canonical form

$$x'(t) = f(t,x_t) = F(t,x(t),x(t-\rho(t))) + \int_{-h}^{0} b(t,s,x(t+s))ds, \qquad (5)$$

$$x_{t_0} = \psi \in \mathcal{C}, \qquad (6)$$

where $F\in C(\mathbb{R}^{2n+1};\mathbb{R}^n)$ contains the dependence on the variable delays (for simplicity, we will consider only one delay function $\rho:\mathbb{R}\to[0,h]$, although the analysis can be extended to a more general setting in a straightforward way), and with the distributed delay term described by $b\in C(\mathbb{R}\times[-h,0]\times\mathbb{R}^n;\mathbb{R}^n)$ (which implies that $f\in C(\mathbb{R}\times\mathcal{C};\mathbb{R}^n)$ as can be proved by using the Lebesgue Theorem), and such that the solutions to (5) satisfy (4).

According to Remark 2, if in addition f is such that uniqueness of solutions holds, then the standard single-valued process can be defined as follows:

$$\mathbb{R}_d \times \mathcal{C} \ni (t, t_0, \psi) \mapsto U(t, t_0, \psi) = x_t \in \mathcal{C}, \tag{7}$$

where $x(\cdot)$ is the unique solution of (5)-(6). However, when f is such that the uniqueness of the problem does not hold or cannot be guaranteed, the process will not necessarily be single-valued but multi-valued in general.

In this respect, the definition given by (7) becomes

$$U(t, t_0, \psi) = \bigcup \{x_t : x(\cdot) \text{ is a solution to (5)-(6) defined globally}\}.$$

But, owing to some realistic reasons related to the models under study (e.g, biological, physical, etc), we may be interested just in solutions which remain in a closed subset $X \subset \mathcal{C}$, what motivates the construction of a multi-valued semiflow in X instead of in the whole space $\mathcal{C}$. To this end, we assume that for any $\psi \in X$ there exists at least one solution to (5)-(6) defined globally in time and that remains in X for all $t \geq t_0$, and denote by $D(t_0, \psi)$ the set of all solutions of (5)-(6) defined for all $t \geq t_0$ which remain in X for all $t \geq t_0$. Then, we can define the multi-valued process generated by (5)-(6) as

$$U(t, t_0, \psi) = \bigcup_{x(\cdot) \in D(t_0, \psi)} \{x_t\}. \tag{8}$$

Let us recall this concept and some of its properties more precisely (cf. [8]). Consider a complete metric space X which in our situation will be a closed subset of $\mathcal{C}$.

Definition 7 *The map $U : \mathbb{R}_d \times X \to P(X)$ is said to be a multi-valued dynamical process (MDP) on X if*
(1) $U(t, t, \cdot) = Id$ (identity map);
(2) $U(t, s, x) \subset U(t, \tau, U(\tau, s, x))$, for all $x \in X$, $s \leq \tau \leq t$,
where $U(t, \tau, U(\tau, s, x)) = \cup_{y \in U(\tau, s, x)} U(t, \tau, y)$.
The MDP U is said to be strict if

$$U(t, s, x) = U(t, \tau, U(\tau, s, x)), \text{ for all } x \in X,\ s \leq \tau \leq t.$$

Lemma 8 *Under the previous assumptions the multi-valued mapping U defined by (8) is a strict MDP.*

Proof. It is easy to check that U is well defined and satisfies (1) in Definition 7. Let us now prove that (2) also holds. Indeed, consider $\phi \in U(t, s, \psi)$. Then from the definition of U, there exists a solution $x(\cdot)$ to (5) with initial datum $x_s = \psi$ and $x_t = \phi$. If $\tau \geq s$, then $x_\tau \in U(\tau, s, \psi)$, and as

$$U(t, \tau, x_\tau) = \{z_t : z(\cdot) \text{ is solution to (5) with } z_\tau = x_\tau\},$$

obviously $x_t = \phi \in U(t, \tau, x_\tau) \subset U(t, \tau, U(\tau, s, \psi))$.

To prove that the MDP is strict, let us consider $\phi \in U(t,\tau,U(\tau,s,\psi))$. Then there exists a solution $x(\cdot)$ to (5) such that $x_\tau = y_\tau$, where $y(\cdot)$ is another solution to (5) with initial value $y_s = \psi$. We now define

$$z(r) = \begin{cases} y(r), & \text{if } \ s-h \le r \le \tau \\ x(r), & \text{if } \ \tau \le r \le t. \end{cases}$$

It is clear that $z(\cdot)$ is solution to equation (5), and it also holds that $z_s = y_s = \psi$, and $z_t = x_t = \phi$, which means that $\phi \in U(t,s,\psi)$. ■

Lemma 9 *The map $U(t,s,\cdot)$ is bounded for all $s \le t$ if and only if*

$$\begin{array}{l} \forall\,(t,t_0) \in \mathbb{R}_d, \quad \forall B' \in \mathcal{B}(X), \quad \exists \ \ B(t,t_0,B') \in \mathcal{B}(X) \ \ \textit{such that} \\ \forall x(\cdot) \in \mathrm{D}\,(t_0,B') \ \ \textit{it follows that } x_{t'} \in B(t,t_0,B') \ \ \forall t' \in [t_0,t], \end{array} \tag{9}$$

where $\mathrm{D}\,(t_0,B') = \cup_{\psi \in B'} \mathrm{D}\,(t_0,\psi)$.

Proof. It is clear that (9) implies that $U(t,s,\cdot)$ is bounded. The converse is a consequence of the fact that the sets

$$U\,(t_0 + kh, t_0, B')$$

are bounded in $C\,([-h,0]\,;\mathbb{R}^n)$ for any $k \in \mathbb{N}$. Indeed, for any $x\,(\cdot) \in D\,(t_0,B')$ and any $s \in [t_0,t]$ we have

$$|x\,(s)| \le \|x_{t_0+k_s h}\| \le C_0,$$

where k_s is the minimum integer such that $s \le t_0 + k_s h$ and

$$C_0 = \max_{0 \le k \le k_t} \sup_{y \in U(t_0+kh,t_0,B')} \|y\|\,.$$

■

For the multi-valued map $F : X \to 2^X$ we shall denote

$$\mathcal{D}(F) = \{x \in X \mid F\,(x) \in P\,(X)\}\,.$$

The multi-valued map F is said to be upper semicontinuous if for any $x \in \mathcal{D}\,(F)$ and any neighborhood $\mathcal{O}$ of $F\,(x)$ there exists $\delta > 0$ such that $F\,(y) \subset \mathcal{O}$, provided that $\rho\,(x,y) < \delta$.

Once again the Ascoli-Arzelà Theorem allows us to prove the following useful result:

Proposition 10 *Let $f \in C(\mathbb{R} \times \mathcal{C};\mathbb{R}^n)$ be a bounded map, and assume that the solutions to (5) satisfy condition (9). Consider the process $U : \mathbb{R}_d \times X \to P(X)$ generated by (5), which is therefore bounded. Then the next properties hold:*

i) *If* $t \geq s+h$, *the process* $U(t,s,\cdot): X \to P(X)$ *is compact, that is, for all* $D \in \mathcal{B}(X)$, *one has that* $\overline{U(t,s,D)} \in K(X)$.

ii) *Given a time* $s \in \mathbb{R}$, *if* $\xi_n \to \xi \in X$, *and* $x^n : [s-h,\infty) \to \mathbb{R}^n$ *is a sequence of solutions to (5) with* $x^n_s = \xi_n$, *then there exists a subsequence* $\{x^\mu\}_\mu$ *such that*

$$x^\mu_t \to x_t \quad in \quad X, \ \forall t \geq s,$$

where $x : [s,\infty) \to \mathbb{R}^n$ *satisfies* $x_s = \xi$ *and equation (5).*

iii) *For any* $s \leq t$ *the map* $U(t,s,\cdot)$ *is upper semicontinuous and has compact values and closed graph.*

Proof. Let $D \in \mathcal{B}(X)$, and consider any sequence of points $\varphi^m \in U(t,s,D)$. Thus, there exists a sequence of solutions of (5), $x^m : [s-h,t] \to \mathbb{R}^n$, with $x^m_t = \varphi^m$.
As $t \geq s+h$, the solutions are differentiable and their derivatives are bounded by

$$C = \sup_{[t-h,t]\times\tilde{D}} |f(s,\eta)|$$

where

$$\tilde{D} = \bigcup_{\theta\in[-h,0]} U(t+\theta,s,D) \in \mathcal{B}(X).$$

The uniform bound and the equicontinuity allow us to apply the Ascoli-Arzelà Theorem and conclude the compactness of $U(t,s,\cdot)$ for all $t \geq s+h$.
To prove ii) we proceed analogously. Observe that the condition $t \geq s+h$ is not necessary now (in $[s-h,s]$ the convergence is assumed). We have initial data ξ^m at time s (converging to ξ in X) and solutions from there $x^m : [s-h,\infty) \to \mathbb{R}^n$. The Ascoli-Arzelà Theorem applied in successive steps on length h (and a diagonal Cantor argument) implies the existence of a converging subsequence x^μ to a function $x : [s,\infty) \to \mathbb{R}^n$ (the convergence is uniform on compact intervals of time).
Using

$$x^\mu(t) = \xi^\mu(0) + \int_s^t f(r,x^\mu_r)dr,$$

by Lebesgue Theorem, we pass to the limit and obtain that

$$\tilde{x}(r) = \begin{cases} \xi(r), & r \in [s-h,s], \\ x(r), & r \geq s \end{cases}$$

solves (5) with initial data ξ at time s.
Point iii) is a consequence of ii). Indeed, if the map $U(t,s,\cdot)$ is not upper semicontinuous at some $\xi \in X$, then there exist a neighbourhood $\mathcal{O}$ of $U(t,s,\xi)$ and sequences $\xi_n \to \xi$, $y_n \in U(t,s,\xi_n)$ such that $y_n \notin \mathcal{O}$, for all n. But ii) implies that for some subsequence $y_{n_k} \to y \in U(t,s,\xi)$, which is a contradiction. The compactness of the values and the graph of $U(t,s,\cdot)$ is proved in a similar way. ■

Remark 11 *The above result also shows that the MDP U is, in the autonomous multi-valued case, a generalized semigroup in the sense of Ball [4].*

In the autonomous framework, all the previous results hold true but now for the multi-valued semiflow (MSF) $G : \mathbb{R}_+ \times X \to P(X)$ generated by the autonomous functional differential equation

$$x'(t) = F(x(t), x(t-h)) + \int_{-h}^{0} b(s, x(t+s))\, ds = f(x_t), \quad t \geq 0, \tag{10}$$

$$x_0 = \psi \in X, \tag{11}$$

which is defined, roughly, as

$$G(t,\psi) = \{x_t : x(\cdot) \text{ is solution of (10)-(11) defined globally in time}\}.$$

In general, we have the following definition of MSF.

Definition 12 *The multi-valued map $G : \mathbb{R}_+ \times X \to P(X)$ is called a multi-valued semiflow (MSF or m-semiflow) if the next conditions are satisfied:*

(1) $G(0,x) = \{x\}$, *for all* $x \in X$;
(2) $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, *for all* $t_1, t_2 \in \mathbb{R}_+$, $x \in X$,

where $G(t,B) = \underset{x \in B}{\cup} G(t,x)$, $B \subset X$.

This definition generalizes the concept of semigroup to the case where an equation can admit more than one solution for a fixed initial value. This approach has already been used for some differential equations and inclusions (cf. [2], [13], [14], [27], [28], [36], [42]). Another definition of generalized semigroup (using trajectories instead of multi-valued maps) is given in [4], [21], with applications to three-dimensional Navier-Stokes and parabolic degenerate equations. We note that this semigroup satisfies in fact the conditions of Definition 12, so that it is a particular case (see [10] for a comparison of both theories). A different method for treating the problem of non-uniqueness is used in [17], [40].

For our equation, the map G is defined in the following way which is analogous to the non-autonomous case. Let X be a closed subset of $\mathcal{C}$ such that for any $\psi \in X$ there exists at least one solution $x(\cdot)$ of (10)-(11) such that $x(t) \in X$, for all $t \geq 0$. We denote by $D(\psi)$ the set of all solutions of (10)-(11) defined for all $t \geq 0$ which remain in X for all $t \geq 0$. Then

$$G(t,\psi) = \bigcup_{x(\cdot) \in D(\psi)} x_t(\cdot).$$

In any case, we can always define the multi-valued process U but for this autonomous situation. However, it is easy to check that $U(t,s,\psi) = G(t-s,\psi)$.

Taking into account this fact, one can obtain autonomous versions of the results in this section in a straightforward manner. Then we have:

Lemma 13 *The map G is an m-semiflow, and, moreover,*

$$G(t+s,x) \equiv G(t,G(s,x)), \text{ for all } x \in X, \ t \geq 0.$$

Proof. It is a consequence of Lemma 8 and the previous comment. ■

4 Autonomous and non-autonomous attractors for MSF and MDP

In this section we shall collect the main definitions and results involving multi-valued semiflows and processes and their attractors. Let us consider a complete metric space X with metric ρ.

4.1 Autonomous attractor for a MSF

4.1.1 Abstract theory of attractors for multi-valued semiflows

Let also denote by $dist(A,B)$ the Hausdorff semi-metric, i.e., for given subsets A and B we have

$$dist(A,B) = \sup_{x\in A} \inf_{y\in B} \rho(x,y).$$

Definition 14 *It is said that the set $\Re \subset X$ is a global attractor of the m-semiflow G if:*

(1) It is attracting, i.e,

$$dist(G(t,B),\Re) \to 0 \text{ as } t \to +\infty, \text{ for all } B \in \mathcal{B}(X);$$

(2) $\Re$ is negatively semi-invariant, i.e., $\Re \subset G(t,\Re)$, for all $t \geq 0$;
(3) It is minimal, that is, for any closed attracting set Y, we have $\Re \subset Y$.

In applications it is desirable for the global attractor to be compact and invariant (i.e. $\Re = G(t,\Re)$, for all $t \geq 0$).

Let us denote $\gamma_t^+(B) = \cup_{\tau\geq t} G(\tau,B)$. The MSF G is called asymptotically upper semi-compact if for all $B \in \mathcal{B}(X)$ such that for some $T(B) \in \mathbb{R}_+$, $\gamma_{T(B)}^+(B) \in \mathcal{B}(X)$, any sequence $\xi_n \in G(t_n,B)$, $t_n \to +\infty$, is precompact in X.

The m-semiflow G is called pointwise dissipative if there exists $B_0 \in \mathcal{B}(X)$ such that $dist\,(G(t,x), B_0) \to 0$, as $t \to +\infty$, for all $x \in X$.

The following two results can be found in [36] (see also [13,14]).

Proposition 15 *Let X be a Banach space and let $G(t,\cdot) = S(t,\cdot) + K(t,\cdot)$ be an m-semiflow, where $K(t_0,\cdot) : X \to P(X)$ is a compact map for some $t_0 > 0$ and $S(t,\cdot) : X \to P(X)$ is a contraction on bounded sets, that is,*

$$dist(S(t,x), S(t,y)) \leq m_1(t)\, m_2(\rho(x,y)),\ \forall\, x,y \in B \in \mathcal{B}(X),\ t \in \mathbb{R}_+, \tag{12}$$

where $m_2 : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $m_1 : \mathbb{R}_+ \to \mathbb{R}$ is a decreasing map such that $m_1(t) \to 0$, as $t \to \infty$. Then G is asymptotically upper semicompact.

Theorem 16 *Let G be a pointwise dissipative and asymptotically upper semicompact m-semiflow. Suppose that $G(t,\cdot) : X \to C(X)$ is upper semicontinuous for any $t \in \mathbb{R}_+$. If for all $B \in \mathcal{B}(X)$, there exists $T(B) \in \mathbb{R}_+$ such that $\gamma^+_{T(B)}(B) \in \mathcal{B}(X)$, then G has the compact global attractor $\Re$, which is the minimal closed attracting set. If $G(t_1, G(t_2,x)) = G(t_1+t_2, x)$, then the attractor is invariant.*

4.1.2 Existence of the global attractor

First we shall prove an abstract result, which will be verified later for some particular situations.

Lemma 17 *Let b and F be continuous and let for each initial condition at least one solution to (10)-(11) be globally defined in X. We assume that $G(t,\cdot)$ is a bounded map for any $t \geq 0$. Then the map $G(t,\cdot)$ has closed values and is upper semicontinuous.*

Proof. It is a consequence of Proposition 10 and Lemma 9. ■

Theorem 18 *Let b and F be continuous and let for each initial condition at least one solution to (10)-(11) be globally defined in X. Suppose that $G(t,\cdot)$ maps bounded sets into bounded ones and that there exists a bounded absorbing set. Then G has a global compact invariant attractor.*

Proof. It is a consequence of Proposition 10, Lemma 9, Proposition 15 and Theorem 16. ■

We now recall (in a general metric space X and for an abstract MDP U) some of the basic concepts and results from the theory of pullback attractors, as developed in Kloeden and Stonier [30], Kloeden and Schmalfuß [29], and Crauel *et al.* [18]. As it has already been mentioned, in the case of non-autonomous differential equations the initial time is as important as the final time, and the classical semigroup property of autonomous dynamical systems is no longer suitable. Therefore, the notions of the classical theory need to be adapted to deal with MDP.

Definition 19 *Let $t \in \mathbb{R}$. The set $D(t)$ is said to attract (in the pullback sense) the set $B \in \mathcal{B}(X)$ at time t if*

$$\lim_{s \to -\infty} dist(U(t,s,B), D(t)) = 0. \tag{13}$$

If (13) is satisfied for all $B \in \mathcal{B}(X)$, then $D(t)$ is said to be (pullback) uniformly attracting at time t.

The pullback attracting property considers the state of the system at time t when the initial time s goes to $-\infty$ (cf. Chepyzhov and Vishik [16]).

Now, the concepts of (shift) orbit until s and ω-limit set at time t are formulated respectively by:

$$\gamma^s(t,B) = \bigcup_{\tau \le s} U(t,\tau,B), \quad \omega(t,B) = \bigcap_{s \le t} \overline{\gamma^s(t,B)}.$$

Clearly, any element y of $\omega(t,B)$ is characterized by the existence of a sequence (τ^m, ξ^m) such that $\xi^m \in U(t,\tau^m,B)$ and $\xi^m \to y$ in X, $\tau^m \to -\infty$. The basic result ensuring the existence of a minimal attracting set is the following:

Theorem 20 *(cf. [8]) Suppose that for $t \in \mathbb{R}$ and $B \in \mathcal{B}(X)$ there exists $D(t,B) \in K(X)$ such that*

$$\lim_{s \to -\infty} dist(U(t,s,B), D(t,B)) = 0. \tag{14}$$

Then, $\omega(t,B)$ is nonempty, compact and the minimal set attracting B at time t.

For any bounded set, we need the following notion:

Definition 21 *The MDP U is called (pullback) asymptotically upper semicompact if for any $B \in \mathcal{B}(X)$ such that for each $t \in \mathbb{R}$ there exists $t_0(t,B)$ such that $\gamma^{t_0}(t,B) \in \mathcal{B}(X)$, any sequence $\xi^m \in U(t,s^n,B)$, where $s^n \to -\infty$, is precompact.*

Then one has:

Lemma 22 *(cf. [8]) The following conditions are equivalent:*

(1) The MDP U is asymptotically upper semicompact and any $B \in \mathcal{B}(X)$ satisfies that for each $t \in \mathbb{R}$ there exists $t_0(t,B)$ such that $\gamma^{t_0}(t,B) \in \mathcal{B}(X)$;
(2) For any $t \in \mathbb{R}$ and $B \in \mathcal{B}(X)$ there exists $D(t,B) \in K(X)$ satisfying (14).

The concept of pullback attractor is the following:

Definition 23 *The family $\{A(t)\}_{t\in\mathbb{R}}$ is said to be a non-autonomous or pullback attractor of the MDP U if:*

(1) $A(t)$ is pullback uniformly attracting at time t for all $t \in \mathbb{R}$;
(2) It is negatively invariant, that is,

$$A(t) \subset U(t,s,A(s)), \text{ for any } (t,s) \in \mathbb{R}_d;$$

(3) It is minimal, that is, for any closed attracting set Y at time t, we have $A(t) \subset Y$.

In the applications it is desirable for $A(t)$ to be compact (in such a case we shall say that the attractor is compact). It would be also of interest to obtain the invariance of $A(t)$ (i.e. $A(t) = U(t,s,A(s))$). However, in order to prove this we need to assume that the map $U(t,s,\cdot)$ is lower semicontinuous (see [8, Proposition 19]), which is a strong assumption.

Remark 24 *It is worth mentioning that it would be possible to present the theory within the more general framework of cocycle dynamical systems (see, e.g. [29]) since, in our canonical formulation, we do explicitly know the dependence of our mapping $f(t,x_t)$ on the delay features, and we can therefore construct the parameter space as the hull of some appropriate functions (being this hull a compact set under suitable assumptions). However, in order to develop a theory for a general functional differential equation $x' = f(t,x_t)$ which can allow, in a unified way, the treatment of several kinds of delay (without a previous explicit knowledge of the hereditary characteristics), it is not clear how we can construct such a parameter set. Nevertheless, we would like to point out that this cocycle formulation has proven extremely fruitful particularly in the case of random dynamical systems (see [6], [7], [18], [19], [39]). For this reason, pullback attractors are often referred to as 'cocycle attractors' (see [29] or [40] for various examples using this general setting).*

We shall use the following general result for the existence of non-autonomous attractors.

Theorem 25 *(cf. [8, Th.18]) Suppose that for all* $(t,s) \in \mathbb{R}_d$, $U(t,s,\cdot)$ *has closed graph and that there exists a family* $D(t) \in K(X)$ *satisfying (14). Then, the set*

$$A(t) = \overline{\bigcup_{B \in \mathcal{B}(X)} \omega(t,B)}$$

is the minimal compact global attractor of U*.*

Finally, we obtain a sufficient result for the existence of pullback attractors for our problem, i.e., differential and integro-differential equations with delay, which extends Theorem 4.1 in [9] to the case of non-uniqueness.

Theorem 26 *Suppose the next assumptions for problem (5)-(6):*

i) b *and* F *are continuous and for each initial condition at least one solution is globally defined in* X;

ii) $U(t,s,\cdot) : X \to X$ *is a bounded map for all* $(t,s) \in \mathbb{R}_d$;

iii) There exists a family $\{B(t)\}_{t \in \mathbb{R}}$ *of bounded absorbing sets for* U*.*

Then there exists the minimal pullback global attractor $A(t)$ *for the MDP* U*, which is also compact.*

Proof. It is a straightforward application of Proposition 10, Lemma 9 and Theorem 25. We note that $D(t) = \overline{U(t, t-t_0, B(t-t_0))}$, where $t_0 \geq h$. ■

5 Applications and examples

The objective from now on is to show that the previous theory can be applied to several situations coming from applications. But, we first prove a Gronwall lemma which will be useful in our proofs.

Lemma 27 *Suppose that* $g(t) \geq 0$ *belongs to* $L^1(0,T)$ *and* $M \geq 0$, $0 < \alpha \leq 2$*. Let* $y(t)$ *be a non-negative continuous function on* $[0,T]$ *such that*

$$y^2(t) \leq M^2 + 2\int_0^t g(\tau)\, y^\alpha(\tau)\, d\tau, \quad \text{for all } t \in [0,T].$$

Then

$$\begin{aligned} &y(t) \leq \left(M^{2-\alpha} + (2-\alpha)\textstyle\int_0^t g(s)\, ds\right)^{\frac{1}{2-\alpha}}, && \text{if} \quad \alpha < 2, \\ &y(t) \leq M \exp\left(\textstyle\int_0^t g(s)\, ds\right), && \text{if} \quad \alpha = 2, \end{aligned} \tag{15}$$

for all $t \in [0,T]$.

Proof. Denote $U(s)=\sqrt{M^2+2\int_0^s g(\tau)\,y^\alpha(\tau)\,d\tau}$, which is a non-decreasing function. Differentiating $U^2(t)$ we have

$$2U(s)\frac{dU(s)}{ds}=2g(s)\,y^\alpha(s)\leq 2g(s)\,U^\alpha(s). \tag{16}$$

Since $U(t)$ is nondecreasing there exists $0\leq\beta\leq T$ such that $U(t)=M$, for all $t\in[0,\beta]$, and $U(t)>M$, for all $t\in[\beta,T]$. Clearly, (15) is satisfied for $t\in[0,\beta]$. If $t>\beta$, then integrating over (β,t) we obtain

$$\begin{array}{ll}\frac{U^{2-\alpha}(t)}{2-\alpha}\leq\frac{M^{2-\alpha}}{2-\alpha}+\int_0^t g(s)\,ds, & \text{if}\quad \alpha<2,\\ U(t)\leq M\exp\left(\int_0^t g(s)\,ds\right), & \text{if}\quad \alpha=2.\end{array}$$

It follows that

$$\begin{array}{ll}y(t)\leq U(t)\leq\left(M^{2-\alpha}+(2-\alpha)\int_0^t g(s)\,ds\right)^{\frac{1}{2-\alpha}}, & \text{if}\quad \alpha<2,\\ y(t)\leq U(t)\leq M\exp\left(\int_0^t g(s)\,ds\right), & \text{if}\quad \alpha=2.\end{array}$$

■

5.1 Autonomous case

Recall that $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the scalar product and norm in $\mathbb{R}^n$, respectively. Consider now the system of equations:

$$\begin{aligned}&x'(t)=F(x(t),x(t-h))=f(x_t),\ t>0,\\ &x_0=\psi\in X,\end{aligned} \tag{17}$$

where $F=(F_1,...,F_n)$, $F_i:\mathbb{R}^{2n}\to\mathbb{R}$, $h>0$, $X=C([-h,0],L)\subset\mathcal{C}$ (L is a closed subset of $\mathbb{R}^n$), and F is continuous.

Let us now introduce the following conditions:

- $(H1)$ For each $\psi\in X$ there exists at least one solution $x(t)$ of (17) such that $x(t)\in X$ for all $t\geq 0$;
- $(H2)$ There exists a constant $K>0$ such that for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ for which

$$\langle F(x,y),x\rangle\leq-\delta \quad\text{if}\quad |x|,|y|\geq K+\varepsilon,\quad x,y\in L;$$

- $(H3)$ There exist constants $C>0$, $0<\alpha\leq 2$, such that

$$\langle F(x,y),x\rangle\leq C(1+|x|^\alpha),\quad\text{for all}\quad x,y\in L.$$

Remark 28

(1) *The following property is a consequence of* $(H2)$:

$$\langle F(x,y), x\rangle < 0 \quad \text{as soon as} \quad |x|, |y| > K, \quad x, y \in L.$$

(2) *If* $X = \mathcal{C}$, *then condition* $(H1)$ *is not necessary.*
(3) *The most usual case in the applications seems to be* $L = \mathbb{R}^n_+$.

Proposition 29 *Let* $(H1)$ *and* $(H3)$ *hold. Then, the m-semiflow* G *is well defined and bounded for any* $t \geq 0$.

Proof. Condition $(H1)$ and Lemma 13 imply that G is well defined.
Let $x(t)$ be an arbitrary solution. We shall obtain an estimate on any interval $[0,T]$. Multiplying (17) by $x(t)$ and using $(H3)$ we get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \leq C\left(1 + |x(t)|^\alpha\right),$$

so that

$$|x(t)|^2 \leq |x(0)|^2 + 2CT + 2\int_0^t C\,|x(s)|^\alpha\, ds, \text{ for all } t \in [0,T].$$

Lemma 27 applied to $y(t) = |x(t)|$ gives

$$\begin{aligned} |x(t)| &\leq \left(\left(|x(0)|^2 + 2CT\right)^{\frac{2-\alpha}{2}} + (2-\alpha)\,CT\right)^{\frac{1}{2-\alpha}}, \quad \text{if} \quad \alpha < 2, \\ |x(t)| &\leq \left(|x(0)|^2 + 2CT\right)^{\frac{1}{2}} \exp(CT), \quad \text{if} \quad \alpha = 2, \end{aligned} \tag{18}$$

for all $t \in [0,T]$. We have obtained that any solution exists globally in time in view of Corollary 6 (note that the continuity of F implies that f is bounded). Hence, in the case $X = \mathcal{C}$ the semiflow G is well defined without using $(H1)$. Finally, it follows from (18) that $G(t,\cdot)$ is bounded for any $t \geq 0$. ■

Corollary 30 *Let* $L = \mathbb{R}^n_+$, $(H3)$ *hold and let for each* i *either* $F_i(x,y) = 0$, *for all* x, $y \in L$ *such that* $x_i = 0$, *or* $F_i(x,y) > 0$, *for all* x, $y \in L$ *such that* $x_i = 0$. *Then* $(H1)$ *holds.*

Proof. In the proof of Proposition 29 we have showed that each solution is defined globally in time. We have to obtain that for each initial condition $\psi \in \mathcal{C}\left([-h,0];\mathbb{R}^n_+\right)$ there exists at least one solution such that $x_i(t) \geq 0$, for all $t \geq 0, i = 1, ..., n$.
If $F_i(x,y) > 0$, for all x, $y \in L$ such that $x_i = 0$, then $x_i(t) \geq 0$, for all $t \geq 0$. Indeed, let $x(t)$ be such that $x_i(t_1) = 0$ and $x_i(t) < 0$ in $(t_1, t_2]$. Then by continuity of F there exists an interval $[t_1, t_3] \subset [t_1, t_2]$ such that

$F_i\left(x\left(t\right),x\left(t-h\right)\right)>0$, $\forall t\in\left[t_1,t_3\right]$, so that

$$\frac{d}{dt}x_i\left(t\right)=F_i\left(x\left(t\right),x\left(t-h\right)\right)>0,\quad\forall t\in\left[t_1,t_3\right],$$

and after integration we obtain

$$x_i\left(t_3\right)>0,$$

which is a contradiction.
If $F_i\left(x,y\right)=0$, for all x, $y\in L$ such that $x_i=0$, and $x_i\left(t_1\right)=0$, for some $t_1\geq 0$, then we can put $x_i\left(t\right)=0$, for all $t\geq t_1$, and continue solving the system of equations for the rest of the components x_j. Hence, with this procedure we obtain the desired solution $x\left(t\right)$. ■

Theorem 31 *Let conditions $(H1)-(H3)$ hold. Then the m-semiflow G has a bounded absorbing set.*

Proof. Define

$$R\left(\alpha\right)=\begin{cases}\left(\left(K^2+2Ch\right)^{\frac{2-\alpha}{2}}+\left(2-\alpha\right)Ch\right)^{\frac{1}{2-\alpha}}, & \text{if}\quad\alpha\neq 2,\\ \left(K^2+2Ch\right)^{\frac{1}{2}}\exp\left(Ch\right), & \text{if}\quad\alpha=2.\end{cases}$$

Let us first assume that $\left|\psi\left(0\right)\right|\leq K$. Then, it is clear from (18) that any solution $x\left(t\right)$ with $\left|\psi\left(0\right)\right|\leq K$ remains in the ball

$$B_0=B_0\left(\alpha\right)=\left\{x\in\mathbb{R}:\left|x\right|\leq R\left(\alpha\right)\right\},$$

for all $t\in\left[0,h\right]$.
We shall prove that in fact this is true for all $t\geq 0$. Suppose the opposite. Then there exists a solution $x\left(t\right)$ and times $T_1\geq 0$, $T_2>T_1+h$ such that $\left|x\left(T_1\right)\right|=K$, $\left|x\left(t\right)\right|>K$, for all $t\in\left(T_1,T_2\right]$, and $\left|x\left(T_2\right)\right|>R\left(\alpha\right)$. We can find then an interval $\left[T_0,T_2\right]$, $T_0\geq T_1+h$, for which $\left|x\left(T_0\right)\right|=R\left(\alpha\right)$ and $\left|x\left(t\right)\right|>R\left(\alpha\right)$ if $t\in\left(T_0,T_2\right]$. But in such a case we have $\left|x\left(t\right)\right|,\left|x\left(t-h\right)\right|>K$, for all $t\in\left(T_0,T_2\right]$, and condition $(H2)$ implies that $\left|x\left(t\right)\right|$ is decreasing in $\left(T_0,T_1\right]$, which is a contradiction.
We have proved that the set B_0 is absorbing for any bounded set with initial conditions satisfying $\left|\psi\left(0\right)\right|\leq K$. Consider now an arbitrary bounded set B of initial conditions satisfying $\left|\psi\left(0\right)\right|>K$. We claim the existence of t_0 such that for any solution $x\left(t\right)$ we have $\left|x\left(t\right)\right|\leq R\left(\alpha\right)$, for all $t\geq t_0$. If this is not the case, then there exists a sequence of times $t^k\nearrow+\infty$ and a sequence of solutions $x^k\left(t\right)$ such that $\left|x^k(t^k)\right|>R\left(\alpha\right)$.
We know from the previous arguments that if $\left|x^k\left(t_1\right)\right|=K$ for some $t_1<t^k$, then $x^k\left(t\right)\leq R\left(\alpha\right)$, for all $t\geq t_1$. Then for any k it is clear that $\left|x^k\left(t\right)\right|>K$, for all $t\in\left[0,t^k\right]$. Hence, $(H2)$ implies that any solution $\left|x^k\left(t\right)\right|$ is decreasing

on the interval $\left[h, t^k\right]$, and then $x^k(t) > R(\alpha)$ on $\left[h, t^k\right]$. If we denote $\varepsilon = R(\alpha) - K > 0$, then using $(H2)$ we have

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \leq -\delta,$$

so that $\left|x^k(t)\right|^2 \leq \left|x^k(2h)\right|^2 - 2\delta(t-2h)$, for all $t \in \left[2h, t^k\right]$. Since $t^k \nearrow +\infty$ and $x^k(0) \in B$ (which is bounded), the sequence $\{x^k(2h)\}_k$ is also bounded (see (18)), and there exists k_0 such that

$$\left|x^k\left(t^k\right)\right|^2 \leq \left|x^k(2h)\right|^2 - 2\delta\left(t^k - 2h\right) < R(\alpha), \text{ for any } k \geq k_0.$$

We have obtained then a contradiction. Hence, B_0 is a bounded absorbing set for the bounded set B. It follows immediately that B_0 is absorbing for any bounded subset. ■

Therefore, it holds the following result, which is a consequence of Theorem 18:

Theorem 32 *Let conditions* $(H1)$-$(H3)$ *hold. Then the m-semiflow* G *has a global invariant compact attractor.*

We shall now consider some examples of interest arising in real applications.

Example 1. Consider the retarded logistic model

$$\frac{dx(t)}{dt} = r|x(t)|^{\alpha-1}\left(1 - \left(\frac{|x(t-h)|}{A}\right)^{\beta}\right),$$

where $r, A > 0$, $1 < \alpha \leq 2$, $\beta > 0$ and $L = \mathbb{R}_+$. In the case where $\alpha = 2$ and $\beta = 1$, the function $x(t)$ describes the evolution of the number of a population (the classical logistic model). In view of Corollary 30, condition $(H1)$ holds, and conditions $(H2) - (H3)$ are obviously satisfied with $L = \mathbb{R}_+$, $K = A$, $C = r$, $-\delta(\varepsilon) = r(A+\varepsilon)^{\alpha}\left(1 - \left(1 + \frac{\varepsilon}{A}\right)^{\beta}\right)$. The constant K is the biological threshold of the population.

It is worth pointing out that in this example there are solutions that can leave the set X. If we choose $\alpha = 5/3$, $A = \beta = r = h = 1$, $x_0(\theta) = -2\theta$, then $x(t) = (t(t-1)/3)^3$ is a solution on $t \in [0,1]$, and it takes negative values. Hence, not all solutions are taken into account to define the m-semiflow G.

Example 2. Consider now the following model

$$\frac{dx(t)}{dt} = p - b|x(t)|^{\alpha-1}\frac{|x(t-h)|^m}{a^m + |x(t-h)|^m},$$

where $a, p, b > 0, 1 < \alpha \leq 2, m > 0$ and $L = \mathbb{R}_+$. In the case $\alpha = 2$ this equation is a model for the concentration x of CO_2 in blood (see [37, p.16]). Again, condition $(H1)$ is a consequence of Corollary 30.
As for condition $(H2)$, let K be the solution of the equation

$$p - bK^{\alpha-1}\frac{K^m}{a^m + K^m} = 0.$$

Since the functions $x^{\alpha-1}$ and $\frac{y^m}{a^m+y^m}$ are increasing for positive x, y, we have that $(H2)$ holds with $-\delta(\varepsilon) = p(K+\varepsilon) - b(K+\varepsilon)^{\alpha}\frac{(K+\varepsilon)^m}{a^m+(K+\varepsilon)^m}$.
Finally, $(H3)$ is straightforward to prove.

Example 3. Consider a model of human respiratory

$$\frac{du(t)}{dt} = -au(t) + av(t) + c,$$
$$\frac{dv(t)}{dt} = bu(t) - bv(t) - p(v(t) - g)\varphi(v(t-h)),$$

where a, b, c, g, p are positive constants, u, v are the CO_2 partial pressures in tissues and lungs, respectively, $x(t) = (u(t), v(t)), y(t) = (u(t-h), v(t-h))$, and φ is a continuous function such that $\varphi(y_2) = 0$, for $y_2 \leq x_0$ (for some $x_0 \geq 0$), and strictly increasing for $y_2 > x_0$. For the physical meaning of the constants see [11]. We set $L = \{x \in \mathbb{R}^2 : u \geq g, v \geq g\}$.

Lemma 33 *All the solutions starting in $X = C([-h, 0]; L)$ remain in this space. Consequently, condition $(H1)$ holds.*

Proof. Let $x(t) = (u(t), v(t))$ be an arbitrary solution with $x_{t_0} = \psi(0) \in \partial L$. The case $u = v = g$ is trivial, so we suppose one of the components is not null. We shall suppose that $x(t) \notin L$ in some interval (t_0, t_1). If $u(t_0) = g$ and less than g in (t_0, t_1), then there exists $t_0 < t_2 \leq t_1$ such that $-au(t) + av(t) + c > 0$, for all $t \in (t_0, t_2)$. Hence, $u(t) \geq g$, for all $t \in (t_0, t_2)$. Then we have to assume that $v(t) < g$, for all $t \in (t_0, t_2)$. But in such a case $bu(t) - bv(t) - p(v(t) - g)\varphi(v(t-h)) > 0$, for all $t \in (t_0, t_2)$. Hence, $v(t) \geq g$, for all $t \in (t_0, t_2)$, and we obtain a contradiction. The solution $x(t)$ cannot leave the set L. On the other hand, condition $(H3)$ follows directly from

$$\langle F(x, y), x\rangle \leq -au^2 + (a+b)uv - bv^2 + cu \leq C\left(1 + |x|^2\right).$$

Thus, by the proof of Proposition 29, we have that each solution is defined globally in time. This implies that $(H1)$ holds. ■

In order to check $(H2)$ we need additional assumptions on the constants of the model.

Lemma 34 *Let $g > x_0$ and let $\gamma p > (a-b)^2/4a$, where $\gamma = \varphi(g)$. Then $(H2)$ holds.*

Proof. First let $a \geq b$. We note that since $\varphi(y_2) \geq \gamma$, for all $y_2 \geq g$, and $-au^2 + (a+b)uv - bv^2 = -b(u-v)\left(\frac{a}{b}u - v\right)$ we have

$$\begin{aligned}\langle F(x,y), x\rangle &= -b(u-v)\left(\frac{a}{b}u - v\right) + cu - pv(v-g)\varphi(y_2) \\ &\leq -b(u-v)\left(\frac{a}{b}u - v\right) + cu - \gamma pv(v-g).\end{aligned}$$

We shall consider the following three regions:

$$\begin{aligned}R^1 &= \left\{x \in L : v \geq \tfrac{a}{b}u\right\}, \\ R^2 &= \left\{x \in L : \tfrac{b}{a}v \leq u \leq v\right\}, \\ R^3 &= \{x \in L : v \leq u\}.\end{aligned}$$

Note that the function $\psi(u,v) = -b(u-v)\left(\frac{a}{b}u - v\right)$ takes positive values only in R^2.
Let $x \in R^1$. Since $v \geq u$, $|x| \geq K_1$ implies $2v^2 \geq u^2 + v^2 \geq K_1^2$, using that $\psi(u,v) \leq 0$, we obtain

$$\begin{aligned}\langle F(x,y), x\rangle &\leq cv - \gamma pv(v-g) \\ &\leq C_1 - \gamma pv^2/2 \\ &\leq -1, \quad \text{if} \quad |x| \geq K_1 = \left(4(C_1+1)/(\gamma p)\right)^{1/2}.\end{aligned}$$

Let now $x \in R^2$. Denote $\xi = \gamma p - (a-b)^2/4a$. Using that $v^2(a-b)^2/4a = \max\limits_{u \in [\frac{b}{a}v, v]} \psi(u,v)$ and $u \leq v$, we have

$$\begin{aligned}\langle F(x,y), x\rangle &\leq v^2(a-b)^2/4a + cv - \gamma pv(v-g) \\ &\leq C_2 - \xi v^2/2 \\ &\leq -1, \quad \text{if} \quad |x| \geq K_2 = \left(4(C_2+1)/\xi\right)^{1/2}.\end{aligned}$$

Finally, let $x \in R^3$. In this case, we have $\psi(u,v) \leq -b(u-v)^2$. Hence,

$$\begin{aligned}\langle F(x,y), x\rangle &\leq -b(u-v)^2 + cu - \gamma pv(v-g) \\ &\leq C_3 - b(u-v)^2/2 - \gamma pv^2/2.\end{aligned}$$

Denote $\eta = \left(2(C_3+1)/b\right)^{1/2}$. If $u - v \geq \eta$, then $\langle F(x,y), x\rangle \leq -1$. Otherwise, $|x| \geq K_3$ implies $3v^2 + 2\eta^2 \geq u^2 + v^2 \geq K_3^2$. It follows that $\langle F(x,y), x\rangle \leq -1$, if $|x| \geq K_3 = \left(2\eta^2 + 6(C_3+1)/(\gamma p)\right)^{1/2}$.

Now let $a < b$ and define the regions:

$$\begin{aligned} R^4 &= \{x \in L : v \geq u\}, \\ R^5 &= \left\{x \in L : v \leq u \leq \tfrac{b}{a} v\right\}, \\ R^6 &= \left\{x \in L : v \leq \tfrac{a}{b} u\right\}. \end{aligned}$$

Region R^4 is treated in the same way as R^1.
For R^5, since $v \geq \frac{a}{b} u$, $|x| \geq K_5$ implies $(1 + b^2/a^2) v^2 \geq u^2 + v^2 \geq K_5^2$, using $v^2 (a-b)^2 / 4a = \max\limits_{u \in [v, \frac{b}{a} v]} \psi(u, v)$, we have

$$\begin{aligned} \langle F(x,y), x \rangle &\leq v^2 (a-b)^2 / 4a + cbv/a - \gamma p v (v - g) \\ &\leq C_5 - \xi v^2 / 2 \\ &\leq -1, \quad \text{if} \quad |x| \geq K_5 = \left(2\left(1 + (b/a)^2\right)(C_5 + 1)/\xi\right)^{1/2}. \end{aligned}$$

Finally, consider $x \in R^6$. In this case, we have $\psi(u, v) \leq -b(au/b - v)^2$. Hence,

$$\begin{aligned} \langle F(x,y), x \rangle &\leq -b(au/b - v)^2 + cu - \gamma p v (v - g) \\ &\leq C_6 - b(au/b - v)^2 / 2 - \gamma p v^2 / 2. \end{aligned}$$

Denote $\eta = (2(C_6 + 1)/b)^{1/2}$. If $au/b - v \geq \eta$, then $\langle F(x,y), x \rangle \leq -1$. Otherwise, $|x| \geq K_6$ implies $(1 + 2b^2/a^2) v^2 + 2b^2 \eta^2 / a^2 \geq u^2 + v^2 \geq K_6^2$. It follows that $\langle F(x,y), x \rangle \leq -1$ if

$$|x| \geq K_6 = \left(2b^2 \eta^2 / a^2 + 2\left(1 + 2b^2/a^2\right)(C_6 + 1)/(\gamma p)\right)^{1/2}.$$

Taking $K = \max\{K_1, K_2, K_3\}$ (or $\max\{K_4, K_5, K_6\}$) condition $(H2)$ holds for $\delta(\varepsilon) = 1$. ■

One of the typical functions used in such models is the Hill controller $\varphi(y_2) = \sigma y_2^n / (\theta^n + y_2^n)$, with $\sigma, n, \theta > 0$. In this case it is clear that $x_0 = 0 < g$. We also note that the condition $\gamma p > (a-b)^2 / 4a$ implies a strong enough effect of the term $p(v(t) - g)\varphi(v(t-h))$, which controls the air flow in the lungs.

5.2 Non-autonomous case

5.2.1 Dissipative and sub-linear terms

We are now interested in considering a situation which takes into account the possible appearance of variable and distributed delays together.

Consider the equation

$$x'(t) = F_0(t, x(t)) + F_1(t, x(t - \rho(t))) + \int_{-h}^{0} b(t, s, x(t + s))ds = f(t, x_t) \quad (19)$$

with $F_0, F_1 \in C(\mathbb{R}\times\mathbb{R}^n; \mathbb{R}^n), \rho \in C^1(\mathbb{R}; [0, h])$ and $b \in C(\mathbb{R}\times[-h, 0]\times\mathbb{R}^n; \mathbb{R}^n)$. Assume the following conditions:

(1) There exist positive scalar functions $m_0, m_1 \in L^1([-h, 0])$ such that

$$|b(t, s, x)| \leq m_0(s) + m_1(s)|x|, \qquad \forall t \in \mathbb{R}, \quad (20)$$

(2) There exist positive constants k_1, k_2, α and a positive function $\beta(\cdot)$ such that

$$\langle x, F_0(t, x)\rangle \leq -\alpha|x|^2 + \beta(t), \quad \forall t \in \mathbb{R},\, x \in \mathbb{R}^n, \quad (21)$$

$$|F_1(t, x)|^2 \leq k_1^2 + k_2^2|x|^2, \quad \forall t \in \mathbb{R},\, x \in \mathbb{R}^n, \quad (22)$$

$$|\rho'(t)| \leq \rho_* < 1, \quad \forall t \in \mathbb{R}, \quad (23)$$

where β is such that

$$\int_{-\infty}^{t} \beta(r)e^{\delta r}dr < \infty, \qquad \forall t \in \mathbb{R}, \forall \delta > 0. \quad (24)$$

Also we denote

$$m_i = \int_{-h}^{0} m_i(s)ds, \quad i = 0, 1.$$

We take as the phase space $X = C([-h, 0]; \mathbb{R}^n)$.
Then, we have the following result:

Theorem 35 *Let conditions (20)-(24) hold. Also, assume that*

$$2m_1eh < 1, \quad (25)$$

and

$$k_2^2 < e^{-1}(1 - \rho^*)\,\alpha\,(\alpha - \lambda^*), \quad (26)$$

where $\lambda^* \in (\lambda_0, \lambda_1)$*, being* $\lambda_0 < \lambda_1$ *the solutions of the equation* $\lambda e^{-\lambda h} = 2m_1$*, and let*

$$\lambda^* < \alpha. \quad (27)$$

Then, Eq. (19) generates a MDP which has the global compact non-autonomous attractor $\{A(t)\}_{t\in\mathbb{R}}$*.*

Remark 36

i) We note that the equation $\lambda e^{-\lambda h} = 2m_1$ *has two solutions* $\lambda_0(h) < \lambda_1(h)$ *if condition (25) holds, and* $\lambda_0(h) \to 0$ *as* $h \to 0$ *(or* $m_1 \to 0$*). Hence (25) holds for* h *(or* m_1*) small, whereas (27) holds if* h *(or* m_1*) is small enough,* *or if α is large. On the other hand, (26) is satisfied for k_2 small or α large. These conditions can be read as: a combination of strong dissipativity and small effects of the delay (in terms of h, m_1 or k_2 small) ensure the existence of the attractor.*

ii) Conditions (26) and (27) are stronger than what is really needed in the proof below, that is, there exist positive values λ and ε such that $\lambda - 2\alpha + \varepsilon + \frac{e^{\lambda h} k_2^2}{\varepsilon(1-\rho_*)} < 0$ and $\lambda \in (\lambda_0, \lambda_1)$.

Proof. Let $x(t)$ be an arbitrary solution with $\|x_{t_0}\| \leq d$. First,we fixed two positive parameters λ and ε to be chosen later on. Then, we have

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda t}|x(t)|^2\right)\\
&= \lambda e^{\lambda t}|x(t)|^2 + 2e^{\lambda t}\langle x(t), F_0(t,x(t)) + F_1(t, x(t-\rho(t)))\rangle\\
&\quad + 2e^{\lambda t}\left\langle x(t), \int_{-h}^{0} b(t,r,x(t+r))dr\right\rangle\\
&\leq \lambda e^{\lambda t}|x(t)|^2 + 2e^{\lambda t}(-\alpha|x(t)|^2 + \beta(t)) + \varepsilon e^{\lambda t}|x(t)|^2 + \frac{e^{\lambda t}}{\varepsilon}|F_1(x(t-\rho(t)))|^2\\
&\quad + 2e^{\lambda t}\left\langle x(t), \int_{-h}^{0} b(t,r,x(t+r))dr\right\rangle\\
&\leq (\lambda - 2\alpha + \varepsilon)e^{\lambda t}|x(t)|^2 + 2e^{\lambda t}\beta(t) + \frac{e^{\lambda t}}{\varepsilon}(k_1^2 + k_2^2|x(t-\rho(t))|^2)\\
&\quad + 2e^{\lambda t}\left\langle x(t), \int_{-h}^{0} b(t,r,x(t+r))dr\right\rangle.
\end{aligned}$$

Integrating between t_0 and t, we obtain

$$\begin{aligned}
e^{\lambda t}|x(t)|^2 \leq\ & e^{\lambda t_0}|x(t_0)|^2 + (\lambda - 2\alpha + \varepsilon)\int_{t_0}^{t} e^{\lambda s}|x(s)|^2 + 2\int_{t_0}^{t} e^{\lambda s}\beta(s)ds\\
&+ \frac{1}{\varepsilon}\int_{t_0}^{t} e^{\lambda s}(k_1^2 + k_2^2|x(s-\rho(s))|^2)ds\\
&+ 2\int_{t_0}^{t} e^{\lambda s}\left\langle x(s), \int_{-h}^{0} b(s,r,x(s+r))dr\right\rangle ds.
\end{aligned}$$

Now we estimate the integral containing the variable delay by using the change of variables $s - \rho(s) = u$, taking into account that ρ takes values in $[0,h]$, and $\frac{1}{1-\rho'(s)} \leq \frac{1}{1-\rho_*}$:

$$\begin{aligned}
&\int_{t_0}^{t} e^{\lambda s}|x(s-\rho(s))|^2 ds\\
&\leq \int_{t_0-h}^{t} e^{\lambda u}\frac{e^{\lambda h}}{1-\rho_*}|x(u)|^2 du\\
&= \frac{e^{\lambda h}}{1-\rho_*}\left[\int_{t_0-h}^{t_0} e^{\lambda u}|x(u)|^2 du + \int_{t_0}^{t} e^{\lambda u}|x(u)|^2 du\right]\\
&\leq \frac{e^{\lambda h} d^2}{1-\rho_*}\int_{t_0-h}^{t_0} e^{\lambda u} du + \frac{e^{\lambda h}}{1-\rho_*}\int_{t_0}^{t} e^{\lambda u}|x(u)|^2 du,
\end{aligned}$$

where we have used that $\|x_{t_0}\| \leq d$ for some $d > 0$. Thus, we obtain

$$
\begin{aligned}
e^{\lambda t}|x(t)|^2 \leq e^{\lambda t_0}|x(t_0)|^2 &+ \left(\lambda - 2\alpha + \varepsilon + \frac{e^{\lambda h}k_2^2}{\varepsilon(1-\rho_*)}\right)\int_{t_0}^t e^{\lambda s}|x(s)|^2 ds \\
&+ 2\int_{t_0}^t e^{\lambda s}\beta(s)ds + \frac{k_1^2}{\varepsilon\lambda}(e^{\lambda t} - e^{\lambda t_0}) + \frac{k_2^2 e^{\lambda h}d^2}{\lambda\varepsilon(1-\rho_*)}(e^{\lambda t_0} - e^{\lambda(t_0-h)}) \\
&+ 2\int_{t_0}^t e^{\lambda s}\left\langle x(s), \int_{-h}^0 b(s,r,x(s+r))dr\right\rangle ds, \qquad (28)
\end{aligned}
$$

and it follows for the last integral in (28) that

$$
\begin{aligned}
&2\int_{t_0}^t e^{\lambda s}\left\langle x(s), \int_{-h}^0 b(s,r,x(s+r))dr\right\rangle ds \\
&\leq 2m_0\int_{t_0}^t |x(s)|e^{\lambda s}ds + 2m_1\int_{t_0}^t |x(s)|\|x_s\|e^{\lambda s}ds \\
&\leq \bar{\varepsilon}\int_{t_0}^t e^{\lambda s}|x(s)|^2 ds + \frac{m_0^2}{\bar{\varepsilon}}\int_{t_0}^t e^{\lambda s}ds + 2m_1\int_{t_0}^t e^{\lambda s}\|x_s\|^2 ds,
\end{aligned}
$$

where $\bar{\varepsilon}$ is another positive constant to be determined later on. Therefore,

$$
\begin{aligned}
e^{\lambda t}|x(t)|^2 \leq e^{\lambda t_0}|x(t_0)|^2 &+ \left(\lambda - 2\alpha + \varepsilon + \frac{e^{\lambda h}k_2^2}{\varepsilon(1-\rho_*)} + \bar{\varepsilon}\right)\int_{t_0}^t e^{\lambda s}|x(s)|^2 ds \\
&+ 2\int_{t_0}^t e^{\lambda s}\beta(s)ds + \left(\frac{k_1^2}{\varepsilon\lambda} + \frac{m_0^2}{\lambda\bar{\varepsilon}}\right)(e^{\lambda t} - e^{\lambda t_0}) \\
&+ \frac{k_2^2 e^{\lambda h}d^2}{\lambda\varepsilon(1-\rho_*)}(e^{\lambda t_0} - e^{\lambda(t_0-h)}) + 2m_1\int_{t_0}^t e^{\lambda s}\|x_s\|^2 ds.
\end{aligned}
$$

Choosing $\varepsilon = \alpha, \lambda = \lambda^*$ and using (26) we obtain

$$
\lambda - 2\alpha + \varepsilon + \bar{\varepsilon} + \frac{e^{\lambda h}k_2^2}{\varepsilon(1-\rho_*)} < 0,
$$

for $\bar{\varepsilon}$ small enough. Then, it holds

$$
\begin{aligned}
e^{\lambda^* t}|x(t)|^2 \leq e^{\lambda^* t_0}|x(t_0)|^2 &+ 2\int_{t_0}^t e^{\lambda^* s}\beta(s)ds + \left(\frac{k_1^2}{\alpha\lambda^*} + \frac{m_0^2}{\lambda^*\bar{\varepsilon}}\right)(e^{\lambda^* t} - e^{\lambda^* t_0}) \\
&+ \frac{k_2^2 e^{\lambda^* h}d^2}{\lambda^*\alpha(1-\rho_*)}(e^{\lambda^* t_0} - e^{\lambda^*(t_0-h)}) + 2m_1\int_{t_0}^t e^{\lambda^* s}\|x_s\|^2 ds.
\end{aligned}
$$

Setting now $t+\theta$ instead of t (where $\theta \in [-h, 0]$), multiplying by $e^{-\lambda^*(t+\theta)}$ and using standard estimates, it follows

$$
\begin{aligned}
&|x(t+\theta)|^2 \\
&\leq e^{-\lambda^*(t+\theta)}e^{\lambda^* t_0}|x(t_0)|^2 + 2e^{-\lambda^*(t+\theta)}\int_{t_0}^{t+\theta} e^{\lambda^* s}\beta(s)ds \\
&\quad + \left(\frac{k_1^2}{\lambda^*\alpha} + \frac{m_0^2}{\lambda^*\bar{\varepsilon}}\right)(1 - e^{\lambda^*(t_0-t-\theta)}) + e^{-\lambda^*(t+\theta)}\frac{k_2^2 e^{\lambda^* h} d^2}{\lambda^*\alpha(1-\rho_*)}(e^{\lambda^* t_0} - e^{\lambda^*(t_0-h)}) \\
&\quad + 2m_1 e^{-\lambda^*(t+\theta)}\int_{t_0}^{t+\theta} e^{\lambda^* s}\|x_s\|^2 ds \\
&\leq e^{-\lambda^* t}e^{\lambda^*(t_0+h)}|x(t_0)|^2 + 2e^{-\lambda^* t}e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* s}\beta(s)ds \\
&\quad + \left(\frac{k_1^2}{\lambda^*\alpha} + \frac{m_0^2}{\lambda^*\bar{\varepsilon}}\right)(1 - e^{\lambda^*(t_0-t+h)}) + e^{-\lambda^* t}\frac{k_2^2 e^{2\lambda^* h} d^2}{\lambda^*\alpha(1-\rho_*)}(e^{\lambda^* t_0} - e^{\lambda^*(t_0-h)}) \\
&\quad + 2m_1 e^{-\lambda^* t}e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* s}\|x_s\|^2 ds.
\end{aligned}
$$

Neglecting the negative terms we deduce

$$
\begin{aligned}
e^{\lambda^* t}\|x_t\|^2 \leq e^{\lambda^*(t_0+h)}d^2 + 2e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* s}\beta(s)ds \\
+ C_1 e^{\lambda^* t} + C_2 d^2 e^{\lambda^* t_0} + L\int_{t_0}^{t} e^{\lambda^* s}\|x_s\|^2 ds
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
C_1 &= \frac{k_1^2}{\lambda^*\alpha} + \frac{m_0^2}{\lambda^*\bar{\varepsilon}}, \\
C_2 &= \frac{k_2^2 e^{2\lambda^* h}}{\lambda^*\alpha(1-\rho_*)}, \\
L &= 2m_1 e^{\lambda^* h}.
\end{aligned}
$$

Gronwall's Lemma and the Fubini Theorem yield

$$
\begin{aligned}
&e^{\lambda^* t}\|x_t\|^2 \\
&\leq e^{\lambda^*(t_0+h)}d^2 + 2e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* s}\beta(s)ds + C_1 e^{\lambda^* t} + C_2 d^2 e^{\lambda^* t_0} \\
&\quad + Le^{Lt}\int_{t_0}^{t} e^{-Ls}\left[e^{\lambda^*(t_0+h)}d^2 + 2e^{\lambda^* h}\int_{t_0}^{s} e^{\lambda^* r}\beta(r)dr + C_1 e^{\lambda^* s} + C_2 d^2 e^{\lambda^* t_0}\right] ds \\
&\leq e^{\lambda^*(t_0+h)}d^2 + 2e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* s}\beta(s)ds + C_1 e^{\lambda^* t} + C_2 d^2 e^{\lambda^* t_0} + Le^{Lt} \\
&\quad \times \Big[\frac{e^{\lambda^*(t_0+h)}d^2 + C_2 d^2 e^{\lambda^* t_0}}{L}(e^{-Lt_0} - e^{-Lt}) + 2e^{\lambda^* h}\int_{t_0}^{t} e^{\lambda^* r}\beta(r)\frac{e^{-Lr} - e^{-Lt}}{L}dr \\
&\qquad + \frac{C_1}{\lambda^* - L}e^{(\lambda^* - L)t}\Big].
\end{aligned}
$$

Therefore

$$\begin{aligned}\|x_t\|^2 \leq{}& e^{\lambda^*(t_0+h)}d^2 e^{-\lambda^* t} + 2e^{\lambda^* h}e^{-\lambda^* t}\int_{t_0}^{t} e^{\lambda^* s}\beta(s)ds + C_1 + C_2 d^2 e^{\lambda^* t_0}e^{-\lambda^* t}\\ &+ e^{-\lambda^* t + L(t-t_0)}(e^{\lambda^*(t_0+h)}d^2 + C_2 d^2 e^{\lambda^* t_0})\\ &+ 2e^{\lambda^*(h-t)+Lt}\int_{t_0}^{t} e^{\lambda^* r}\beta(r)(e^{-Lr} - e^{-Lt})dr\\ &+ \frac{LC_1}{\lambda^* - L}. \end{aligned} \tag{29}$$

We can see that (29) and Corollary 6 imply that all solutions exist globally in time (so, U is well defined), and also that the maps $U(t, t_0, \cdot)$ are bounded. On the other hand, condition (25) implies that $\lambda^* - L > 0$. Then it follows from the previous inequality that

$$B(t) = \left\{ y \in \mathcal{C} \,\middle|\, \begin{array}{l} \|y\|^2 \leq 2e^{\lambda^* h}e^{-\lambda^* t}\int_{-\infty}^{t} e^{\lambda^* s}\beta(s)ds + \frac{LC_1}{\lambda^*-L} + \eta \\ \qquad + C_1 + 2e^{\lambda^*(h-t)+Lt}\int_{-\infty}^{t} e^{\lambda^* r}\beta(r)(e^{-Lr} - e^{-Lt})dr \end{array} \right\},$$

where $\eta > 0$, is a family of bounded absorbing sets. We conclude the proof by applying Theorem 26. ■

We can consider also the case where F_0, F_1 and b do not depend on the time variable, that is, we have an autonomous equation. We note that in such a situation $\beta(t) \equiv \beta$, and then neglecting negative terms (29) becomes

$$\begin{aligned}\|x_t\|^2 \leq{}& e^{\lambda^*(t_0+h)}d^2 e^{-\lambda^* t} + \frac{2\beta e^{\lambda^* h}}{\lambda^*} + C_1 + C_2 d^2 e^{\lambda^* t_0}e^{-\lambda^* t}\\ &+ e^{-\lambda^* t + L(t-t_0)}(e^{\lambda^*(t_0+h)}d^2 + C_2 d^2 e^{\lambda^* t_0})\\ &+ \frac{2\beta e^{\lambda^* h}}{\lambda^* - L} + \frac{LC_1}{\lambda^* - L}. \end{aligned}$$

Now $t_0 = 0$ is fixed and $t \to +\infty$. It follows that the set

$$B_0 = \left\{ y \in \mathcal{C} : \|y\|^2 \leq C_1 + 2\beta e^{\lambda^* h}\left(\frac{1}{\lambda^*} + \frac{1}{\lambda^* - L}\right) + \frac{LC_1}{\lambda^* - L} \right\}$$

is attracting for the m-semiflow G, which is also bounded. We obtain the same result as in Theorem 32.

Remark 37 *The statement of Theorem 35 remains valid if* $X = C([-h, 0], L)$, *being* L *a closed subset of* $\mathbb{R}^n$, *supposing that for each* $\psi \in X$ *and* $t_0 \in \mathbb{R}$ *there exists at least one solution such that* $x(t) \in X$, *for all* $t \geq t_0$.

Corollary 38 *If* $L = \mathbb{R}^n_+$ *and for each* $i = 1, ..., n$ *one of the following conditions holds:*

(1) $F_0^i(t,\psi(0)) + F_1^i(t,\psi(-\rho(t))) + \int_{-h}^0 b^i(t,s,\psi(s))ds = 0$, *for all* $t \in \mathbb{R}$, *and* $\psi \in C([-h,0];L)$, *with* $\psi^i(0) = 0$*;*
(2) $F_0^i(t,\psi(0)) + F_1^i(t,\psi(-\rho(t))) + \int_{-h}^0 b^i(t,s,\psi(s))ds > 0$, *for all* $t \in \mathbb{R}$, *and* $\psi \in C([-h,0];\mathbb{R}^n)$, *with* $\psi^i(0) < 0$.

Then for each $\psi \in X$, $t_0 \in \mathbb{R}$ *there exists at least one solution* $x(t)$ *of (19) such that* $x(t) \in X$ *for all* $t \geq t_0$.

Proof. It is similar to the proof of Corollary 30. ■

Remark 39 *We can consider two simpler situations:*

Assume that b *satisfies*

$$|b(t,s,x)| \leq m(s)|x|, \tag{30}$$

and that $F_0(t,x) = -\alpha x$, $F_1(t,x) = 0$. *Then, we can deduce (obtaining an estimate rather similar to (29)) that the pullback attractor exists and is just one point (the null solution in* $\mathcal{C}$*). Also, this attractor is a global attractor in the usual forward sense (as* $t \to +\infty$*), and implies that the null solution is asymptotically stable (what means extinction in a biological model).*

On the other hand, if we delete the bound of b *on* x*, we can reproduce the same calculus but extending the dependence of the bound on* t*. More exactly, given*

$$b \in C(\mathbb{R} \times [-h,0] \times \mathbb{R}^n; \mathbb{R}^n)$$

verifying

$$|b(t,s,x)| \leq K(t,s) \qquad \forall x \in \mathbb{R}^n, \tag{31}$$

an analogous result is obtained if

$$R(t) := \int_{-h}^0 K(t,s)ds$$

satisfies

$$\int_{-\infty}^r R^2(t)e^{2\tilde{\alpha}t}dt < \infty \qquad \forall r \in \mathbb{R}, \quad \text{for some} \quad \tilde{\alpha} \in (0,\alpha). \tag{32}$$

5.2.2 Weaker assumptions on the dissipativity

In a similar way, we can also weaken the dissipativity assumption on the function F_0 in the sense that more general non-autonomous situations can be covered by our results. To this end, we start again from equation (19), but with $F(t,x_t) = F_0(t,x(t)) + F_1(t,x(t-\rho(t)))$ where $\rho \in C(\mathbb{R};[-h,0])$ and

$$\langle F_0(t,x),x\rangle \leq (-\alpha + \gamma_1(t))|x|^2 + \gamma_2(t), \quad |F_1(t,x)| \leq \gamma_3(t), \tag{33}$$

being γ_i positive functions satisfying

$$\int_{-\infty}^{t} [\gamma_1(s) + e^{\delta s}(\gamma_2(s) + \gamma_3^2(s))]ds < \infty, \qquad \forall t \in \mathbb{R}, \forall \delta > 0, \tag{34}$$

and b verifying conditions (31)–(32). Then

$$\begin{aligned}\frac{d}{dt}|x(t)|^2 &= 2\langle x(t), x'(t)\rangle = 2\langle x(t), F(t, x_t)\rangle + 2\langle x(t), \int_{-h}^{0} b(t, s, x(t+s))ds\rangle \\ &\leq 2(\gamma_1(t) - \alpha)|x(t)|^2 + 2\gamma_2(t) + 2|x(t)| \left(\gamma_3(t) + \int_{-h}^{0} K(t, s)ds\right) \\ &\leq [2(\gamma_1(t) - \alpha) + \varepsilon]|x(t)|^2 + \frac{\gamma_3^2(t)}{\varepsilon} + 2\gamma_2(t) + \frac{1}{\varepsilon}R^2(t).\end{aligned}$$

By Gronwall Lemma, it holds

$$\begin{aligned}|x(t)|^2 &\leq e^{\int_{t_0}^{t}[2(\gamma_1(s)-\alpha)+\varepsilon]ds}|x(t_0)|^2 \\ &\quad + \int_{t_0}^{t} \left(2\gamma_2(s) + \frac{\gamma_3^2(s) + R^2(s)}{\varepsilon}\right) e^{\int_{s}^{t}[2(\gamma_1(r)-\alpha)+\varepsilon]dr}ds \\ &= e^{(2\alpha-\varepsilon)(t_0-t)}e^{2\int_{t_0}^{t}\gamma_1(s)ds}|x(t_0)|^2 \\ &\quad + e^{(\varepsilon-2\alpha)t}\int_{t_0}^{t} e^{(2\alpha-\varepsilon)s}e^{2\int_{s}^{t}\gamma_1(r)dr}\left(2\gamma_2(s) + \frac{\gamma_3^2(s) + R^2(s)}{\varepsilon}\right) ds \\ &\leq e^{(2\alpha-\varepsilon)(t_0-t)}e^{M_t}|x(t_0)|^2 \\ &\quad + e^{(\varepsilon-2\alpha)t}e^{M_t}\int_{t_0}^{t} e^{(2\alpha-\varepsilon)s}\left(2\gamma_2(s) + \frac{\gamma_3^2(s) + R^2(s)}{\varepsilon}\right) ds,\end{aligned}$$

where we have denoted

$$M_t = 2\int_{-\infty}^{t} \gamma_1(r)dr.$$

The existence of a family of bounded absorbing sets in $\mathcal{C}$ is already standard (choosing some $\varepsilon < 2\alpha$). Hence, we obtain the same result as in Theorem 35.

5.2.3 Examples

Let us consider now some examples from real applications. In all of these examples we shall consider that $L = \mathbb{R}_+$, hence $X = C([-h, 0], L)$. Also, in all the examples Corollary 38 implies that the semiprocess U is well defined.

Example 1. Mackey-Glass model of production of blood cells [35]:

$$\frac{dx(t)}{dt} = \frac{\beta(t)}{1 + |x(t-h)|^n} - \delta x(t),$$

where $n, \delta > 0$, $\beta(t) > 0$ is continuous and $\int_{-\infty}^{t} e^{\varepsilon s}\beta^2(s)\,ds < +\infty$, for any $t \in \mathbb{R}$ and $\varepsilon > 0$. Conditions (33)-(34) are fulfilled.

A generalization of this model is the following [37, p.20], [38]:

$$\frac{dx(t)}{dt} = \frac{\beta(t)|x(t-h)|^m}{1+|x(t-h)|^n} - \delta x(t),$$

where we suppose that $0 \leq m \leq n+1$ and $|\beta(t)| \leq k$, for all $t \in \mathbb{R}$. Let also

$$k^2 < \frac{\delta^2}{e}, \quad \text{if} \quad m = n+1.$$

For conditions (25)-(27) note that $\lambda^* = \rho^* = m_1 = 0$, $\alpha = \delta$. So, (26) follows from the condition on k if $m = n+1$ and by Young inequality if $m < n+1$.

Another generalization of this model appears if we consider an integral term:

$$\frac{dx(t)}{dt} = \int_{-h}^{0} \frac{\beta(t,s)}{1+|x(t-h)|^n} ds - \delta x(t),$$

where $\beta(t,s) > 0$ satisfies $|\beta(t,s)| \leq m_0(s)$, $m_0 \in L^1([-h,0])$ (or $|\beta(t,s)| \leq K(t,s)$ with K satisfying (32)). All the conditions of Theorem 35 are satisfied (note that $m_1 = k_2 = \lambda^* = 0$).

Example 2. Lasota and Wazewska model of production of blood cells [33]:

$$\frac{dx(t)}{dt} = \beta(t)\exp(-\xi(t)x(t-h)) - \delta x(t),$$

where $\delta > 0$, $\beta(t) > 0, \xi(t) \geq 0$ are continuous and $\int_{-\infty}^{t} e^{\varepsilon s}\beta^2(s)\,ds < +\infty$, for any $t \in \mathbb{R}$ and $\varepsilon > 0$. It is clear again that (33)-(34) are satisfied.

Consider also the following generalization of the model:

$$\frac{dx(t)}{dt} = \int_{-h}^{0} \beta(t,s)\exp(-\xi(t+s)x(t+s))\,ds - \delta x(t),$$

where β satisfies the same conditions of the previous example.

Example 3. Consider the model:

$$\frac{dx(t)}{dt} = \beta(t)|x(t-h)|^n \exp(-\xi(t)x(t-h)) - \delta x(t),$$

where $\delta > 0$, $0 < n \leq 1$, $\beta(t) > 0, \xi(t) \geq 0$ are continuous and $|\beta(t)|^2 \leq k^2$, for any $t \in \mathbb{R}$. If $n = 1$ we have to assume also that $k^2 < \frac{\delta^2}{e}$. For conditions (25)-(27) note that $\lambda^* = m_1 = 0$, $\alpha = \delta$. So, (26) follows from the condition on k if $n = 1$ and by Young inequality if $n < 1$.

In the particular case where $n = 1$ this is the Nicholson model of blowflies [22].

We can consider also the following model:

$$\frac{dx(t)}{dt} = \int_{-h}^{0} \beta(t) |x(t-h)|^n \exp(-\xi(t+s)x(t+s))\,ds - \delta x(t),$$

where $\delta > 0$, $0 < n \leq 1$, $\beta(t) > 0, \xi(t) \geq 0$ are continuous and $|\beta(t)|^2 \leq k^2$, for any $t \in \mathbb{R}$. If $n = 1$ we have to assume that $m_1 = kh$ is small enough, so that (25) and (27) hold. If $n < 1$ we can make m_1 as small as we want using the Young inequality. Hence, (25)-(27) are satisfied with $\alpha = \delta,\ k_2 = \rho^* = 0$.

An analysis of persistence and extinction for these models is given in [34].

6 Conclusions

The existence of an attractor (autonomous or pullback) has been proved in several situations arising in real applications, when some hereditary features appear in the models. The use of multi-valued semiflows and processes allowed us to provide results covering also the cases in which non-uniqueness of solutions can take place. However, our analysis has been done by considering only finite delays. Therefore, it is an interesting task to study the framework with unbounded (infinite) delays, as well as the situations modelled by differential inclusions rather than differential equations (whose importance comes, for instance, from viability reasons in biological problems). We plan to investigate these points in subsequent papers.

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