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ATTRACTORS FOR DIFFERENTIAL EQUATIONS WITH MULTIPLE VARIABLE DELAYS

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Abstract. We establish some results on the existence of pullback attractors for non–autonomous delay differential equations with multiple delays. In particular, we generalise some recent works on the existence of pullback attractors for delay differential equations.

1. **Introduction.**

The state of many real–world problems at a given moment may depend on their previous state(s). When the rate of change of state variables depends on past values of states, independently of their derivatives at previous instants, delay differential equations are often used in the mathematical modeling of the process in question. Furthermore, if the dependence on the past is independent of time and states, the equations take the following form

$$\dot{x} = f(x_t) \tag{1}$$

where $f : C \to \mathbb{R}^n$, $C := C([-h, 0]; \mathbb{R}^n)$, $h > 0$ is a positive delay, and x_t denotes the segment of the solution defined by $x_t(\theta) := x(t + \theta),\ \theta \in [-h, 0]$. We have quite a detailed, but far from complete, knowledge about the non–linear dynamics of these type of equations, see [12, 18, 23]. When equations with many or distributed delays

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are considered, the available knowledge is rather restricted. Even the linear stability analysis of these equations can be a challenging task. The stability properties of certain equations with distributed delays compared to associated equations with one delay were studied in [8, 9]. Using topological fixed point theory, the existence of periodic solutions to non–linear equations with two delays was studied in [17]. A recent work, using computer assisted proofs, proved the coexistence of three periodic solutions to a non–linear equation with two delays [10]. The aforementioned results are about autonomous equations, i.e. of type (1). The theory to study the long time behaviour of these systems in terms of attracting sets was established in [6]. One of the most recent works in this direction is presented in [13].

Usually, the assumption on time invariant delays makes the treatment less involved. However, in some cases, mathematical models with time varying delays can provide better explanations of many real–world phenomena. For example, an interesting numerical experiment on gravitational force with time varying delays was reported in [14]. In these cases, equations modeling the phenomena become explicitly non-autonomous, i.e.,

$$\dot{x} = f(t, x_t), \tag{2}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$. Although, the theoretical foundation for studying the attracting sets of non–autonomous dynamical systems was laid in [21, 22], our particular knowledge about the dynamics of non–autonomous delay differential equations is less developed.

Recently, the concept of pullback attractor was introduced and is being used to analyse non-autonomous dynamics (see, e.g., Caraballo et al. [1] and Kloeden [11], and the references therein). Some aspects of the bifurcation theory for non–autonomous ordinary differential equations were discussed in [19].

The objective of the present work is to contribute to the above mentioned fields of studies, mainly focusing on the existence of pullback attractors. Namely, we generalise recent results presented in [2, 3] on attractors for delay differential equations with time varying delay. We will consider delay differential equations of the form

$$\dot{x}(t) = F_0(t, x(t)) + \sum_{i=1}^{m} F_i(x(t - \rho_i(t))) \tag{3}$$

under the following assumption on F_0:

(A1): $F_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and there exist $\alpha_0 > 0,\ \beta_0 \geq 0$ such that

$$\langle F_0(t, x), x\rangle \leq -\alpha_0 |x|^2 + \beta_0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Here $\rho_i : \mathbb{R} \to [0, h]$ are functions representing the variable delays of the model; additional restrictions are imposed on them in Sections 3.1 and 3.2, as well as on the terms $F_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \dots, m$. Furthermore, $< \cdot, \cdot >$ stands for the inner product in $\mathbb{R}^n$. Under additional conditions, Section 3.1 generalise a result of [2] for equations with multiple delays under some sublinearity condition. Section 3.2 generalises a result of [3] for multiple, measurable delays.

The rest of the paper is organised as follows. Section 2 summarizes the necessary theory of processes. In section 3, we state and prove our main results.

2. **Preliminaries.** To start with, recall that $C = C([-h,0];\mathbb{R}^n)$ denotes the Banach space of continuous functions $\phi : [-h,0] \to \mathbb{R}^n$ equipped with the usual supremum norm. Equation (3) can be rewritten as a non–autonomous differential equation of form

$$\dot{x}(t) = f(t, x_t) \tag{4}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous and maps bounded sets into bounded sets. Furthermore, given a continuous function $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we denote by $x_t(\cdot)$ the element in C given by

$$x_t(\theta) = x(t+\theta),\ \ \theta \in [-h,0].$$

When $x(\cdot)$ is a solution of (4), then $x_t(\cdot)$ is said to be the solution segment at time t.

If an initial function $\phi \in C$ is prescribed at the initial time $s \in \mathbb{R}$, the basic theory of delay differential equations (see [7]) implies, under standard assumptions, the existence of the unique solution $x(\cdot; s, \phi)$ of (4) on $[s-r, \infty)$, which satisfies, in addition, the initial condition $x_s(\cdot) = \phi$, in other words, $x_s(\theta)=x(s+\theta)=\phi(\theta)$ for all $\theta \in [-h,0]$.

The unique solution of the initial value problems associated to (4) defines the solution map $U(t,s) : C \ni \phi \mapsto x_t(\cdot; s, \phi) \in C$ for $t \geq s$, which is, in fact, a process (also called a two-parameters semigroup), i.e.

- $U(t,s) : C \to C$ is a continuous map for all $t \geq s$;
- $U(s,s) = id_C$, the identity on C, for all $s \in \mathbb{R}$,
- $U(t,s) = U(t,\tau)U(\tau,s)$ for $t \geq \tau \geq s$.

As in the autonomous case, we look for invariant attracting sets. First, we introduce the Hausdorff semi–distance between subsets A and B in a metric space (X,d) as

$$dist(A,B) = \sup_{a\in A} \inf_{b \in B} d(a,b).$$

Definition 2.1. Let U be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a (global) pullback attractor for U if, for all $s \in \mathbb{R}$, it satisfies

- $U(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for all $t \geq s$, and
- $\lim_{s\to\infty} dist(U(t,t-s)D, \mathcal{A}(t)) = 0$, for all bounded subsets D of X.

Definition 2.2. $\{B(t)\}_{t\in\mathbb{R}}$ is said to be absorbing with respect to the process U if, for $t \in \mathbb{R}$ and $D \subset X$ bounded, there exist $T_D(t) > 0$ such that for all $\tau \geq T_D(t)$

$$U(t, t-\tau)D \subset B(t).$$

The following result (see [5, 20]) shows that the existence of a family of compact absorbing sets implies the existence of a pullback attractor.

Theorem 2.3. *Let $U(t,s)$ be a process on a complete metric space X. If there exists a family $\{B(t)\}_{t\in\mathbb{R}}$ of compact absorbing sets then, there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ such that $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\subset X\\ bounded}} \Lambda_D(t)}$$

where

$$\Lambda_D(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{s\geq n} U(t,t-s)D}.$$

Theorem 2.4. *Suppose that $U(t,s)$ maps bounded sets into bounded sets and there exists a family $\{B(t)\}_{t\in\mathbb{R}}$ of bounded absorbing sets for U. Then there exists a pullback attractor for problem (4).*

We emphasize that it is possible to consider a more general definition of pullback attractor which attracts family of sets in a universe instead of only bounded sets (see [1],[16] for a detailed analysis of this theory). However, the present concept is enough for our interests.

3. **Main results. Sublinear non-linearities.** In this section, in addition to (A1), we assume that the delay terms are sub-linear in the following sense:

(A2): $F_i,\ i=1,\dots,m$ is sublinear, i.e., there exist $k_i>0,\ i=1,\dots,m$, such that

$$|F_i(x)|^2 \le k_i^2(1+|x|^2),\ x\in\mathbb{R}^n.$$

We will split our analysis into two cases. In the first one, we will consider smooth variable delay functions and uniform (with respect to time) dissipativity on the term F_0 (in order words, independent of t). In the second, we will weaken the hypotheses on the delay functions requiring only the measurability of them, and will allow the dissipativity condition not necessarily be uniform. However, we will need to impose a bit more restrictions on the functions F_i for $i=1,\dots,m$.

3.1. **Smooth delay functions.** When the delay functions are continuously differentiable we can prove the following result which is a natural extension of Theorem 4.3 in [3]. Although the proof follows the same idea with necessary modifications, we provide it here to keep our presentation as much self-contained as possible.

Theorem 3.1. *Assume that assumptions (A1) and (A2) are satisfied. Furthermore, suppose that each delay function $\rho_i(\cdot)$ is continuously differentiable with $\rho_i'(t)\le\rho_{i*}<1$ for all $t\in\mathbb{R}$. Then, if $m^2\,k_i^2<\alpha_0^2(1-\rho_{i*})$, for all $i=1,\dots,m$, there exists a family of bounded absorbing sets, $\{B(t)\}_{t\in\mathbb{R}}$, and consequently, there exists a pullback attractor for this problem.*

Proof. Let $\lambda>0$ be a constant to be determined later on, and denote (for the sake of simplicity) $\varepsilon=\frac{\alpha_0}{m}$. Denote $x(\tau)=x(\tau;t_0-t,\psi),\ \tau\in[t_0-t,t_0]$, for any $\psi\in C$ such that $\|\psi\|\le d$. Then, by a suitable application of the Young inequality ($2ab\le\varepsilon a^2+\varepsilon^{-1}b^2$), it follows that

$$\begin{aligned}
\frac{d}{d\tau}e^{\lambda\tau}|x(\tau)|^2 &= \lambda e^{\lambda\tau}|x(\tau)|^2+2e^{\lambda\tau}\langle x(\tau),f(\tau,x_\tau)\rangle\\
&= \lambda e^{\lambda\tau}|x(\tau)|^2+2e^{\lambda\tau}\langle x(\tau),F_0(\tau,x(\tau))\rangle\\
&\quad+2e^{\lambda\tau}\sum_{i=1}^m\langle x(\tau),F_i(x(\tau-\rho_i(\tau)))\rangle\\
&\le(\lambda-2\alpha_0)e^{\lambda\tau}|x(\tau)|^2+2\beta_0e^{\lambda\tau}+e^{\lambda\tau}|x(\tau)|^2\sum_{i=1}^m\varepsilon\\
&\quad+e^{\lambda\tau}\sum_{i=1}^m\varepsilon^{-1}|F_i(x(\tau-\rho_i(\tau)))|^2
\end{aligned}$$

$$\leq (\lambda - \alpha_0)\, e^{\lambda\tau}|x(\tau)|^2 + \left(2\beta_0 + \varepsilon^{-1}\sum_{i=1}^{m} k_i^2\right) e^{\lambda\tau}$$
$$+ e^{\lambda\tau}\varepsilon^{-1}\sum_{i=1}^{m} k_i^2|x(\tau - \rho_i(\tau))|^2.$$

Integration on the interval $[t_0 - t, \tau]$ yields that

$$\begin{aligned} e^{\lambda\tau}|x(\tau)|^2 - e^{\lambda(t_0-t)}|x(t_0-t)|^2 \leq{}& (\lambda - \alpha_0)\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds \\ &+ \frac{2\beta_0 + \varepsilon^{-1}\sum_{i=1}^{m} k_i^2}{\lambda}\left[e^{\lambda\tau} - e^{\lambda(t_0-t)}\right] \\ &+ \varepsilon^{-1}\sum_{i=1}^{m} k_i^2 \int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho_i(s))|^2 ds. \end{aligned} \tag{5}$$

Now we compute the integrals for the addends in the last sum.

$$\begin{aligned} \int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho_i(s))|^2 ds &\leq \frac{1}{1-\rho_{i*}}\int_{t_0-t-h}^{\tau} e^{\lambda u + \lambda h}|x(u)|^2 du \\ &\leq \frac{e^{\lambda h}}{1-\rho_{i*}}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|x(u)|^2 du + \int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right] \\ &\leq \frac{e^{\lambda h}}{1-\rho_{i*}}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|\psi(u)|^2 du + \int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right] \\ &\leq \frac{d^2 e^{\lambda h}}{\lambda(1-\rho_{i*})}\left[e^{\lambda(t_0-t)} - e^{\lambda(t_0-t-h)}\right] \\ &\quad + \frac{e^{\lambda h}}{1-\rho_{i*}}\int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du. \end{aligned}$$

It follows that

$$\begin{aligned} e^{\lambda\tau}|x(\tau)|^2 \leq{}& e^{\lambda(t_0-t)}d^2 + \frac{2\beta_0 + \varepsilon^{-1}\sum_{i=1}^{m} k_i^2}{\lambda}\left[e^{\lambda\tau} - e^{\lambda(t_0-t)}\right] \\ &+ \frac{d^2 e^{\lambda h}}{\lambda}\left[e^{\lambda(t_0-t)} - e^{\lambda(t_0-t-h)}\right]\sum_{i=1}^{m}\frac{k_i^2\varepsilon_i^{-1}}{1-\rho_{i*}} \\ &+ \left(\lambda - \alpha_0 + e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right)\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds. \end{aligned}$$

Now, observe that

$$\begin{aligned} -\alpha_0 + \varepsilon^{-1}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}} &= -\alpha_0 + \frac{m}{\alpha_0}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}} \\ &\leq -\alpha_0 + \frac{1}{m\alpha_0}\sum_{i=1}^{m}\frac{m^2k_i^2}{1-\rho_{i*}} \\ &< -\alpha_0 + \frac{1}{m\alpha_0}m\alpha_0^2 \\ &= 0. \end{aligned}$$

Consequently, we can choose a positive, but small enough, λ such that

$$\lambda - \alpha_0 + e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}} < 0.$$

This implies that

$$|x(\tau)|^2 \le d^2\left[1+\frac{e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]e^{\lambda(t_0-t-\tau)} + \frac{2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}.$$

Setting $\tau = t_0+\theta,\ \theta\in[-h,0]$ we get

$$|x(t_0+\theta)|^2 \le d^2\left[1+\frac{e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]e^{-\lambda(t+\theta)} + \frac{2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}$$

$$\begin{aligned}\sup_{\theta\in[-h,0]}|x(t_0+\theta)|^2 &\le d^2\left[1+\frac{e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]e^{-\lambda t+\lambda h} + \frac{2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}\\ &\le 1+\frac{2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}\end{aligned}$$

provided that

$$t \ge T_D = \lambda^{-1}\log\left\{\left[1+\frac{e^{\lambda h}\varepsilon^{-1}}{\lambda}\sum_{i=1}^{m}\frac{k_i^2}{1-\rho_{i*}}\right]d^2e^{\lambda h}\right\}.$$

Consequently, the family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ given by $B(t) := B$, for all $t\in\mathbb{R}$, where B denotes the ball in $C([-h,0];\mathbb{R}^n)$ centered at zero with radius $r = 1+\frac{2\beta_0+\varepsilon^{-1}\sum_{i=1}^{m}k_i^2}{\lambda}$, is absorbing. On the other hand, as the associated process maps bounded sets of $C([-h,0];\mathbb{R}^n)$ into bounded sets, then Theorem 2.4 (see also Theorem 4.1 in [3]) ensures the existence of the pullback attractor. □

3.2. **Measurable delay functions.** Observe that the differentiability assumption on the delay functions has been a key point in the method used in the previous section. Our aim now is to relax this hypothesis and prove a similar result on the existence of pullback attractor. Our analysis will be carried out assuming that the delay functions $\rho_i(\cdot)$ are only measurable. But we would like to point out that there exists another technique which can be used when the variable delays are continuous, the so-called Razumikhin method (see, for instance [15]). This will be analysed in a subsequent paper.

In this section, we assume that our non-delay term satisfies a non-autonomous dissipativity condition, i.e. we impose the following assumption on F_0:

(A1'): $F_0 : \mathbb{R}^{n+1}\to\mathbb{R}^n$ is continuous and there exist $\alpha_0>0$, and a non-negative measurable function $\beta(\cdot)$ such that

$$\langle F_0(t,x),x\rangle \le -\alpha_0|x|^2+\beta(t),\ \ t\in\mathbb{R}, x\in\mathbb{R}^n. \tag{6}$$

As for F_i we assume Lipschitz continuity, i.e.,

(A2'): There exists $L_i>0,\ i=1,\dots,m$ such that for any $x,y\in\mathbb{R}^n$

$$|F_i(x)-F_i(y)| \le L_i|x-y|,$$

and $F_i(0)=0$.

Now we can establish our main result in this section.

Theorem 3.2. *Assume that assumptions (A1') and (A2') are satisfied and that there exists* $0 < \lambda < \alpha_0$ *such that*

$$e^{\lambda h} \frac{m}{\alpha_0} \sum_{i=1}^{m} L_i^2 < \lambda \quad \text{and} \quad C_t := \sup_{\sigma \geq 0} \int_{t-\sigma}^{t} e^{\lambda(s-t+\sigma)} \beta(s)\, ds < +\infty, \forall t \in \mathbb{R}. \tag{7}$$

Then, if $\rho_i,\ i = 1, \ldots, m$, *is measurable, there exists a family of bounded absorbing sets,* $\{B(t)\}_{t \in \mathbb{R}}$, *and consequently, there exists a pullback attractor for the process generated by (3).*

Proof. Let us consider the number $\lambda > 0$ from (7), for the sake of clarity write $\varepsilon = \dfrac{\alpha_0}{m}$, and denote $x(\tau) = x(\tau; t_0 - t, \psi),\ \tau \in [t_0 - t, t_0]$, for any $\psi \in C$ such that $\|\psi\| \leq d$, and $t_0 \in \mathbb{R}$. Then, applying again the Young inequality in the delay terms below,

$$\begin{aligned}
\frac{d}{d\tau} e^{\lambda\tau} |x(\tau)|^2 &= \lambda e^{\lambda\tau} |x(\tau)|^2 + 2e^{\lambda\tau} \langle x(\tau), f(\tau, x_\tau) \rangle \\
&= \lambda e^{\lambda\tau} |x(\tau)|^2 + 2e^{\lambda\tau} \langle x(\tau), F_0(\tau, x(\tau)) \rangle \\
&\quad + 2e^{\lambda\tau} \sum_{i=1}^{m} \langle x(\tau), F_i(x(\tau - \rho_i(\tau))) \rangle \\
&\leq (\lambda - 2\alpha_0) e^{\lambda\tau} |x(\tau)|^2 + 2e^{\lambda\tau} \beta(\tau) + e^{\lambda\tau} |x(\tau)|^2 m\varepsilon \\
&\quad + e^{\lambda\tau} \varepsilon^{-1} \sum_{i=1}^{m} |F_i(x(\tau - \rho_i(\tau)))|^2 \\
&\leq \big(\lambda - \alpha_0) e^{\lambda\tau} |x(\tau)|^2 + 2e^{\lambda\tau} \beta(\tau) \\
&\quad + e^{\lambda\tau} \varepsilon^{-1} \sum_{i=1}^{m} L_i^2 |x(\tau - \rho_i(\tau))|^2.
\end{aligned}$$

Integration on the interval $[t_0 - t, \tau]$ yields that

$$\begin{aligned}
e^{\lambda\tau} |x(\tau)|^2 - e^{\lambda(t_0 - t)} |x(t_0 - t)|^2 \leq (\lambda - \alpha_0) &\int_{t_0 - t}^{\tau} e^{\lambda s} |x(s)|^2 ds \\
&+ 2 \int_{t_0 - t}^{\tau} e^{\lambda s} \beta(s) ds \\
&+ \varepsilon^{-1} \sum_{i=1}^{m} L_i^2 \int_{t_0 - t}^{\tau} e^{\lambda s} |x(s - \rho_i(s))|^2 ds.
\end{aligned} \tag{8}$$

The integrand in the third sum can be estimated

$$\int_{t_0 - t}^{\tau} e^{\lambda s} |x(s - \rho_i(s))|^2 ds \leq \int_{t_0 - t}^{\tau} e^{\lambda s} \sup_{\theta \in [-h, 0]} |x(s + \theta)|^2 ds$$

Thus we have

$$\begin{aligned}
|x(\tau)|^2 \leq & e^{\lambda(t_0 - t - \tau)} |x(t_0 - t)|^2 + (\lambda - \alpha_0) \int_{t_0 - t}^{\tau} e^{\lambda(s - \tau)} |x(s)|^2 ds \\
&+ e^{-\lambda\tau} \left(2 \int_{t_0 - t}^{\tau} e^{\lambda s} \beta(s) ds + \varepsilon^{-1} \sum_{i=1}^{m} L_i^2 \int_{t_0 - t}^{\tau} e^{\lambda s} \sup_{\theta \in [-h, 0]} |x(s + \theta)|^2 ds \right).
\end{aligned} \tag{9}$$

Now, since $\lambda - \alpha_0 < 0$,

$$|x(\tau)|^2 \leq \left(e^{\lambda(t_0-t)}d^2 + 2\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{-\lambda\tau} + \varepsilon^{-1}\sum_{i=1}^m L_i^2 \int_{t_0-t}^{\tau} e^{-\lambda(\tau-s)} \sup_{\theta\in[-h,0]} |x(s+\theta)|^2 ds.$$

Taking the supremum after substituting $\tau+\theta,\ \theta\in[-h,0]$, we obtain

$$\sup_{\theta\in[-h,0]} |x(\tau+\theta)|^2 \leq e^{\lambda h}\left(e^{\lambda(t_0-t)}d^2 + 2\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{-\lambda\tau} + e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2 e^{-\lambda\tau}\int_{t_0-t}^{\tau} e^{\lambda s} \sup_{\theta\in[-h,0]} |x(s+\theta)|^2 ds.$$

That is

$$e^{\lambda\tau}\sup_{\theta\in[-h,0]} |x(\tau+\theta)|^2 \leq e^{\lambda h}\left(e^{\lambda(t_0-t)}d^2 + 2\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right) + e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2 \int_{t_0-t}^{\tau} e^{\lambda s} \sup_{\theta\in[-h,0]} |x(s+\theta)|^2 ds.$$

Thanks to the Gronwall lemma, we have that

$$\sup_{\theta\in[-h,0]} |x(\tau+\theta)|^2 \leq e^{\lambda h}\left(e^{\lambda(t_0-t)}d^2 + 2\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2(\tau-t_0+t)}e^{-\lambda\tau}$$

The substitution $\tau = t_0$ and the fact that $e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^n L_i^2 < \lambda$ imply that

$$\begin{aligned}
&\sup_{\theta\in[-h,0]} |x(t_0+\theta)|^2 \\
&\leq e^{\lambda h}\left(e^{\lambda(t_0-t)}d^2 + 2\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right)e^{e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2(t-t_0)} \\
&\leq e^{\lambda h}\left(e^{-\left(\lambda - e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2\right)(t-t_0)}d^2 + 2e^{\lambda(t-t_0)}\int_{t_0-t}^{t_0} e^{\lambda s}\beta(s)ds\right) \\
&\leq e^{\lambda h}\left(e^{-\left(\lambda - e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2\right)(t-t_0)}d^2 + 2\sup_{t\geq 0}\int_{t_0-t}^{t_0} e^{\lambda(s-t_0+t)}\beta(s)ds\right)
\end{aligned}$$

Thus, we finally obtain that

$$\begin{aligned}
\sup_{\theta\in[-h,0]} |x(t_0+\theta)|^2 &\leq e^{\lambda h}\left(e^{-\left(\lambda - e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2\right)(t-t_0)}d^2 + 2C_{t_0}\right) \\
&\leq (1+2C_{t_0})\,e^{\lambda h}
\end{aligned}$$

provided

$$t \geq T_D = \left(\lambda - e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2\right)^{-1}\log\left(d^2 e^{\left(\lambda - e^{\lambda h}\varepsilon^{-1}\sum_{i=1}^m L_i^2\right)t_0}\right).$$

Consequently, the family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ in $C([-h,0];\mathbb{R}^n)$ given by $B(t) := B(0;r(t))$, for all $t\in\mathbb{R}$, where $B(0;r(t))$ denotes the ball centered at zero with radius $r(t) = (1+2C_t)\,e^{\lambda h}$, is absorbing. Taking into account again that the associated process maps bounded sets into bounded sets of $C([-h,0];\mathbb{R}^n)$, the existence of the pullback attractor is ensured again by Theorem 2.4. □

Remark 1. It is worth noticing that, although assumption (7) may seem rather artificial, it is somehow quite natural since it holds provided that either the delay parameter h or the Lipschitz constants L_i are small in comparison with the dissipativity constant α_0.

4. **Conclusion.** In this paper we presented two non–trivial extensions of previous results on the asymptotic behaviour of solutions to non–linear delay differential equations. These earlier results established existence of pullback attractors for delay differential equations with one delay function. Our work generalises these results to the case of multiple delay functions and also weakened the assumptions on the delays. Thus our study contributes to the available knowledge on the long–time behaviour of delay differential equations. To proceed even further, we intend to study the structure of existing attractors to the above studied equations.

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