Asymptotic Exponential Stability of Stochastic Partial Differential Equations with Delay

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ABSTRACT:
Sufficient conditions for pathwise asymptotic exponential stability of the solution of the stochastic PDE with delay $dx_t = Ax_t\, dt + B(x_{\rho(t)})\, dw_t$ *are given. The assumptions on the operators* A *and* B *are essentially the same that in the case without delay. In addition, our deduction also shows an alternative proof for some of the results in this case. In fact, the crucial difference is that we do not use the operator* P *from Haussmann [8] and Ichikawa [11].*

KEY WORDS: Stochastic partial differential equation with delay, semigroups, Wiener process, pathwise asymptotic stability.

1. INTRODUCTION

The main aim of this paper is to prove some results on asymptotic stability for the linear Itô equation with delay in infinite dimension. First, sufficient conditions for exponential decay of the second moment are obtained. Next, asymptotic stability of sample paths wp1 (with probability 1) is deduced.

We consider a Stochastic Partial Differential Equation with delay which includes the case treated by Haussmann [8]. We develop a similar theory for strong solutions. However, we use a different technique to prove the results: we do not use the operator P from [8] and [11].

In order to illustrate and justify our work, let us show the following example:

Consider a one-dimensional rod of length π whose ends are maintained at 0° and whose sides are isolated. Assume that there is an exothermic reaction taking place inside the rod and that the heat produced at time t is proportional to the temperature at time $t-h$, $(h \geq 0)$. It is well known that the temperature in the rod may be modeled to satisfy

$$(P)\quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + ru(t-h,x)\ ,\ \ 0<x<\pi\ ,\ \ t>0 \\ u(t,0)=u(t,\pi)=0\ ,\ \ t>0, \\ u(t,x)=\psi(t,x)\ ,\ \ t\in[-h,0]\ ,\ \ x\in[0,\pi], \end{cases}$$

where r depends on the rate of reaction and $\psi : [-h,0]\times[0,\pi] \longrightarrow \mathbb{R}$ is a given function. If we assume $r = r_0$, a constant, it is not difficult to obtain asymptotic stability if $r_0^2 < 1$. In fact, we obtain exponential stability (see [2]).

However, it happens that, in many situations, r is random. Indeed, if we suppose that r is modeled as $r = r_0 \dot{w}$, where w is a one-dimensional Wiener process, then the PDE appearing in problem (P) can be written as an Itô equation:

$$du_t = Au_t\,dt + Bu_{t-h}\,dw_t\,,$$

where $A(= d^2/dx^2) \in \mathcal{L}(V,V')$, $B(= r_0 I) \in \mathcal{L}(H,H)$, with $V = H_0^1([0,\pi])$, $H = L^2([0,\pi])$ (see [1]). (Observe that with this election of Hilbert spaces the condition $u(t,0) = u(t,\pi) = 0\ ,\ \ t>0,$ is automatically fulfilled if $u_t \in V\ ,\ \ \forall t>0$).

Haussmann proves in [8] that, in the case without delay (i.e. $h=0$), pathwise asymptotic stability occurs if $r_0^2 < 2$. We shall obtain the same result not only for constant delays but for variable ones (see Sections 2, 3, 4). In fact, we study a more general equation:

$$du_t = Au_t\,dt + Bu_{\rho(t)}\,dw_t\ ,\ \ t>0\,,$$

where A, B are linear operators on suitable Hilbert spaces, w_t is a Hilbert-valued Wiener process and ρ is a function of delay. We present an alternative method for obtaining the stability results from [8] and [11] with similar, and in some cases identical, hypothesis.

In Section 2 we state the conditions under which strong solutions exists and coincides with the mild solution. In Sections 3, 4 the asymptotic exponential stability of second moment and the pathwise asymptotic stability wp1 are deduced. Last, some examples are given in Section 5 in order to illustrate our theory.

2. STATEMENT OF THE PROBLEM

Let V and H be two separable Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$ respectively. Assume that V is densely and continuously imbedded in H. We identify H with its dual H':

$$V \hookrightarrow H \hookrightarrow V',$$

and we denote by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality between V' and V ($\langle x\in V', y\in V\rangle$).

Let w_t be a Wiener process defined over the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values into the separable Hilbert space K, with incremental covariance operator W. Let $(\mathcal{F}_t)_{t\geq 0}$ be the σ-field generated by $\{w_s, 0 \leq s \leq t\}$, then w_t is a martingale relative to $(\mathcal{F}_t)_{t\geq 0}$ and we have the following representation of w_t :

$$w_t = \sum_{i=1}^{\infty} \beta_t^i e_i,$$

where (e_i) is an orthonormal set of eigenvalues of W, β_t^i are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $We_i = \lambda_i e_i$ and $\operatorname{tr} W = \sum_{i=1}^{\infty} \lambda_i$ (tr denotes the trace of an operator, see [10], [13,14]).

We assume that $A : V \longrightarrow V'$ is a linear continuous operator (i.e. $A \in \mathcal{L}(V, V')$), B is an element of $\mathcal{L}(V, \mathcal{L}(K, H))$ and

$$(c1): \qquad \exists\, \nu \in \mathbb{R}, \varepsilon > 0 : \quad -2\langle Ax, x\rangle + \nu|x|^2 \geq \varepsilon\|x\|^2 + \langle \Delta(I)x, x\rangle \quad \forall x \in V,$$

where $\Delta(I)$ (I is the identity operator) is an operator in $\mathcal{L}(V, V')$: for $P \in \mathcal{L}(H)$, $\Delta(P) \in \mathcal{L}(V, V')$ is defined by the relation

$$\langle \Delta(P)x, y\rangle = \operatorname{tr}\left(B(x)^* P B(y) W\right), \qquad \forall x, y \in V.$$

If $B \in \mathcal{L}(H, \mathcal{L}(K, H))$ then $\Delta(P) \in \mathcal{L}(H)$ and

$$(\Delta(P)x, y) = \operatorname{tr}\left(B(x)^* P B(y) W\right), \qquad \forall x, y \in H.$$

Let $\rho : [0, +\infty) \longrightarrow \mathbb{R}$ be a continuously differentiable function (of delay) such that

$$\exists\, h > 0 : \qquad -h \leq \rho(t) \leq t \quad \forall t \geq 0, \quad \rho'(t) \geq 1, \quad \forall t \geq 0, \tag{2.1}$$

which obviously implies that there exists a positive constant k with

$$\rho^{-1}(t) \leq t + k, \quad \forall t \geq -h, \tag{2.2}$$

and let ψ be a function such that

$$\begin{cases} \psi \in L^2(\Omega \times [-h, 0], \mathcal{F}_0 \otimes \mathcal{B}([-h, 0]), d\mathbf{P} \otimes dt; V), \\ \psi(0) = x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H). \end{cases} \tag{2.3}$$

Under the preceeding hypotheses (see Real [15,16]) there exists an unique process, x_t, adapted to $\mathcal{F}_t$, $\forall t \in [-h, T]$, $\forall T > 0$ (where $\mathcal{F}_t = \mathcal{F}_0$, $\forall t \in [-h, 0]$), solution of

$$(PC) \quad \begin{cases} x_t \in L^2(\Omega \times (0, T); V) \cap L^2(\Omega; C(0, T; H)), \;\; \forall T > 0, \\ x_t = x_0 + \int_0^t Ax_s\, ds + \int_0^t B(x_{\rho(s)})\, dw_s, \;\; \mathbf{P} - a.s., \; \forall t \in [0, T], \;\; \text{(equality in } V'), \\ x_t = \psi(t), \quad \forall t \in [-h, 0] \;\; \text{(equality in } H). \end{cases}$$

In the sequel, we shall write (PC) in the following abreviate form:

$$(PC)' \quad \begin{cases} x_t \in L^2(\Omega \times (0,T);V) \cap L^2(\Omega; C(0,T;H)), & \forall T > 0, \\ dx_t = Ax_t\,dt + B(x_{\rho(t)})\,dw_t\,, & \forall t \in [0,T], \\ x_t = \psi(t)\,, \quad \forall t \in [-h,0]\,, \end{cases}$$

where

$$L^2(\Omega \times (0,T);V) = L^2(\Omega \times (0,T), \mathcal{F} \otimes \mathcal{B}([0,T]), d\mathbf{P} \otimes dt; V),$$

$$L^2(\Omega; C(0,T;H)) = L^2(\Omega, \mathcal{F}, \mathbf{P}; C(0,T;H)).$$

Such a process is called the strong solution of (PC). Note that when $B \in \mathcal{L}(H, \mathcal{L}(K,H))$, we can assume the following hypotheses $(c2)$ and (2.4) instead of $(c1)$ and (2.3), respectively:

$$(c2): \qquad \exists \nu \in \mathbb{R}, \varepsilon > 0: \quad -2\langle Ax, x\rangle + \nu|x|^2 \geq \varepsilon\|x\|^2 \quad \forall x \in V,$$

$$\begin{cases} \psi \in L^2(\Omega \times [-h,0], \mathcal{F}_0 \otimes \mathcal{B}([-h,0]), d\mathbf{P} \otimes dt; H), \\ \psi(0) = x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H). \end{cases} \tag{2.4}$$

Moreover, if condition $(c2)$ is satisfied (observe that $(c1)$ implies $(c2)$), A generates a strongly continuous semigroup U_t (see Dautray-Lions [6]), and the strong solution of (PC) is also the mild solution. In other words, the strong solution x_t satisfies

$$(PCG) \quad \begin{cases} x_t = U_t\psi(0) + \int_0^t U_{t-s}B(x_{\rho(s)})\,dw_s\,, & \mathbf{P}-a.s.\,, \forall t \in [0,T] \text{ (equality in } H) \\ x_t = \psi(t)\,, \quad t \in [-h,0] \end{cases}$$

(see [2,3]).

In order to obtain asymptotic stability of paths wp1, we shall divide our study into two cases:

a) The first case: when $B \in \mathcal{L}(H, \mathcal{L}(K,H))$;

b) The second case: when $B \in \mathcal{L}(V, \mathcal{L}(K,H))$.

In the first case, we impose essentially the same hypotheses than those in [8]. In the second, our conditions are rather restrictive when A and B do not conmute. In both cases, we first get exponential stability of the second moment of x_t. Next, we use this result to deduce the pathwise asymptotic stability wp1.

3. ASYMPTOTIC STABILITY: THE FIRST CASE

We begin with the exponential stability for the second moment. We shall also denote by $|\cdot|$ the norm in $\mathcal{L}(H,H)$.

THEOREM 3.1. *Let* $A \in \mathcal{L}(V,V')$, $B \in \mathcal{L}(H, \mathcal{L}(K,H))$. *Assume* (2.1), (2.2), (2.4), $(c2)$ *and*

$$(H_1): \qquad \exists \gamma > 0, c > 0: \quad |U_t| \leq ce^{-\gamma t}, \quad \forall t \geq 0,$$

$$(H_2): \qquad \left| \int_0^\infty \Delta(U_t^* U_t)\,dt \right| < 1.$$

Then, there exist positive constants λ, K *such that the strong solution* x_t *of* (PC) *satisfies*

$$\mathbf{E}|x_t|^2 \le K\|\psi\|_1^2 e^{-\lambda t}, \quad \forall t \ge 0, \tag{3.1}$$

where $\|\psi\|_1^2 = \max\{\mathbf{E}|\psi(0)|^2, \int_{-h}^0 \mathbf{E}|\psi(t)|^2\, ds\}$.

Proof: We shall split the proof in two steps. In the first one, we deduce that there exist positive constants λ, K_1, such that

$$\int_0^\infty e^{\lambda t}\mathbf{E}|x_t|^2\, dt \le K_1\|\psi\|_1^2. \tag{3.2}$$

In the second, using (3.2) and Itô's formula we obtain (3.1).

First step: Since x_t satisfies (PCG), it follows

$$\begin{cases} x_t = U_t x_0 + \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s, \ \mathbf{P}-a.s. \quad \forall t \ge 0 \\ x_t = \psi(t), \quad t \in [-h, 0]. \end{cases}$$

Hence,

$$\begin{aligned} |x_t|^2 = |U_t x_0|^2 &+ \left| \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s \right|^2 \\ &+ 2\left(U_t x_0, \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s \right), \ \mathbf{P}-a.s., \ \forall t \ge 0. \end{aligned}$$

¿From here, we obtain

$$\mathbf{E}|x_t|^2 = \mathbf{E}|U_t x_0|^2 + \mathbf{E}\left| \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s \right|^2, \ \forall t \ge 0,$$

since $U_t x_0$ is $\mathcal{F}_0$-measurable, and consequently

$$\mathbf{E}\left(U_t x_0, \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s \right) = 0.$$

By [10, Proposition 1.4]

$$\begin{aligned} \mathbf{E}\left| \int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s \right|^2 &= \int_0^t \mathbf{E}\left[\mathrm{tr}\left((U_{t-s}B(x_{\rho(s)}))^*(U_{t-s}B(x_{\rho(s)}))W \right) ght \right] ds \\ &= \int_0^t \mathbf{E}\left[\mathrm{tr}\ \left((B(x_{\rho(s)}))^* U_{t-s}^* U_{t-s} B(x_{\rho(s)})W \right) \right]\, ds \\ &= \int_0^t \mathbf{E}\left(\Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)} \right)\, ds. \end{aligned}$$

Then it follows

$$\mathbf{E}|x_t|^2 = \mathbf{E}|x_0|^2 + \int_0^t \mathbf{E}(\Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)})\, ds.$$

We take $\lambda > 0$ (still undetermined). From the last equation,

$$\begin{aligned} \int_0^\infty e^{\lambda t}\mathbf{E}|x_t|^2\, dt = &\int_0^\infty e^{\lambda t}\mathbf{E}|U_t x_0|^2\, dt \\ &+ \int_0^\infty e^{\lambda t} \int_0^t \mathbf{E}\left(\Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)} \right)\, ds\, dt. \end{aligned} \tag{3.3}$$

Evaluating the first term on the right-hand side of (3.3), we obtain:

By (H_1),

$$\int_0^\infty e^{\lambda t}\mathbf{E}|U_t x_0|^2\,dt \le \frac{c^2}{2\gamma-\lambda}\|\psi\|_1^2, \tag{3.4}$$

if λ is such that $0<\lambda<2\gamma$.

Also, Fubini's Theorem and the change of variables $u=\rho(s)$ yield

$$\begin{aligned}
&\int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(\Delta(U_{t-s}^*U_{t-s})x_{\rho(s)},x_{\rho(s)}\right)\,ds\,dt\\
&\qquad= \int_0^\infty e^{\lambda s}\int_0^\infty e^{\lambda t}\mathbf{E}\left(\Delta(U_t^*U_t)x_{\rho(s)},x_{\rho(s)}\right)\,dt\,ds\\
&\qquad\le \left|\int_0^\infty e^{\lambda t}\Delta(U_t^*U_t)\,dt\right|\int_0^\infty e^{\lambda s}\mathbf{E}|x_{\rho(s)}|^2\,ds\\
&\qquad\le f(\lambda)e^{\lambda k}\int_{-h}^\infty e^{\lambda s}\mathbf{E}|x_s|^2\,ds\\
&\qquad\le f(\lambda)e^{\lambda k}\|\psi\|_1^2 + f(\lambda)e^{\lambda k}\int_0^\infty e^{\lambda s}\mathbf{E}|x_s|^2\,ds,
\end{aligned} \tag{3.5}$$

where

$$f(\lambda)=\left|\int_0^\infty e^{\lambda t}\Delta(U_t^*U_t)\,dt\right|.$$

Using (3.3)–(3.5),

$$\begin{aligned}
&\int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(\Delta(U_{t-s}^*U_{t-s})x_{\rho(s)},x_{\rho(s)}\right)\,ds\,dt\\
&\qquad\le f(\lambda)e^{\lambda k}\left(1+\frac{c^2}{2\gamma-\lambda}\right)\|\psi\|_1^2\\
&\qquad\quad + f(\lambda)e^{\lambda k}\int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(\Delta(U_{t-s}^*U_{t-s})x_{\rho(s)},x_{\rho(s)}\right)\,ds\,dt.
\end{aligned} \tag{3.6}$$

The continuity of functions defined by integrals depending on parameters and (H_2) yield

$$\lim_{\lambda\to 0+} f(\lambda)e^{\lambda k}<1.$$

Then, we can take λ verifying $0<\lambda<2\gamma$, $f(\lambda)e^{\lambda k}<1$. Consequently, there exists a constant C_1 (which only depends on λ) such that

$$\int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(\Delta(U_{t-s}^*U_{t-s})x_{\rho(s)},x_{\rho(s)}\right)\,ds\,dt \le C_1\|\psi\|_1^2. \tag{3.7}$$

¿From (3.4) and (3.7) it follows that for each $\lambda>0$, small enough, there exists a positive constant $K_1=K_1(\lambda)$ such that

$$\int_0^\infty e^{\lambda t}\mathbf{E}|x_t|^2\,dt \le K_1\|\psi\|_1^2. \tag{3.8}$$

Second step: Applying Itô's formula for the process $e^{\lambda t}|x_t|^2$, we obtain

$$\begin{aligned}
e^{\lambda t}|x_t|^2 = |x_0|^2 &+ \lambda\int_0^t e^{\lambda s}|x_s|^2\,ds + 2\int_0^t e^{\lambda s}\langle Ax_s,x_s\rangle\,ds\\
&+\int_0^t e^{\lambda s}(\Delta(I)x_{\rho(s)},x_{\rho(s)})\,ds + 2\int_0^t e^{\lambda s}(x_s,B(x_{\rho(s)})dw_s).
\end{aligned} \tag{3.9}$$

¿From $(c2)$,

$$\begin{aligned}e^{\lambda t}\mathbf{E}|x_t|^2 &= \mathbf{E}|x_0|^2 + \lambda\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2 ds + 2\int_0^t e^{\lambda s}\mathbf{E}\langle Ax_s, x_s\rangle\, ds \\ &\quad + \int_0^t e^{\lambda s}\mathbf{E}\left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right)\, ds \\ &\leq \|\psi\|_1^2 + (\lambda+\nu)\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\, ds + |\Delta(I)|\int_0^t e^{\lambda s}\mathbf{E}|x_{\rho(s)}|^2 ds\,,\ \forall t\geq 0. \end{aligned} \tag{3.10}$$

By (3.8) and the change of variables used before, we get

$$e^{\lambda t}\mathbf{E}|x_t|^2 \leq K\|\psi\|_1^2\,, \quad \forall t\geq 0, \tag{3.11}$$

and (3.1) follows. ∎

The following Lemma shows that, in fact, the strong solution x_t lies in $L^2(\Omega; C(0,+\infty; H))$.

LEMMA 3.1. *Assume* $(c2)$. *If* x_t *satisfies* (3.1), *then there exists a constant* $K_2 \in \mathbb{R}$ *such that*

$$\mathbf{E}\left[\sup_{0\leq t<\infty}|x_t|^2\right] \leq K_2\|\psi\|_1^2. \tag{3.12}$$

Proof: We shall use a technique which is similar the one used by Haussmann in [8]. The Energy equality (see [14, Theorem 3.1]) and $(c2)$ yield:

$$\begin{aligned}|x_t|^2 &= |x_0|^2 + 2\int_0^t \langle Ax_s, x_s\rangle\, ds + \int_0^t \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right)\, ds + 2\int_0^t \left(x_s, B(x_{\rho(s)})\, dw_s\right) \\ &\leq |x_0|^2 + \nu\int_0^t |x_s|^2\, ds + |\Delta(I)|\int_0^t |x_{\rho(s)}|^2\, ds + 2\int_0^t \left(x_s, B(x_{\rho(s)})\, dw_s\right). \end{aligned} \tag{3.13}$$

Now, we fix $T>0$. From (3.13),

$$\begin{aligned}\mathbf{E}\left[\sup_{0\leq t\leq T}|x_t|^2\right] &\leq \|\psi\|_1^2 + \nu\int_0^T \mathbf{E}|x_s|^2\, ds + |\Delta(I)|\int_0^T \mathbf{E}|x_{\rho(s)}|^2\, ds \\ &\quad + 2\mathbf{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t (x_s, B(x_{\rho(s)})\, dw_s)\right|\right]. \end{aligned} \tag{3.14}$$

Again, the change $u=\rho(s)$ yields

$$\int_0^T \mathbf{E}|x_{\rho(s)}|^2\, ds \leq \|\psi\|_1^2 + \int_0^T \mathbf{E}|x_s|^2\, ds. \tag{3.15}$$

Then,

$$\begin{aligned}E\left[\sup_{0\leq t\leq T}|x_t|^2\right] &\leq (1+|\Delta(I)|)\|\psi\|_1^2 + (\nu+|\Delta(I)|)\int_0^T \mathbf{E}|x_s|^2\, ds \\ &\quad + \mathbf{E}\left[2\sup_{0\leq t\leq T}\left|\int_0^t \left(x_s, B(x_{\rho(s)})\, dw_s\right)\right|\right]. \end{aligned} \tag{3.16}$$

Equation (3.1) implies

$$\int_0^T \mathbf{E}|x_s|^2\, ds \le \left[1 - e^{-\lambda T}\right] \frac{K_1\|\psi\|_1^2}{\lambda} \le \frac{K_1\|\psi\|_1^2}{\lambda}, \quad \forall T > 0. \tag{3.17}$$

Also, Burkhölder–Davis–Gundy's inequality, the preceeding change and (3.15) conduce to the following chain of inequalities (for any $l > 0$):

$$\begin{aligned}
2\mathbf{E}&\left[\sup_{0\le t\le T}\left|\int_0^t \left(x_s, B(x_{\rho(s)})\, dw_s\right)\right|\right] \\
&\le 6\mathbf{E}\left[\int_0^T |x_s|^2 \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right)\, dsght]^{1/2} \\
&\le 6\mathbf{E}\left[\sup_{0\le t\le T}|x_t|\left(\int_0^T \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds\right)^{1/2} ight] \\
&\le 3l\mathbf{E}\left[\sup_{0\le t\le T}|x_t|^2\right] + \frac{3}{l}\int_0^T \mathbf{E}\left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \\
&\le 3l\mathbf{E}\left[\sup_{0\le t\le T}|x_t|^2\right] + \frac{3}{l}|\Delta(I)|\int_0^T \mathbf{E}|x_{\rho(s)}|^2\, ds.
\end{aligned} \tag{3.18}$$

If we take $l = \frac{1}{6}$ and substitute into (3.16) after using (3.15), (3.17), (3.18),

$$\mathbf{E}\left[\sup_{0\le t\le T}|x_t|^2\right] \le K_2\|\psi\|_1^2, \quad \forall T > 0. \tag{3.19}$$

Since K_2 is independent of T, Lebesgue's Theorem yields (3.12). ∎

Now we state the pathwise asymptotic stability for the solution of (PC) using a technique close to Haussmann's.

THEOREM 3.2. *Assume* (3.1) *holds for the strong solution of* (PC) *(for example, the conditions of Theorem* 3.1 *are fulfilled), and* $-A$ *is coercive (in other words,* $(c2)$ *is satisfied). Then, there exist positive constants* α, β *and there exists* $\Lambda \subset \Omega$ *with* $\mathbf{P}(\Lambda) = 0$, *such that*

$$\forall \omega \in \Omega \setminus \Lambda \quad \exists T(\omega) \in \mathbb{R}: \quad |x_t(\omega)|^2 \le \alpha\|\psi\|_1^2 e^{-\beta t},\ \forall t \ge T(\omega). \tag{3.20}$$

Proof: Let N_0 be the first natural number such that $\rho(N_0) \ge 0$. Since $\rho'(t) \ge 1 > 0$ then $\rho(N) > 0$, $\forall N > N_0$. Let N be a natural number with $N \ge N_0$. ¿From Itô's formula,

$$\begin{aligned}
|x_t|^2 = |x_N|^2 &+ 2\int_N^t \langle Ax_s, x_s\rangle\, ds + \int_N^t \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \\
&+ 2\int_N^t \left(x_s, Bx_{\rho(s)}\, dw_s\right), \quad \forall t \ge N
\end{aligned} \tag{3.21}$$

and by $(c2)$

$$\begin{aligned}
|x_t|^2 \le |x_N|^2 &+ \nu\int_N^t |x_s|^2\, ds + \int_N^t \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \\
&+ 2\int_N^t \left(x_s, Bx_{\rho(s)}\, dw_s\right), \quad \forall t \ge N.
\end{aligned} \tag{3.22}$$

Let I_N denote the interval $[N, N+1]$. For each $\varepsilon > 0$ we have

$$\begin{aligned} &\mathbf{P}\left[\sup_{t\in I_N} |x_t|^2 \geq \varepsilon^2\right] \\ &\leq \mathbf{P}\left[|x_N|^2 \geq \frac{\varepsilon^2}{4}\right] + \mathbf{P}\left[\nu \int_N^{N+1} |x_s|^2 ds \geq \frac{\varepsilon^2}{4}\right] \\ &\quad + \mathbf{P}\left[\int_N^{N+1} \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \geq \frac{\varepsilon^2}{4}\right] \\ &\quad + \mathbf{P}\left[2 \sup_{t\in I_N} \left|\int_N^t \left(x_s, Bx_{\rho(s)}\, dw_s\right)\right| \geq \frac{\varepsilon^2}{4}\right]. \end{aligned} \tag{3.23}$$

Kolmogorov's inequality and (3.1) give

$$\mathbf{P}\left[|x_N|^2 \geq \frac{\varepsilon^2}{4}\right] \leq \frac{4}{\varepsilon^2}\mathbf{E}|x_N|^2 \leq \frac{4K\|\psi\|_1^2}{\varepsilon^2} e^{-\lambda N}. \tag{3.24}$$

$$\mathbf{P}\left[\nu \int_N^{N+1} |x_s|^2 ds \geq \frac{\varepsilon^2}{4}\right] \leq \frac{4\nu}{\varepsilon^2} \int_N^{N+1} \mathbf{E}|x_s|^2 ds \leq \frac{4\nu K\|\psi\|_1^2}{\lambda\varepsilon^2} e^{-\lambda N}. \tag{3.25}$$

In view of the change $u = \rho(s)$ and (3.1),

$$\begin{aligned} \mathbf{P}\left[\int_N^{N+1} \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \geq \frac{\varepsilon^2}{4}\right] &\leq \frac{4}{\varepsilon^2} \int_N^{N+1} \left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds \\ &\leq \frac{4|\Delta(I)|}{\varepsilon^2} \int_{\rho(N)}^{\rho(N+1)} \mathbf{E}|x_u|^2\, du \\ &\leq \frac{4|\Delta(I)|Ke^{\lambda h}\|\psi\|_1^2}{\lambda\varepsilon^2} e^{-\lambda N}. \end{aligned} \tag{3.26}$$

Finally, using successively the inequalities of Kolmogorov, Burkhölder–Davis–Gundy and Hölder, Lemma 3.1 and the usual change of variables,

$$\begin{aligned} &\mathbf{P}\left[2 \sup_{t\in I_N} \left|\int_N^t \left(x_s, B(x_{\rho(s)}\, dw_s\right)\right| \geq \frac{\varepsilon^2}{4}\right] \\ &\leq \frac{8}{\varepsilon^2}\mathbf{E}\left[\sup_{t\in I_N} \left|\int_N^t \left(x_s, B(x_{\rho(s)}\, dw_s\right)\right|\right] \\ &\leq \frac{24}{\varepsilon^2}\mathbf{E}\left[\sup_{t\in I_N} |x_t|^2\right]^{1/2} \left[\int_N^{N+1} \mathbf{E}\left(\Delta(I)x_{\rho(s)}, x_{\rho(s)}\right) ds\right]^{1/2} \\ &\leq \frac{24|\Delta(I)|K_2^{1/2}\|\psi\|_1}{\varepsilon^2} \left[\int_N^{N+1} \mathbf{E}|x_{\rho(s)}|^2 ds\right]^{1/2} \\ &\leq \frac{24|\Delta(I)|K_2^{1/2}K^{1/2}\|\psi\|_1^2 e^{\lambda h/2}}{\lambda^{1/2}\varepsilon^2} e^{-\lambda N/2}. \end{aligned} \tag{3.27}$$

For every natural number $N \geq N_0$, we set $\varepsilon = \varepsilon_N = \|\psi\|_1 e^{-\lambda N/8}$. Then, it follows

$$\mathbf{P}\left[\sup_{t\in I_N} |x_t|^2 \geq \varepsilon_N^2\right] \leq Me^{-\lambda N/4}, \tag{3.28}$$

where M is independent of N.

Borel-Cantelli's Lemma and (3.28) imply (3.20). ∎

Remark 3.1. However, there exists a condition stronger than $(H_1)-(H_2)$. This condition is

$$(H_{12}): \qquad \exists\, \mu>0: \quad -2\langle Ax,x\rangle \geq \mu|x|^2+(\Delta(I)x,x)\,, \quad \forall x\in V.$$

Now we shall prove that (H_{12}) and $(c2)$ yield (3.1).

THEOREM 3.3. *If (H_{12}) and $(c2)$ hold, then the solution of (PC) verifies* (3.1).

Proof: Since $\lim_{\lambda\to 0^+}[\lambda-\mu+|\Delta(I)|(e^{\lambda k}-1)]=-\mu<0$, we can take $\lambda>0$ such that

$$\lambda-\mu+|\Delta(I)|(e^{\lambda k}-1)<0.$$

Hence, (H_{12}) and Itô's formula for $e^{\lambda t}|x_t|^2$ give

$$\begin{aligned} e^{\lambda t}\mathbf{E}|x_t|^2-\mathbf{E}|x_0|^2 &= \lambda\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds+2\int_0^t e^{\lambda s}\mathbf{E}\langle Ax_s,x_s\rangle\,ds \\ &\quad +\int_0^t e^{\lambda s}\mathbf{E}\left(\Delta(I)x_{\rho(s)},x_{\rho(s)}\right)\,ds \\ &\leq (\lambda-\mu)\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds+\int_0^t e^{\lambda s}\mathbf{E}\left(\Delta(I)x_{\rho(s)},x_{\rho(s)}\right)\,ds \\ &\quad -\int_0^t e^{\lambda s}\mathbf{E}\left(\Delta(I)x_s,x_s\right)\,ds. \end{aligned} \tag{3.29}$$

Then

$$e^{\lambda t}\mathbf{E}|x_t|^2\leq (1+e^{\lambda k}|\Delta(I)|)||\psi||_1^2+\left[\lambda-\mu+|\Delta(I)|(e^{\lambda k}-1)\right]\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds. \tag{3.30}$$

Clearly, (3.1) follows from (3.30). ∎

Remark 3.2. Consequently, we can obtain (3.20) under the hypothesis (H_{12}) and (3.1), using a proof analogous to that of Theorem 3.2 above.

4. ASYMPTOTIC STABILITY: THE SECOND CASE

We denote by $||\cdot||$ the norm in $\mathcal{L}(V,V')$, by $\bar{c}$ the norm of the injection $V\hookrightarrow H$ (then, $|x|\leq\bar{c}||x|| \quad \forall x\in V$), and by $||\psi||_1^2=\max\{\mathbf{E}|\psi(0)|^2,\int_{-h}^0\mathbf{E}||\psi(s)||^2\,ds\}$.

The following result is similar to Theorem 3.3.

THEOREM 4.1. *If x_t is the solution of (PC) and $(c1)$ holds with $\nu\bar{c}^2-\varepsilon<0$, then* (3.1) *also holds.* ∎

Remark 4.1. Observe that $\nu\bar{c}^2-\varepsilon<0$ if, for example, $\nu\leq 0$. In this case, $(c1)$ implies (H_{12}).

Next, we shall consider the case $\nu > 0$ and $\nu\bar{c}^2 - \varepsilon \geq 0$. We carry out our analysis of stability in two steps:

i) If B and U_t conmute,

ii) If B and U_t do not conmute.

In the case i), our hypotheses are exactly the same as in [8], while in ii) we shall consider a case somehow less general than in [8].

Remark 4.2. Observe that under $(c1)$ (or $(c2)$), U_t maps H into V and there is a constant c_0 satisfying

$$\int_0^\infty e^{-2\nu t}\|U_t x\|^2\, dt \leq c_0|x|^2\,, \qquad \forall x \in V$$

(see [12, Cap. IV, Theorem 1.1]). Then, the operators BU_t and U_tB, defined by

$$(BU_t)(x)(y) = B(U_t x)(y)\,, \quad \forall x \in V \quad \forall y \in K;$$

$$(U_tB)(x)(y) = U_t(B(x)(y))\,, \quad \forall x \in V \quad \forall y \in K,$$

belong to $\mathcal{L}(V, \mathcal{L}(K, H))$.

We say that B and U_t conmute if

$$(BU_t)(x) = (U_tB)(x)\ , \qquad \forall x \in V.$$

THEOREM 4.2. *Assume* $A \in \mathcal{L}(V, V')$, $B \in \mathcal{L}(V, \mathcal{L}(K, H))$, $(2.1) - (2.3), (c1), (H_1)$, (H_2'), *where*

$$(H_2'): \qquad \left|\int_0^\infty U_t^*\Delta(I)U_t\, dt\right| < 1.$$

Assume also that B *conmute with* U_t. *Then* (3.1) *holds for the solution* x_t *of* (PC).

Proof: Our proof is similar to that of Theorem 3.1. According to our hypotheses, we can take $\lambda > 0$ such that

$$0 < \lambda < 2\gamma\,,\ e^{\lambda k} - 1 - \frac{\varepsilon}{\|\Delta(I)\|} < 0\,,\ e^{\lambda k} g(\lambda) < 1,$$

where

$$g(\lambda) = \left|\int_0^\infty e^{\lambda t}U_t^*\Delta(I)U_t\, dt\right|.$$

Then, as in Theorem 3.1,

$$\mathbf{E}|x_t|^2 = \mathbf{E}|U_t x_0|^2 + \mathbf{E}\left|\int_0^t U_{t-s}B(x_{\rho(s)})\, dw_s\right|^2. \tag{4.1}$$

Since B and U_t conmute, we have

$$\begin{aligned}\mathbf{E}\left|\int_0^t U_{t-s}B(x_{\rho(s)}\, dw_s\right|^2 &= \int_0^t \mathbf{E}\left[\mathrm{tr}\ \left[(U_{t-s}B(x_{\rho(s)}))^*(U_{t-s}B(x_{\rho(s)}))W\right]\right]\, ds \\ &= \int_0^t \mathbf{E}\left[\mathrm{tr}\ \left[(B(U_{t-s}x_{\rho(s)}))^*(B(U_{t-s}x_{\rho(s)}))W\right]\right]\, ds \\ &= \int_0^t \mathbf{E}\left(U_{t-s}^*\Delta(I)U_{t-s}x_{\rho(s)}, x_{\rho(s)}\right)\, ds.\end{aligned} \tag{4.2}$$

Now, (4.1) yields

$$\int_0^\infty e^{\lambda t}\mathbf{E}|x_t|^2\,dt = \int_0^\infty e^{\lambda t}\mathbf{E}|U_t x_0|^2\,dt + \int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(U_{t-s}^*\Delta(I)U_{t-s}x_{\rho(s)}, x_{\rho(s)}\right)\,ds\,dt. \tag{4.3}$$

The first term on the right-hand side of (4.3) is bounded as in Theorem 3.1. To obtain a bound for the second one, it is sufficient to observe that

$$\begin{aligned}\int_0^\infty e^{\lambda t}\int_0^t \mathbf{E}\left(U_{t-s}^*\Delta(I)U_{t-s}x_{\rho(s)}, x_{\rho(s)}\right)\,ds\,dt \\ = \int_0^\infty e^{\lambda s}\int_0^\infty e^{\lambda t}\mathbf{E}\left(U_t^*\Delta(I)U_t x_{\rho(s)}, x_{\rho(s)}\right)\,dt\,ds \\ \leq g(\lambda)\int_0^\infty e^{\lambda s}\mathbf{E}|x_{\rho(s)}|^2\,ds.\end{aligned}$$

The proof can be completed as in Theorem 3.1, using the estimates obtained for λ at the beginning.
∎

Let us now turn our attention to the nonconmutative case. At first, we shall prove a technical Lemma, to be used in the sequel:

LEMMA 4.1. *Assume* $(c1)$. *Then, for every* $\lambda > 0$ *such that* $(e^{\lambda k}-1)||\Delta(I)||-\varepsilon<0$, *the strong solution of* (PC) *satisfies*

$$\frac{e^{\lambda t}\mathbf{E}|x_t|^2}{\varepsilon-(e^{\lambda k}-1)||\Delta(I)||} + \int_0^t e^{\lambda s}\mathbf{E}||x_s||^2\,ds \leq \frac{(1+||\Delta(I)||e^{\lambda k})}{\varepsilon-(e^{\lambda k}-1)||\Delta(I)||}||\psi||_1^2 + \frac{\lambda+\nu}{\varepsilon-(e^{\lambda k}-1)||\Delta(I)||}\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds\,,\ \forall t\geq 0. \tag{4.4}$$

Proof: According to Itô's formula, the usual change of variables and $(c1)$, we obtain

$$\begin{aligned}e^{\lambda t}\mathbf{E}|x_t|^2 \leq (1+||\Delta(I)||e^{\lambda k})||\psi||_1^2 + (\lambda+\nu)\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds \\ + ((e^{\lambda k}-1)||\Delta(I)||-\varepsilon)\int_0^t e^{\lambda s}\mathbf{E}||x_s||^2\,ds\,.\end{aligned}$$ ∎

THEOREM 4.3. *If* $(2.1)-(2.3), (c1), (H_1), (H_2'')$ *hold, where*

$$(H_2''):\qquad \left\|\int_0^\infty \Delta(U_t^*U_t)\,dt\right\| < \frac{\varepsilon}{\nu}\,,$$

then (3.1) *holds for the strong solution* x_t *of* (PC).

Proof: As in Theorem 4.2 we can take λ such that $0<\lambda<2\gamma$, $(e^{\lambda k}-1)||\Delta(I)||-\varepsilon<0$, and

$$\frac{e^{\lambda k}(\lambda+\nu)f(\lambda)}{\varepsilon-(e^{\lambda k}-1)||\Delta(I)||} < 1,$$

where

$$f(\lambda) = \left\| \int_0^\infty e^{\lambda t} \Delta(U_t^* U_t)\, dt \right\|.$$

(Note that from (H_2''), it follows

$$\lim_{\lambda \to 0^+} \frac{e^{\lambda k}(\lambda + \nu) f(\lambda)}{\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|} < 1).$$

Now, we obtain

$$\begin{aligned} \int_0^\infty e^{\lambda t} \mathbf{E}|x_t|^2\, dt = & \int_0^\infty e^{\lambda t} \mathbf{E}|U_t x_0|^2\, dt \\ & + \int_0^\infty e^{\lambda t} \int_0^t \mathbf{E}\langle \Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)}\rangle\, ds\, dt. \end{aligned} \tag{4.5}$$

Again, we have

$$\int_0^\infty e^{\lambda t} \mathbf{E}|U_t x_0|^2\, dt \le \frac{c^2}{2\gamma - \lambda} \|\psi\|_1^2. \tag{4.6}$$

Moreover, Lemma 4.1 and (4.6) yield

$$\begin{aligned} & \int_0^\infty e^{\lambda t} \int_0^t \mathbf{E}\langle \Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)}\rangle\, ds\, dt \\ & = \int_0^\infty e^{\lambda s} \int_0^\infty e^{\lambda t} \mathbf{E}(\Delta(U_t^* U_t) x_{\rho(s)}, x_{\rho(s)})\, dt\, ds \\ & \le f(\lambda) \int_0^\infty e^{\lambda s} \mathbf{E}\|x_{\rho(s)}\|^2\, ds \\ & \le e^{\lambda k} f(\lambda) \|\psi\|_1^2 + e^{\lambda k} f(\lambda) \int_0^t e^{\lambda t} \mathbf{E}\|x_t\|^2\, dt \\ & \le e^{\lambda k} f(\lambda) \left[1 + \frac{1 + \|\Delta(I)\| e^{\lambda k}}{\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|} \right] \|\psi\|_1^2 \\ & \quad + \frac{e^{\lambda k} f(\lambda)(\lambda + \nu)}{\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|} \int_0^\infty e^{\lambda s} \mathbf{E}|x_s|^2\, ds \\ & \le e^{\lambda k} f(\lambda) \left[1 + \frac{1 + \|\Delta(I)\| e^{\lambda k}}{\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|} + \frac{(\lambda + \nu) c^2}{(2\gamma - \lambda)(\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|)} \right] \|\psi\|_1^2 \\ & \quad + \frac{e^{\lambda k} f(\lambda)(\lambda + \nu)}{\varepsilon - (e^{\lambda k} - 1)\|\Delta(I)\|} \int_0^\infty e^{\lambda t} \int_0^t \mathbf{E}(\Delta(U_{t-s}^* U_{t-s}) x_{\rho(s)}, x_{\rho(s)})\, ds\, dt. \end{aligned} \tag{4.7}$$

Hence from (4.5)–(4.7) there exists a constant K_4 (which only depends on λ), such that

$$\int_0^\infty e^{\lambda t} \mathbf{E}|x_t|^2\, dt \le K_4 \|\psi\|_1^2. \tag{4.8}$$

Applying Lemma 4.1 again, we conclude that

$$e^{\lambda t} \mathbf{E}|x_t|^2 \le K_5 \|\psi\|_1^2, \quad \forall t \ge 0. \quad \blacksquare \tag{4.9}$$

LEMMA 4.2. *Under assumptions of Theorem* 4.1 *(or Theorem* 4.2 *or Theorem* 4.3*), there exist positive constants* $\overline{K}_1$ *and* $\overline{K}_2$ *such that*

$$\int_\alpha^t \mathbf{E}\langle \Delta(I) x_s, x_s\rangle\, ds \le \overline{K}_1 \|\psi\|_1^2 e^{-\lambda\alpha}, \ \forall t \ge \alpha, \quad \forall \alpha \ge 0, \tag{4.10}$$

$$\mathbf{E}\left[\sup_{0\le t<\infty}|x_t|^2\right]\le \overline{K}_2||\psi||_1^2, \tag{4.11}$$

where λ *is the stability parameter appearing in* (3.1).

Proof: According to $(c1)$ and Itô's formula, we obtain

$$\begin{aligned} e^{\lambda t}\mathbf{E}|x_t|^2 &\le (1+e^{\lambda k}||\Delta(I)||)||\psi||_1^2+(\lambda+\nu)\int_0^t e^{\lambda s}\mathbf{E}|x_s|^2\,ds \\ &+(e^{\lambda k}-1)\int_0^t e^{\lambda s}\mathbf{E}\langle\Delta(I)x_s,x_s\rangle\,ds-\varepsilon\int_0^t e^{\lambda s}\mathbf{E}||x_s||^2\,ds. \end{aligned} \tag{4.12}$$

Let us now assume that the hypotheses of Theorem 4.1 hold. Then, (4.12) implies

$$\begin{aligned} e^{\lambda t}\mathbf{E}|x_t|^2 &\le (1+e^{\lambda k}||\Delta(I)||)||\psi||_1^2 \\ &+\left[(\lambda+\nu)\overline{c}^2-\varepsilon+(e^{\lambda k}-1)||\Delta(I)||\right]\int_0^t e^{\lambda s}\mathbf{E}||x_s||^2\,ds \end{aligned} \tag{4.13}$$

and hence,

$$\int_0^t e^{\lambda s}\mathbf{E}||x_s||^2\,ds\le \frac{(1+e^{\lambda k}||B||^2)}{\varepsilon-(\lambda+\nu)\overline{c}^2-(e^{\lambda k}-1)||B||^2}||\psi||_1^2\equiv C_1||\psi||_1^2. \tag{4.14}$$

Then

$$\int_0^t e^{\lambda s}\mathbf{E}\langle\Delta(I)x_s,x_s\rangle\,ds\le||\Delta(I)||C_1||\psi||_1^2\equiv\overline{K}_1||\psi||_1^2\quad\forall t>0. \tag{4.15}$$

Consequently, for $0\le\alpha<t$, we have

$$\int_\alpha^t e^{\lambda s}\mathbf{E}\langle\Delta(I)x_s,x_s\rangle\,ds\le\overline{K}_1||\psi||_1^2,\quad\forall t\ge\alpha,\quad\forall\alpha\ge0, \tag{4.16}$$

or

$$e^{-\lambda\alpha}\int_\alpha^t e^{\lambda s}\mathbf{E}\langle\Delta(I)x_s,x_s\rangle\,ds\le\overline{K}_1||\psi||_1^2e^{-\lambda\alpha},\quad\forall t\ge\alpha,\quad\forall\alpha\ge0. \tag{4.17}$$

Since $e^{\lambda(s-\alpha)}>1$, $\forall\alpha\le s$, then

$$\int_\alpha^t \mathbf{E}\langle\Delta(I)x_s,x_s\rangle\,ds\le\overline{K}_1||\psi||_1^2e^{-\lambda\alpha}. \tag{4.18}$$

¿From the other hand, under conditions of Theorems 4.2 and 4.3, the proof is similar. In fact, the crucial point is that, in any case, always exists a constant $\overline{K}$ such that

$$\int_0^\infty e^{\lambda t}\mathbf{E}|x_t|^2\,dt\le\overline{K}||\psi||_1^2,$$

where λ also satisfies

$$(e^{\lambda k}-1)||\Delta(I)||-\varepsilon<0.$$

The proof of (4.11) is deduced as in Lemma 3.1, using (4.10). ∎

THEOREM 4.4. *Assume* (4.10), (4.11). *Then, there exist positive constants* α,β *and there exists* $\Lambda\subset\Omega$ *with* $\mathbf{P}(\Lambda)=0$, *such that for every* $\omega\in\Omega\setminus\Lambda$ *exists* $T(\omega)>0$ *verifying*

$$|x_t(\omega)|^2\le\alpha||\psi||_1^2e^{-\beta t},\quad\forall t\ge T(\omega).$$

Proof: Apply Lemma 4.2 in the proof of Theorem 3.2. ∎

5. EXAMPLES

The most interesting applications for our theory arise when A and B are differential operators. In particular, when A is an elliptic operator and B is a first order differential operator. Let us see some examples.

In this Section we suppose that ρ, satisfies (2.1) (and so (2.2)), and the initial function ψ verifies (2.4) (if $B \in \mathcal{L}(H, \mathcal{L}(K, H))$), or (2.3) (if $B \in \mathcal{L}(V, \mathcal{L}(K, H))$).

EXAMPLE 5.1. Let V be the Sobolev space $H_0^1([0,\pi];\mathbb{R})$ and let H denote $L^2([0,\pi];\mathbb{R})$. It is well known (see, for example, Brezis [1]) that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

Moreover, the seminorm in $H_0^1([0,\pi];\mathbb{R})$ given by

$$||u|| = \left[\int_0^\pi \left[\frac{du}{dx}\right]^2 dx\right]^{1/2}$$

is in fact a norm equivalent to the usual one in this space, $||\cdot||_{H_0^1([0,\pi];\mathbb{R})}$. For the sake of simplicity we shall use the norm defined above as the norm in V. In this case, the norm of the canonical injection in H is $\bar{c} = 1$ (see [1]).

Let $|\cdot|$ and $(\cdot,\cdot)$ be the norm and the associated inner product in $L^2([0,\pi];\mathbb{R})$, and let $||\cdot||$ and $((\cdot,\cdot))$ the norm in $H_0^1([0,\pi];\mathbb{R})$ defined before, and its associated scalar product, respectively. Let A be the operator $\dfrac{d^2}{dx^2}$ and consider the following heat equation with delay which coincides with the one in Section 1 when $\rho(t) = t - h$:

$$(P1) \qquad \begin{cases} du(t,x) = \left(\dfrac{\partial^2 u}{\partial x^2}\right)(t,x) + r_1 u(\rho(t), x)\, dw_t\,, \\ u(t,0) = u(t,\pi) = 0\ , \forall t \geq 0\,, \quad u(t,x) = \psi(x) \quad \forall t \in [-h, 0], \end{cases}$$

where w_t is a real and standard Wiener process (i.e. $K = \mathbb{R}$) and r_1 is a constant .

Let $B = r_1 I$. Obviously $A \in \mathcal{L}(V, V')$ and $B \in \mathcal{L}(H, \mathcal{L}(\mathbb{R}, H)) \equiv \mathcal{L}(H)$.

We observe that

$$\langle Au, v\rangle = -\int_0^\pi \frac{du}{dx}\frac{dv}{dx}\, dx, \quad \forall u, v \in V, \tag{5.1}$$

$$\langle \Delta(I)u, u\rangle = |Bu|^2 = r_1^2 \int_0^\pi u^2\, dx = r_1^2 |u|^2, \quad \forall u \in H, \tag{5.2}$$

In particular,

$$-2\langle Au, u\rangle = 2\int_0^\pi \left(\frac{du}{dx}\right)^2 dx = 2||u||^2, \quad \forall u \in V. \tag{5.3}$$

Hence we can write $(c2)$ in the form

$$2||u||^2 + \nu|u|^2 \geq \varepsilon||u||^2, \quad \forall u \in V, \tag{5.4}$$

and this condition holds if $\nu = 0, \varepsilon \leq 2$. Since

$$|Bu|^2 + \mu|u|^2 = (r_1^2 + \mu)|u|^2 \leq 2||u||^2 = -2\langle Au, u\rangle, \quad \forall v \in V,$$

then, if we take r_1 and μ such that $r_1^2 < 1, 0 < \mu \leq 1 - r_1^2$, (H_{12}) follows. Thus we can apply Theorem 3.3 and Remark 3.2. Consequently (3.1) and (3.20) hold.

Now, we shall prove that there exists r_1 with $r_1 \geq 1$ such that (3.1) (and so, (3.20)) still holds. We use Theorem 3.1 to obtain this result. It is enough to verify the hypotheses (H_1), (H_2). Let us consider at first that A generates a strongly continuous semigroup U_t satisfying (H_1) with $\gamma = c = 1$. Indeed, Weinberger [17] proves that:

"If $u_0 \in H$, then $u(t,x) = U_t u_0(x)$ is the solution of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0\,, \quad 0 < x < \pi\,, \quad t > 0, \\ u(t,0) = u(t,\pi) = 0, \quad t > 0, \\ u(0,x) = u_0(x), \quad 0 < x < \pi, \end{cases}$$

and moreover $|u(t,.)| \leq e^{-t}|u_0(.)|$".

Next, we observe that $\Delta(P) = B^*PB$ and

$$\left|\int_0^\infty \Delta(U_t^* U_t)\, dt\right| = \left|\int_0^\infty B^* U_t^* U_t B\, dt\right| \leq |B|^2 \int_0^\infty |U_t|^2\, dt \leq r_1^2 \int_0^\infty e^{-2t}\, dt = \frac{r_1^2}{2}.$$

If $r_1^2 < 2$, then (H_2) holds and according to Theorems 3.1–3.2, the conditions (3.1) and (3.20) are satisfied.

In conclusion, if $r_1 \in (-\sqrt{2}, \sqrt{2})$ we obtain pathwise asymptotic stability wp1 for the solution of $(P1)$.

Remark 5.1. Let r_1 be a function in $L^\infty([0,\pi];\mathbb{R})$ instead of a constant. The same conclusions as in Example 5.1 are obtained if

$$||r_1||^2_{L^\infty(0,\pi;\mathbb{R})} < 2.$$

EXAMPLE 5.2. Let V and H as in the precedent Example, $A = \dfrac{d^2}{dx^2}$, $B = r_1 \dfrac{d}{dx}$; now we have $A \in \mathcal{L}(V,V')$ and $B \in \mathcal{L}(V, \mathcal{L}(K,H)) \equiv \mathcal{L}(V,H)$, $\forall r_1 \in \mathbb{R}$, and

$$|Bu|^2 = r_1^2 \int_0^\pi \left(\frac{du}{dx}\right)^2 dx = r_1^2||u||^2, \quad \forall u \in V.$$

Then $(c1)$ takes the form

$$2||u||^2 + \nu|u|^2 \geq \varepsilon||u||^2 + r_1^2||u||^2, \quad \forall u \in V,$$

and if $\nu = 0,\ r_1^2 < 2,\ \varepsilon = 2 - r_1^2$, this condition is verified.

According to this, the hypothesis of Theorem 4.1 hold, since $\nu\bar{c}^2 - \varepsilon < 0$. Then Theorem 4.4 follows. As in Example 5.1 we can suppose that $r_1 \in L^\infty([0,\pi];\mathbb{R})$ with $||r_1||^2_{L^\infty([0,\pi];\mathbb{R})} < 2$ and the precedent conclusions are also true.

This case is the analogous to Example 5.1 when we observe heat diffusion in a rod relative to an origin moving with velocity $r_1\dot{w}$ (see [8]).

EXAMPLE 5.3. Finally we are going to study an example with spatial dimension greater than 1. Let $\mathcal{O}$ be a bounded open subset in $\mathbb{R}^N$. Let $V = H_0^1(\mathcal{O};\mathbb{R})$, $H = L^2(\mathcal{O};\mathbb{R})$, $A = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, $B = \sum_{i=1}^N r_i(x)\frac{\partial}{\partial x_i}$ where $r_i(\cdot) \in L^\infty(\mathcal{O};\mathbb{R})$.

We also know that the seminorm $||u||^2 = \int_{\mathcal{O}} \sum_{i=1}^N \left[\frac{\partial u}{\partial x_i}\right]^2 dx$ is, in fact, a norm in V, equivalent to the usual one.

Now,

$$-\langle Au, u\rangle = \int_{\mathcal{O}} \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\, dx = ||u||^2, \quad \forall u \in V,$$

$$\begin{aligned}\langle \Delta(I)u, u\rangle = |Bu|^2 &= \int_{\mathcal{O}} \left[\sum_{i=1}^N r_i(x)\frac{\partial u}{\partial x_i}\right]^2 dx \\ &\le \sum_{i=1}^N ||r_i||^2_{L^\infty(\mathcal{O};\mathbb{R})} \int_{\mathcal{O}} \left[\frac{\partial u}{\partial x_i}\right]^2 dx \le Nr^2||u||^2,\end{aligned}$$

where $r^2 = \max\{||r_i||^2_{L^\infty(\mathcal{O};\mathbb{R})} : 1 \le i \le N\}$. Hypothesis $(c1)$ will be true if we can find ε and ν such that

$$Nr^2||u||^2 + \varepsilon||u||^2 \le \nu|u|^2 + 2||u||^2.$$

If we take r_i such that $r^2 < 2/N$ and $\varepsilon = 2 - Nr^2$, $\nu = 0$, then $(c1)$ holds. Since, moreover, $\nu\bar{c}^2 - \varepsilon < 0$ then Theorems 4.1 and 4.4 follow.

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