

## ASYMPTOTIC BEHAVIOUR OF A STOCHASTIC SEMILINEAR DISSIPATIVE FUNCTIONAL EQUATION WITHOUT UNIQUENESS OF SOLUTIONS

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ABSTRACT. In this work we present the existence and uniqueness of pullback and random attractors for stochastic evolution equations with infinite delays when the uniqueness of solutions for these equations is not required. Our results are obtained by means of the theory of set-valued random dynamical systems and their conjugation properties.

*Dedicated to Peter E. Kloeden on his 60<sup>th</sup> birthday.*

1. **Introduction.** In this work, we investigate the asymptotic behaviour of a stochastic functional evolution equation in a separable Hilbert space  $H$ ,

$$\begin{cases} du(t) + Au(t)dt = F(u_t)dt + dW(t), & \text{if } t \geq 0, \\ u(t) = \tilde{x}_0(t), & \text{if } t \leq 0, \end{cases} \quad (1)$$

where  $-A$  is the infinitesimal generator of a  $C_0$  contraction semigroup,  $F$  is a non-linear term, and  $W$  is an infinite dimensional two-sided Wiener process. Our aim is to study the qualitative properties of this problem by analyzing the existence of the so-called pullback and random attractors associated to the multivalued non-autonomous and random dynamical systems (MNDS and MRDS for short) generated by the solutions of (1). For interesting papers related to the theory of these attractors we cite the works by Caraballo *et al.* [4], [5] and [6], Chueshov [8], Flandoli and Schmalfuß [9], Kloeden [13], Kloeden and Schmalfuß [14], Robinson [16], Schmalfuß [17], amongst many others.

In the recent work [3], the existence of pullback and random attractors for MNDS and MRDS generated by the solutions of delayed random semilinear equations has been analyzed, where the delay could be infinite. Motivated by this paper, our intention in this work is to prove analogous results dealing with the equation (1), where non-uniqueness of solutions may happen (and for that we consider multivalued systems). As in [3] we allow the delay to be equal to infinite. To establish our results, we will transform the stochastic equation (1) into an evolution equation without noise but with random coefficients by means of the Ornstein-Uhlenbeck process.

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It is worth pointing out that, in this article, we not only extend the results obtained in [3] to the stochastic situation, but also we develop a specific technique for the equation with random coefficients, since, once we have transformed the stochastic equation into a random one, some continuity problems appear, and, therefore, we cannot apply directly the results in [3].

We organize this paper as follows. In Section 2, we include basic concepts on multivalued non-autonomous and random dynamical systems as well as on their associated attractors. In Section 3, we introduce the Ornstein-Uhlenbeck process and discuss some important properties of this process. We then transform (1) into a random equation, and show that we can restrict ourselves to the study of such a random evolution equation provided that we consider a particular metric dynamical system, in which some crucial properties for the involved processes hold.

**2. Definitions and preliminaries.** In the interest of brevity, in what follows we only recall some basic definitions for set-valued non-autonomous and random dynamical systems, and formulate very general sufficient conditions for the existence of a pullback attractor for these systems, which is a random set if the non-autonomous perturbation is a noise (for a more comprehensive presentation of random dynamical systems see Arnold [1] in the single-valued case, and Caraballo *et al.* [5] for a generalization of this concept to the case of multivalued mappings).

A *metric dynamical system*  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$  with two-sided time  $\mathbb{R}$  consists of a probability space  $\mathcal{P} := (\Omega, \mathcal{F}, \mathbb{P})$  and a family of transformations  $\{\theta_t\}_{t \in \mathbb{R}}$  such that:

1. It is a one-parameter group, i.e.

$$\theta_0 = \text{id}_\Omega, \quad \theta_{t+s} = \theta_t \theta_s, \quad \forall t, s \in \mathbb{R},$$

2.  $(t, \omega) \in \mathbb{R} \times \Omega \mapsto \theta_t \omega$  is measurable,
3.  $\mathbb{P}$  is *invariant* with respect to  $\theta$ , i.e.,  $\theta_t \mathbb{P} = \mathbb{P}$ , for all  $t \in \mathbb{R}$ , which means that  $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$ , for all  $A \in \mathcal{F}$  and all  $t \in \mathbb{R}$ .

In addition, we assume that the metric dynamical system is *ergodic* (we also say that  $\mathbb{P}$  is ergodic with respect to  $\theta$ ), i.e, for any  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set  $B \in \mathcal{F}$ , which means that  $\theta_t B = B$  for all  $t \in \mathbb{R}$ , we have either  $\mathbb{P}(B) = 0$  or  $\mathbb{P}(B) = 1$ .

If we replace in the definition of a metric dynamical system the probability space  $\mathcal{P}$  by its completion  $\mathcal{P}^c := (\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  the above measurability property given in the second point is not true in general, see Arnold [1] Appendix A. But for fixed  $t \in \mathbb{R}$  we have that the mapping

$$\theta_t : (\Omega, \bar{\mathcal{F}}) \mapsto (\Omega, \bar{\mathcal{F}})$$

is measurable.

Assume that  $\Omega'$  is a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set of full measure, which is not in  $\mathcal{F}$  but in the completed  $\sigma$ -algebra  $\bar{\mathcal{F}}$ . This set  $\Omega'$  could be given by all  $\omega \in \Omega$  having a particular property. The next result, which has interest by itself, shows how then we could consider on  $\Omega'$  a restricted metric dynamical system.

**Lemma 1.** *Let  $\Omega'$  be a element of  $\mathcal{P}^c$  such that  $\bar{\mathbb{P}}(\Omega') = 1$ . In addition,  $\Omega'$  is supposed to be  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant. Let  $\mathcal{F}_{\Omega'}$  be the trace- $\sigma$ -algebra of  $\mathcal{F}$  with respect to  $\Omega'$ ,  $\mathbb{P}'(A') := \bar{\mathbb{P}}(A')$  for  $A' \in \mathcal{F}_{\Omega'}$ , and  $\theta'$  the restriction of  $\theta$  to  $\Omega' \times \mathbb{R}$ . Then  $(\Omega', \mathcal{F}_{\Omega'}, \mathbb{P}', \theta')$  forms a metric dynamical system such that for every  $A' \in \mathcal{F}_{\Omega'}$  and every  $A \in \mathcal{F}$  with  $A' = A \cap \Omega'$  we have that  $\mathbb{P}'(A') = \mathbb{P}(A)$ . In addition, if  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$  is ergodic then  $(\Omega', \mathcal{F}_{\Omega'}, \mathbb{P}', \theta')$  is also ergodic.*

*Proof.* Let  $\mathcal{F} \times \mathcal{B}(\mathbb{R})$  be the set of rectangles  $F \times R$ ,  $F \in \mathcal{F}$  and  $R \in \mathcal{B}(\mathbb{R})$  and let  $\sigma(\mathcal{G})$  denote the  $\sigma$ -algebra generated by a set system  $\mathcal{G}$  over  $\Omega \times \mathbb{R}$ . Then, for the trace  $\sigma$ -algebra with respect to  $\Omega' \times \mathbb{R}$ , we have that

$$\begin{aligned} \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \cap (\Omega' \times \mathbb{R}) &= \sigma(\mathcal{F} \times \mathcal{B}(\mathbb{R})) \cap (\Omega' \times \mathbb{R}) = \sigma((\mathcal{F} \times \mathcal{B}(\mathbb{R})) \cap (\Omega' \times \mathbb{R})) \\ &= \sigma(\mathcal{F}_{\Omega'} \times \mathcal{B}(\mathbb{R})) = \mathcal{F}_{\Omega'} \otimes \mathcal{B}(\mathbb{R}), \end{aligned}$$

see Halmos [10], Theorem 5.E, where the expression on the right hand side is a  $\sigma$ -algebra over  $\Omega' \times \mathbb{R}$ .

We have  $\theta'(\Omega' \times \mathbb{R}) = \Omega'$ . Hence, by the above equality,

$$(\theta')^{-1}(A') = (\theta')^{-1}(A \cap \Omega') = \theta^{-1}(A) \cap (\Omega' \times \mathbb{R}) \in \mathcal{F}_{\Omega'} \otimes \mathcal{B}(\mathbb{R})$$

since  $\theta^{-1}(A) \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$ . Therefore,  $\theta'$  is measurable.

Because  $\mathbb{P}'(\Omega') = 1$  and  $\mathcal{F}_{\Omega'} \subset \mathcal{F}$  the function  $\mathbb{P}'$  is a probability measure on  $\mathcal{F}_{\Omega'}$ . Clearly, if  $A \in \mathcal{F}$  and  $A' = A \cap \Omega'$ , then  $\mathbb{P}'(A') = \bar{\mathbb{P}}(A \cap \Omega') = \bar{\mathbb{P}}(A) + \bar{\mathbb{P}}(\Omega') - \bar{\mathbb{P}}(A \cup \Omega') = \bar{\mathbb{P}}(A) = \mathbb{P}(A)$ .

In addition,  $\mathbb{P}'$  is invariant since, for  $t \in \mathbb{R}$  and  $A' \in \mathcal{F}_{\Omega'}$ , we have

$$\begin{aligned} \mathbb{P}'(\theta'_t A') &= \mathbb{P}'(\theta'_t(A \cap \Omega')) = \mathbb{P}'(\theta_t A \cap \Omega') = \bar{\mathbb{P}}(\theta_t A \cap \Omega') = \bar{\mathbb{P}}(\theta_t A) \\ &= \mathbb{P}(\theta_t A) = \mathbb{P}(A) = \bar{\mathbb{P}}(A) = \bar{\mathbb{P}}(A \cap \Omega') = \bar{\mathbb{P}}(A') = \mathbb{P}'(A'). \end{aligned}$$

Now let  $A'$  be a  $\{\theta'_t\}_{t \in \mathbb{R}}$ -invariant set in  $\mathcal{F}_{\Omega'}$ :  $A' = \theta'_t A'$ , for all  $t \in \mathbb{R}$ . Suppose that  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$  is ergodic. Then a simple calculation shows that

$$0 = \mathbb{P}'(A' \Delta \theta_t A') = \bar{\mathbb{P}}((A \Delta \theta_t A) \cap \Omega') = \bar{\mathbb{P}}(A \Delta \theta_t A) = \mathbb{P}(A \Delta \theta_t A),$$

so that  $A$  is a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set modulo  $\mathbb{P}$ . Hence  $\mathbb{P}(A) = \bar{\mathbb{P}}(A \cap \Omega') = \bar{\mathbb{P}}(A') = \mathbb{P}'(A') = 1$  or  $0$  by the ergodicity of  $\mathbb{P}$  and by Arnold [1], pages 537 and 539.  $\square$

**Remark 2.** If  $\Omega' \in \mathcal{F}$  then the statements of Lemma 1 are trivial because  $\mathcal{F}_{\Omega'} \subset \mathcal{F}$ . If some property holds  $\mathbb{P}$  almost surely on a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set  $\Omega' \subset \Omega$  of full measure, then we say that this property holds  $\theta$ -almost surely and we suppose that we change the metric dynamical system in the above sense so that the property holds for all  $\omega \in \Omega'$ . For this new metric dynamical system we will always keep the old notation  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ .

From now on, let  $(X, d_X)$  be a Polish space.

Let  $D : \omega \mapsto D(\omega)$  be a multivalued mapping in  $X$  over  $\mathcal{P}$ . The set of multivalued mappings  $D$  with closed and non-empty images is denoted by  $C(X)$ . Let also denote by  $P_f(X)$  the set of all non-empty closed subsets of the space  $X$ .

A multivalued mapping is called a *random set* if

$$\omega \mapsto \inf_{y \in D(\omega)} d_X(x, y)$$

is a random variable for every  $x \in X$ . It is well known that a mapping is a random set if and only if for every open set  $\mathcal{O}$  in  $X$  the inverse image  $\{\omega : D(\omega) \cap \mathcal{O} \neq \emptyset\}$  is measurable, i.e., it belongs to  $\mathcal{F}$  (see Hu and Papageorgiou [12, Proposition 2.1.4]).

We now introduce non-autonomous and random dynamical systems.

**Definition 3.** A multivalued map  $U : \mathbb{R}^+ \times \Omega \times X \rightarrow P_f(X)$  is called a *multivalued non-autonomous dynamical system (MNDS)* if

- i)  $U(0, \omega, \cdot) = \text{id}_X$ ,
- ii)  $U(t+\tau, \omega, x) \subset U(t, \theta_\tau \omega, U(\tau, \omega, x))$  (*cocycle property*), for all  $t, \tau \in \mathbb{R}^+, x \in X, \omega \in \Omega$ .

It is called a *strict MNDS* if, moreover,  $U(t + \tau, \omega, x) = U(t, \theta_\tau \omega, U(\tau, \omega, x))$ , for all  $t, \tau \in \mathbb{R}^+$ ,  $x \in X, \omega \in \Omega$ .

An MNDS is called a *multivalued random dynamical system (MRDS)* if the multivalued mapping  $(t, \omega, x) \mapsto U(t, \omega, x)$  is  $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$  measurable, i.e.

$$\{(t, \omega, x) : U(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$$

for every open set  $\mathcal{O}$  of the space  $X$ .

For the above composition of multivalued mappings we use that, for any non-empty set  $Y \subset X$ ,  $U(t, \omega, Y)$  is defined by

$$U(t, \omega, Y) = \bigcup_{x \in Y} U(t, \omega, x).$$

Our first aim is to formulate a general sufficient condition ensuring that an MNDS defines an MRDS. To do that, we introduce an appropriate concept of continuity for multivalued mappings: we say that  $U(t, \omega, \cdot)$  is *upper-semicontinuous at  $x_0$*  if for every neighborhood  $\mathcal{U}$  of the set  $U(t, \omega, x_0)$  there exists  $\delta > 0$  such that if  $d_X(x_0, y) < \delta$  then  $U(t, \omega, y) \subset \mathcal{U}$ . In general  $U(t, \omega, \cdot)$  is called *upper-semicontinuous* if it is upper-semicontinuous at every  $x_0$  in  $X$ .

Assuming that  $\Omega$  is a Polish space, it is not difficult to extend this definition and obtain upper-semicontinuity with respect to all variables.

The proof of the following result can be found in Caraballo *et al.* [3].

**Lemma 4.** *Let  $\Omega$  be a Polish space and let  $\mathcal{F}$  be the associated Borel- $\sigma$ -algebra. Suppose that  $(t, \omega, x) \mapsto U(t, \omega, x)$  is upper-semicontinuous. Then this mapping is measurable in the sense of Definition 3.*

In order to define the concept of attractor (both pullback and random) we need to recall the definitions of invariance, absorption and attraction.

A multivalued mapping  $D$  is said to be *negatively (resp. strictly) invariant* for the MNDS  $U$  if  $D(\theta_t \omega) \subset U(t, \omega, D(\omega))$  (resp.  $=$ ), for  $\omega \in \Omega$ ,  $t \in \mathbb{R}^+$ .

Let  $\mathcal{D}$  be a family of multivalued mappings in  $C(X)$ . We say that a family  $K$  is *pullback  $\mathcal{D}$ -attracting* if for every  $D \in \mathcal{D}$

$$\lim_{t \rightarrow +\infty} \text{dist}_X(U(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), K(\omega)) = 0, \text{ for all } \omega \in \Omega,$$

where by  $\text{dist}_X(A, B)$  we denote the Hausdorff semi-distance of two non-empty sets  $A, B$ :

$$\text{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} d_X(x, y).$$

$B$  is said to be *pullback  $\mathcal{D}$ -absorbing* if for every  $D \in \mathcal{D}$  and  $\omega \in \Omega$  there exists  $T = T(\omega, D) > 0$  such that

$$U(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega), \text{ for all } t \geq T. \quad (2)$$

Throughout this work we always consider a particular system of sets (see Schmalfuß [18]). Namely, let  $\mathcal{D}$  be a set of multivalued mappings in  $C(X)$  satisfying the *inclusion closed property*: suppose that  $D \in \mathcal{D}$  and let  $D'$  be a multivalued mapping in  $C(X)$  such that  $D'(\omega) \subset D(\omega)$  for  $\omega \in \Omega$ , then  $D' \in \mathcal{D}$ . The reason to consider such a system of sets is that then we will have a unique attractor in  $\mathcal{D}$ .

The following two concepts are the main objects to be studied in this work.

**Definition 5.** *A family  $A \in \mathcal{D}$  is said to be a global pullback  $\mathcal{D}$ -attractor for the MNDS  $U$  if it satisfies:*

- i)  $\mathcal{A}(\omega)$  is compact for any  $\omega \in \Omega$ ;
- ii)  $\mathcal{A}$  is pullback  $\mathcal{D}$ -attracting;
- iii)  $\mathcal{A}$  is negatively invariant.

$\mathcal{A}$  is said to be a strict global pullback  $\mathcal{D}$ -attractor if the invariance property in the third item is strict.

A natural modification of this definition for MRDS is the following.

**Definition 6.** Suppose that  $U$  is an MRDS and suppose that the properties of Definition 5 are satisfied. In addition, we suppose that  $\mathcal{A}$  is a random set with respect to  $\mathcal{P}^c$ . Then  $\mathcal{A}$  is called a random global pullback  $\mathcal{D}$ -attractor.

**Remark 7.** In particular, the global random attractor attracts in the pullback sense all the random sets in  $\mathcal{D}$ .

**Remark 8.** In contrast to the theory of random attractors for single valued random dynamical systems we have weaker assumptions on the measurability of  $\mathcal{A}$ . Of course, it is desirable to obtain that  $\mathcal{A}$  is a random set with respect to  $\mathcal{P}$ , but usually we need stronger assumptions in the applications to obtain this property.

To analyze the asymptotic behaviour of an abstract delay evolution equation without uniqueness of solutions and with additive noise, we will use in this paper the following two general results on the existence and uniqueness of pullback and random attractors associated to MNDS and MRDS respectively. For the proofs we refer the reader to Caraballo *et al.* [3].

**Theorem 9.** Suppose that the MNDS  $U(t, \omega, \cdot)$  is upper-semicontinuous for  $t \geq 0$  and  $\omega \in \Omega$ . Let  $B \in \mathcal{D}$  be a multivalued mapping such that the MNDS is asymptotically compact with respect to  $B$ , i.e. for every sequence  $t_n \rightarrow +\infty$  and  $\omega \in \Omega$ , it holds that every sequence  $y_n \in U(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$  is pre-compact. In addition, suppose that  $B$  is pullback  $\mathcal{D}$ -absorbing. Then, the set  $\mathcal{A}$  given by

$$\mathcal{A}(\omega) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} U(t, \theta_{-t}\omega, B(\theta_{-t}\omega))} \quad (3)$$

is a pullback  $\mathcal{D}$ -attractor. Furthermore,  $\mathcal{A}$  is the unique element from  $\mathcal{D}$  with these properties. In addition, if  $U$  is a strict MNDS, then  $\mathcal{A}$  is strictly invariant.

**Lemma 10.** Under the assumptions in Theorem 9, let  $\omega \mapsto U(t, \omega, B(\omega))$  be a random set with respect to  $\bar{\mathcal{F}}$  for  $t \geq 0$ . Assume also that  $U(t, \omega, B(\omega))$  is closed for all  $t \geq 0$  and  $\omega \in \Omega$ . Then the set  $\mathcal{A}$  defined by (3) is a random set with respect to  $\mathcal{P}^c$ .

**3. An abstract stochastic functional evolution equation.** Let us consider  $H = (H, \|\cdot\|, (\cdot, \cdot))$  a separable Hilbert space and let  $-A$  be the generator of a  $C_0$  contraction semigroup  $(S(t))_{t \geq 0}$  on  $H$ :

$$\|S(t)x\| \leq \|x\|e^{-\alpha t}, \quad \text{for some } \alpha > 0 \quad \text{and every } t \geq 0.$$

In particular, we assume that  $A$  is a symmetric operator, positive on  $H$  and that has a compact inverse. Then the eigenvectors of  $A$  form a complete orthonormal system  $\{e_i\}_{i \in \mathbb{N}}$  on  $H$  with spectrum  $\lambda_1 < \lambda_2 < \dots$  of finite multiplicity. The first

eigenvalue is denoted by  $\alpha$ . Assume that the operators  $S(t)$  are compact for  $t > 0$ . We also consider the space  $V := D(A^{\frac{1}{2}})$  given by

$$V = \left\{ u = \sum_{i \in \mathbb{N}} u_i e_i \in H : \sum_{i \in \mathbb{N}} \lambda_i |\hat{u}_i|^2 < \infty \right\},$$

equipped with the norm  $\|u\|_V^2 := \sum_{i \in \mathbb{N}} \lambda_i |\hat{u}_i|^2$ , where  $\hat{u}_i = (u, e_i)$ .

In what follows we consider a two sided Wiener process  $W$  with covariance  $Q = Q^* \geq 0$  such that  $\text{tr}(A^{\frac{1}{2}} Q A^{\frac{1}{2}}) < \infty$ . Then, by the Kolmogorov test, see Kunita [15], Theorem 1.4.1, there exists a Wiener process  $W$  with trajectories in  $\Omega = C_0(\mathbb{R}, V)$ .  $\Omega$  is a topological space with respect to the compact open topology. The associated Borel- $\sigma$ -algebra is denoted by  $\mathcal{F}$ . In addition we set

$$\mathcal{F}_t = \sigma\{W(\tau, \cdot) - W(s, \cdot) : \tau, s \leq t\} \subset \mathcal{F}.$$

The canonical version of this Wiener process is a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$  where  $\mathbb{P}$  is the Wiener measure related to  $W$ . Over this filtered probability space we also consider the measurable flow  $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$  given by  $\theta_t \omega(\cdot) := \omega(t + \cdot) - \omega(t)$ , for  $\omega \in \Omega$  and  $t \in \mathbb{R}$ . Note that the metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$  is ergodic with respect to  $\theta$  [2].

**Lemma 11.** *There exists a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set  $\Omega' \subset \Omega$  of full  $\mathbb{P}$ -measure such that for all  $\varepsilon > 0$  and  $\omega \in \Omega'$ , there exists a constant  $C(\varepsilon, \omega)$  such that*

$$\|W(t, \omega)\|_V^2 \leq \varepsilon |t|^3 + C(\varepsilon, \omega), \quad (4)$$

for all  $t \in \mathbb{R}$ .

*Proof.* For simplicity we only consider the case  $t > 0$ . First of all notice that we have

$$\begin{aligned} & \|W(t, \omega)\|_V \\ &= \|W(t, \omega) - W([t], \omega) + W([t], \omega) - W([t] - 1, \omega) + W([t] - 1, \omega) + \cdots - 0\|_V \\ &\leq \|W(t, \omega) - W([t], \omega)\|_V + \|W([t], \omega) - W([t] - 1, \omega)\|_V + \cdots + \|W(1, \omega)\|_V \\ &\leq \sum_{i=0}^{[t]} \sup_{s \in [0, 1]} \|\theta_i W(s, \omega)\|_V. \end{aligned} \quad (5)$$

Since  $\{W(t, \omega)\}_{t \geq 0}$  is a square integrable  $V$ -valued martingale with respect to  $\{\mathcal{F}_t\}_{t \geq 0}$  (and the same is true when considering  $\{W(t, \omega)\}_{t < 0}$  with respect to  $\{\mathcal{F}_t\}_{t < 0}$ ), then  $\mathbb{E} \sup_{t \in [-T, T]} \|W(t, \omega)\|_V^2 < \infty$ , for all  $T > 0$ . Therefore,

$$\mathbb{E} \sup_{t \in [0, 1]} \sup_{s \in [0, 1]} \|\theta_t W(s, \omega)\|_V^2 \leq 4 \mathbb{E} \sup_{t \in [0, 2]} \|W(t, \omega)\|_V^2 < \infty.$$

Applying Arnold [1], Proposition 4.1.3. (ii), we have that  $\sup_{s \in [0, 1]} \|\theta_t W(s, \omega)\|_V^2$  is sublinear growing in  $\Omega'$ , a  $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set, measurable with respect to  $\mathcal{F}$ , and of full  $\mathbb{P}$ -measure. Then, from (5), for  $\varepsilon > 0$  and  $\omega \in \Omega'$ , we have

$$\begin{aligned} \|W(t, \omega)\|_V^2 &\leq ([t] + 1) \sum_{i=0}^{[t]} \sup_{s \in [0, 1]} \|\theta_i W(s, \omega)\|_V^2 \leq ([t] + 1) \left( \sum_{i=0}^{[t]} (K(\varepsilon, \omega) + \varepsilon i) \right) \\ &\leq ([t] + 1)^2 K(\varepsilon, \omega) + \frac{[t]}{2} ([t] + 1)^2 \varepsilon, \end{aligned}$$

and then the result follows.  $\square$

From (4) it follows that  $\|W(t, \omega)\|_V$ ,  $t \in \mathbb{R}$ , has subexponential growth as  $|t| \rightarrow \infty$ , so for every  $\varepsilon > 0$  and  $\omega \in \Omega'$ , there exists a positive constant  $\tilde{C}(\varepsilon, \omega)$  such that

$$\|W(t, \omega)\|_V \leq \tilde{C}(\varepsilon, \omega)e^{\varepsilon|t|}, \quad t \in \mathbb{R}.$$

Therefore, thanks to Lemma 1 and Remark 2, we can construct a restricted metric dynamical system, which, in the sequel will be denoted with the old symbol  $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ , so that, from now on

$$\Omega := \left\{ \omega \in C_0(\mathbb{R}, V) : \lim_{t \rightarrow \pm\infty} \frac{\log^+ \|W(t, \omega)\|_V}{t} = 0 \right\}. \quad (6)$$

We are interested in studying the asymptotic behaviour of the following stochastic functional partial differential equation

$$\begin{cases} du(t) + Au(t)dt = F(u_t)dt + dW(t), & \text{if } t \geq 0, \\ u(t) = \tilde{x}_0(t), & \text{if } t \leq 0. \end{cases} \quad (7)$$

We define  $u_t$  in the following way:

$$u_t(s) := \begin{cases} u(t+s) & \text{for } s \in [-t, 0], \\ \tilde{x}_0(s+t) & \text{for } s < -t, \end{cases}$$

where  $t \geq 0$ , and  $\tilde{x}_0$  is a given continuous function on  $\mathbb{R}^-$  with values in  $H$ .

We assume that the non-linear term  $F : C_\gamma \rightarrow H$  is a continuous mapping such that

$$\|F(u)\| \leq c_1 + c_2\|u\|_\gamma \quad (8)$$

for appropriate positive constants  $c_1, c_2$ , where  $C_\gamma$  is the space given by

$$C_\gamma := \left\{ y \in C((-\infty, 0]; H) : \lim_{\tau \rightarrow -\infty} y(\tau) e^{\gamma\tau} \text{ exists} \right\},$$

with  $\gamma > \alpha$ , and  $\|y\|_\gamma := \sup_{\tau \in (-\infty, 0]} e^{\gamma\tau} \|y(\tau)\| < \infty$ . It is known that  $(C_\gamma, \|\cdot\|_\gamma)$  is a separable Banach space (see Hino *et al.* [11, p.15]).

Now we consider the following linear stochastic differential equation

$$dz + Az dt = dW.$$

It is known that this equation has a unique stationary solution denoted by  $z^*$ . From  $z^*$  we can define the well-known stationary Ornstein-Uhlenbeck process  $\bar{z} : \mathbb{R} \times \Omega \rightarrow H$  given by  $\bar{z}(t, \omega) := z^*(\theta_t \omega)$  which can be written in the form

$$\bar{z}(t, \omega) = \int_{-\infty}^t S(t-\tau) dW(\omega, \tau) = \int_{-\infty}^0 S(-\tau) dW(\theta_t \omega, \tau), \quad \text{for } t \in \mathbb{R}.$$

We would like to study the Cauchy problem (7) by means of the classical change of variables  $y(t) = u(t) - z^*(\theta_t \omega)$ , for which we firstly need to obtain some properties for  $z^*(\omega)$ . Indeed, for  $z^*(\omega)$  we choose the following version

$$\begin{aligned} z^*(\omega) &:= \int_{-\infty}^0 S(-\tau) dW(\omega, \tau) \\ &= \lim_{t \rightarrow \infty} \left( -S(t)W(-t) + A \int_{-t}^0 S(-\tau) \omega(\tau) d\tau \right), \end{aligned} \quad (9)$$

where the last expression is obtained applying the integration by parts formula for stochastic integrals.

**Lemma 12.** *For  $\omega \in \Omega$  the limit in (9) exists in  $H$ .*

*Proof.* From (6) and the properties of  $S$  it is straightforward that the first term of the right hand side of (9) tends to zero.

Let us prove that there exists (in  $H$ )  $\lim_{t \rightarrow -\infty} A \int_t^0 S(-\tau)\omega(\tau)d\tau$ . Consider  $t_1 < t_2 \leq 0$ ,  $0 < \mu < \lambda_1$  and

$$C_\mu := \sup_{i \in \mathbb{N}} \frac{\lambda_i}{2(\lambda_i - \mu)}. \quad (10)$$

Then, denoting  $\hat{\omega}_i(t) = (\omega(t), e_i)$ , we have

$$\begin{aligned} \|A \int_{t_1}^{t_2} S(-\tau)\omega(\tau)d\tau\|^2 &= \sum_{i \in \mathbb{N}} \left( A \int_{t_1}^{t_2} S(-\tau)\omega(\tau)d\tau, e_i \right)^2 \\ &= \sum_{i \in \mathbb{N}} \left( \int_{t_1}^{t_2} S(-\tau)\omega(\tau)\lambda_i d\tau, e_i \right)^2 \\ &= \sum_{i \in \mathbb{N}} \left( \int_{t_1}^{t_2} e^{\lambda_i \tau} \hat{\omega}_i(\tau) \lambda_i^{1/2} \lambda_i^{1/2} e^{-\mu \tau} e^{\mu \tau} d\tau \right)^2 \\ &\leq \sum_{i \in \mathbb{N}} \left( \int_{t_1}^{t_2} e^{2(\lambda_i - \mu)\tau} \lambda_i e^{\mu \tau} d\tau \right) \left( \int_{t_1}^{t_2} (\hat{\omega}_i(\tau))^2 \lambda_i e^{\mu \tau} d\tau \right) \\ &\leq \sup_{i \in \mathbb{N}} \frac{\lambda_i}{2(\lambda_i - \mu)} \int_{t_1}^{t_2} \sum_{i \in \mathbb{N}} |\hat{\omega}_i(\tau)|^2 \lambda_i e^{\mu \tau} d\tau = C_\mu \int_{t_1}^{t_2} \|\omega(\tau)\|_V^2 e^{\mu \tau} d\tau. \end{aligned}$$

As  $\int_{-\infty}^0 \|\omega(\tau)\|_V^2 e^{\mu \tau} d\tau < \infty$  in  $\Omega$  (where  $\Omega$  is given by (6)), the conclusion follows.  $\square$

Notice that, as an immediate consequence of the previous estimate, we have that

$$\|z^*(\omega)\|^2 = \|A \int_{-\infty}^0 S(-\tau)\omega(\tau)d\tau\|^2 \leq C_\mu \int_{-\infty}^0 \|\omega(\tau)\|_V^2 e^{\mu \tau} d\tau. \quad (11)$$

As we have mentioned, we study the Cauchy problem (7) by means of the change of variables  $y(t) = u(t) - z^*(\theta_t \omega)$ . Then  $y(t)$  satisfies the random evolution equation

$$\frac{dy}{dt} + Ay = f(\theta_t \omega, y_t), \quad (12)$$

where  $f(\omega, \xi) := F(\xi + Z^*(\cdot, \omega))$  ( $Z^*(\cdot, \omega)$  is the element of  $C_\gamma$  defined as  $Z^*(s, \omega) := z^*(\theta_s \omega)$ , for  $s \leq 0$ ), with initial condition

$$y(s) = \tilde{x}_0(s) - z^*(\theta_s \omega) =: x_0(s), \quad \text{for } s \leq 0. \quad (13)$$

Indeed, we can express the transformation of our stochastic equation (7) into the random evolution equation (12) by means of the following operator: for every  $\omega \in \Omega$ , let  $L(\omega) : C_\gamma \rightarrow C_\gamma$  be given by

$$L(\omega)\xi := \xi - Z^*(\cdot, \omega). \quad (14)$$

Then, it is clear that  $y_t(s) = L(\theta_t \omega)u_t(s) = u_t(s) - z^*(\theta_{t+s} \omega)$ .

We observe that the map  $f$  is Caratheodory. The continuity of  $\xi \mapsto f(\omega, \xi)$  is obvious, and the measurability of  $\omega \mapsto f(\omega, \xi)$  follows from the measurability of the map  $\omega \mapsto Z^*(\cdot, \omega) \in C_\gamma$  (which can be proved using the representation (9) with standard arguments).



For  $\omega \in \Omega$  we say that a function  $[0, T] \ni t \rightarrow y_t \in C_\gamma$  is a mild solution of (12)-(13) if

$$y_t(s) = \begin{cases} S(t+s)x_0(0) + \int_0^{s+t} S(t+s-\tau)f(\theta_\tau\omega, y_\tau)d\tau & : s \in [-t, 0], \\ x_0(s+t) & : s < -t. \end{cases} \quad (15)$$

In [3], by using the compactness of the semigroup  $S(t)$ ,  $t > 0$ , the existence of a mild solution to the random equation (12)-(13) in  $C_\gamma$  has been established. Also it is shown that every mild solution can be extended to a global one, that is, defined for all  $T \geq 0$ . Here, by a mild solution of (7) we understand the process  $u_t$  defined by means of the transformation (14) with  $y_t$  given in (15), that is,  $u_t = y_t + Z^*(\cdot, \theta_t\omega)$ , for every  $t \in [0, T]$ .

As we have already mentioned, we aim to show the existence of a random attractor for the corresponding dynamical system generated by the solutions of (7). Our next objective is to prove that, in order to do that, we can restrict ourselves to the same analysis but for the random evolution equation (12).

We first establish the next result which is used to obtain a conjugated MNDS to a given MNDS.

**Lemma 13.** *Let  $X$  be a separable Banach space, and let  $U$  be an MNDS. Suppose that the mapping  $\mathcal{T} : \Omega \times X \rightarrow X$  satisfies the following properties: for fixed  $\omega \in \Omega$  the mapping  $\mathcal{T}(\omega)(\cdot)$  is a homeomorphism on  $X$ , and for fixed  $x \in X$ , the mappings  $\mathcal{T}(\cdot)(x)$ ,  $\mathcal{T}^{-1}(\cdot)(x)$  are measurable. Then, the mapping*

$$(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times X \mapsto \mathcal{T}^{-1}(\theta_t\omega)U(t, \omega, \mathcal{T}(\omega)x) =: \tilde{U}(t, \omega, x) \in C(X)$$

is also an MNDS.

Moreover, if  $U$  is strict, then  $\tilde{U}$  is also strict. And, if  $U$  is a MRDS, then  $\tilde{U}$  is also a MRDS.

*Proof.* The result follows from the following facts:

- i)  $\tilde{U}(0, \omega, x) = \mathcal{T}^{-1}(\omega)U(0, \omega, \mathcal{T}(\omega)x) = \mathcal{T}^{-1}(\omega)\mathcal{T}(\omega)x = x$ .
- ii) For all  $t, \tau \in \mathbb{R}^+$ ,  $x \in X$  and  $\omega \in \Omega$  it holds

$$\begin{aligned} \tilde{U}(t+\tau, \omega, x) &= \mathcal{T}^{-1}(\theta_{t+\tau}\omega)U(t+\tau, \omega, \mathcal{T}(\omega)x) \\ &\subset \mathcal{T}^{-1}(\theta_{t+\tau}\omega)U(t, \theta_\tau\omega, U(\tau, \omega, \mathcal{T}(\omega)x)) \\ &= \mathcal{T}^{-1}(\theta_t\theta_\tau\omega)U(t, \theta_\tau\omega, \mathcal{T}(\theta_\tau\omega)\tilde{U}(\tau, \omega, x)) \\ &= \tilde{U}(t, \theta_\tau\omega, \tilde{U}(\tau, \omega, x)), \end{aligned}$$

so the cocycle property holds.

The strict property follows by the same arguments.

Finally, we have to prove that

$$(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times X \mapsto \tilde{U}(t, \omega, x)$$

is measurable if  $U$  is. Since  $U$  is measurable there exists a sequence of measurable selections of  $U$  such that

$$U(t, \omega, x) = \overline{\cup_{n \in \mathbb{N}} u_n(t, \omega, x)},$$

see [12], page 155. Hence

$$\tilde{U}(t, \omega, x) = \mathcal{T}^{-1}(\theta_t\omega)\overline{\cup_{n \in \mathbb{N}} u_n(t, \omega, \mathcal{T}(\omega)x)} = \overline{\cup_{n \in \mathbb{N}} \mathcal{T}^{-1}(\theta_t\omega)u_n(t, \omega, \mathcal{T}(\omega)x)}$$

is measurable, as well, where we have used standard theorems for multivalued measurable maps (see e.g. [12, Chapter 2.2]). We note that the map

$$(t, \omega, x) \mapsto (t, \omega, \mathcal{T}(\omega)x) \in \mathbb{R}^+ \times \Omega \times X$$

is measurable, and then the composition  $u_n(t, \omega, \mathcal{T}(\omega)x)$  is also measurable.  $\square$

Let us consider the following system of sets:

$$\tilde{\mathcal{D}} := \{\tilde{D} \in C(C_\gamma) : \tilde{D}(\omega) = L^{-1}(\omega)D(\omega), \text{ for } \omega \in \Omega, D \in \mathcal{D}\}, \quad (16)$$

where  $L^{-1}$  is the inverse operator of  $L$  given by (14) (note that  $L^{-1}$  is well defined and is given by  $L^{-1}(\omega)\xi = \xi + Z^*(\cdot, \omega)$ , for  $\xi \in C_\gamma$ ).

**Theorem 14.** *Under the hypotheses in this section the following results hold:*

- i) *The random evolution equation (12) with initial condition (13) possesses globally defined mild solutions which generate a strict MNDS. In particular, the MNDS  $U : \mathbb{R}^+ \times \Omega \times C_\gamma \rightarrow P_f(C_\gamma)$  is defined by*

$$U(t, \omega, x_0) := \cup y_t, \quad (17)$$

*where this union is taken on the set of mild solutions  $[0, +\infty) \ni t \mapsto y_t \in C_\gamma$  such that  $y_0 = x_0$ .*

- ii) *The multivalued mapping  $\tilde{U} : \mathbb{R}^+ \times \Omega \times C_\gamma \rightarrow P_f(C_\gamma)$  defined by*

$$\tilde{U}(t, \omega, \tilde{x}_0) = L^{-1}(\theta_t \omega)U(t, \omega, L(\omega)\tilde{x}_0)$$

*is another strict MNDS such that  $\tilde{U}(t, \omega, \tilde{x}_0) = \cup u_t$ , where  $u_t$  stands for any mild solution of (7) with initial condition  $\tilde{x}_0$ .*

- iii) *There is a one-to-one correspondence between pullback attractors of  $U$  and  $\tilde{U}$ . In particular, if  $\mathcal{A}(\omega)$  is a (strictly invariant) pullback  $\mathcal{D}$ -attractor of  $U$  then  $\tilde{\mathcal{A}}(\omega) := L^{-1}(\omega)\mathcal{A}(\omega)$  is a (strictly invariant) pullback  $\tilde{\mathcal{D}}$ -attractor of  $\tilde{U}$ . Conversely, if  $\tilde{\mathcal{E}}(\omega)$  is a (strictly invariant) pullback  $\tilde{\mathcal{D}}$ -attractor for  $\tilde{U}$ , then  $\mathcal{E}(\omega) := L(\omega)\tilde{\mathcal{E}}(\omega)$  is a (strictly invariant) pullback  $\mathcal{D}$ -attractor for  $U$ .*

*Moreover, if  $\mathcal{A}(\omega)$  is random with respect to  $\mathcal{P}^c$ , then  $\tilde{\mathcal{A}}(\omega)$  is as well, and conversely.*

*Proof.* The first part of this result follows from Caraballo *et al.* [3], since from (8) it immediately holds that

$$\|f(\omega, \xi)\| \leq \tilde{c}_1(\omega) + \tilde{c}_2(\omega)\|\xi\|_\gamma,$$

where  $\tilde{c}_1 : \Omega \rightarrow \mathbb{R}$ , defined by  $\tilde{c}_1(\omega) := c_1 + c_2\|Z^*(\cdot, \omega)\|_\gamma$ , is measurable with respect to  $\mathcal{F}$ , integrable with respect to every finite interval and subexponentially growing for  $t \rightarrow \pm\infty$ . Here  $\tilde{c}_2(\omega)$  is simply  $c_2$ .

The second part of this theorem follows from Lemma 13 taking as  $\mathcal{T}$  the mapping  $L$  given by (14), so that  $U$  and  $\tilde{U}$  are conjugated MNDS, and from the definition of mild solutions for both random and stochastic problems.

Finally we show the relation between the attractors associated to  $U$  and  $\tilde{U}$ . Assume that  $\mathcal{A}(\omega)$  is a pullback  $\mathcal{D}$ -attractor for  $U$ . Then, from the continuity of  $L$  it follows the compactness of  $\tilde{\mathcal{A}}$ . From the negative invariance property of  $\mathcal{A}$  we can deduce the negative invariance for  $\tilde{\mathcal{A}}$ :

$$\begin{aligned} \tilde{\mathcal{A}}(\theta_t \omega) &= L^{-1}(\theta_t \omega)\mathcal{A}(\theta_t \omega) \subset L^{-1}(\theta_t \omega)U(t, \omega, \mathcal{A}(\omega)) \\ &= L^{-1}(\theta_t \omega)U(t, \omega, L(\omega)\tilde{\mathcal{A}}(\omega)) = \tilde{U}(t, \omega, \tilde{\mathcal{A}}(\omega)). \end{aligned}$$

In the same way one can check that if  $\mathcal{A}$  is strictly invariant, then  $\tilde{\mathcal{A}}$  also is. Consider now  $\tilde{D} \in \tilde{\mathcal{D}}$ , where  $\tilde{\mathcal{D}}$  is defined by (16). It follows

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \text{dist}_{C_\gamma}(\tilde{U}(t, \theta_{-t}\omega, \tilde{D}(\theta_{-t}\omega)), \tilde{\mathcal{A}}(\omega)) \\ &= \lim_{t \rightarrow +\infty} \text{dist}_{C_\gamma}(L^{-1}(\omega)U(t, \theta_{-t}\omega, L(\theta_{-t}\omega)\tilde{D}(\theta_{-t}\omega)), L^{-1}(\omega)\mathcal{A}(\omega)) \\ &= \lim_{t \rightarrow +\infty} \text{dist}_{C_\gamma}(U(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) + Z^*(t, \omega), \mathcal{A}(\omega) + Z^*(t, \omega)) \\ &= \lim_{t \rightarrow +\infty} \text{dist}_{C_\gamma}(U(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \text{ for all } \omega \in \Omega, \end{aligned}$$

and thus, as the set of multivalued mappings  $\mathcal{D}$  satisfying the inclusion closed property is attracted by  $\mathcal{A}$ , then  $\tilde{\mathcal{D}}$  is attracted by  $\tilde{\mathcal{A}}$ .

The measurability property of the attractor follows from

$$\begin{aligned} \tilde{\mathcal{A}}(\omega) &= L^{-1}(\omega)\mathcal{A}(\omega) = L^{-1}(\omega)\overline{\cup_{n \in \mathbb{N}} a_n(\omega)} \\ &= \overline{\cup_{n \in \mathbb{N}} L^{-1}(\omega)a_n(\omega)}, \end{aligned}$$

where  $a_n(\omega)$  are measurable selections of  $\mathcal{A}(\omega)$  with respect to  $\overline{\mathcal{F}}$ .  $\square$

On account of Theorem 14, we can restrict ourselves to the analysis of the existence of attractors to the MNDS generated by the random equation (12).

In the sequel, let us consider the system  $\mathcal{D}$  given by the multivalued mappings  $D$  with  $D(\omega) \subset B_{C_\gamma}(0, \varrho(\omega))$ , the closed ball with center zero and radius  $\varrho$ , which is supposed to have a subexponential growth:

$$\lim_{t \rightarrow \pm\infty} \frac{\log^+ \varrho(\theta_t\omega)}{t} = 0 \quad \text{for } \omega \in \Omega.$$

Then, it is easily checked that, in this particular situation, the system of sets  $\tilde{\mathcal{D}}$  given by (16) is exactly the same as  $\mathcal{D}$ .

To start with, we should emphasize that it is also shown in [3] that  $U$  given by (17) satisfies that, for fixed  $t \geq 0$  and  $\omega \in \Omega$ , the mapping  $x_0 \mapsto U(t, \omega, x_0)$  is upper-semicontinuous. In addition, assuming that

$$c_2 < \alpha, \tag{18}$$

$U$  is pullback asymptotically compact with respect to  $B(\omega) = B_{C_\gamma}(0, R(\omega))$ , the random ball in  $C_\gamma$  with center zero and random radius

$$R(\omega) := 2 \int_{-\infty}^0 e^{(\alpha - c_2)\tau} (c_1 + c_2 \|Z^*(\cdot, \theta_\tau\omega)\|_\gamma) d\tau. \tag{19}$$

Furthermore, assuming (18), it is proved that  $B(\omega)$  belongs to the sets system  $\mathcal{D}$ , and it is a pullback  $\mathcal{D}$ -absorbing set in the sense of (2). Hence, applying Theorem 9, the MNDS generated by (12) has a pullback strictly invariant  $\mathcal{D}$ -attractor  $\mathcal{A}$  in  $C(C_\gamma)$ .

We want to show that the MNDS is an MRDS and that its corresponding pullback attractor is in fact a random attractor. To prove the first assertion, by Lemma 4, it is enough to prove that  $U$  is upper-semicontinuous with respect to all its variables. And to prove that (see the proof of Theorem 6.1 in [3]) a sufficient condition is given in the following result:

**Theorem 15.** *Assume that*

$$\Omega \times C_\gamma \ni (\omega, \xi) \mapsto F(\xi + Z^*(\cdot, \omega)) \in H \quad \text{is continuous,} \tag{20}$$

and also that, for every  $\omega_0 \in \Omega$  and  $t_0 \in \mathbb{R}$ , there exists a neighborhood  $\mathcal{V} = \mathcal{V}(\omega_0, t_0)$  such that, for some  $\beta > 1$ ,

$$\int_0^t \left( (c_1 + c_2 \|Z^*(\cdot, \theta_\tau \omega)\|_\gamma)^\beta + c_2^\beta \right) d\tau \leq C(\omega_0, t_0) < \infty, \quad (21)$$

for all  $(\omega, t) \in \mathcal{V}$ . Then, the mapping

$$(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times C_\gamma \mapsto U(t, \omega, x) \in C(C_\gamma)$$

is upper-semicontinuous.

Moreover, according to Lemma 10, in order to prove that  $\mathcal{A}$  is also a random attractor associated to  $U$ , we would need to check that  $\omega \mapsto U(t, \omega, B(\omega))$  is a random set with respect to  $\bar{\mathcal{F}}$  for  $t \geq 0$ , for which, in view of the next lemma, a sufficient condition is the continuous dependence of the radius  $R(\omega)$  of the absorbing multivalued set  $B(\omega)$  (see the proof in [3]):

**Lemma 16.** *In addition to (18), (20) and (21), assume that*

$$\limsup_{\omega \rightarrow \omega_0} R(\omega) \leq R(\omega_0) \quad \text{for } \omega_0 \in \Omega. \quad (22)$$

Then, the multivalued mapping  $\omega \mapsto U(t, \omega, B(\omega))$  is  $\bar{\mathcal{F}}$  measurable for  $t \geq 0$ . In addition,  $U(t, \omega, B(\omega)) \in C(C_\gamma)$ .

But the problem is that over the chosen metric dynamical system the mapping  $\Omega \ni \omega \mapsto Z^*(\cdot, \omega) \in C_\gamma$  is not continuous, so none of the conditions (20) and (22) are satisfied. However, we will overcome this continuity problem by considering  $Z^*(\cdot, \omega)$  over the following subsets  $\Omega_N$ : let us fix a small constant  $\zeta$  satisfying that

$$0 < 2\zeta < \mu := \alpha - c_2. \quad (23)$$

Consider, for  $N \in \mathbb{N}$ , the sets

$$\Omega_N := \{\omega \in \Omega : \|\omega(t)\|_V \leq N e^{\zeta|t|}, \text{ for } t \in \mathbb{R}\}.$$

Then, it is easily checked that  $\Omega = \cup_N \Omega_N$  and  $\Omega_N \in \mathcal{F}$ .

We are interested in establishing some topological properties of the sets  $\Omega_N$ .

**Lemma 17.** *For every  $N$  the space  $\Omega_N$  is a Polish space.*

*Proof.*  $\Omega_N$  as a subspace of  $\Omega$  is separable. Let  $\omega_n \rightarrow \omega_0$  where  $\omega_n \in \Omega_N$  for  $n \in \mathbb{N}$ . Then  $\omega_n$  tends uniformly on every interval  $[-T, T]$  to  $\omega_0$ . Hence,  $\|\omega_0(t)\|_V \leq N e^{\zeta|t|}$  for  $t \in \mathbb{R}$ , so that  $\omega_0 \in \Omega_N$  which gives the completeness of  $\Omega_N$ .  $\square$

**Lemma 18.** *For any  $N \in \mathbb{N}$ , there exists an  $M = M(N)$  such that*

$$\|z^*(\theta_t \omega)\| \leq M e^{\zeta|t|}$$

for every  $\omega \in \Omega_N$ , where  $\zeta$  is given in (23).

*Proof.* From (11) we also have

$$\left\| A \int_{-\infty}^0 S(-\tau) \theta_t \omega(\tau) d\tau \right\|^2 \leq C_\mu \int_{-\infty}^0 \|\theta_t \omega(\tau)\|_V^2 e^{\mu\tau} d\tau,$$

for  $\mu = \lambda_1 - c_2$  and the corresponding  $C_\mu$  given by (10). Therefore, we get

$$\begin{aligned} \|A \int_{-\infty}^0 S(-\tau) \theta_t \omega(\tau) d\tau\|^2 &\leq 2C_\mu \int_{-\infty}^0 e^{\mu\tau} \|\omega(t)\|_V^2 d\tau + 2C_\mu \int_{-\infty}^0 e^{\mu\tau} \|\omega(t+\tau)\|_V^2 d\tau \\ &\leq \frac{2C_\mu}{\mu} N^2 e^{2\zeta|t|} + 2C_\mu \int_{-\infty}^0 e^{\mu\tau} N^2 e^{2\zeta|\tau|} d\tau e^{2\zeta|t|} \leq N^2 \frac{4C_\mu}{\mu - 2\zeta} e^{2\zeta|t|} \end{aligned}$$

which gives the conclusion.  $\square$

**Lemma 19.** *The mapping  $\Omega_N \ni \omega \mapsto Z^*(\cdot, \omega) \in C_\gamma$  is continuous.*

*Proof.* Note that with our chosen constants, see (23), we have that  $2\zeta < \mu < \alpha < \gamma$ . Suppose that  $\omega_n \rightarrow \omega_0$ , for  $\omega_n, \omega_0 \in \Omega_N$ .

Given an  $\varepsilon > 0$ , consider  $t_0 = t_0(\varepsilon) < 0$  and  $T = T(\varepsilon) < 0$  such that

$$\frac{8N^2 C_\mu}{\mu - 2\zeta} e^{2(\gamma - \zeta)t_0} < \varepsilon^2, \quad \frac{8N^2 C_\mu}{\mu - 2\zeta} e^{2(\mu - \zeta)T} < \varepsilon^2,$$

and, for these  $t_0, T$  an  $n_0(\varepsilon)$  such that for  $n \geq n_0(\varepsilon)$ ,

$$\sup_{t \in [t_0 + T, 0]} \|\omega_n(t) - \omega_0(t)\|_V^2 \leq \frac{\varepsilon^2 \mu}{2C_\mu}.$$

Thanks to (9) and Lemma 12 we have

$$\begin{aligned} \sup_{t \leq 0} e^{2\gamma t} \|z^*(\theta_t \omega_n) - z^*(\theta_t \omega_0)\|^2 &= \sup_{t \leq 0} e^{2\gamma t} \|A \int_{-\infty}^0 S(-\tau) (\theta_t \omega_n(\tau) - \theta_t \omega_0(\tau)) d\tau\|^2 \\ &\leq 2C_\mu \sup_{t \leq 0} e^{2\gamma t} \int_{-\infty}^0 e^{\mu\tau} \|\omega_n(t) - \omega_0(t)\|_V^2 d\tau \\ &\quad + 2C_\mu \sup_{t \leq 0} e^{2\gamma t} \int_{-\infty}^0 e^{\mu\tau} \|\omega_n(t+\tau) - \omega_0(t+\tau)\|_V^2 d\tau =: E_1 + E_2, \end{aligned}$$

where  $C_\mu$  is given by (10). As  $\|\omega_n(t) - \omega_0(t)\|_V^2 \leq 2\|\omega_n(t)\|_V^2 + 2\|\omega_0(t)\|_V^2 \leq 4N^2 e^{2\zeta|t|}$ ,  $E_1$  is estimated by

$$\begin{aligned} E_1 &\leq \sup_{t \leq t_0} 2C_\mu e^{2\gamma t} 4N^2 e^{-2\zeta t} \int_{-\infty}^0 e^{\mu\tau} d\tau + \frac{2C_\mu}{\mu} \sup_{t \in [t_0, 0]} \|\omega_n(t) - \omega_0(t)\|_V^2 \\ &\leq \frac{8N^2 C_\mu}{\mu} e^{2(\gamma - \zeta)t_0} + \frac{2C_\mu}{\mu} \sup_{t \in [t_0, 0]} \|\omega_n(t) - \omega_0(t)\|_V^2. \end{aligned}$$

Similarly we obtain for  $E_2$ :

$$\begin{aligned} E_2 &\leq \sup_{t \leq t_0} 2C_\mu e^{2\gamma t} \int_{-\infty}^0 e^{\mu\tau} \|\omega_n(\tau+t) - \omega_0(\tau+t)\|_V^2 d\tau \\ &\quad + \sup_{t_0 \leq t \leq 0} 2C_\mu e^{2\gamma t} \int_{-\infty}^T e^{\mu\tau} \|\omega_n(\tau+t) - \omega_0(\tau+t)\|_V^2 d\tau \\ &\quad + \sup_{t_0 \leq t \leq 0} 2C_\mu e^{2\gamma t} \int_T^0 e^{\mu\tau} \|\omega_n(\tau+t) - \omega_0(\tau+t)\|_V^2 d\tau \\ &\leq \frac{8N^2 C_\mu}{\mu - 2\zeta} e^{2(\gamma - \zeta)t_0} + \frac{8N^2 C_\mu}{\mu - 2\zeta} e^{(\mu - 2\zeta)T} + \frac{2C_\mu}{\mu} \sup_{t \in [t_0 + T, 0]} \|\omega_n(t) - \omega_0(t)\|_V^2. \end{aligned}$$

Hence, for  $n \geq n_0(\varepsilon)$ ,

$$\|Z^*(\cdot, \omega_n) - Z^*(\cdot, \omega_0)\|_\gamma < \sqrt{5}\varepsilon.$$

□

Let  $\mathcal{F}_{\Omega_N}$  be the trace  $\sigma$ -algebra of  $\mathcal{F}$  with respect to  $\Omega_N$ . Let  $B_{\Omega_N}(a, r)$ ,  $a \in \Omega_N$ ,  $r > 0$  be a ball in  $\Omega_N$ . These balls can be generated by  $B_{\Omega}(a, r) \cap \Omega_N$  where  $B_{\Omega}(a, r)$  is a ball in  $\Omega$ . The same is true for all open sets in  $\Omega_N$ . Hence  $\mathcal{F}_{\Omega_N}$  is just the Borel- $\sigma$ -algebra of  $\Omega_N$ . Moreover, since  $\Omega_N \in \mathcal{F}$  we have  $\mathcal{F}_{\Omega_N} \subset \mathcal{F}$ .

Let us define

$$\mathbb{P}_{\Omega_N}(A) := \mathbb{P}(A), \text{ for } A \in \mathcal{F}_{\Omega_N},$$

that is,  $\mathbb{P}_{\Omega_N}$  is just the restriction of  $\mathbb{P}$  to  $\mathcal{F}_{\Omega_N}$ .

**Lemma 20.**  $\mathbb{P}_{\Omega_N}$  is a finite measure on  $(\Omega_N, \mathcal{F}_{\Omega_N})$ .

*Proof.* It follows immediately, due to the fact that  $\mathbb{P}$  is a probability measure and  $\mathcal{F}_{\Omega_N} \subset \mathcal{F}$ , since

i)  $\mathbb{P}_{\Omega_N}(\emptyset) = \mathbb{P}(\emptyset) = 0$ .

ii) If  $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_{\Omega_N}$  are such that  $F_n \cap F_m = \emptyset$ , for  $n \neq m$ , then it holds

$$\mathbb{P}_{\Omega_N}\left(\bigcup_{n=1}^{\infty} F_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(F_n) = \sum_{n=1}^{\infty} \mathbb{P}_{\Omega_N}(F_n).$$

□

However, we cannot claim that  $\mathbb{P}_{\Omega_N}$  is a probability measure, because  $\mathbb{P}_{\Omega_N}(\Omega_N) < 1$ . Nevertheless, for the objectives in this paper, this is not a restriction, since we only need to work with a finite measure.

Let  $\bar{\mathcal{F}}_{\Omega_N}$  be the completion of  $\mathcal{F}_{\Omega_N}$  with respect to  $\mathbb{P}_{\Omega_N}$ . We denote by  $\Omega_N^c$  the complement of  $\Omega_N$ .

**Lemma 21.** If  $A \in \bar{\mathcal{F}}_{\Omega_N}$ , then  $A \in \bar{\mathcal{F}}$ .

*Proof.* We can find  $\check{A}, \hat{A} \in \mathcal{F}_{\Omega_N}$  such that

$$\check{A} \subset A \subset \hat{A}, \quad \mathbb{P}_{\Omega_N}(\check{A}) = \mathbb{P}_{\Omega_N}(\hat{A}).$$

Define  $\check{B} = \check{A} \cup \Omega_N^c$  and  $\hat{B} = \hat{A} \cup \Omega_N^c$ . These sets are in  $\mathcal{F}$  and

$$\mathbb{P}(\check{B}) = \mathbb{P}(\check{A}) + \mathbb{P}(\Omega_N^c) = \mathbb{P}(\hat{B}) = \mathbb{P}(\hat{A}) + \mathbb{P}(\Omega_N^c).$$

Because of  $\check{B} \subset A \cup \Omega_N^c \subset \hat{B}$  we have that  $A \cup \Omega_N^c \in \bar{\mathcal{F}}$ , hence  $A = (A \cup \Omega_N^c) \cap \Omega_N \in \bar{\mathcal{F}}$ . □

We now show how over  $\Omega_N$  we can deduce the continuous dependence of the radius  $R(\omega)$  of the absorbing set given by (19).

**Lemma 22.** The mapping  $\omega \mapsto R(\omega)$  is continuous on  $\Omega_N$ .

*Proof.* Suppose  $\omega_n, \omega_0 \in \Omega_N$ ,  $\omega_n \rightarrow \omega_0$ . Due to Lemma 19, we have

$$\lim_{n \rightarrow \infty} 2e^{(\alpha - c_2)\tau} (c_1 + c_2 \|Z^*(\cdot, \theta_\tau \omega_n)\|_\gamma) = 2e^{(\alpha - c_2)\tau} (c_1 + c_2 \|Z^*(\cdot, \theta_\tau \omega_0)\|_\gamma).$$

On the other hand, thanks to Lemma 18 and since we have chosen  $0 < \zeta < \alpha - c_2$ , there exists a non-negative integrable majorant function:

$$|2e^{(\alpha - c_2)\tau} (c_1 + c_2 \|Z^*(\cdot, \theta_\tau \omega_n)\|_\gamma)| \leq 2c_1 e^{(\alpha - c_2)\tau} + 2Mc_2 e^{(\alpha - c_2 - \zeta)\tau}.$$

Therefore, the convergence of  $R(\omega_n)$  follows by Lebesgue's theorem. □

Finally, we can establish the main result in this section:

**Theorem 23.** *Under the hypotheses in this section, the MNDS  $U$  given by (17) is also an MRDS. In addition, the pullback attractor  $\mathcal{A}$  is  $\bar{\mathcal{F}}$  measurable, that is,  $\mathcal{A}$  is also the random attractor associated to the MRDS  $U$ .*

*Proof.* The first part of this result follows from Lemma 19 and the fact that, for every  $\omega_0 \in \Omega_N$  and  $t_0 \in \mathbb{R}$ , there exists  $\mathcal{V} = \mathcal{V}(\omega_0, t_0)$  such that for some  $\beta > 1$  we have

$$\begin{aligned} & \int_0^t \left( (c_1 + c_2 \|Z^*(\cdot, \theta_\tau \omega)\|_\gamma)^\beta + c_2^\beta \right) d\tau \\ & \leq c_2^\beta t + \int_0^t (c_1 + c_2 \sup_{s \leq 0} e^{\gamma s} \|z^*(\theta_{\tau+s} \omega)\|)^\beta d\tau \\ & \leq c_2^\beta t + \int_0^t (c_1 + c_2 M \sup_{s \leq 0} e^{\gamma s} e^{c|\tau+s|})^\beta d\tau \leq c_2^\beta t + (c_1 + c_2 M e^{c|t|})^\beta t \leq C(t_0), \end{aligned}$$

for all  $(\omega, t) \in \mathcal{V}$ , so that the uniform integrability condition (21) holds. Then, applying Theorem 15 and Lemma 4, for every open set  $\mathcal{O}$  of the space  $C_\gamma$ ,

$$\{(t, \omega, x) \in \mathbb{R}^+ \times \Omega_N \times C_\gamma : U(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} := A_{N, \mathcal{O}} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_{\Omega_N} \otimes \mathcal{B}(C_\gamma),$$

and also  $A_{N, \mathcal{O}} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(C_\gamma)$ . Hence

$$\begin{aligned} & \{(t, \omega, x) \in \mathbb{R}^+ \times \Omega \times C_\gamma : U(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \\ & = \bigcup_{N=1}^{\infty} \{(t, \omega, x) \in \mathbb{R}^+ \times \Omega_N \times C_\gamma : U(t, \omega, x) \cap \mathcal{O} \neq \emptyset\} \\ & = \bigcup_{N=1}^{\infty} A_{N, \mathcal{O}} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(C_\gamma) \end{aligned}$$

and then,  $U$  is an MRDS.

On the other hand, from Lemma 16 and Lemma 22, the mapping

$$\Omega_N \ni \omega \mapsto U(t, \omega, B(\omega)) \in C(C_\gamma)$$

is  $\bar{\mathcal{F}}_{\Omega_N}$  measurable for  $t \geq 0$ , that is, we have the measurability of  $U$  in the sense that, for every open set  $\mathcal{O}$  of  $C_\gamma$ , the set

$$C_{N, \mathcal{O}} := \{\omega \in \Omega_N : U(t, \omega, B(\omega)) \cap \mathcal{O} \neq \emptyset\} \in \bar{\mathcal{F}}_{\Omega_N},$$

and then, due to Lemma 21,

$$\{\omega \in \Omega : U(t, \omega, B(\omega)) \cap \mathcal{O} \neq \emptyset\} = \bigcup_{N=1}^{\infty} C_{N, \mathcal{O}} \in \bar{\mathcal{F}}_{\Omega_N} \subset \bar{\mathcal{F}}$$

and, as a consequence of Lemma 10, the pullback attractor  $\mathcal{A}$  for the MRDS associated to the random equation (12) is also the random attractor.  $\square$

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