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ASYMPTOTIC BEHAVIOUR OF A LOGISTIC LATTICE SYSTEM

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Dedicated to Professor José Real, in Memoriam

Abstract. In this paper we study the asymptotic behaviour of solutions of a lattice dynamical system of a logistic type. Namely, we study a system of infinite ordinary differential equations which can be obtained after the spatial discretization of a logistic equation with diffusion. We prove that a global attractor exists in suitable weighted spaces of sequences.

1. **Introduction.**

Lattice differential equations arise naturally in many real situations where the spatial structure possesses a discrete character. These systems are used to model, for instance, cellular neural networks with applications to image processing, pattern recognition, and brain science [15, 16, 17, 18]. They also appear in modeling the propagation of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites (see for example, [5], [6], [35], [34], [26, 27]). We can also find lattice differential equations in problems related to chemical reaction theory [20, 25, 30] as well. Additionally, it can appear after a spatial discretization of a differential equation, as is the case we will analyze in the present paper. More specifically, we are interested in a spatial discretization of a logistic reaction-diffusion equation to be described later on.
In the mathematical literature, one can find many works on deterministic lattice dynamical systems. Some studies on traveling waves can be found in [11, 31, 12, 41, 1, 2] and the references therein. The chaotic properties of solutions for such systems have been investigated by [11] and [14, 36, 13, 19]. The existence and properties of the global attractor for lattice differential equations have been established, for example, in [4], [7], [32], [33], [38], [39], [42], [43], [44].

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Also, one can find several papers considering stochastic versions of lattice dynamical systems (see, e.g., [3], [8], [9] [10], [23], [24]).
In the paper [38], Wang considered the following lattice differential equation:

$$\begin{cases} \dfrac{du_i}{dt} - \nu\left(u_{i-1} - 2u_i + u_{i+1}\right) - h(u_i) - g_i = 0,\ t > 0,\ i \in \mathbb{Z}, \\ \\ u_i(0) = u_i^0,\ i \in \mathbb{Z}, \end{cases} \tag{1}$$

where $(u_i)_{i\in\mathbb{Z}}$ is a sequence, ν is a positive constant, $g = (g_i)_{i\in\mathbb{Z}}$ is given, the nonlinear term described by h is a smooth function which satisfies, for some positive constants α, β and γ

$$h(u)u \leq -\alpha u^2 + \beta, \quad h'(u) \leq \gamma, \quad \text{for } u \in \mathbb{R}. \tag{2}$$

It is then proved that this system generates a lattice dynamical system in a suitable weighted space of sequences. However, the assumptions imposed on the nonlinearity h is restrictive enough so that some interesting examples cannot be covered by the results proved in [38]. For instance, one simple case which needs a separate study is the one given by the following logistic equation

$$\begin{cases} \dfrac{du_i}{dt} - \left(u_{i-1} - 2u_i + u_{i+1}\right) - ru_i\left(t\right)\left(1 - \dfrac{1}{b}u_i\left(t\right)\right) = 0,\ t > 0,\ i \in \mathbb{Z}, \\ \\ u_i(0) = u_i^0,\ i \in \mathbb{Z}, \end{cases} \tag{3}$$

where $r, b > 0$ are constants. System (3) can be considered as a discrete approximation of the logistic equation with diffusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - ru\left(t\right)\left(1 - \dfrac{1}{b}u\left(t\right)\right) = 0,\ t > 0,\ x \in \mathbb{R}, \\ \\ u(0) = u^0\left(x\right),\ x \in \mathbb{R}, \end{cases}$$

It is easy to see that the function $h(u) = ru(1 - \dfrac{1}{b}u)$ does not satisfy (2), and this motivates the analysis carried out in the present paper. Although our analysis can be extended for more general nonlinear term h, we have preferred to develop this case due to the importance of this logistic model in applications.

The content of the paper is as follows.
First, in Section 2 we state the problem in a suitable framework and prove the existence and uniqueness of non-negative solutions in the space of sequences ℓ^2, which is important due to the biological meaning of the variable u_i as a population. However, the space ℓ^2 is not appropriate to study the long-time dynamics of the system, as the equation is not dissipative enough in this space. We observe also that the fixed point $\overline{u}$ given by $\overline{u}_i = b$, for all i, does not belong to ℓ^2, although it plays a relevant role in the dynamics. Then, following [38], we consider a suitable weighted space ℓ_δ^2. We then extend the semigroup defined in ℓ^2 to a new one in the weighted space.
Next we prove the existence of the global attractor of our problem in Section 3. It is important to remark here that, using an appropriate weighted space, the quadratic term $\frac{r}{b}u^2$ dominates the linear term ru, which is not the case in the space ℓ^2. The reason is that in non-weighted spaces we have the chain of inclusions $\ell^1 \subset \ell^2 \subset \ell^3 \subset$ ..., whereas in our weighted spaces ℓ_δ^p we have the opposite: $\ell_\delta^1 \supset \ell_\delta^2 \supset \ell_\delta^3 \supset$ ...

In Section 4 we prove some properties on the regularity of the solutions and the attractor, proving that the attractor is compact in any ℓ^p_δ, $p \geq 2$, and also that the attracting property holds with respect to the norm of these spaces. Finally, some open problems concerning the stability properties of the stationary points are stated.

2. **Statement of the logistic lattice system.** Our aim in this paper is to study the following logistic system

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) - ru_i(t)(1 - \dfrac{1}{b}u_i(t)) = 0, \ t > 0, \ i \in \mathbb{Z}, \\ \\ u_i(0) = u_i^0, \ i \in \mathbb{Z}, \end{cases} \tag{4}$$

where $r, b > 0$ are constants.

We consider the separable Hilbert space $\ell^2 = \{v = (v_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} v_i^2 < \infty\}$ with norm $\|v\| = \sqrt{\sum_{i\in\mathbb{Z}} v_i^2}$ and scalar product $(w, v) = \sum_{i\in\mathbb{Z}} w_i v_i$.

We define the operator $f : \ell^2 \to \ell^2$ by $(f(v))_i = f_i(v_i) = -rv_i(1 - \frac{1}{b}v_i)$ for $i \in \mathbb{Z}$. For any $v \in \ell^2$, as $|v_i| \leq \|v\|$ for all $i \in \mathbb{Z}$, we have

$$\begin{aligned} |f_i(v_i)| &\leq r\left|v_i(1 - \frac{1}{b}v_i)\right| \\ &\leq r\left(1 + \frac{1}{b}\|v\|\right)|v_i|. \end{aligned}$$

This implies that f is well defined and

$$\|f(v)\| \leq r\left(1 + \frac{1}{b}\|v\|\right)\|v\|.$$

Also,

$$\begin{aligned} \|f(v) - f(w)\|^2 &\leq \frac{2r^2}{b^2}\sum_{i\in\mathbb{Z}}\left(b^2(v_i - w_i)^2 + (v_i^2 - w_i^2)^2\right) \qquad (5) \\ &\leq 2r^2\|v - w\|^2 + \frac{2r^2}{b^2}\max_{i\in\mathbb{Z}}|v_i + w_i|^2\|v - w\|^2 \\ &\leq 2\frac{r^2}{b^2}\left(b^2 + 2\|v\|^2 + 2\|w\|^2\right)\|v - w\|^2. \end{aligned}$$

This implies that the map f is Lipschitz in bounded sets of ℓ^2.

We define the operator $A : \ell^2 \to \ell^2$ by

$$(Av)_i := -v_{i-1} + 2v_i - v_{i+1}, \ i \in \mathbb{Z}.$$

Also, we define the operators $\overline{B}$, $\overline{B}^*: \ell^2 \to \ell^2$ by

$$\left(\overline{B}v\right)_i := v_{i+1} - v_i, \ \left(\overline{B}^*v\right)_i := v_{i-1} - v_i.$$

It is easy to check that

$$A = \overline{B}^*\,\overline{B} = \overline{B}\,\overline{B}^*,$$
$$\left(\overline{B}^*w, v\right) = \left(w, \overline{B}v\right),$$

and also that A is a globally Lipschitz operator.

Then the operator $F : \ell^2 \to \ell^2$ is defined by

$$F(v) = -Av - f(v)$$

and (4) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} = F(u), \ t > 0, \\ \\ u(0) = u^0. \end{cases} \tag{6}$$

Hence, the operator F is also Lipschitz in bounded sets of ℓ^2. Standard results for differential equations in Banach spaces [40] ensure the existence of a unique local solution $u \in C^1([0,\alpha),\ell^2)$ of problem (4) for every initial datum $u^0 \in \ell^2$, where $[0,\alpha)$ is the maximal interval of existence of the solution. Moreover, for two solutions $u(\cdot), v(\cdot)$ defined in the interval $[0,a]$ (for $0 < a < \alpha$) we can obtain that

$$\frac{1}{2}\frac{d}{dt}\|u(t) - v(t)\|^2 \leq \beta(M)\|u(t) - v(t)\|^2,$$

for some $\beta(M)$, where $M > 0$ is such that $\|u(t)\|, \|v(t)\| \leq M$ for all $t \in [0,a]$. Then, using Gronwall's lemma we obtain that

$$\|u(t) - v(t)\| \leq e^{\beta(M)t}\|u(0) - v(0)\| \text{ for all } t \in [0,a]. \tag{7}$$

Due to the biological meaning of the variable u (which is a population) we need to consider only non-negative values of u_i, so that we shall not define a semigroup in the space ℓ^2. Therefore, we have to establish first that for any initial value u^0 satisfying $u_i^0 \geq 0$, for all $i \in \mathbb{Z}$, the solution $u(t)$ is also non-negative for any $t \geq 0$.

Lemma 1. *Let $u^0 \in \ell^2$ be such that $u_i^0 \geq 0$ for all $i \in \mathbb{Z}$. Then, the unique solution $u(\cdot)$ of (4) satisfies $u_i(t) \geq 0$ for all $i \in \mathbb{Z}$ and $t \in [0,\alpha)$. Moreover, $u(\cdot)$ is globally defined in time, that is, $\alpha = +\infty$.*
If $u^0, v^0 \in \ell^2$ are such that $u_i^0, v_i^0 \geq 0$ for all $i \in \mathbb{Z}$, then the corresponding solutions satisfy

$$\|u(t) - v(t)\| \leq e^{rt}\left\|u^0 - v^0\right\| \textit{ for all } t \geq 0. \tag{8}$$

Proof. Let $z^+ = \max\{z,0\}$, $z^- = \max\{-z,0\}$. For $v \in \ell^2$, $v^\pm = \left(v_i^\pm\right)_{i\in\mathbb{Z}}$. Obviously, $z = z^+ - z^-$.
We note that $u(\cdot) \in C^1\left([0,\alpha),\ell^2\right)$ implies that $\left(\dfrac{du}{dt}, (-u)^+\right) = -\dfrac{1}{2}\dfrac{d}{dt}\|(-u)^+\|^2$ (see [21, Lemma 2.2]). Also,

$$\begin{aligned} \left(A(-v),(-v)^+\right) &= \left(A(-v)^+,(-v)^+\right) - \left(A(-v)^-,(-v)^+\right) \\ &= \left(\overline{B}(-v)^+,\overline{B}(-v)^+\right) \\ &\quad - \sum_{i\in\mathbb{Z}}\left((-v_{i+1})^- - (-v_i)^-\right)\left((-v_{i+1})^+ - (-v_i)^+\right) \\ &= \left(\overline{B}(-v)^+,\overline{B}(-v)^+\right) \\ &\quad + \sum_{i\in\mathbb{Z}}(-v_i)^-(-v_{i+1})^+ + \sum_{i\in\mathbb{Z}}(-v_{i+1})^-(-v_i)^+ \geq 0, \ \forall v \in \ell^2, \end{aligned} \tag{9}$$

and

$$\begin{aligned}\left(f\left(u\right),\left(-u\right)^{+}\left(t\right)\right) &= -\frac{r}{b}\sum_{i\in\mathbb{Z}}u_{i}\left(t\right)\left(b-u_{i}\left(t\right)\right)\left(-u_{i}\right)^{+}\left(t\right)\\ &= r\left\Vert\left(-u\right)^{+}\left(t\right)\right\Vert^{2}-\frac{r}{b}\sum_{i\in\mathbb{Z}}u_{i}\left(t\right)\left(\left(-u_{i}\right)^{+}\left(t\right)\right)^{2}\\ &\leq\frac{r}{b}\left(b+\max_{i\in\mathbb{Z}}\left|u_{i}\left(t\right)\right|\right)\left\Vert\left(-u\right)^{+}\left(t\right)\right\Vert^{2}.\end{aligned}$$

Notice that in any compact interval $[0,\overline{\alpha}]\subset[0,\alpha)$ there exists $M_{\overline{\alpha}}$ such that $\max_{i\in\mathbb{Z}}\left|u_{i}\left(t\right)\right|\leq M_{\overline{\alpha}}$ for all $t\in[0,\overline{\alpha}]$. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\Vert(-u)^{+}\right\Vert^{2} &= \left(-\frac{du}{dt},(-u)^{+}\right)=-\left(A(-u),(-u)^{+}\right)+\left(f\left(u\right),(-u)^{+}\right)\\ &\leq\frac{r}{b}(b+M_{\overline{\alpha}})\left\Vert(-u)^{+}\right\Vert^{2},\ \forall t\in[0,\overline{\alpha}],\end{aligned}$$

and

$$\left\Vert\left(-u\right)^{+}\left(t\right)\right\Vert^{2}\leq\left\Vert\left(-u\right)^{+}\left(0\right)\right\Vert^{2}+\frac{2r}{b}(b+M_{\overline{\alpha}})\int_{0}^{t}\left\Vert\left(-u\right)^{+}\left(s\right)\right\Vert^{2}ds \text{ if } 0\leq t\leq\overline{\alpha}.$$

Hence, from the Gronwall lemma we have

$$\left\Vert\left(-u\right)^{+}\left(t\right)\right\Vert^{2}\leq\left\Vert\left(-u\right)^{+}\left(0\right)\right\Vert^{2}e^{\frac{2r}{b}(b+M_{\overline{\alpha}})t}\text{ if }0\leq t\leq\overline{\alpha}.$$

Since $u_{i}\left(0\right)\geq0$, we obtain that $\left\Vert\left(-u\right)^{+}\left(0\right)\right\Vert=0$, and therefore $\left\Vert\left(-u\right)^{+}\left(t\right)\right\Vert=0$ for $0\leq t\leq\overline{\alpha}$. Thus, $u_{i}\left(t\right)\geq0$ for all $i\in\mathbb{Z}$ and $t\in[0,\overline{\alpha}]$. As $[0,\overline{\alpha}]\subset[0,\alpha)$ is arbitrary, it follows that $u_{i}\left(t\right)\geq0$ for all $i\in\mathbb{Z}$ and $t\in[0,\alpha)$.
Further, we shall obtain an estimate of non-negative solutions. From $u_{i}\left(t\right)\geq0$ we have that

$$-\left(f\left(u\left(t\right)\right),u\left(t\right)\right)=r\sum_{i\in\mathbb{Z}}u_{i}\left(t\right)\left(1-\frac{1}{b}u_{i}\left(t\right)\right)u_{i}\left(t\right)\leq r\left\Vert u\left(t\right)\right\Vert^{2}.$$

Then

$$\frac{1}{2}\frac{d}{dt}\left\Vert u\right\Vert^{2}=\left(-Au\left(t\right),u\left(t\right)\right)-\left(f\left(u\left(t\right)\right),u\left(t\right)\right)\leq r\left\Vert u\left(t\right)\right\Vert^{2}$$

and

$$\left\Vert u\left(t\right)\right\Vert^{2}\leq\left\Vert u\left(0\right)\right\Vert^{2}e^{2rt}. \tag{10}$$

Now, a standard result [40, p. 79] implies that $\alpha=+\infty$.
Finally, for two solutions $u\left(\cdot\right),v\left(\cdot\right)$ corresponding to non-negative initial data we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\Vert u\left(t\right)-v\left(t\right)\right\Vert^{2} &\leq-\left(f\left(u\left(t\right)\right)-f\left(v\left(t\right)\right),u\left(t\right)-v\left(t\right)\right)\\ &= r\left\Vert v-w\right\Vert^{2}-\frac{r}{b}\sum_{i\in\mathbb{Z}}\left(v_{i}+w_{i}\right)\left(v_{i}-w_{i}\right)^{2}\\ &\leq r\left\Vert v-w\right\Vert^{2},\end{aligned}$$

and (8) follows from the Gronwall lemma. □

Denote now $E=\ell^{2}$ and $E^{+}=\{v\in E:v_{i}\geq0,\ \forall i\in\mathbb{Z}\}$. Thanks to Lemma 1 we can define a semigroup of operators $S:\mathbb{R}^{+}\times E^{+}\rightarrow E^{+}$ by the rule

$$S\left(t,u^{0}\right)=u\left(t\right),$$

where $u(\cdot)$ is the unique solution to (4) with initial datum $u(0)=u^0\in E^+$. Also, (8) implies that S is continuous with respect to the initial value u^0.

As it can be easily seen in our analysis below (see also [38]), we will not be able to deduce the existence of a global attractor for the semigroup S working in the space ℓ^2. In order to solve the problem, we will need to extend this semigroup to a new one in a suitable weighted space.
In a similar way as it was done in [38], we consider the following weighted space

$$\ell_\delta^2=\{v=(v_i)_{i\in\mathbb{Z}}:\sum_{i\in\mathbb{Z}}\left(1+|\delta i|^2\right)^{-1}|v_i|^2<\infty\},$$

where $0<\delta\leq 1$, with norm

$$\|v\|_\delta=\left(\sum_{i\in\mathbb{Z}}\left(1+|\delta i|^2\right)^{-1}|v_i|^2\right)^{\frac{1}{2}}$$

and scalar product

$$(v,w)_\delta=\sum_{i\in\mathbb{Z}}\left(1+|\delta i|^2\right)^{-1}v_iw_i.$$

We define the function $g_\delta:\mathbb{R}\to\mathbb{R}$ by

$$g_\delta(x)=(1+|\delta x|^2)^{-1}.$$

Then, by straightforward computations, one can check that

$$1+|\delta(i\pm 1)|^2\leq 3\left(1+|\delta i|^2\right),\quad\text{for all } i, \tag{11}$$

whence

$$1+|\delta i|^2\leq 3\left(1+|\delta(i\pm 1)|^2\right),\quad\text{for all } i. \tag{12}$$

Thanks to (11)-(12), it is not difficult to prove that the function g satisfies:

$$\left|\frac{d}{dx}g_\delta(x)\right|\leq 2\delta^2|x|(1+|\delta x|^2)^{-2}\leq\delta g_\delta(x),\quad\text{for all } x\in\mathbb{R}, \tag{13}$$

$$3^{-1}g_\delta(i)\leq g_\delta(i\pm 1)\leq 3g_\delta(i),\ \text{for all } i\in\mathbb{Z}, \tag{14}$$

$$|g_\delta(i\pm 1)-g_\delta(i)|=|g'_\delta(\xi)|\leq\delta g_\delta(\xi)\leq 3\delta g_\delta(i),\ \text{for some } \xi \text{ and all } i\in\mathbb{Z}, \tag{15}$$

$$\left|(\overline{B}(g_\delta(i))_{i\in\mathbb{Z}})_i\right|\leq 3\delta g_\delta(i),\ |(\overline{B}^*(g_\delta(i))_{i\in\mathbb{Z}})_i|\leq 3\delta g_\delta(i),\ \text{for all } i\in\mathbb{Z}. \tag{16}$$

We note that the operators A and $\overline{B}$ can be defined also in the space ℓ_δ^2, although we will keep the same notation for these operators. For any $v\in\ell_\delta^2$ it is proved in [38, Eq. (5.4) in p. 237] that

$$(Av,v)_\delta\geq\frac{1}{2}\left\|\overline{B}v\right\|_\delta^2-\frac{27}{2}\delta^2\|v\|_\delta^2. \tag{17}$$

Consider also the weighted spaces $\ell_\delta^p=\{v=(v_i)_{i\in\mathbb{Z}}:\sum_{i\in\mathbb{Z}}(1+|\delta i|)^{-2}|v_i|^p<\infty\}$, $p\geq 1$, where $0<\delta\leq 1$, with norm

$$\|v\|_{\ell_\delta^p}=\left(\sum_{i\in\mathbb{Z}}\left(1+|\delta i|^2\right)^{-1}|v_i|^p\right)^{\frac{1}{p}}.$$

We define now the operator $\overline{f}:\ell^2_\delta\to l^1_\delta$ by $\left(\overline{f}\left(v\right)\right)_i=f_i\left(v_i\right)=-rv_i(1-\dfrac{1}{b}v_i)$ for $i\in\mathbb{Z}$. Note that $\ell^2_\delta\subset\ell^1_\delta$ with continuous embedding, and denote by C_δ the constant satisfying

$$\|v\|_{\ell^1_\delta}\le C_\delta\|v\|_\delta,\quad \text{for all } v\in\ell^2_\delta.$$

Since

$$\left|f_i\left(v_i\right)\right|\le r\left|v_i\right|+\frac{r}{b}v_i^2,$$

this operator is well defined, bounded and

$$\left\|\overline{f}\left(v\right)\right\|_{\ell^1_\delta}\le r\left\|v\right\|_{\ell^1_\delta}+\frac{r}{b}\left\|v\right\|^2_\delta.$$

Then, the operator $\overline{F}:\ell^2_\delta\to\ell^1_\delta$ is defined by

$$\overline{F}\left(v\right)=-Av-\overline{f}\left(v\right)$$

and (4) can be rewritten as

$$\begin{cases}\dfrac{du}{dt}=\overline{F}\left(u\right),\ t>0,\\[2ex] u\left(0\right)=u^0.\end{cases}\tag{18}$$

Lemma 2. *The maps $\overline{f}:\ell^2_\delta\to\ell^1_\delta$ and $\overline{F}:\ell^2_\delta\to\ell^1_\delta$ are continuous.*

Proof. The continuity of $\overline{f}$ follows from

$$\begin{aligned}\left\|\overline{f}\left(v\right)-\overline{f}\left(w\right)\right\|_{\ell^1_\delta}&\le r\left\|v-w\right\|_{l^1_\delta}+\frac{r}{b}\sum_{i\in\mathbb{Z}}g_\delta\left(i\right)\left(\left|v_i\right|+\left|w_i\right|\right)\left|v_i-w_i\right| \\ &\le rC_\delta\left\|v-w\right\|_\delta+\frac{r}{b}C_\delta\left\|v-w\right\|_\delta\left(\left\|v\right\|_\delta+\left\|w\right\|_\delta\right).\end{aligned}\tag{19}$$

Also, notice that $A:\ell^2_\delta\to\ell^1_\delta$ satisfies

$$\begin{aligned}\left\|Av-Aw\right\|_{\ell^1_\delta}&\le\sum_{i\in\mathbb{Z}}g_\delta\left(i\right)\left(\left|v_{i+1}-w_{i+1}\right|+2\left|v_i-w_i\right|+\left|v_{i-1}-w_{i-1}\right|\right) \\ &\le 8\left\|v-w\right\|_{\ell^1_\delta} \\ &\le 8C_\delta\left\|v-w\right\|_\delta,\end{aligned}\tag{20}$$

so that it is continuous. Then, the operator $\overline{F}:\ell^2_\delta\to\ell^1_\delta$ is also continuous. □

Now, we establish a Lipschitz property for the solutions in the space ℓ^2_δ.

Lemma 3. *Let $u\left(\cdot\right),v\left(\cdot\right)$ be two solutions of (4) with corresponding initial data $u^0,v^0\in E^+$, respectively. Then, there exists a constant $\beta\left(r,\delta\right)>0$ such that*

$$\left\|u\left(t\right)-v\left(t\right)\right\|_\delta\le e^{\beta(r,\delta)t}\left\|u^0-v^0\right\|_\delta\ \textit{for all } t\ge 0.\tag{21}$$

Proof. First, for $v,w\in E^+$, we have that

$$\begin{aligned}\left(f\left(v\right)-f\left(w\right),v-w\right)_\delta&=-\frac{r}{b}\sum_{i\in\mathbb{Z}}g_\delta\left(i\right)\left(b\left(v_i-w_i\right)-\left(v_i^2-w_i^2\right)\right)\left(v_i-w_i\right) \\ &=-r\left\|v-w\right\|^2_\delta+\frac{r}{b}\sum_{i\in\mathbb{Z}}g_\delta\left(i\right)\left(v_i+w_i\right)\left(v_i-w_i\right)^2 \\ &\ge -r\left\|v-w\right\|^2_\delta.\end{aligned}$$

Thus, by (17) we obtain

$$\frac{1}{2}\frac{d}{dt}\left\Vert u\left(t\right)-v\left(t\right)\right\Vert_{\delta}^{2}\leq\left(\frac{27}{2}\delta^{2}+r\right)\left\Vert u\left(t\right)-v\left(t\right)\right\Vert_{\delta}^{2},$$

and the result follows by applying the Gronwall lemma. □

Let us now denote $E_{\delta}=\ell_{\delta}^{2}$ and $E_{\delta}^{+}=\{v\in E_{\delta}:v_{i}\geq0,\ \forall i\in\mathbb{Z}\}$.

Theorem 4. *The semigroup S can be extended to a semigroup $S_{\delta}:\mathbb{R}^{+}\times E_{\delta}^{+}\rightarrow E_{\delta}^{+}$ satisfying*

$$\left\Vert S_{\delta}\left(t,u^{0}\right)-S_{\delta}\left(t,v^{0}\right)\right\Vert_{\delta}\leq e^{\beta(r,\delta)t}\left\Vert u^{0}-v^{0}\right\Vert_{\delta}\ \text{for all } t\geq0. \tag{22}$$

Thus, it is continuous with respect to the initial data.

Proof. Let $T>0$ and let $\mathcal{G}:E_{\delta}^{+}\rightarrow C([0,T],E_{\delta}^{+})$, with domain $D(\mathcal{G})=E^{+}$, be the map defined by $\mathcal{G}(u^{0})=u\left(\cdot\right)$, where $u\left(\cdot\right)$ is the unique solution of problem (4). In view of (21), this map is continuous. It is easy to see that E^{+} is dense in E_{δ}^{+}, so that $\mathcal{G}$ can be extended uniquely to a map $\widetilde{\mathcal{G}}:E_{\delta}^{+}\rightarrow C([0,T],E_{\delta}^{+})$ with domain $D(\mathcal{G})=E_{\delta}^{+}$, and such that $\widetilde{\mathcal{G}}\left(u^{0}\right)=\mathcal{G}\left(u^{0}\right)$ for $u^{0}\in E^{+}$. Moreover, if $\widetilde{\mathcal{G}}\left(u^{0}\right)=u\left(\cdot\right)$ and $\widetilde{\mathcal{G}}\left(v^{0}\right)=v\left(\cdot\right)$, then (21) holds. Hence, we set $S_{\delta}\left(t,u^{0}\right)=\widetilde{\mathcal{G}}\left(u^{0}\right)\left(t\right).$ □

Corollary 5. *For any initial data $u^{0}\in E_{\delta}^{+}$ we have*

$$\left\Vert S(t,u^{0})\right\Vert_{\delta}\leq e^{\beta(r,\delta)t}\left\Vert u^{0}\right\Vert_{\delta}\ \text{for all } t\geq0. \tag{23}$$

Proof. The result follows from (22) by taking the constant solution $S_{\delta}\left(t,v^{0}\right)=S_{\delta}\left(t,0\right)\equiv0.$ □

A natural question which arises is the following: is the function $u\left(\cdot\right)=S_{\delta}\left(\cdot,u^{0}\right)$ a solution of (4) is some sense? The following theorem provides a positive answer.

Theorem 6. *For any $u^{0}\in E_{\delta}^{+}$, the map $u\left(\cdot\right)=S_{\delta}\left(\cdot,u^{0}\right)$ satisfies the following properties:*

(1) $u\left(\cdot\right)\in C([0,\infty),\ell_{\delta}^{2})$.
(2) $u\left(\cdot\right)\in C^{1}([0,\infty),\ell_{\delta}^{1})$.
(3) *The equality*

$$u\left(t\right)=u^{0}+\int_{0}^{t}\overline{F}\left(u\left(\tau\right)\right)d\tau \tag{24}$$

holds in ℓ_{δ}^{1} for all $t\geq0$. Hence, $\dfrac{du}{dt}=\overline{F}\left(u\left(t\right)\right)$ in ℓ_{δ}^{1}.

Conversely, if a function $\widetilde{u}\left(\cdot\right)$ satisfies properties (1)–(3) and $\widetilde{u}\left(0\right)=u^{0}\in E_{\delta}^{+}$, then $\widetilde{u}\left(\cdot\right)=u\left(\cdot\right)=S_{\delta}\left(\cdot,u^{0}\right).$

Proof. Let $u\left(\cdot\right)=S_{\delta}\left(\cdot,u^{0}\right)$. First, the fact that $u\left(\cdot\right)\in C([0,\infty),\ell_{\delta}^{2})$ was proved in Theorem 4. We take a sequence $u^{0,n}\in E^{+}$ such that $u^{0,n}\rightarrow u^{0}$ in ℓ_{δ}^{2}. It follows from Theorem 4 that $u^{n}\left(\cdot\right)=S\left(\cdot,u^{0,n}\right)$ converges to $u\left(\cdot\right)$ in $C\left([0,T],\ell_{\delta}^{2}\right)$ for any

$T>0$. Hence, by (19) and (20) there exist $K_1, K_2(T)>0$ such that

$$\begin{aligned}
&\left\| \int_0^t \overline{F}\left(u\left(\tau\right)\right) d\tau - \int_0^t \overline{F}\left(u^n\left(\tau\right)\right) d\tau \right\|_{\ell^1_\delta} \\
&\quad \leq \int_0^t \left\| \overline{F}\left(u\left(\tau\right)\right) - \overline{F}\left(u^n\left(\tau\right)\right) \right\|_{\ell^1_\delta} d\tau \\
&\quad \leq K_1 \int_0^t \left\| u\left(\tau\right) - u^n\left(\tau\right) \right\|_\delta \left(1 + \left\| u\left(\tau\right) \right\|_\delta + \left\| u^n\left(\tau\right) \right\|_\delta \right) d\tau \\
&\quad \leq K_2\left(T\right) \int_0^t \left\| u\left(\tau\right) - u^n\left(\tau\right) \right\|_\delta d\tau \to 0 \text{ for } t \leq T.
\end{aligned}$$

Thus, (24) is proved and $u(\cdot) \in C^1([0,\infty), \ell^1_\delta)$ follows.
Let now $\widetilde{u}\left(\cdot\right)$ satisfy properties (1)–(3) and $\widetilde{u}\left(0\right) = u^0 \in E^+_\delta$. Since $u\left(\cdot\right) = S_\delta\left(\cdot, u^0\right)$ also satisfies (1)–(3), we have

$$\begin{aligned}
\left\| \widetilde{u}\left(t\right) - u\left(t\right) \right\|_{\ell^1_\delta} &\leq \int_0^t \left\| \overline{F}\left(\widetilde{u}\left(\tau\right)\right) - \overline{F}\left(u\left(\tau\right)\right) \right\|_{\ell^1_\delta} d\tau \\
&\leq K_2\left(T\right) \int_0^t \left\| \widetilde{u}\left(\tau\right) - u\left(\tau\right) \right\|_\delta d\tau.
\end{aligned}$$

The Gronwall lemma implies that $\widetilde{u}\left(\cdot\right) = u\left(\cdot\right)$. □

3. **Existence of the global attractor.** In this section we will prove the existence and topological properties of the global attractor for the semigroup S_δ. For this aim, we first recall some well–known results of the general theory of attractors for semigroups of operators in metric spaces.
Let $S : \mathbb{R}^+ \times X \to X$ be a semigroup in the complete metric space X with metric ρ. The set $B_0 \subset X$ is called *absorbing* for the semigroup S if for any bounded set B there is a time $T\left(B\right) > 0$ such that $S\left(t, B\right) \subset B_0$ for any $t \geq T$.
The semigroup S is *asymptotically compact* if for any bounded set B such that $\cup_{t \geq T(B)} S(t, B)$ is bounded for some $T(B)$, any arbitrary sequence $y_n \in S\left(t_n, B\right)$, where $t_n \to \infty$, is relatively compact.
Recall that $dist(C, B) = \sup_{x \in C} \inf_{y \in B} \rho(x - y)$ is the Hausdorff semi-distance from the set C to the set B.
The set $\mathcal{A}$ is called a *global attractor* of S if it is invariant ($S(t, \mathcal{A}) = \mathcal{A}$ for any $t \geq 0$) and attracts any bounded set B, that is, $dist\left(S(t, B), \mathcal{A}\right) \to 0$ as $t \to \infty$.
The function $x\left(\cdot\right) : \mathbb{R} \to X$ is said to be a *complete trajectory* of S if $x\left(t+s\right) = S\left(t, x\left(s\right)\right)$ for any $s \in \mathbb{R}$, $t \geq 0$. A complete trajectory is said to be bounded if $\cup_{s \in \mathbb{R}} x\left(s\right)$ is a bounded set.
We state a well–known result about the existence and properties of global attractors.

Theorem 7. *([29] and [22]) Let $X \ni x \mapsto S(t, x)$ be continuous for any $t \geq 0$. Assume that S is asymptotically compact and possesses a bounded absorbing set B_0. Then there exists a global compact attractor $\mathcal{A}$, which is the minimal closed set attracting any bounded set. The attractor $\mathcal{A}$ is the union of all bounded complete trajectories of S. If, moreover, the space X is connected and the map $t \mapsto S\left(t, x\right)$ is continuous for any $x \in X$, then the set $\mathcal{A}$ is connected.*

Therefore, in order to prove the existence of the global attractor we need to obtain a bounded absorbing set and the asymptotic compactness of the semigroup S_δ in the complete metric space $X = E^+_\delta$.

Lemma 8. *For any initial data $u^0 \in E_\delta^+$, it follows that*

$$\left\|S_\delta\left(t,u^0\right)\right\|_\delta^2 \le \left\|u^0\right\|_\delta^2 e^{-t} + K_0 \text{ for all } t\ge 0, \tag{25}$$

where $K_0 = K_0(\delta,r,b)$ is a constant. Thus, the ball

$$B_0 = \{v\in\ell_\delta^2 : \|v\|_\delta \le \sqrt{1+K_0}\}$$

is an absorbing bounded set and the set $\gamma_0^+(B) = \cup_{t\ge0}S_\delta(t,B)$ is bounded in E_δ^+ for any bounded set $B\subset E_\delta^+$.

Proof. First, let $u^0\in E^+$ and $u(t) = S(t,u^0) = S_\delta(t,u^0)$. We multiply the equation in (4) by $u(t)$ in the space ℓ_δ^2. Hence, by (17) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_\delta^2 &= -\left(A(u(t)),u(t)\right)_\delta + \frac{r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\left(bu_i^2(t) - u_i^3(t)\right)\\ &\le -\frac{1}{2}\left\|\overline{B}u(t)\right\|_\delta^2 + \left(\frac{27}{2}\delta^2 + r\right)\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^2(t) - \frac{r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t).\end{aligned}$$

By Young's inequality $ab\le \varepsilon a^p + C_\varepsilon b^q$ with $p=\frac{3}{2}$, $q=3$ and $\varepsilon = \frac{r}{2b\left(\frac{27}{2}\delta^2+r+\frac{1}{2}\right)}$, we obtain

$$\begin{aligned}\left(\frac{27}{2}\delta^2 + r + \frac{1}{2}\right)\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^2(t) &\le \frac{r}{2b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t) \\ &\quad + \left(\frac{27}{2}\delta^2 + r + \frac{1}{2}\right)C_\varepsilon\sum_{i\in\mathbb{Z}} g_\delta(i).\end{aligned} \tag{26}$$

Denote $K_0 = K_0(\delta,r,b) = 2\left(\frac{27}{2}\delta^2 + r + \frac{1}{2}\right)C_\varepsilon\sum_{i\in\mathbb{Z}} g_\delta(i)$. Then

$$\frac{d}{dt}\|u\|_\delta^2 + \|u(t)\|_\delta^2 + \frac{r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t) \le K_0 \tag{27}$$

and by from Gronwall's lemma

$$\|u(t)\|_\delta^2 \le e^{-t}\|u(0)\|_\delta^2 + K_0\left(1-e^{-t}\right),$$

so that (25).
If $u^0\in E_\delta^+$, then we take a sequence $u_n^0\to u^0$ in ℓ_δ^2, $u_n^0\in E^+$. Since (22) implies that $S_\delta(t,u_n^0)\to S_\delta(t,u^0)$ in ℓ_δ^2, we obtain that (25) holds. □

To prove the asymptotic compactness we need some estimates of the tails. Let us take a smooth function $\theta(s)$ satisfying $0\le\theta(s)\le 1$, for $s\ge 0$, and

$$\begin{aligned}\theta(s) &= 0, \text{ if } 0\le s\le 1,\\ \theta(s) &= 1, \text{ if } s\ge 2.\end{aligned}$$

Lemma 9. *For any bounded subset $B\subset E_\delta^+$ and $\varepsilon>0$, there exist $K(\varepsilon,B)>0$ and $T(\varepsilon,B)>0$ such that*

$$\sum_{|i|\ge 2K} g_\delta(i)\left(S_\delta\left(t,u^0\right)\right)_i^2 \le \varepsilon \text{ if } t\ge T, \tag{28}$$

for any $u^0\in B$.

Proof. First, let $u^0 \in E^+$ and $u(t) = S(t,u^0) = S_\delta(t,u^0)$. We multiply the equation in (4) by $(v_i(t))_{i\in\mathbb{Z}} = \left(\theta\left(\frac{|i|}{k}\right)u_i(t)\right)_{i\in\mathbb{Z}}$ in the space ℓ^2_δ. Then

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) + \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,(A(u(t)))_i\,u_i(t) = \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,(bu_i^2(t) - u_i^3(t)). \tag{29}$$

For the second term we use the following estimate for an arbitrary $u \in \ell^2_\delta$:

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,(Au)_i\,u_i \\
&= \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\left(-u_{i+1}u_i + 2u_i^2 - u_{i-1}u_i\right) \\
&\geq -\frac{1}{2}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_{i+1}^2 - \frac{1}{2}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_{i-1}^2 \qquad (30)\\
&\quad + \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2 \\
&\geq -2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2 + \frac{3}{2}\sum_{i\in\mathbb{Z}}\left(\theta\left(\frac{|i|}{k}\right) - \theta\left(\frac{|i-1|}{k}\right)\right)g_\delta(i)\,u_i^2 \\
&\quad - \frac{3}{2}\sum_{i\in\mathbb{Z}}\left(\theta\left(\frac{|i+1|}{k}\right) - \theta\left(\frac{|i|}{k}\right)\right)g_\delta(i)\,u_i^2 \\
&\geq -2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2 - \frac{3}{2k}\sum_{i\in\mathbb{Z}}\left(|\theta'(\xi_i)| + |\theta'(\xi_{i+1})|\right)g_\delta(i)\,u_i^2 \\
&\geq -2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2 - \frac{3C_1}{k}\|u\|_\delta^2, \qquad (31)
\end{aligned}$$

where we have used (14) and $|\theta'(s)| \leq C_1$ for all s. Using (31) in (29) we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) &\leq (2+r)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) \\
&\quad - \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^3(t) + \frac{3C_1}{k}\|u(t)\|_\delta^2.
\end{aligned}$$

Arguing as in (26) we obtain

$$\begin{aligned}
\left(2 + r + \frac{1}{2}\right)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) &\leq \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^3(t) \\
&\quad + D_{r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i),
\end{aligned}$$

where $D_{r,b}$ is a positive constant depending on r and b. Also, by (25) there exists $C(B)$ such that $3C_1\|u(t)\|_\delta^2 \leq C(B)$ for all $t \geq 0$. Thus, we have

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) + \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^2(t) \leq 2D_{r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i) + \frac{2C(B)}{k}.$$

Thus, for any $\varepsilon > 0$, there exists $K(\varepsilon, B)$ such that

$$2D_{r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_{\delta}(i) \leq \frac{\varepsilon}{4},$$

$$\frac{2C(B)}{k} \leq \frac{\varepsilon}{4}, \text{ if } k \geq K,$$

The Gronwall lemma implies now that

$$\begin{aligned}\sum_{|i|\geq 2K} g_{\delta}(i)\, u_i^2(t) &\leq \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{K}\right)g_{\delta}(i)\, u_i^2(t)\\ &\leq e^{-t}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{K}\right)g_{\delta}(i)\, u_i^2(0) + \frac{\varepsilon}{2}\\ &\leq \varepsilon,\end{aligned}$$

if $t \geq T(\varepsilon, B)$.
Finally, if $u^0 \in E_{\delta}^{+}$, we take a sequence $u_n^0 \to u^0$ in ℓ_{δ}^2, $u_n^0 \in E^{+}$. Since (22) implies that $S_{\delta}(t, u_n^0) \to S_{\delta}(t, u^0)$ in ℓ_{δ}^2, we obtain that (28) holds. □

Lemma 10. *The semigroup S_{δ} is asymptotically compact.*

Proof. Let $\xi^n \in S_{\delta}(t_n, B)$, where $t_n \to \infty$ and B is a bounded set. In view of Lemma 8, the sequence $\{\xi^n\}$ is bounded. Then, passing to a subsequence, $\xi^n \to \xi$ weakly in ℓ_{δ}^2. Lemma 9 implies that, for any $\varepsilon > 0$, there exist $K(\varepsilon, B)$ and $N(\varepsilon, B)$ such that

$$\sum_{|i|\geq 2K} g_{\delta}(i)(\xi_i)^2 \leq \varepsilon,$$

$$\sum_{|i|\geq 2K} g_{\delta}(i)(\xi_i^n)^2 \leq \varepsilon \text{ if } n \geq N.$$

Then, a standard argument implies that $\xi^n \to \xi$ strongly in ℓ_{δ}^2, and the lemma is proved. □

On account of Lemma 8, Lemma 10 and Theorem 7 we obtain the following.

Theorem 11. *The semigroup S_{δ} possesses a global compact connected attractor $\mathcal{A}$, which is the union of all bounded complete trajectories of S_{δ}.*

4. **Regularity of solutions and the attractor.** Further, we shall prove some regularity properties of the solutions and the attractor in the space ℓ_{δ}^p.

Theorem 12. *For any $u^0 \in E_{\delta}^{+}$, the solution $u(\cdot) = S_{\delta}(\cdot, u^0)$ satisfies*

$$\|u(t+\tau)\|_{\ell_{\delta}^p}^p \leq K_p\left(\frac{1}{\tau^{p-2}}\left(1 + \|u^0\|_{\delta}^2\right) + 1\right), \text{ for all } t \geq 0, \tag{32}$$

for some constant $K_p > 0$, where $p \geq 2, p \in \mathbb{N}$, and $0 < \tau \leq 1$ are arbitrary. Also, $u(\cdot) \in C^1((0,\infty), \ell_{\delta}^p)$ for all $p \geq 2$.

Proof. For $p=2$, inequality (32) was proved in Lemma 8. We will prove it for $p\geq 3$, $p\in\mathbb{N}$.
Let $u^0\in E^+$. It follows from (27) that

$$\int_s^t \|u(x)\|^3_{\ell^3_\delta}\,dx \leq K_1(t-s)+\|u(s)\|^2_\delta\,, \tag{33}$$

for some $K_1>0$. We note that $u(\cdot)\in C^1\left([0,\infty),\ell^2\right)\subset C^1\left([0,\infty),\ell^p_\delta\right)$, for all $p\geq 1$, so that we can multiply the equation in (4) by $u^2(\cdot)$ in ℓ^2_δ. Then

$$\frac{1}{3}\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t)+\sum_{i\in\mathbb{Z}} g_\delta(i)\left(Au(t)\right)_i u_i^2(t)=r\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t)-\frac{r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^4(t)\,.$$

Now, for any $u\in\ell^3_\delta$, from the Young inequality and (14) we can deduce

$$\begin{aligned}\sum_{i\in\mathbb{Z}} g_\delta(i)\left(Au_i\right)u_i^2 &= \sum_{i\in\mathbb{Z}} g_\delta(i)\left(-u_{i+1}u_i^2+2u_i^3-u_{i-1}u_i^2\right)\\ &\geq -\frac{1}{3}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_{i+1}^3-\frac{1}{3}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_{i-1}^3+\frac{2}{3}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3\\ &\geq -2\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3.\end{aligned}$$

Then

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t)+\frac{3r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^4(t)\leq 3(r+2)\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t)\,.$$

Also, by applying again the Young inequality $ab\leq\varepsilon a^p+C_\varepsilon b^q$ with $p=\frac{4}{3}$, $q=4$ and $\varepsilon=\frac{r}{b3(r+2)}$ we obtain

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^3(t)+\frac{2r}{b}\sum_{i\in\mathbb{Z}} g_\delta(i)\,u_i^4(t)\leq C_{r,b}\sum_{i\in\mathbb{Z}} g_\delta(i)=K_2, \tag{34}$$

for some $K_2>0$. For $0<\tau\leq 1$, let $0\leq t\leq s\leq t+\tau$. Integrating the last inequality over $(s,t+\tau)$, it follows

$$\|u(t+\tau)\|^3_{\ell^3_\delta}\leq\|u(s)\|^3_{\ell^3_\delta}+K_2\tau.$$

Integrating now over $(t,t+\tau)$, and using (33) and (25), we obtain

$$\begin{aligned}\|u(t+\tau)\|^3_{\ell^3_\delta} &\leq \frac{1}{\tau}\left(K_1\tau+K_2\tau^2+\|u(s)\|^2_\delta\right)\\ &\leq K_3\left(\frac{1}{\tau}\left(1+\left\|u^0\right\|^2_\delta\right)+1\right),\ \forall t\geq 0,\end{aligned} \tag{35}$$

for some constant $K_3>0$. After another integration in (34) over $(t+\tau,t+2\tau)$, and using (35), we have

$$\begin{aligned}\int_{t+\tau}^{t+2\tau}\|u(s)\|^4_{l^4_\delta}\,ds &\leq \frac{b}{2r}\left(\|u(t+\tau)\|^3_{\ell^3_\delta}+K_2\tau\right)\\ &\leq \frac{bK_3}{2r}\left(\frac{1}{\tau}\left(1+\left\|u^0\right\|^2_\delta\right)+1\right)+\frac{bK_2}{2r}\tau\\ &\leq \overline{K}_3\left(\frac{1}{\tau}\left(1+\left\|u^0\right\|^2_\delta\right)+1\right).\end{aligned} \tag{36}$$

Let $p \geq 4$, $p \in \mathbb{N}$. Assume that

$$\|u\,(t+(p-3)\,\tau)\|_{\ell_\delta^{p-1}}^{p-1} \leq K_{p-1}\left(\frac{1}{\tau^{p-3}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right), \tag{37}$$

$$\int_{t+(p-3)\tau}^{t+(p-2)\tau} \|u\,(s)\|_{\ell_\delta^p}^p\, ds \leq \overline{K}_{p-1}\left(\frac{1}{\tau^{p-3}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right),\ \forall t \geq 0, \tag{38}$$

as the induction hypothesis, which is satisfied for $p = 4$. We will prove that (37)-(38) holds if we substitute p by $p+1$ for all natural numbers $p \geq 4$.
We multiply the equation in (4) by $u^{p-1}\,(\cdot)$ in ℓ_δ^2. Then

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p\,(t) + \sum_{i\in\mathbb{Z}} g_\delta\,(i)\,(Au\,(t))_i\, u_i^{p-1}\,(t) &= r\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p\,(t) \\ &\quad - \frac{r}{b}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^{p+1}\,(t)\,.\end{aligned}$$

First, for any $u \in \ell_\delta^p$, by the Young inequality once more and (14), we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\,(Au_i)\, u_i^{p-1} &= \sum_{i\in\mathbb{Z}} g_\delta\,(i)\left(-u_{i+1}u_i^{p-1}+2u_i^p-u_{i-1}u_i^{p-1}\right) \\ &\geq -\frac{1}{p}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_{i+1}^p - \frac{1}{p}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_{i-1}^p + \frac{2}{p}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p \\ &\geq -\frac{4}{p}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p. \end{aligned} \tag{39}$$

Then

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p\,(t) + \frac{pr}{b}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^{p+1}\,(t) \leq p\left(r+\frac{4}{p}\right)\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p\,(t)\,.$$

Also, by Young's inequality $ab \leq \varepsilon a^p + C_\varepsilon b^q$ with $p = \frac{p+1}{p}$, $q = p+1$ and $\varepsilon = \frac{r}{bp\left(r+\frac{4}{p}\right)}$ we obtain

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^p\,(t) + \frac{(p-1)r}{b}\sum_{i\in\mathbb{Z}} g_\delta\,(i)\, u_i^{p+1}\,(t) \leq C_{r,b,p}\sum_{i\in\mathbb{Z}} g_\delta\,(i) = \widetilde{K}_p. \tag{40}$$

Let $t+(p-3)\,\tau \leq s \leq t+(p-2)\,\tau$. Integrating the last inequality over $(s, t+(p-2)\,\tau)$ we have

$$\|u\,(t+(p-2)\,\tau)\|_{\ell_\delta^p}^p \leq \|u\,(s)\|_{\ell_\delta^p}^p + \widetilde{K}_p\tau.$$

Integrating now over $(t+(p-3)\,\tau, t+(p-2)\,\tau)$, and using (37) and (38), we arrive at

$$\begin{aligned}\|u\,(t+(p-2)\,\tau)\|_{\ell_\delta^p}^p &\leq \overline{K}_{p-1}\frac{1}{\tau}\left(\frac{1}{\tau^{p-3}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right)+\widetilde{K}_p \\ &\leq K_p\left(\frac{1}{\tau^{p-2}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right),\ \forall t \geq 0,\end{aligned} \tag{41}$$

for some constant $K_p > 0$. Also, integrating in (40) over $(t+(p-2)\tau, t+(p-1)\tau)$, and using (41), we deduce

$$\begin{aligned}\int_{t+(p-2)\tau}^{t+(p-1)\tau} \|u(s)\|_{\ell_\delta^{p+1}}^{p+1} ds &\le \frac{b}{(p-1)r}\left(\|u(t+(p-2)\tau)\|_{l_\delta^p}^p + \widetilde{K}_p\tau\right) \\ &\le \frac{bK_p}{(p-1)r}\left(\frac{1}{\tau^{p-2}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right)+\frac{b\widetilde{K}_p}{(p-1)r}\tau \\ &\le \overline{K}_p\left(\frac{1}{\tau^{p-2}}\left(1+\left\|u^0\right\|_\delta^2\right)+1\right).\end{aligned} \tag{42}$$

Thus, (41) is satisfied for all $p \ge 3$, $p \in \mathbb{N}$. Subtituting τ by $\frac{\tau'}{p-2}$ we obtain (32).
Now, let $u^0 \in E_\delta^+$ and $u^{0,n} \to u^0$ in ℓ_δ^2, where $u^{0,n} \in E^+$. Since $u^n(t) = S_\delta(t, u_n^0) \to S_\delta(t, u^0)$ in ℓ_δ^2, (41) implies that $u^n(t) \to u(t)$ weakly in ℓ_δ^p for any $t > 0$. Thus, (32) follows.
Finally, we will verify that $u(\cdot) \in C^1((0,\infty), \ell_\delta^p)$. Let $[\varepsilon, T] \subset (0,\infty)$ be arbitrary. In view of (32), we have that $u^n(\cdot)$ is bounded in $L^\infty(\varepsilon, T; \ell_\delta^q)$ for all $q \ge 2$. When $q \notin \mathbb{N}$, this follows by taking $\overline{q} \ge q, \overline{q} \in \mathbb{N}$, and using the continuous embedding $\ell_\delta^{\overline{q}} \subset \ell_\delta^q$. Also,

$$\left\|\frac{du^n}{dt}(t)\right\|_{\ell_\delta^p} = \left\|\overline{F}(u^n(t))\right\|_{\ell_\delta^p} \le r\|u^n(t)\|_{\ell_\delta^p} + \frac{r}{b}\|u^n(t)\|_{\ell_\delta^{2p}}^2,$$

so that $\frac{du^n}{dt}$ is bounded in $L^\infty(\varepsilon, T; \ell_\delta^p)$ as well. Then $u^n \to u$, $\frac{du^n}{dt} \to \frac{du}{dt}$ weakly star in $L^\infty(\varepsilon, T; \ell_\delta^p)$, and $u(\cdot) \in W^{1,\infty}(\varepsilon, T; \ell_\delta^p) \subset C([r,T], \ell_\delta^p)$ for all $p \ge 2$. Since the map $v \mapsto \overline{F}(v)$ is continuous from ℓ_δ^{2p} onto ℓ_δ^p (this can be proved similarly as in Lemma 2), $\frac{du}{dt} = \overline{F}(u(\cdot)) \in C([r,T], \ell_\delta^p)$. Hence, $u(\cdot) \in C^1((0,\infty), \ell_\delta^p)$. □

Corollary 13. *The global attractor $\mathcal{A}$ given in Theorem 11 is bounded in ℓ_δ^p for any $p \ge 2$. Hence, it is the union of all complete trajectories of S_δ which are bounded in ℓ_δ^p.*

Proof. First, let $p \ge 2, p \in \mathbb{N}$. Let $y \in \mathcal{A}$. Since $\mathcal{A} = \mathcal{S}_\delta(1, \mathcal{A})$, there exists $z \in \mathcal{A}$ such that $y = u(1)$ and $u(\cdot) = S_\delta(\cdot, z)$. In view of (32) with $\tau = 1$ and $t = 0$ we have

$$\|u(1)\|_{\ell_\delta^p}^p \le K_p\left(\left(1+\|z\|_\delta^2\right)+1\right) \le C,$$

since $\mathcal{A}$ is bounded in ℓ_δ^2.
Now, if $p \ge 2$ is arbitrary, then we take $\overline{p} \ge p, \overline{p} \in \mathbb{N}$, and the boundedness of $\mathcal{A}$ follows from the continuous embedding $\ell_\delta^{\overline{p}} \subset \ell_\delta^p$. □

We can prove in fact that $\mathcal{A}$ is compact in ℓ_δ^p. To this end, we will obtain an estimate of the tails in the space ℓ_δ^p.

Lemma 14. *For any bounded set $B \subset E_\delta^+$, $p \ge 2$ and $\varepsilon > 0$, there exists $R(\varepsilon, B, p) > 0$ and $T(\varepsilon, B, p) > 0$ such that*

$$\sum_{|i| \ge 2R} g_\delta(i)\left(S_\delta\left(t, u^0\right)\right)_i^p \le \varepsilon \text{ if } t \ge T, \tag{43}$$

for any $u^0 \in B$.

Proof. Let $u^0 \in B$ and $u(t) = S_\delta(t, u^0)$. We multiply the equation in (4) by $(v_i(t))_{i\in\mathbb{Z}} = \left(\theta\left(\frac{|i|}{k}\right)u_i^{p-1}(t)\right)_{i\in\mathbb{Z}}$ in the space ℓ_δ^2. Then

$$\frac{1}{p}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t) + \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\left(A(u(t))\right)_i u_i^{p-1}(t) \tag{44}$$
$$= \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\left(bu_i^p(t) - u_i^{p+1}(t)\right).$$

Since, by Theorem 12, we know that $u(\cdot) \in C^1\left((0,\infty), \ell_\delta^q\right)$ for all $q \geq 2$, these computations are correct for $t > 0$.
We note that, using (14) and $|\theta'(s)| \leq C_1$, for all s, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\left(Au_i\right)u_i^{p-1}\\
&\quad= \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\left(-u_{i+1}u_i^{p-1} + 2u_i^p - u_{i-1}u_i^{p-1}\right)\\
&\quad\geq -\frac{1}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_{i+1}^p - \frac{1}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_{i-1}^p + \frac{2}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p\\
&\quad\geq -\frac{4}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p + \frac{3}{p}\sum_{i\in\mathbb{Z}}\left(\theta\left(\frac{|i|}{k}\right) - \theta\left(\frac{|i-1|}{k}\right)\right)g_\delta(i)\,u_i^p\\
&\qquad\quad - \frac{3}{p}\sum_{i\in\mathbb{Z}}\left(\theta\left(\frac{|i+1|}{k}\right) - \theta\left(\frac{|i|}{k}\right)\right)g_\delta(i)\,u_i^p\\
&\quad\geq -\frac{4}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p - \frac{3}{pk}\sum_{i\in\mathbb{Z}}\left(|\theta'(\xi_i)| + |\theta'(\xi_{i+1})|\right)g_\delta(i)\,u_i^p\\
&\quad\geq -\frac{4}{p}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p - \frac{6C_1}{pk}\left\|u\right\|_{\ell_\delta^p}^p.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t) &\leq \left(r + \frac{4}{p}\right)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t)\\
&\quad - \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^{p+1}(t) + \frac{6C_1}{pk}\left\|u(t)\right\|_{\ell_\delta^p}^p.
\end{aligned}$$

By applying once again the Young inequality $ab \leq \varepsilon a^p + C_\varepsilon b^q$ with $p = \frac{p+1}{p}$, $q = p+1$ and $\varepsilon = \frac{r}{b\left(r+\frac{4}{p}+\frac{1}{p}\right)}$, it follows

$$\begin{aligned}
\left(r + \frac{4}{p} + \frac{1}{p}\right)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t) &\leq \frac{r}{b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^{p+1}(t)\\
&\quad + D_{p,r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i).
\end{aligned}$$

Also, (32) implies that there exists $C(B)$ such that $6C_1 \|u(t)\|_{\ell_\delta^p}^p \leq C(B)$ for all $t \geq 1$. Thus,

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t)+\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\,u_i^p(t)\leq pD_{p,r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i) + \frac{C(B)}{k},$$

if $t \geq 1$. For any $\varepsilon > 0$, there exists $R(\varepsilon, p)$ such that

$$pD_{\delta,r,b}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_\delta(i)\leq\frac{\varepsilon}{4},$$
$$\frac{C(B)}{k}\leq\frac{\varepsilon}{4},\text{ if } k\geq R.$$

Then, Theorem 12 and the Gronwall lemma imply

$$\begin{aligned}\sum_{|i|\geq 2R}g_\delta(i)\,u_i^p(t) &\leq \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{R}\right)g_\delta(i)\,u_i^p(t)\\ &\leq e^{-(t-1)}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{R}\right)g_\delta(i)\,u_i^p(1)+\frac{\varepsilon}{2}\\ &\leq \widetilde{C}(B)e^{-(t-1)}+\frac{\varepsilon}{2}\\ &\leq \varepsilon,\end{aligned}$$

if $t \geq T(\varepsilon, B, p)$. □

Theorem 15. *The global attractor $\mathcal{A}$ given in Theorem 11 is compact in ℓ_δ^p for any $p \geq 2$. Moreover, for any bounded subset $B \subset E_\delta^+$, we have*

$$dist_{\ell_\delta^p}\left(S_\delta(t,B),\mathcal{A}\right)\to 0 \text{ as } t\to+\infty, \tag{45}$$

where $dist_{\ell_\delta^p}$ denotes the Hausdorff semi-distance with respect to the metric in ℓ_δ^p.

Proof. Arguing by contradiction, if (45) did not hold, then there would exist $\varepsilon > 0$ and a sequence $\xi^n \in S_\delta(t_n, B)$, with $t_n \to +\infty$, such that

$$dist_{\ell_\delta^p}\left(\xi^n,\mathcal{A}\right)>\varepsilon. \tag{46}$$

By Theorem 12 and Corollary 13, $\{\xi^n\}$ and $\mathcal{A}$ are bounded in ℓ_δ^p. Since $dist(\xi^n, \mathcal{A}) \to 0$, we can assume that $\xi^n \to \xi \in \mathcal{A}$ in ℓ_δ^2. By Lemma 14, for any $\sigma > 0$, there exist $n_0(\sigma)$ and $K(\sigma)$ such that

$$\sum_{|i|\geq K}(\xi_i^n)^p\leq\sigma,\ \sum_{|i|\geq K}(\xi_i)^p\leq\sigma.$$

Then, reasoning in a standard way, one can check that $\xi^n \to \xi$ in ℓ_δ^p, which is a contradiction with (46).

In order to prove the compactness of $\mathcal{A}$ in ℓ_δ^p, we take an arbitrary sequence $\{\xi^n\} \subset \mathcal{A}$. Since $\xi^n \in S_\delta(t_n, \mathcal{A})$, where $t_n \to +\infty$, the same argument proves that $\xi^n \to \xi \in \mathcal{A}$ in ℓ_δ^p. Thus, $\mathcal{A}$ is compact in ℓ_δ^p. □

Finally, we will state some facts concerning the fixed points (equilibria) of the system (4).
The point $\overline{u} \in \ell^2$ is said to be a *fixed point* (also *stationary point* or *equilibrium*) for S_δ if $S_\delta(t, \overline{u}) = \overline{u}$ for any $t \geq 0$. Denote by Z the set of stationary points of S.

Lemma 16. *A point $\overline{u} \in E_\delta^+$ is a fixed point of S_δ if and only if*

$$\overline{F}(\overline{u}) = A\overline{u} + \overline{f}(\overline{u}) = 0 \text{ in } \ell_\delta^1. \tag{47}$$

Proof. It is obvious that if (47) holds, then $u(t) \equiv \overline{u}$ satisfies properties (1)–(3) in Theorem 6. Hence, $S_\delta(t, \overline{u}) = \overline{u}$ for all $t \geq 0$.
Conversely, if $S_\delta(t, \overline{u}) = \overline{u}$ for all $t \geq 0$, then, by Theorem 6, we have that $u(\cdot) = \overline{u}$ satisfies

$$\int_0^t \overline{F}(\overline{u}) d\tau = t\overline{F}(\overline{u}) = 0 \text{ for } t > 0.$$

Thus, (47) holds. □

Lemma 17. *The semigroup S_δ possesses, at least, the following fixed points:*

$$u^1 \equiv 0,$$
$$u^2 \equiv b.$$

We note that $u^1 \in E^+$, but $u^2 \notin E^+$.

Proof. It follows from $Au^j + \overline{f}\left(u^j\right) = 0$ and Lemma 16. □

Remark 18. *It is worth mentioning that there are some interesting open problems related to these fixed points. However, it seems that a more sophisticated analysis is needed in order to solve them. We plan to study those in the near future.*

1. *Are u^1, u^2 the only fixed points in the space ℓ_δ^2?*
2. *Are the points u^1, u^2 stable or unstable? The natural conjecture (inspired in the finite dimensional case) is that u^1 could be unstable, whereas u^2 stable.*

An answer to these questions would provide a very useful information about the structure of the global attractor.

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