

ALGORITHMIC INVARIANTS FOR ALEXANDER MODULES

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ABSTRACT. Let G be a group given by generators and relations. It is possible to compute a presentation matrix of a module over a ring through Fox's differential calculus. We show how to use Gröbner bases as an algorithmic tool to compare the chains of elementary ideals defined by the matrix. We apply this technique to classical examples of groups and to compute the elementary ideals of Alexander matrix of knots up to 11 crossings with the same Alexander polynomial.

1. INTRODUCTION

Let $G = \langle \mathbf{x} : \mathbf{r} \rangle$ be a group given by generators and relations, where $\mathbf{x} = (x_1, \dots, x_n)$ is a base of the free group F and $\mathbf{r} = (r_1, \dots, r_m)$ are the relations. Through Fox's differential calculus [Crowell et al.(1977)] it is possible to compute the presentation matrix of the Alexander module of the group. We review briefly these concepts.

We build from G the ring of the group $\mathbb{Z}G$. A derivation over the group ring is a map $D : \mathbb{Z}G \rightarrow \mathbb{Z}G$ such that

$$\begin{aligned} D(\nu_1 + \nu_2) &= D\nu_1 + D\nu_2, \\ D(\nu_1\nu_2) &= (D\nu_1)\mathfrak{t}(\nu_2) + \nu_1 D\nu_2, \end{aligned}$$

where \mathfrak{t} is the trivializer and $\nu_1, \nu_2 \in \mathbb{Z}G$. For elements in G , the second condition is

$$D(g_1g_2) = Dg_1 + g_1Dg_2.$$

Then a derivation can be seen as the unique linear extension to $\mathbb{Z}G$ of a map $D : G \rightarrow \mathbb{Z}G$ that verifies the previous condition.

It is known that each generator x_j in the group G defines a unique derivation $D_j = \partial/\partial x_j$ in $\mathbb{Z}G$, such that

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

Let H be the abelianized group of G . Considering the group rings we have the composition of maps

$$\mathbb{Z}F \xrightarrow{D_j} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\mathbf{a}} \mathbb{Z}H,$$

where γ is the projection and \mathbf{a} is the abelianizer. The Alexander matrix from G is $A = (a_{ij})$, where

$$a_{ij} = \mathbf{a}\gamma\left(\frac{\partial r_j}{\partial x_i}\right).$$

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Note that this matrix is the transposed of the matrix defined by [Crowell et al.(1977)]. The Alexander matrix presents a module over the ring $\mathbb{Z}H$. If two groups are isomorphic then the modules are isomorphic.

A finite presentation for M is an exact sequence

$$R^n \xrightarrow{\alpha} R^m \xrightarrow{\Phi} M \rightarrow 0$$

where R^n and R^m are free R -modules with respective bases $\mathbf{f}_1, \dots, \mathbf{f}_n$ and $\mathbf{e}_1, \dots, \mathbf{e}_m$. If α is represented by the matrix A with respect to these bases then the $m \times n$ matrix A is a presentation matrix for M .

Theorem 1. [Lickorish(1998), Thm. 6.1] *If A_1 and A_2 are presentation matrices of a module M then they are related by a sequence of matrix transformations of the following form and their inverses:*

- (1) *Permutation of rows and columns.*
- (2) *Replacement of the matrix A_1 by $\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}$.*
- (3) *Addition of an extra column of zeros to the matrix A_1 .*
- (4) *Addition of a scalar multiple of a row (column) to another row (column).*

We say that A_1 and A_2 are Fitting equivalents.

Definition 1. Let M be a R module, with an $m \times n$ presentation matrix A . The r -th elementary ideal F_r of M is the ideal generated by all the $(m-r+1) \times (m-r+1)$ minors of A .

By convention, $F_r(M) = R$ when $r > m$ and $F_r(M) = 0$ if $r \leq 0$. They form an ascending chain $F_k(M) \subset F_{k+1}(M)$. The elementary ideals are independent of the presentation matrix chosen to evaluate them.

2. ALGORITHMS IN THE RING GROUP

The ring $\mathbb{Z}H$ is commutative, because H is an abelian group, and it has a special form.

Proposition 1. *The ring $\mathbb{Z}H$ is isomorphic to $\mathbb{Z}[x_1^\pm, \dots, x_n^\pm]/J$, where J is the ideal generated by the relations r_1, \dots, r_m under commutativity.*

Proof. Through the abelianizer, all the relations have the form $\prod x_i^{e_i} = 1$, so J is generated by the elements $\prod x_i^{e_i} - 1$. \square

Corollary 1. *There is an algorithm to compare ideals in $\mathbb{Z}H$.*

Proof. Through the bijection between ideals in $\mathbb{Z}[x_1^\pm, \dots, x_n^\pm]/J$ and ideals in $R = \mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$ that contains J , the problem is reduced to compare ideals in R . In this ring we can compute Gröbner bases [Sims(1994), Pauer et al.(1999)], or by the isomorphism $R \simeq \mathbb{Z}[x_1, \dots, x_n, w]/\langle x_1 \cdots x_n w - 1 \rangle$ [Adams et al.(1994)]. \square

There is no known algorithm to decide whether two matrices present isomorphic modules. There are other invariants as the ideal row (column) class [Fox et al.(1964)] or the Nakanishi index [Kawauchi(1996)]. However we do not know algorithms to compute them and *ad hoc* arguments are needed to give their values for specific matrices [Fox et al.(1964), Kearton et al.(2003)].

Example 1. Let consider the groups given by the presentations

$$D_8 = \langle x, y | x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle, Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle.$$

D_8 is the dihedral group of order 8 (symmetry group of the square) and Q_8 is the quaternion group. A classical exercise in group theory is to show that these two groups are not isomorphic. Let see how can this be accomplished with elementary ideals. Let r_i be the the relations in D_8 :

$$r_1 : x^4 = 1, r_2 : y^2 = 1, r_3 : yxy^{-1}x = 1.$$

Then

$$\begin{aligned} \frac{\partial r_1}{\partial x} &= 1 + x + x^2 + x^3, & \frac{\partial r_2}{\partial x} &= 0, & \frac{\partial r_3}{\partial x} &= y + x^{-1}, \\ \frac{\partial r_1}{\partial y} &= 0, & \frac{\partial r_2}{\partial y} &= 1 + y, & \frac{\partial r_3}{\partial y} &= 1 - x^{-1} \end{aligned}$$

In the abelianized group we add the relation $xy = yx$, so $x^2 = 1, y^2 = 1$. The Alexander module of the group has a presentation matrix

$$M(D_8) = \begin{pmatrix} 2 + 2x & 0 & x + y \\ 0 & y + 1 & 1 - x \end{pmatrix},$$

over the ring $\mathbb{Z}[x^\pm, y^\pm]/\langle x^2 - 1, y^2 - 1 \rangle$.

We proceed in an analogous way with Q_8 . We write the relations

$$s_1 : x^4 = 1, s_2 : x^2y^{-2} = 1, s_3 : xy^{-1}xy = 1,$$

and

$$\begin{aligned} \frac{\partial s_1}{\partial x} &= 1 + x + x^2 + x^3, & \frac{\partial s_2}{\partial x} &= 1 + x, & \frac{\partial s_3}{\partial x} &= 1 + xy^{-1}, \\ \frac{\partial s_1}{\partial y} &= 0, & \frac{\partial s_2}{\partial y} &= -x^2(1 + y), & \frac{\partial s_3}{\partial y} &= -xy^{-1} + y^{-1} \end{aligned}$$

As before, in the abelianized group the relations are reduced to $x^2 = 1, y^2 = 1$ and a presentation matrix of the Alexander module is

$$M(Q_8) = \begin{pmatrix} 2 + 2x & 1 + x & 1 + xy \\ 0 & -1 - y & -xy + y \end{pmatrix}$$

over the ring $\mathbb{Z}[x^\pm, y^\pm]/\langle x^2 - 1, y^2 - 1 \rangle$.

We compute a Gröbner basis in $\mathbb{Z}[x^\pm, y^\pm]$ of the second elementary ideal. Adding the polynomials $x^2 - 1, y^2 - 1$, we get

$$F_2(M(D_8)) = \langle 4, 1 + y, 1 - x \rangle, F_2(M(Q_8)) = \langle 2, 1 + x, 1 + y \rangle.$$

They are different so the groups are not isomorphic.

Example 2. In [Kanenobu(1986)] it is defined a class of knots $K_{p,q}$, with $p, q \in \mathbb{N}$, that has the Alexander matrix

$$A_{p,q} = \begin{pmatrix} t^2 - 3t + 1 & (p - q)t \\ 0 & t^2 - 3t + 1 \end{pmatrix}.$$

Lemma 2 of [Kanenobu(1986)] asserts that $K_{p,q}$ and $K_{p',q'}$ have isomorphic Alexander modules if and only if $|p - q| = |p' - q'|$. Let us show how to apply our approach to give a new proof of this lemma. If the modules are isomorphic then the second elementary ideals F_2 must coincide. A Gröbner basis of the ideal is equal to $\{t^2 - 3t + 1, p - q\}$, so $F_2(A_{p,q}) = F_2(A_{p',q'})$ if and only if $|p - q| = |p' - q'|$.

3. AN APPLICATION TO KNOT THEORY

One of the main invariants in knot theory is the fundamental group of the knot complement. The Alexander matrix can be computed from the Seifert matrix, and with the tables listed in [Burde et al.(1985)] and [Livingston(2004)] we can get the Alexander matrix of knots up to 11 crossings. As an application of the algorithm described before we give a list of knots with the same Alexander polynomial (grouped by boxes) and where the elementary ideals give more information to distinguish knots (see Table 1). For example, from Table 1 we deduce that $11a_{102}$ and $11a_{181}$ are different, but we cannot say anything about $11a_{102}$ and $11a_{199}$.

The first step was to compute the Alexander matrix of the knot and reduce it through the transformations given by Theorem 1. In all cases we have got at most a 2×2 presentation matrix, so F_3 is always equal to R . The Gröbner bases were computed over the ring $\mathbb{Z}[t, w]$, adjoining to the ideals the polynomial $tw - 1$.

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TABLE 1. Elementary ideals

Name	Elem. ideal F_2	Name	Elem. ideal F_2	Name	Elem. ideal F_2
11n ₁₀₀ 9 ₃₇	R $\langle 3, t - 2 \rangle$	10 ₁₄₀ 11n ₇₃ 11n ₇₄	$\langle 2, t^2 - t + 1 \rangle$ $\langle t^2 - t + 1 \rangle$ $\langle t^2 - t + 1 \rangle$	10 ₆₀ 11n ₁₆₅	R $\langle 2, t^2 + t + 1 \rangle$
11n ₉₇ 6 ₁ 9 ₄₆	R R $\langle 3, t + 1 \rangle$	11n ₁₁₆ 11n ₄₉	R $\langle 2, t^2 + t + 1 \rangle$	11a ₂₂₃ 11n ₁₄₈	R $\langle 5, t^2 + 2t + 1 \rangle$
11a ₁₀₂ 11a ₁₈₁ 11a ₁₉₉	R $\langle 3, t - 2 \rangle$ R	11n ₁ 9 ₄₈	R $\langle 3, t + 1 \rangle$	11a ₁₀₈ 11a ₁₃₉ 11a ₂₃₁ 11a ₅₇ 11a ₈₈	R R $\langle t^2 - t + 1 \rangle$ $\langle t^2 - t + 1 \rangle$ R
10 ₁₁₃ 11a ₁₀₇ 11a ₃₄₇	R $\langle 2, t^2 + t + 1 \rangle$ $\langle 2, t^2 + t + 1 \rangle$	10 ₁₅₅ 11n ₃₇ 8 ₉	$\langle t + 1, 5 \rangle$ R R	11a ₁₀₉ 11a ₄₄ 11a ₄₇	R $\langle t^2 - t + 1 \rangle$ $\langle t^2 - t + 1 \rangle$
11a ₁₈₇ 11a ₂₄₉ 11a ₃₈ 11a ₈	R $\langle 3, t - 2 \rangle$ R R	11n ₁₆₄ 11n ₈₅ 8 ₁₈ 9 ₂₄	$\langle t^2 - t + 1 \rangle$ R $\langle t^2 - t + 1 \rangle$ R	10 ₁₂₃ 11a ₂₈	$\langle t^4 - 3t^3 + 3t^2 - 3t + 1 \rangle$ R
11a ₁₃₂ 11a ₃₅₂ 11a ₆	$\langle 2, t^2 + t + 1 \rangle$ $\langle 3, t^2 - t + 1 \rangle$ R	10 ₁₆₃ 11n ₈₇ 9 ₂₈ 9 ₂₉	$\langle 2, t^2 + t + 1 \rangle$ R R R	10 ₈₇ 10 ₉₈ 11a ₁₆₅ 11a ₅₈ 11n ₇₂	R $\langle 1 - t + t^2 \rangle$ $\langle 2, 1 - t + t^2 \rangle$ R $\langle 1 - t + t^2 \rangle$
10 ₆₅ 10 ₇₇ 11n ₇₁ 11n ₇₅	$\langle 2, t^2 + t + 1 \rangle$ R $\langle -t^2 + t - 1 \rangle$ $\langle -t^2 + t - 1 \rangle$	10 ₅₉ 11n ₆₆ 9 ₄₀	R R $\langle t^2 - 3t + 1 \rangle$	10 ₁₄₄ 11n ₉₉	$\langle 2, 1 - t + t^2 \rangle$ R
10 ₁₀₃ 10 ₄₀	$\langle 5, t + 1 \rangle$ R	10 ₄₂ 10 ₇₅	R $\langle 3, t + 1 \rangle$	11n ₈₃ 9 ₄₁	$\langle 2, t^2 + t + 1 \rangle$ $\langle 7, 1 + t \rangle$

Name	Elem. ideal F_2
11n ₁₆₂ 9 ₃₉	$\langle 2, t^2 + t + 1 \rangle$ R
11a ₃₁ 11a ₃₁₇	R $\langle t + 1, 5 \rangle$
10 ₆₇ 10 ₇₄ 11n ₆₈	R $\langle 3, t + 1 \rangle$ R
11a ₁₅₇ 11a ₂₆₄ 11a ₃₀₅ 11a ₈₀	$\langle 2, t^4 + t^2 + 1 \rangle$ R R R
10 ₆₃ 9 ₃₈	$\langle 2, t^2 + t + 1 \rangle$ R
11a ₂₇₇ 11a ₉₉	$\langle t + 1, 3 \rangle$ R
11a ₁₉₆ 11a ₂₁₆ 11a ₂₈₆	$\langle 7, t + 1 \rangle$ R R