

## LINKS AMONG CHARACTERISTICALLY NILPOTENT, $C$ -GRADED AND DERIVED FILIFORM LIE ALGEBRAS

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**ABSTRACT.** The aim of this paper is to study the interconnectivity among Lie algebras that are characteristically nilpotent, derived and  $c$ -graded filiform, establishing some characterization theorems involving them.

At the same time we offer an algorithm that will allow us to know, given a complex filiform Lie algebra, which of those properties hold.

**Introduction.** Although there is a lot of research on Lie algebras, it often studies those that are characteristically nilpotent, derived or  $c$ -graded filiform in isolation and characterization theorems do not usually consider possible connections among these kinds of Lie algebras. In an attempt to fill this gap this paper tries to give some characterization theorems of characteristically nilpotent Lie algebras involving derived Lie algebras or  $c$ -graded filiform Lie algebras. Indeed, the main results obtained are Theorems 3.2 and 4.1.

We also show an algorithmic procedure to check if, given a complex filiform Lie algebra, it is, under certain conditions, characteristically nilpotent, a derived Lie algebra or a  $c$ -graded filiform Lie algebra.

The structure of this paper is the following. In Section 1 we briefly show the historical evolution of these three kinds of Lie algebras, since they were first introduced. These historical notes could explain how the main difficulties in the study of these Lie algebras appeared and how they were solved. Section 2 recalls the definitions and main properties that will be used in the following sections of the paper. Section 3, on the other hand, is devoted to the study of the connection between characteristically nilpotent filiform Lie algebras and derived Lie algebras. First we give two nilpotent Lie algebras (those of the lowest possible dimension) which are neither derived nor

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characteristically nilpotent. Both algebras show that the union of those two classes is not dense in the space of nilpotent Lie algebras and thus, there is no connection between characteristically nilpotent Lie algebras in general and derived Lie algebras, since neither class is contained in the other. Next, we restrict this problem to the particular case of characteristically nilpotent Lie algebras which are also filiform and we give Theorem 3.2. In Section 4 we study the link between characteristically nilpotent Lie algebras and  $c$ -graded filiform Lie algebras. Next we consider in this part the particular case of Lie algebras which are both characteristically nilpotent and filiform. The main result of this section is Theorem 4.1. Finally, Section 5 shows the algorithmic procedure previously mentioned, which allows us to know if a given and fixed complex filiform Lie algebra is characteristically nilpotent, derived or  $c$ -graded filiform Lie algebra, under determined conditions.

**1. Historic notes.** Jacobson proved in 1955, see [11], that a Lie algebra in characteristic 0 that has a nonsingular derivation must be nilpotent, and went on to ask whether the converse held, namely, must a nilpotent Lie algebra  $\mathfrak{g}$  have a nonsingular derivation  $\mathbf{D}$ ? Note that such a Lie algebra  $\mathfrak{g}$  must be a derived algebra since  $[\mathbf{D} + \mathfrak{g}, \mathbf{D} + \mathfrak{g}] = \mathfrak{g}$ .

Two years later, Dixmier and Lister, see [8], answered Jacobson's question in the negative by displaying the following example of a nilpotent Lie algebra  $\mathfrak{g}$  (of dimension 8 and basis  $\{e_1, \dots, e_8\}$ ) with no nonsingular derivations:

$$\begin{aligned} [e_1, e_2] = -[e_3, e_4] &= e_5 \\ [e_1, e_3] = [e_2, e_4] &= e_6 \\ [e_1, e_4] = -[e_2, e_6] = -[e_3, e_5] &= e_7 \\ [e_1, e_5] = -[e_2, e_3] = [e_4, e_6] &= -e_8 \end{aligned}$$

They called such algebras *characteristically nilpotent*, defining them as those whose derivation algebra is nilpotent (note that by Engel's theorem, this implies that such Lie algebras are nilpotent). In fact, they showed that their algebra could not be a *derived algebra*. So they, in turn, asked whether that property is shared by all characteristically nilpotent Lie algebras.

The response seemed to be affirmative and some examples of characteristically nilpotent Lie algebras that are not derived were shown (see those of Bratzlavsky [4], Hakimjanov [12] and Carles [5]). However, 20 years later, Luks (see [14]) answered the Dixmier-Lister question in the negative, showing that an algebra *can* be both characteristically nilpotent and derived. He found a characteristically nilpotent Lie algebra, which was derived from a Lie algebra of dimension 18. Moreover, in fact, a similar response had been already given by Leger and Togo in 1959 (see [13]), although in a partial way. They gave a necessary condition for a characteristically nilpotent Lie algebra to be a derived algebra, but they could not prove the existence of such an algebra. In their research, they proved some characterizations involving characteristically nilpotent Lie algebras and derived Lie algebras. One of them is shown here:

**Theorem 1.1.** *Let  $\mathfrak{g}$  be a nilpotent Lie algebra, over a field  $K$ , and assume  $Z(\mathfrak{g})$  is its center. Then, the following results are verified:*

1. *If  $\mathfrak{g}$  is a characteristically nilpotent Lie algebra, then  $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$  and  $\mathfrak{g}^3 \neq \{0\}$ .*
2. *Let  $D(\mathfrak{g})$  be the Lie algebra of derivations of  $\mathfrak{g}$ . Then  $\mathfrak{g}$  is a characteristically nilpotent Lie algebra if and only if  $D(\mathfrak{g})$  is a nilpotent Lie algebra and  $\dim(\mathfrak{g}) \neq 1$ .*
3. *Suppose  $K$  is an algebraically closed field of characteristic 0. Then,  $\mathfrak{g}$  is a characteristically nilpotent Lie algebra if and only if all semi-simple automorphisms of  $\mathfrak{g}$  are of finite order.*
4. *Suppose  $\mathfrak{g} = \bigoplus_i I_i$ , where  $I_i$  are ideals of  $\mathfrak{g}$ , for  $1 \leq i \leq n$ . Then,  $\mathfrak{g}$  is a characteristically nilpotent Lie algebra if and only if  $I_i$  is characteristically nilpotent, for all  $i$ .*

Moreover, by using these results, Leger and Togo proved the following results, which also link characteristically nilpotent Lie algebras and derived Lie algebras:

**Theorem 1.2.** *Let  $\mathfrak{g}$  be a characteristically nilpotent Lie algebra. Then, the following results are verified:*

1. *If  $D(\mathfrak{g})$  annihilates  $Z(\mathfrak{g})$ , then  $\mathfrak{g}$  is not a derived algebra.*

2. Let  $m$  and  $n$  be the solvability and nilpotency indexes of  $\mathfrak{g}$ , respectively. If  $m - 1 > (n + 1)/2$ , then  $\mathfrak{g}$  is not a derived algebra.
3. If either  $D(\mathfrak{g})(\mathfrak{g}) \subset \mathfrak{g}^2$  or  $\mathfrak{g}^4 = \{0\}$ , then  $\mathfrak{g}$  is not a derived algebra.

On the other hand, it is very easy to give examples of nilpotent Lie algebras which are both, derived and not characteristically nilpotent. In fact, the existence of nilpotent derived algebras that are not characteristically nilpotent is really evident in the set of derived algebras of small-dimensional solvable Lie algebras. So, the history allows to establish that neither property of nilpotent Lie algebras, *characteristically nilpotent* or *not being a derived algebra*, implies the other, and neither excludes the other. Indeed, we give in this paper two examples of nilpotent Lie algebras that are not neither characteristically nilpotent nor derived algebras. We check that these Lie algebras are those of the lowest dimension verifying that property, see Theorem 3.1.

So, in this situation, one is motivated to study classes of nilpotent Lie algebras where both concepts, characteristically nilpotent and derived, are related, that is, in fact are equivalent. The question is then *what classes?*

The class of *filiform* Lie algebras, within the nilpotent Lie algebras, was introduced by Vergne in the late 1960's (in her Ph.D. Thesis, see [19], later published in 1970, see [20]). However, before that, Blackburn, see [3], had already studied the analogous class of finite Lie  $p$ -groups and used the term *maximal class* to call them, which is also now used for Lie algebras. In fact, the terms *filiform* and *maximal class* are synonyms.

Then, as we will see now, the special structure of complex filiform Lie algebras (in fact, they are the most structured subset of nilpotent Lie algebras) allows us to restrict them in a natural way and even to develop special computational techniques within this class since, within the variety of nilpotent Lie multiplications on a fixed vector space, non-filiforms can be relegated to small-dimensional components; thus, from some intuitive point of view, *quite a lot* of nilpotent Lie algebras are filiform, in spite of this last subset not being dense in the variety of nilpotent Lie algebras.

So, one is motivated to consider the particular case of characteristically nilpotent Lie algebras which are also filiform and to try to get some relations between them. In fact, some results related to classifications (in dimensions 9 and 10) and to the study of irreducible components of the variety of these algebras (characteristically nilpotent and filiform) have already been obtained by one of the authors of this paper, see [6] and [7], for instance. Moreover, since it seems to be accepted by experts (like Shalev and Zelmanov, see [18]) that one will not totally classify Lie algebras of finite dimension (filiforms between them), it is reasonable for research workers in Lie theory to impose further conditions to these algebras to obtain, at least, partial results. It is the main reason for our project.

And what happened with  $c$ -graded algebras? In 1966, Vergne got the classification of naturally graded filiform Lie algebras, that is, those admitting a gradation associated with the filtration which the lower central series involves, in a natural way, in a Lie algebra. Vergne also proved that within the class of filiform Lie algebras there exist two of which are graded only when the dimension is even or a unique one for dimension odd, see [20].

Later, Reyes in 1998 (see [17]) studied filiform Lie algebras of maximal length, that is, those admitting a connected gradation having the same number of subspaces as the dimension of the algebra.

Finally, Márquez, in her Ph.D. thesis, under the guidance of two of the authors of this paper, in 2001, introduces the class of  $c$ -graded filiform Lie algebras, which generalize both algebras just mentioned. So, naturally graded filiform Lie algebras by Vergne and filiform Lie algebras of maximal length by Reyes are  $c$ -graded filiform Lie algebras, for  $c = 1$  and  $c = 2$ , respectively, see [15].

**2. Definitions and notations.** In this paper, all the Lie algebras which appear will be considered over the complex field  $\mathbf{C}$ .

As we have just mentioned in the previous section, Dixmier and Lister defined in 1957 the characteristically nilpotent Lie algebras as those in which all their derivations are nilpotent. Remember that if  $\mathfrak{g} = (\mathbf{C}^n, [,])$  is a Lie algebra of dimension  $n$ , with  $[,]$  the associated law, a *derivation* of  $\mathfrak{g}$  is a map  $D : \mathfrak{g} \mapsto \mathfrak{g}$  such that  $D([x, y]) = [Dx, y] + [x, Dy]$  for all  $x, y \in \mathfrak{g}$ . The historic evolution of characteristically

nilpotent Lie algebras has already been dealt with in the previous section.

*2.1 Filiform Lie algebras.* Let  $\mathfrak{g} = (\mathbf{C}^n, [\cdot, \cdot])$  be a Lie algebra of dimension  $n$ , with  $[\cdot, \cdot]$  the Lie bracket. We consider the *lower central series* of  $\mathfrak{g}$  defined by  $\mathcal{C}^1\mathfrak{g} = \mathfrak{g}$ ,  $\mathcal{C}^i\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{i-1}\mathfrak{g}]$ . In the case of groups, this term goes back at least as far as Hall in the 1930's and in the case of algebras, Vergne also used it in her paper, although in fact the term may appear in the work of Hall and Witt, also in the 1930's. It was used by Ancochea and Goze to classify complex nilpotent Lie algebras of dimension 7 and complex filiform Lie algebras of dimension 8, since it is an invariant of these algebras, in the sense of not depending on the chosen basis, see [1].

A Lie algebra  $\mathfrak{g}$  of dimension  $n$  is *filiform* if  $\dim \mathcal{C}^i\mathfrak{g} = n - i$  for  $2 \leq i \leq n$ . If  $x \in \mathfrak{g}$  we denote by  $\text{ad}(x)$  the adjoint mapping associated to  $x$ , i.e., the map  $y \mapsto [x, y]$ . As we already said, filiform Lie algebras were defined by Vergne in 1966, see [19].

Let  $\mathfrak{g}$  be a filiform Lie algebra of dimension  $n$ . Then there exists a basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of  $\mathfrak{g}$  such that  $e_1 \in \mathfrak{g} \setminus \mathcal{C}^2\mathfrak{g}$ , the matrix of  $\text{ad}(e_1)$  with respect to  $\mathcal{B}$  has a Jordan block of order  $n - 1$  and  $\mathcal{C}^i\mathfrak{g}$  is the vector space generated by  $\{e_2, \dots, e_{n-i+1}\}$  with  $2 \leq i \leq n - 1$ .

Note that the previous conditions involve  $[e_1, e_h] = e_{h-1}$  for  $3 \leq h \leq n$ . Besides, as  $[e_2, e_n] = 0$  ( $\{e_2\}$  is the center of  $\mathfrak{g}$ ) and  $[e_1, e_3] = e_2$ , we can conclude that  $[e_3, e_n] = \alpha e_2$  and thus, the change of basis  $e'_n = e_n + \alpha e_1$ ,  $e'_k = e_k$  ( $k \neq n$ ) gives  $[e'_3, e'_n] = 0$  and it does not change the rest of the vectors of the basis. Such a basis  $\mathcal{B}$  is called an *adapted basis*.

It is easy to deduce that, with respect to such a basis, it is verified

$$\mathfrak{g}^2 \equiv \{e_2, \dots, e_{n-1}\}, \mathfrak{g}^3 \equiv \{e_2, \dots, e_{n-2}\}, \dots, \mathfrak{g}^{n-1} \equiv \{e_2\}, \mathfrak{g}^n \equiv \{0\}.$$

So, if  $h > 2$ , we have

$$[e_h, e_n] = c_{h,n}^{h-1} e_{h-1} + \dots + c_{h,n}^2 e_2$$

where some or all of these coefficients equalling 0. These coefficients  $c_{h,n}^{h-1}$  verify  $c_{4,n}^3 = c_{5,n}^4 = \dots = c_{n-1,n}^{n-2}$ . They are called *first coefficients*.

Moreover, it is already proved in [9] that if  $c_{h,n}^{h-1} \neq 0$ , then the dimension  $n$  of the algebra is even and that the base change

$$e'_1 = e_1, \quad e'_n = \frac{e_n}{c_{h,n}^{h-1}}$$

with the filiformity imply that all of these coefficients are equal to 1.

A Lie algebra is said to be a *model* filiform Lie algebra if the only nonzero brackets between the elements of an adapted basis are the following:  $[e_1, e_h] = e_{h-1}$ , ( $h = 3, \dots, n$ ).

In the last section, for the sake of simplicity, we will use  $\mu(x, y)$  instead of  $[x, y]$  for the Lie bracket in a Lie algebra. From now on, if  $x, y, z \in \mathfrak{g}$ , we denote by  $J(x, y, z) = 0$  the associated Jacobi identity.

**2.2 Derived algebras.** Let  $\mathfrak{g}$  be a nilpotent Lie algebra of dimension  $n$ . We say that  $\mathfrak{g}$  is a *derived* Lie algebra if there exists a Lie algebra  $\mathfrak{g}'$  of dimension  $n + k$ , for some  $k \geq 1$ , such that  $\mathcal{C}^2 \mathfrak{g}' = \mathfrak{g}$ .

It is already proved in [9] that any complex filiform Lie algebra  $\mathfrak{g}$  is either derived from a solvable Lie algebra  $\mathfrak{h}$  with  $\dim \mathfrak{h} = \dim \mathfrak{g} + 1$  or  $\mathfrak{g}$  is not derived from any Lie algebra. Moreover, the following result is also proved in that paper, which we will use in the proof of Theorem 3.2:

**Theorem 2.1.** *A necessary condition for the complex filiform Lie algebra  $\mathfrak{g}$ , of dimension  $n \geq 4$ , with an adapted basis  $\{e_1, \dots, e_n\}$ , to be derived from the solvable Lie algebra  $\mathfrak{h}$ , with basis  $\{e_1, \dots, e_n, U\}$  is  $a_{1,1} \neq 0$ , where  $[e_h, U] = \sum_{k=1}^h a_{h,k} e_k$ .*

**2.3  $c$ -graded filiform Lie algebras.** Suppose  $c$  is in  $\mathbf{N}$  (the natural numbers). Let  $\mathfrak{g}$  be a complex filiform Lie algebra of dimension  $n$ .  $\mathfrak{g}$  is said to be  $c$ -graded if there exists an adapted basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of  $\mathfrak{g}$  such that

$$(1) \quad [e_h, e_k] = a_{h,k} e_{h+k-n-c}$$

where  $h + k - n - c \geq 2$ , for all  $e_h, e_k \in \mathcal{B}$ , with  $1 < h$  and  $k \leq n$ . In such a case, it is said that  $\mathcal{B}$  is a  $c$ -graded basis of  $\mathfrak{g}$ .

Note that, according to this definition, model complex filiform Lie algebras are  $c$ -graded, whatever  $c$  is. Besides, it is not difficult to check that  $1 \leq c \leq n - 3$ .

With respect to these algebras, the following result is satisfied, see [15]:

**Theorem 2.2.** *Let  $\mathfrak{g}$  be a complex filiform Lie algebra of dimension  $n$ . If  $\mathfrak{g}$  is  $c$ -graded, then  $\mathfrak{g}$  is a derived algebra (from a solvable Lie algebra of dimension  $n + 1$ ).*

Márquez also points out that the converse is not true. In order to get this, she shows the following filiform Lie algebra of dimension 8 (with respect to the adapted basis  $\{e_1, \dots, e_8\}$ ), which is derived but it is not  $c$ -graded for any  $c$ , see [15]:

$$\begin{aligned} [e_1, e_h] &= e_{h-1} & (3 \leq h \leq 8) \\ [e_4, e_7] &= e_2, & [e_4, e_8] &= e_3, \\ [e_5, e_6] &= -e_2, & [e_5, e_8] &= e_4, \\ [e_6, e_8] &= e_5 + e_2, & [e_7, e_8] &= e_6 + e_3. \end{aligned}$$

However, the converse can be true if some hypotheses are added. Indeed, Márquez also proves in [15] that, if the complex filiform Lie algebra has first coefficient zero, then it is derived. Moreover, as a consequence of both results, Márquez proves the following characterization theorem:

**Theorem 2.3.** *A complex filiform Lie algebra is  $c$ -graded for some  $c \geq 0$  if and only if it is either of first coefficient null and derived or it is a model algebra.*

**3. Relation between characteristically nilpotent filiform Lie algebras and derived Lie algebras.** Firstly, we are going to show in this section two examples of nilpotent Lie algebras which are neither derived nor characteristically nilpotent.

3.1 *Examples of nilpotent Lie algebras which are neither derived nor characteristically nilpotent.*

**Theorem 3.1.** *The nilpotent Lie algebras of smallest dimension which are neither derived from another Lie algebra nor are character-*



istically nilpotent are the following two nilpotent Lie algebras of dimension 7:

$$\begin{array}{ll} \mathfrak{g}_1 : [e_1, e_k] = e_{k-1} & 3 \leq k \leq 7 & \mathfrak{g}_2 : [e_1, e_k] = e_{k-1} & 3 \leq k \leq 7 \\ [e_2, e_5] = e_3 & & [e_2, e_6] = e_3 & \\ [e_2, e_6] = e_4 & & [e_2, e_7] = e_4 + e_3 & \\ [e_2, e_7] = e_5 + e_3 & & & \end{array}$$

*Proof.* Let consider the algebra  $\mathfrak{g}_1$ . If  $\mathfrak{g}_1$  was derived from the solvable Lie algebra  $S_1$ , of dimension 8, see subsection 2.2, and basis  $\{e_1, \dots, e_7, U\}$ , then the following brackets would appear in the definition of  $S_1$ :

$$(2) \quad [e_h, U] = \sum_{j=1}^7 a_{h,j} e_j$$

and the rest of nonzero brackets as in  $\mathfrak{g}_1$ .

Then, from  $J(e_1, e_2, U) = 0$ , we obtain  $a_{2,4} = a_{1,5} + a_{1,7}$ ,  $a_{2,5} = a_{1,6}$ ,  $a_{2,6} = a_{1,7}$ ,  $a_{2,7} = 0$ . If we now substitute these expressions in (2), we obtain

$$[e_2, U] = a_{2,1} e_1 + a_{2,2} e_2 + a_{2,3} e_3 + (a_{1,5} + a_{1,7}) e_4 + a_{1,6} e_5 + a_{1,7} e_6$$

Then, by proceeding in a similar way with the identities  $J(e_1, e_3, U) = 0, \dots, J(e_1, e_7, U) = 0, J(e_2, e_3, U) = 0, \dots, J(e_2, e_7, U) = 0, J(e_3, e_4, U) = 0, \dots, J(e_6, e_7, U) = 0$ , if we substitute the expressions obtained in each identity in both (2) and in the result previously obtained for the brackets  $[e_h, U]$ , we will obtain the following results (no expressions are obtained from the identities which do not appear):

1. From  $J(e_1, e_2, U) = 0$ , we obtain  $a_{2,4} = a_{1,5} + a_{1,7}$ ,  $a_{2,5} = a_{1,6}$ ,  $a_{2,6} = a_{1,7}$ ,  $a_{2,7} = 0$ .
2. From  $J(e_1, e_3, U) = 0$ , we obtain  $a_{3,4} = a_{3,5} = a_{3,6} = a_{3,7} = 0$ .
3. From  $J(e_1, e_4, U) = 0$ , we obtain  $a_{3,1} = a_{3,2} = a_{4,5} = a_{4,6} = a_{4,7} = 0$ ,  $a_{4,4} = a_{3,3} - a_{1,1} = 0$ .
4. From  $J(e_1, e_5, U) = 0$ , we obtain  $a_{5,4} = a_{4,3} - a_{1,2}$ ,  $a_{5,5} = a_{3,3} - 2 a_{1,1}$ ,  $a_{4,1} = a_{4,2} = a_{5,6} = a_{5,7} = 0$ .

5. From  $J(e_1, e_6, U) = 0$ , we obtain  $a_{6,5} = a_{4,3} - 2 a_{1,2}$ ,  $a_{6,6} = a_{3,3} - 3 a_{1,1}$ ,  $a_{6,4} = a_{5,3}$ ,  $a_{5,1} = a_{5,2} = a_{6,7} = 0$ .

6. From  $J(e_1, e_7, U) = 0$ , we obtain  $a_{7,4} = a_{6,3} - a_{1,2}$ ,  $a_{7,5} = a_{5,3}$ ,  $a_{7,6} = a_{4,3} - 3 a_{1,2}$ ,  $a_{7,7} = a_{3,3} - 4 a_{1,1}$ ,  $a_{6,1} = a_{6,2} = 0$ .

7. From  $J(e_2, e_3, U) = 0$ , we obtain  $a_{2,1} = 0$ .

8. From  $J(e_2, e_5, U) = 0$ , we obtain  $a_{2,2} = 2 a_{1,1}$ .

9. From  $J(e_2, e_6, U) = 0$ , we obtain  $a_{1,2} = 0$ .

10. From  $J(e_2, e_7, U) = 0$ , we obtain  $a_{1,1} = 0$ .

11. From  $J(e_4, e_7, U) = 0$ , we obtain  $a_{7,1} = 0$ .

12. From  $J(e_5, e_7, U) = 0$ , we obtain  $a_{7,2} = 0$ .

So, the expressions (2) are now reduced to

$$\begin{aligned} [e_1, U] &= a_{1,3} e_3 + a_{1,4} e_4 + a_{1,5} e_5 + a_{1,6} e_6 + a_{1,7} e_7 \\ [e_2, U] &= a_{2,3} e_3 + (a_{1,5} + a_{1,7}) e_4 + a_{1,6} e_5 + a_{1,7} e_6 \\ [e_3, U] &= a_{3,3} e_3 \\ [e_4, U] &= a_{4,3} e_3 + a_{3,3} e_4 \\ [e_5, U] &= a_{5,3} e_3 + a_{4,3} e_4 + a_{3,3} e_5 \\ [e_6, U] &= a_{6,3} e_3 + a_{5,3} e_4 + a_{4,3} e_5 + a_{3,3} e_6 \\ [e_7, U] &= a_{7,3} e_3 + a_{6,3} e_4 + a_{5,3} e_5 + a_{4,3} e_6 + a_{3,3} e_7 \end{aligned}$$

Then, we can easily observe that basis vectors  $e_1$  and  $e_2$  do not appear either in them or among the definitions of  $[e_h, e_k]$ , and thus  $\mathfrak{g}_1$  cannot be the derived algebra of a solvable Lie algebra.

In a similar way, we obtain for the algebra  $\mathfrak{g}_2$  the following expressions:

$$\begin{aligned} [e_1, U] &= a_{1,2} e_2 + a_{1,3} e_3 + a_{1,4} e_4 + a_{1,5} e_5 + a_{1,6} e_6 + a_{1,7} e_7 \\ [e_2, U] &= a_{2,3} e_3 + (a_{1,6} + a_{1,7}) e_4 + a_{1,7} e_5 \\ [e_3, U] &= a_{3,3} e_3 \\ [e_4, U] &= a_{4,3} e_3 + a_{3,3} e_4 \\ [e_5, U] &= a_{5,3} e_3 + a_{4,3} e_4 + a_{3,3} e_5 \\ [e_6, U] &= a_{6,3} e_3 + (a_{5,3} - a_{1,2}) e_4 + a_{4,3} e_5 + a_{3,3} e_6 \\ [e_7, U] &= a_{7,3} e_3 + (a_{6,3} - a_{1,2}) e_4 + (a_{5,3} - 2 a_{1,2}) e_5 + a_{4,3} e_6 + a_{3,3} e_7 \end{aligned}$$

in which the basis vector  $e_1$  does not appear. So,  $\mathfrak{g}_2$  cannot be the derived algebra from a solvable Lie algebra either.

Note that both algebras cannot be characteristically nilpotent either, due to some of the coefficients  $a_{h,h}$  with  $1 \leq h \leq 7$  (in fact, these are the diagonal elements of matrix  $A = (a_{i,j})$  which will be considered in the Theorem 3.2) are nonzero.  $\square$

So, these examples show that there is no relation, in general, between characteristically nilpotent Lie algebras and derived algebras. However, if we suppose that the Lie algebras are both characteristically nilpotent and filiform, then the situation changes and some results can be given. Indeed, we are going to prove in this subsection that a necessary and sufficient condition for a complex filiform Lie algebra to be characteristically nilpotent is not to be a derived algebra. It is also convenient to note that this equivalence was previously proved for dimensions of the algebra less than or equal to 9 and conjectured in the case of greater dimensions by Núñez in [16]. Moreover, other colleagues and ourselves have also given an algorithmic procedure which determines if a complex filiform Lie algebra is characteristically nilpotent and, in the negative case, gives a solvable Lie algebra from which the initial algebra is derived. This algorithm, which involves lots of complicated computations, can be made by using any symbolic computation package (particularly, we used the MAPLE package in our research, see [2]).

So, let us consider in this section and from now on, the particular case of characteristically nilpotent Lie algebras which are also filiform. By using Theorem 2.1, see subsection 2.2, a characterization of characteristically nilpotent filiform Lie algebras is given by the following:

**Theorem 3.2** (Main Theorem). *Let  $\mathfrak{g}$  be a complex filiform Lie algebra of dimension  $n \geq 4$ . Then  $\mathfrak{g}$  is characteristically nilpotent if and only if  $\mathfrak{g}$  is not a derived algebra.*

*Proof.* Let  $D : \mathfrak{g} \rightarrow \mathfrak{g}$  be a derivation of  $\mathfrak{g}$ . Let  $A = (a_{i,j}) \in \text{Mat}(n \times n, \mathbf{C})$  be the matrix of  $D$  with respect to an adapted basis

$\mathcal{B} = \{e_1, \dots, e_n\}$  of  $\mathfrak{g}$ . Since  $D(\mathcal{C}^k \mathfrak{g}) \subset \mathcal{C}^k \mathfrak{g}$  for any  $k$ , we have

$$\begin{aligned} a_{i,j} &= 0, & (\text{for } 2 \leq i \leq n-1, j > i); \\ a_{i,1} &= 0, & (\text{for } 2 \leq i \leq n-1). \end{aligned}$$

Moreover, from  $D[e_3, e_n] = [De_3, e_n] + [e_3, De_n]$  we deduce that  $a_{n,1} = 0$ . Thus, the eigenvalues of  $D$  are  $\{a_{1,1}, \dots, a_{n,n}\}$ .

For each  $(i, j, k)$  with  $1 \leq i < j \leq n$  and  $1 \leq k \leq n$ , we will denote by  $ec(i, j, k)$  the coefficient of  $e_k$  in the expression of  $D[e_i, e_j] - [De_i, e_j] - [e_i, De_j]$  with respect to the basis  $\mathcal{B}$ . The elements of the matrix  $A$  satisfy the homogeneous linear system defined by

$$\mathcal{S} = \{ec(i, j, k) = 0, \quad 1 \leq i < j \leq n, \quad 1 \leq k \leq n\}$$

Conversely, any matrix  $A = (a_{i,j}) \in \text{Mat}(n \times n, \mathbf{C})$  verifying the system  $\mathcal{S}$ , defines a derivation in  $\mathfrak{g}$ .

In [16] and [9] it is already proved that if all the solutions of the system  $\mathcal{S}$  verify  $a_{1,1} = 0$ , then  $a_{i,i} = 0$  for all  $2 \leq i \leq n$ . In this case, in which the Lie algebra  $\mathfrak{g}$  is not derived (see Theorem 2.1), all the derivations of  $\mathfrak{g}$  are nilpotent and thus  $\mathfrak{g}$  is characteristically nilpotent. Conversely, it is obvious that, if there exists a solution of  $\mathcal{S}$  verifying  $a_{1,1} \neq 0$ , which implies that  $\mathfrak{g}$  is a derived algebra, then the corresponding derivation is not nilpotent and  $\mathfrak{g}$  is not characteristically nilpotent.  $\square$

*Note.* In both papers the authors also prove that, with the exception of two kinds of filiform Lie algebras (the model of each dimension and the  $Q_n$  algebra of each even dimension, this last kind of filiform Lie algebra is defined in [15], for instance), any derivation  $D$  of a filiform Lie algebra with matrix  $A$  with respect to an adapted basis, verifies  $a_{i,i} = \lambda_i a_{1,1}$  with  $\lambda_i \in \mathbf{C} \setminus \{0\}$  and  $2 \leq i \leq n$ . Those two kinds of filiform Lie algebras are derived algebras and they are not characteristically nilpotent. Note, besides, that the hypothesis  $n \geq 4$  is needed to avoid the model filiform Lie algebra, which is the trivial case.

We observe that this result allows us to prove, by an easy and efficient way, some results previously obtained by different authors, related to characteristically nilpotent Lie algebras, which in principle, did not seem to have any relation themselves. We consider the following:

**Corollary 3.1.** *The following results are satisfied:*

1. *There are no characteristically nilpotent Lie algebras of dimension  $n \leq 6$ , over an algebraically closed field.*
2. *There are only three filiform Lie algebras of dimension 7 which are characteristically nilpotent Lie algebras.*
3. *The class of characteristically nilpotent Lie algebras is not empty, for dimensions  $n \geq 7$ .*

*Proof.* 1. It is already proved by some of us, see [16], that over an algebraically closed field, every filiform Lie algebra of dimensions  $n \leq 6$  is derived from a solvable Lie algebra of dimension  $n + 1$ . So, according to Theorem 3.2, characteristically nilpotent Lie algebras of dimensions less than or equal to 6 cannot exist.

This result has previously been given by Hakimjanov ([12]). Moreover, Bratzlavsky, see [4], has also shown that there are not characteristically nilpotent Lie algebras of dimensions less than or equal to 5, whatever the characteristic of the underlying field is.

2. Starting from Ancochea and Goze's classification of nilpotent Lie algebras of dimension 7, see [1], Núñez proved in [16] that only 8 of the 125 nilpotent Lie algebras of this dimension are filiform. Moreover, he proved that only three of them are not derived (concretely, numbers 3, 4 and 5 of Ancochea and Goze's classification).

Then, according to the last theorem, these three algebras are characteristically nilpotent. This result has been previously established by Godfrey ([10]).

3. The previous results imply that this one is obvious. In fact, it has been already stated by Carles ([5]).  $\square$

**4. Relation between characteristically nilpotent filiform Lie algebras and  $c$ -graded filiform Lie algebras.** As we recalled in Section 2, Márquez states that if a filiform Lie algebra of dimension  $n$  is  $c$ -graded, then it is a derived algebra (from a solvable Lie algebra of dimension  $n + 1$ ). Moreover, since it is not difficult to find examples of filiform Lie algebras not verifying the converse of this result (see subsection 2.3 of Section 2), it is necessary to reduce this problem to continue progressing on this research. So, Márquez deals with

filiform Lie algebras having the first coefficient zero and proves that the converse is then true. As a conclusion, Márquez sets that a filiform Lie algebra is  $c$ -graded if and only if it is either of first coefficient zero and derived or it is a model algebra.

Therefore, as an immediate consequence of this last result and of Theorem 2.3, the following is proved:

**Theorem 4.1.** *A filiform Lie algebra having its first coefficient zero is characteristically nilpotent if and only if it is not a  $c$ -graded filiform Lie algebra, for any  $c \geq 0$ .  $\square$*

Now, we are going to give in the following section an algorithmic procedure which shows when a given and fixed filiform Lie algebra is characteristically nilpotent, derived or  $c$ -graded Lie algebra. However, since we obtain different results according to the filiform Lie algebra being of first coefficient equal or different from 0 ( $c \neq 1$  or  $c = 1$ , respectively), we give now an algorithm which allows it in the case  $c = 1$ .

**5. The algorithmic procedure.** To give this algorithmic procedure we use the following result, related to Lie algebras: *Let  $\mathfrak{g}$  be a Lie algebra of dimension  $n$ , and let  $\mathcal{B} = \{e_1, \dots, e_n\}$  be a basis of  $\mathfrak{g}$ . It is known that every automorphism  $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}$  is associated with a basis change  $\mathcal{B} \mapsto \mathcal{B}' = \Phi(\mathcal{B}) = \{\Phi(e_1), \dots, \Phi(e_n)\}$  preserving the law of the Lie algebra. Under these conditions, if  $[\cdot, \cdot]_{\mathcal{B}}$  and  $[\cdot, \cdot]_{\mathcal{B}'}$  represent such a law, with respect to basis  $\mathcal{B}$  and  $\mathcal{B}'$ , respectively, then*

$$[X, Y]_{\mathcal{B}} = 0 \iff [\Phi(X), \Phi(Y)]_{\mathcal{B}'} = 0.$$

So, to know if an  $n$ -dimensional filiform Lie algebra having its first coefficient nonzero is  $c$ -graded for some  $c$  with respect to a basis  $\mathcal{B}$ , we have to evaluate if there exists an adapted basis change verifying the condition that every nonzero bracket between basis vectors can be expressed as in (1). Moreover, as the first coefficient of the filiform Lie algebra is nonzero, it implies that  $c = 1$  is a necessary condition for the filiform Lie algebra to be  $c$ -graded. Note that in such a case, the Lie algebra is derived and thus, it is not characteristically nilpotent. Then, we obtain a solvable Lie algebra from which the initial one is derived.

The algorithmic procedure which we are going to show involves lots of computations, which can be made by using any symbolic computation package. Particularly, we have used the MAPLE package to obtain our results.

This algorithm is the following:

### Algorithm

#### Input

1. The integer  $n$ ,  $n > 5$ .
2. The adapted basis  $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$  with respect to which the following conditions are verified:

- $\mu$  is a anti-symmetric bilinear form.

- $\mu$  verifies:

$$\mu(e_1, e_i) = e_{i-1} \quad \text{for } 3 \leq i \leq n$$

$$\mu(e_2, e_h) = 0 \quad \text{for } 1 \leq h \leq n$$

$$\mu(e_3, e_h) = 0 \quad \text{for } 4 \leq h \leq n$$

where  $\{e_1, \dots, e_n\}$  is the canonical basis in  $\mathcal{C}^n$ .

- $\mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) = 0$  for all  $x, y, z \in \mathcal{C}^n$  (Jacobi identities).

3. The known brackets  $\mu(e_h, e_n)$ , for  $1 \leq h < n$ .

#### Output

1. The law  $\mu'$  of the filiform Lie algebra with respect to a new adapted basis  $\mathcal{B}' = \{e'_1, \dots, e'_n\}$ .
2. The coefficients  $c_{h,k}^l$  resulting from the brackets between vectors belonging to  $\mathcal{B}'$ , that is,  $[e'_h, e'_k] = \sum_{l=2}^{h+k-n-1} c_{h,k}^l e'_l$ .
3. If the filiform Lie algebra  $\mathfrak{g}$  is 1-graded (or equivalently, if it is not characteristically nilpotent), by evaluating if  $c_{h,k}^{h+k-n-1} \neq 0$  in every nonzero bracket  $[e_h, e_k]$  and by seeing if the equation system  $c_{h,k}^l = 0$  (with  $4 \leq h < k \leq n$  and  $2 \leq l \leq h+k-n-2$ ) has a solution.

4. In the positive case, the law of the filiform Lie algebra defined over the new basis (that it, with respect to which  $\mathfrak{g}$  is 1-graded).
5. The matrix  $A = (a_{i,j})$  of any derivation  $\mathcal{D}$  with respect to the new adapted basis  $\mathcal{B}'$ , a solution  $a_{1,1} \neq 0$  and a Lie algebra  $\mathfrak{g}'$  such that  $\mathcal{C}^2 \mathfrak{g}' = \mathfrak{g}$ .

### Method

Step 1 Give the law of the filiform Lie algebra with respect to an adapted basis  $\mathcal{B} = \{e_1, \dots, e_n\}$ .

Step 2 Evaluate the coefficients  $c_{h,k}^l$  in the nonzero brackets

$$[e_h, e_k] = \sum_{l=2}^{h+k-n-1} c_{h,k}^l e_l.$$

Step 3 If some of the coefficients  $c_{h,k}^{h+k-n-1}$  are equal to 0, then the filiform Lie algebra is not 1-graded.

Step 4 Else, define a new adapted basis  $\mathcal{B}' = \{e'_1, \dots, e'_n\}$ , by considering  $e'_1 = \sum_{i=1}^n a_{1,i} e_i$  and  $e'_n = \sum_{i=2}^n a_{2,i} e_i$  and by computing, in a recursive way, the rest of vectors  $e'_h = [e'_1, e'_{h+1}]$ .

Step 5 Obtain the new law  $\mu'$  of the filiform Lie algebra with respect to  $\mathcal{B}' = \{e'_1, \dots, e'_n\}$ , given by

$$[e'_h, e'_k] = \sum_{l=2}^{h+k-n-1} d_{h,k}^l e'_l,$$

with  $4 \leq h < k \leq n$ .

Step 6 Solve the equation system  $d_{h,k}^l = 0$  (with  $4 \leq h < k \leq n$  and  $2 \leq l \leq h+k-n-2$ ), in the unknowns  $a_{i,j}$ , with  $1 \leq i \leq 2$  and  $i \leq j \leq n$ .

Step 7 If that system has no a solution, then the filiform Lie algebra is not 1-graded.



- Step 8 If, on the contrary, that system has a solution, then the filiform Lie algebra is 1-graded and the law  $\mu'$  can be expressed by  $[e'_h, e'_k] = d_{h,k}^{h+k-n-1} e'_{h+k-n-1}$ .
- Step 9 Define a derivation  $\mathcal{D}$  of  $\mathfrak{g}$  whose matrix is  $A = (a_{i,j})$  with respect to  $\mathcal{B}$ , where  $a_{i,j}$  are parameters, with  $\mathcal{D}(e'_i) = \sum a_{i,j} e'_j$  and  $a_{i,j} = 0$  for  $n \geq j > i \geq 2$  and  $a_{j,1} = 0$  for  $n \geq j > 1$ .
- Step 10 Define  $ec(i, j, k)$  as the coefficient of the vector  $e_k$  in the expression  $D(\mu(e'_i, e'_j)) - \mu(De'_i, e'_j) - \mu(e'_i, De'_j)$  for all  $1 \leq i, j, k \leq n$ .
- Step 11 Solve the linear system of equations  $ec(i, j, k) = 0$  in the unknowns  $a_{i,j}$ .
- Step 12 Give a solution  $a_{1,1} \neq 0$  and the law  $\mu''$  of a different Lie algebra  $\mathfrak{g}'$  such that  $\mathcal{C}^2 \mathfrak{g}' = \mathfrak{g}$ .

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