Arab J Math (2012) 1:439–463
DOI 10.1007/s40065-012-0044-z

SURVEY PAPER

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Metric fixed point theory on hyperconvex spaces: recent progress

Received: 13 June 2012 / Accepted: 14 July 2012 / Published online: 11 August 2012

Abstract In this survey we present an exposition of the development during the last decade of metric fixed point theory on hyperconvex metric spaces. Therefore we mainly cover results where the conditions on the mappings are metric. We will recall results about proximinal nonexpansive retractions and their impact into the theory of best approximation and best proximity pairs. A central role in this survey will be also played by some recent developments on $\mathbb{R}$-trees. Finally, some considerations and new results on the extension of compact mappings will be shown.

Mathematics Subject Classification 47H10 · 47H09 · 47H04 · 54E40

الملخص

نقدم في هذا المسح استعراضاً للتطور الحاصل في نظرية القيمة الثابتة المترية على الفضاءات المترية فوق المحدبة خلال العقد الماضي. لذلك، نغطي بشكل رئيس النتائج التي تكون الشروط على الرواسم فيها مترية. سوف نستذكر نتائج حول الانكماشات غير التمددية الدنيا وتأثيرها في نظرية أفضل تقريب وأفضل الأزواج الدنيا. تلعب بعض التطورات الحديثة حول أشجار-$\mathbb{R}$ دوراً مركزياً في هذا المسح. أخيراً، سوف يتم إثبات بعض الاعتبارات والنتائج الجديدة حول تمديد الرواسم المتراصة.

1 Introduction

In the monograph *Handbook of Metric Fixed Point Theory* edited by Kirk and Sims, and published in 2001 by Kluwer Academic Publishers, we can find the chapter *Introduction to hyperconvex spaces* [21] by Espínola and Khamsi. This survey was mainly devoted to the study of metric hyperconvexity and its impact on metric fixed point theory which began in late seventies of the twentieth century [60,64]. It is already a decade after the *Handbook on Metric Fixed Point Theory* appeared, and many new facts on hyperconvex metric spaces and fixed points have been stated in these past 10 years. In this survey we will address some of these facts with a special attention to those involving metric conditions on the mappings. Therefore, it is not our intention to recall in detail basic and fundamental facts on hyperconvex metric spaces, which were already very generously explained along the first five sections of [21], but rather to present an update of some of the topics covered by the other last six sections of [21].

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The study of fixed point theory for nonexpansive mappings on hyperconvex spaces was initiated, independently, by Sine [60] and Soardi [60], and very fundamental properties about hyperconvex spaces as well as the structure of the fixed point set of nonexpansive mappings were shown by Baillon in [9]. After the fundamental work of Sine on hyperconvexity and fixed points, which may be compiled in the series of works [35,50,60–63], many authors continued stating new facts during the past decade of the twentieth century. The aforementioned chapter [21] in the *Handbook of Metric Fixed Point Theory* was devoted to collect many of those results. Since then new interesting facts have been obtained about fixed points and best approximation theory in hyperconvex metric spaces. These studies have been especially fruitful when focusing on $\mathbb{R}$-trees. The connection between $\mathbb{R}$-tress and hyperconvexity became evident for most fixed point theorists after the work [40] from 1998 by Kirk; however, it was needed to wait for some time until fixed point results on $\mathbb{R}$-trees began to be studied [22] in a more systematic way. This explains why one of the most fruitful topics in the field nowadays, which is fixed point results on $\mathbb{R}$-trees, was not covered in [21].

Interestingly enough, there has been a whole burst in the attention that the mathematical community has paid on metric spaces and analysis on metric spaces in the past 20 years. There are many metric structures which have called the attention of a large number of mathematicians from different branches. These structures include fractal metric spaces, metric spaces of bounded curvature, hyperbolic geodesic spaces, sub-Riemannian spaces, doubling metric spaces, metric spaces with the Poincaré inequality,… All of them have shown to be good structures to develop analysis from different perspectives. Hyperconvex spaces may also be considered within this class of intriguing metric structures which allow us to obtain results which one would only expect to be possible under certain linear structures. For references on the study of analysis on metric spaces or nonsmooth analysis the interested reader may check [3,30,31].

In this survey we start fixing some notation and giving some basic facts on hyperconvex spaces to make the exposition more self-contained. This will be covered in Sects. 2 and 3. In Sect. 4 we deal with nonexpansive proximinal retracts, best approximation, and best proximity pairs. In [21] a lot of attention was given to the study of nonexpansive proximinal retracts. By then the problem of characterizing such subsets of hyperconvex spaces was still open. In Sect. 4 we find the solution to this problem and its relation to the problem of best approximation and best proximity pairs. Some other related results are also addressed in this section. Section 5 will be devoted to recent progress on $\mathbb{R}$-trees and metric fixed point theory. The very peculiar structure of these spaces has provided a very suitable scenario to develop beautiful and deep fixed point results for single and multivalued mappings. We will walk the history of this topic and show some of the most surprising and far-reaching results on $\mathbb{R}$-trees. One of the most remarkable features of these results is that they do not require the set to be bounded to guarantee existence of fixed points. We will show how dealing with multivalued mappings is especially fruitful on $\mathbb{R}$-trees. In our last section, Sect. 6, we consider some recent advances on the theory of extension of Hölder maps and relate them to extensions of uniformly continuous mappings under $\aleph_0$-hyperconvex conditions. In this section we include some new results on compact extensions of compact mappings.

2 Notation and basic definitions

Metric spaces will be denoted as (M, d), or just M for simplicity, where M is the space and d stands for the distance on M. Main elements for us in a metric space will be closed balls which will be denoted by $B(x, r)$ meaning the closed ball of center x and radius $r \geq 0$. The following notation is also typical when dealing with metric spaces and will be used along all this survey.

Let M be a metric space, $x \in M$ and A and B subsets of M; then

$$\begin{aligned}
r_x(A) &= \sup\{d(x, y) \colon y \in A\}, \\
r(A) &= \inf\{r_x(A) \colon x \in M\}, \\
R(A) &= \inf\{r_x(A) \colon x \in A\}, \\
\operatorname{diam}(A) &= \sup\{d(x, y) \colon x, y \in A\}, \\
\operatorname{dist}(x, A) &= \inf\{d(x, y) \colon y \in A\}, \\
\operatorname{dist}(A, B) &= \inf\{d(x, y) \colon x \in A, y \in B\}, \\
C(A) &= \{x \in M \colon r_x(A) = r(A)\}, \\
C_A(A) &= \{x \in A \colon r_x(A) = R(A)\}, \\
\operatorname{cov}(A) &= \bigcap\{B \colon B \text{ is a closed ball containing } A\},
\end{aligned}$$

where $r(A)$ is the *radius* of A relative to M, $\operatorname{diam}(A)$ is the *diameter* of A, $R(A)$ is the *Chebyshev radius* of A, and $\operatorname{cov}(A)$ is the *admissible cover* of A.

Hyperconvex metric spaces were introduced by Aronszajn and Panitchpakdi to extend Hahn–Banach's theorem from the real line to more general spaces, see [8,21,36]. As a result they determined metric conditions guaranteeing such an extension and named spaces satisfying these conditions *hyperconvex metric spaces*.

Definition 2.1 A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

then

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset.$$

A subset of a metric space with this property will be called sometimes as a hyperconvex set.

The name of hyperconvexity comes from the fact that when the ball intersecting condition holds for each pair of balls then the metric space is said to be (metrically) convex. In the literature they are also known as injective metric spaces and Banach spaces which are hyperconvex are generically denotes as $\mathcal{P}_1$ spaces [48,51]. The study of normed spaces satisfying this property and other related ones was very important during the mid twentieth century, and the interested reader may check [48] in this regard. For more on the injective nature of hyperconvex metric spaces see [8,36] or [21, Section 4].

A fundamental fact of hyperconvex spaces is given by the next proposition which proof can be found in [8,21], see also [36, Proposition 4.4].

Proposition 2.2 *If M is a hyperconvex metric space then it is complete.*

Next we summarize some metric properties of hyperconvex spaces which have played a major role in the structure of hyperconvex spaces and have been extensively applied in metric fixed point theory; for proofs consult [21, Lemma 3.3] or [36, Lemma 4.1].

Lemma 2.3 *Let A be a bounded subset of a hyperconvex metric space M. Then*

1. $\operatorname{cov}(A) = \cap\{B(x, r_x(A)) : x \in M\}$.
2. $r_x(\operatorname{cov}(A)) = r_x(A)$, *for any* $x \in M$.
3. $r(\operatorname{cov}(A)) = r(A)$.
4. $r(A) = \frac{1}{2}\operatorname{diam}(A)$.
5. $\operatorname{diam}(\operatorname{cov}(A)) = \operatorname{diam}(A)$.
6. *If* $A = \operatorname{cov}(A)$ *then* $r(A) = R(A)$.

A very important property in hyperconvexity with immediate impact in fixed point and approximation theory (see [21,46]) is that of being nonexpansive retracts.

Definition 2.4 Let (M, d_1) and (N, d_2) be two metric spaces. A map $T : M \to N$ is said to be nonexpansive if

$$d_2(Tx, Ty) \leq d_1(x, y)$$

for all $x, y \in M$.

Definition 2.5 A subset A of a metric space M is said to be a *nonexpansive retract* (of M) if there exists a nonexpansive retraction from M onto A, that is, a nonexpansive mapping $R : M \to A$ such that $Rx = x$ for each $x \in A$.

Hyperconvex metric spaces are nonexpansive retracts of any metric space where are isometrically embedded [21, Section 4]. Even more, this property characterizes hyperconvex subsets of a hyperconvex metric space.

Lemma 2.6 *Let M be a hyperconvex metric space and $A \subseteq M$. Then A is hyperconvex (as a metric space with the induced metric) if and only if A is a nonexpansive retract of M.*

A step forward in this property is to require the set to be a *proximinal nonexpansive retract*.

Definition 2.7 Let M be a metric space and $A \subseteq M$ nonempty. The metric projection onto A is then defined by

$$P_A(x) = B(x, \operatorname{dist}(x, A)) \cap A.$$

If $P_A(x)$ is nonempty for each $x \in M$, then A is said to be a proximinal subset of M. If A is proximinal and P_A is nonexpansive or admits a nonexpansive selection, then A is said to be a proximinal nonexpansive retract of M.

Proximinal nonexpansive retracts are relevant in the theory of fixed point and best approximation points. To study the nature of such sets for a hyperconvex space is a much more complicated task than that of characterizing nonexpansive retracts. Still some classes of proximinal nonexpansive retracts have been identified. Among them we find the classes of *admissible* and *externally hyperconvex subsets*.

Definition 2.8 Let M be a metric space. $A \subseteq M$ is said to be an admissible subset of M if $A = \operatorname{cov}(A)$. The collection of all admissible subsets of M is then denoted by $\mathcal{A}(M)$.

Admissible subsets of hyperconvex spaces enjoy of a very large number of properties. These properties were very relevant when studying approximation problems and fixed point results; see for instance [21,33,46,61,62] among others. Obviously, the class of admissible sets is closed under arbitrary intersections.

Definition 2.9 A subset E of a metric space M is said to be externally hyperconvex (relative to M) if given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of positive real numbers satisfying

$$d(x_\alpha, x_\beta) \le r_\alpha + r_\beta \quad \text{and} \quad \operatorname{dist}(x_\alpha, E) \le r_\alpha;$$

it follows that

$$\bigcap_\alpha B(x_\alpha; r_\alpha) \cap E \neq \emptyset.$$

The class of externally hyperconvex sets relative to M will be denoted as $\mathcal{E}(M)$.

Externally hyperconvex subsets were first introduced in [8] and used for first time in fixed point theory in [37] by Khamsi, Kirk and Martínez-Yáñez. It was proved in [37] that a nonexpansive multivalued mapping with externally hyperconvex values admits nonexpansive selections. This result was then applied to obtain existence of fixed points for multivalued mappings (see also [21, Section 9]). The next result, which is immediate to prove, relates admissible and externally hyperconvex sets.

Lemma 2.10 *Let M be a hyperconvex space; then any admissible subset of M is also an externally hyperconvex subset of M.*

It is easy to find examples that show that the converse to this lemma is not true. A proof for the next theorem can be found in [21, p. 426].

Theorem 2.11 *Let E be a nonempty externally hyperconvex subset of a metric space M. Then E is a proximinal nonexpansive retract of M.*

3 Further properties and basic fixed point results in hyperconvex spaces

In this section we present a brief description of some of the most fundamental fixed point theorems for single and multivalued nonexpansive mappings; some other properties of hyperconvex metric spaces will also be recalled. A more detailed exposition of these results can be found in [21] (see also [36, Chapter 4]). The first versions of Kirk's fixed point theorem for hyperconvex spaces were independently obtained by Sine [60] and Soardi [64]. These results can be summarized in the next one.

Theorem 3.1 *Let M be a nonempty bounded hyperconvex metric space and $T: M \to M$ a nonexpansive mapping. Then T has a fixed point. Moreover, the set of fixed points of T, that is,*

$$\mathrm{Fix}(T) = \{x \in M : Tx = x\}$$

is a hyperconvex subset of M.

A more general and very celebrated version of this result on ε-fixed points was later proved by Sine [63] or [36, Section 4.6].

Definition 3.2 Given $T: M \to M$ we say that $x \in M$ is an ε-fixed point of T if $d(x, Tx) \leq \varepsilon$. The set of ε-fixed points is denoted by $\mathrm{Fix}_\varepsilon(T)$.

Theorem 3.3 *Let M be a bounded hyperconvex metric space and $T: M \to M$ a nonexpansive mapping. Then, for $\varepsilon \geq 0$, $\mathrm{Fix}_\varepsilon(T)$ is nonempty and hyperconvex.*

The next theorem was proved by Baillon [9] and gives an extended version of Theorem 3.1.

Theorem 3.4 *Let M be a bounded hyperconvex metric space. Any commuting family of nonexpansive mappings $\{T_i\}_{i\in I}$, with $T_i: M \to M$, has a common fixed point. Moreover, the common fixed point set is hyperconvex.*

Baillon proved also in [9] the next very important property on hyperconvex sets.

Theorem 3.5 *Let $\{H_\alpha\}_{\alpha\in\mathcal{A}}$ be decreasing (with respect to the set inclusion) family of nonempty and bounded hyperconvex sets; then*

$$\bigcap_{\alpha\in\mathcal{A}} H_\alpha \neq \emptyset \textit{ and hyperconvex.}$$

To obtain a counterpart of the fixed point theorem for nonexpansive mappings for multivalued nonexpansive mappings, we need first to introduce the Pompeiu–Hausdorff distance (for more on this see [36, p. 24], [26, p. 19] or [3, p. 72]).

Definition 3.6 Let M be a metric space and let $\mathcal{M}$ denote the family of all nonempty bounded closed subsets of M. For $A \in \mathcal{M}$ and $\varepsilon > 0$ define the ε-neighborhood of A to be the set

$$N_\varepsilon(A) = \{\varepsilon > 0 : \mathrm{dist}(x, A) < \varepsilon\}.$$

Now for $A, B \in \mathcal{M}$ set

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}.$$

Then $(\mathcal{M}, H)$ is a metric space and H is called the Pompeiu–Hausdorff distance on $\mathcal{M}$.

A nonexpansive multivalued mapping is then defined as follows:

Definition 3.7 Let M be a metric space and $T: M \to 2^M$ a multivalued mapping with closed values. T is said to be nonexpansive if

$$H(Tx, Ty) \leq d(x, y).$$

The question now is under which conditions a multivalued nonexpansive mapping has fixed points. The study of this problem was initiated by Nadler in [55] for Banach spaces (see also [26, Chapter 15]). It turned out that hyperconvex spaces were very adequate to deal with multivalued mappings. More general versions of the next result can be found in [37] (see also [62] or [21]).

Theorem 3.8 *Let M be a hyperconvex space and $T: M \to \mathcal{E}(M)$ a nonexpansive multivalued mapping. Then there exists a nonexpansive selection $f: M \to M$, that is, a nonexpansive mapping such that $f(x) \in Tx$ for all $x \in M$.*

The same result was proved for admissible valued mappings in [62]; it is an open question posed in [62] whether this theorem remains true if the images are supposed to be hyperconvex instead of admissible sets.

As a corollary of this theorem and Theorem 3.1, we have the fixed point result for nonexpansive multivalued mappings (for more on this see [21, Section 9]).

Corollary 3.9 *Let M be a nonempty bounded hyperconvex space. If $T\colon M \to \mathcal{E}(M)$ is a nonexpansive multivalued map then its set of fixed points is nonempty and hyperconvex.*

Classical fixed point topological results as those of Darbo–Sadovski or Schauder theorem have also found their counterparts in hyperconvex spaces. This survey focuses on metric fixed point theory, so it is not our goal to cover recent advances on topological fixed point theory for hyperconvex spaces. The reader can find an exposition of this kind of results in [21, Section 7].

Theorem 3.10 *Let M be a compact hyperconvex metric space and $T\colon M \to M$ a continuous mapping. Then T has a fixed point.*

Different interesting papers on extensions of the above result have appeared after the publication of [21]; some of them are [11,14,19,34,46]. Also a number of works have been published about KKM mappings on hyperconvex spaces after [33] as, for instance, [15].

4 Proximinal nonexpansive retracts and best proximity pairs

One of the subjects that received more attention in Chapter 13 [21] of the *Handbook in Metric Fixed Point Theory* was the question of characterizing proximinal nonexpansive retracts of hyperconvex spaces. This question was not solved by then, but now we know its solution. This is a kind of result with important connections to several problems in best approximation results and best proximity pairs. We begin this section describing the problems of best approximation and best proximity pairs.

We know that fixed point theory is an important tool for solving equations $Tx = x$ for mappings $T\colon D \to M$ where D is a subset of M. However, if T does not have fixed points, then one often tries to find an element x which is in some sense closest to Tx. A classical result in this direction is the very well known Ky Fan best approximation theorem. Best approximation results for nonself mappings try to find x such that $d(x, Tx) = \mathrm{dist}(Tx, D)$. Notice that these results need not give optimal solutions to the best approximation problem, that is, those minimizing $d(x, Tx)$. In contrast to this, best proximity pair theorems provide approximate solutions that are optimal. A few advances have been obtained for hyperconvex spaces about existence of best proximity pairs in the last ten years [6,20,38,47,52,54]. In this section we collect some of them.

Definition 4.1 Let M be a metric space and let A and B be nonempty subsets of M. Let

$$A_0 = \{x \in A\colon\ d(x, y) = \mathrm{dist}(A, B) \text{ for some } y \in B\};$$
$$B_0 = \{x \in B\colon\ d(x, y) = \mathrm{dist}(A, B) \text{ for some } y \in A\}.$$

A pair $(x, y) \in A_0 \times B_0$ for which $d(x, y) = \mathrm{dist}(A, B)$ is called a best proximity pair for A and B.

The basic questions about the problem of best proximity pairs are

1. When are A and B mutually proximinal in the sense that there exists a best proximity pair $(a, b) \in A_0 \times B_0$?
2. Given that A and B are mutually proximinal and given a mapping $T\colon A \to 2^B$, when does the mapping T have a best proximity pair solution, that is, when does there exist a best proximity pair $(x, y) \in A_0 \times B_0$ such that $y \in Tx$?

It was shown in [47] that this problem can be solved if some of sets under consideration are proximinal nonexpansive retracts. Being proximinal nonexpansive retract is a very unusual property in general spaces but not in hyperconvex spaces. Theorem 2.11 states that externally hyperconvex sets are proximinal nonexpansive retracts. The complete characterization of such retracts was given in [20]. To show this characterization we need to introduce a new class of sets: these sets were first introduced in [23] and received the name of *weakly externally hyperconvex* sets.

Definition 4.2 A subset E of a metric space M is said to be weakly externally hyperconvex (relative to M) if E is externally hyperconvex relative to $E \cup \{z\}$ for each $z \in M$.

More precisely, given any family $\{x_\alpha\}$ of points in M all but at most one of which lies in E, and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta, \text{ with } \operatorname{dist}(x_\alpha, E) \leq r_\alpha \text{ if } x_\alpha \notin E;$$

it follows that

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \cap E \neq \emptyset.$$

It directly follows from the definition that weakly externally hyperconvex subsets are proximinal. At this point it is interesting to note that when the three classes of subsets so far presented are subsets of the same hyperconvex metric space M, then they are related in the following way: let A be a subset of M; then

$$\begin{aligned} A \text{ is admissible (in } M) &\Rightarrow A \text{ is externally hyperconvex (relative to } M) \\ &\Rightarrow A \text{ is weakly externally hyperconvex (relative to } M) \\ &\Rightarrow A \text{ is hyperconvex.} \end{aligned}$$

The next definition was introduced in [63].

Definition 4.3 Let A be a subset of a metric space M. A mapping $R\colon A \to M$ is said to be ε-constant if $d(x, R(x)) \leq \varepsilon$ for each $x \in A$.

For A as above the ε-neighborhood of A is defined as follows:

$$N_\varepsilon(A) = \bigcup_{a \in A} B(a, \varepsilon).$$

The following fact was proved in [23] and already explained in [21, Section 4]:

Lemma 4.4 *Let A be a weakly externally hyperconvex subset of a hyperconvex metric space M; then for any $\varepsilon > 0$ the set $N_\varepsilon(A)$ is weakly externally hyperconvex and there is an ε-constant nonexpansive retraction of $N_\varepsilon(A)$ on A.*

The most it was known about weakly externally hyperconvex spaces and nonexpansive proximinal retractions in [23] is summarized in the following result which was already contained in [21, Section 4].

Theorem 4.5 *Suppose A is a weakly externally hyperconvex subset of a metrically convex metric space M. Then given any $\varepsilon > 0$ there exists a nonexpansive retraction $R\colon M \to A$ with the property that if $u \in M \setminus A$ there exists $v \in M \setminus A$ with $d(v, R(v)) = \operatorname{dist}(v, A)$ and $d(u, v) \leq \varepsilon$.*

For a subset A of a metric space M and $\varepsilon > 0$ we use S_ε to denote the level set

$$S_\varepsilon = \{u \in M : \operatorname{dist}(u, A) = \varepsilon\}.$$

The next two technical results were given in [20]. The first one is a consequence of the proof of Theorem 4.5.

Lemma 4.6 *Let $\varepsilon > 0$, A and M as above, and $S = \bigcup_{n \in \mathbb{N}} S_{n\varepsilon}$. Then the retraction given by Theorem 4.5 can be chosen so that $d(v, R(v)) = \operatorname{dist}(v, A)$ for any $v \in S$.*

We include the proof of the next lemma for completeness.

Lemma 4.7 *Let A be a weakly externally hyperconvex subset of a metrically convex metric space M. For each $n \in \mathbb{N}$ let $\varepsilon_n = \frac{1}{2^n}$; then there exists a nonexpansive retraction r_n (associated with ε_n) as in Corollary 4.6 such that the sequence of retractions $\{r_n\}$ satisfies that*

$$d(r_n(x), r_m(x)) \leq \sum_{j=n+1}^{j=m} \frac{1}{2^j}$$

for $x \in M$ and $n < m$. Moreover, $\{r_n(x)\}$ is convergent for each $x \in M$.

Proof For $n = 1$ we take r_1 as the one given by Lemma 4.6. We prove next that given r_i for $1 \le i \le n$ as in the statement of the lemma we can construct r_{n+1} as required. We consider $S_{\varepsilon_{n+1}}$ and proceed as in Theorem 4.5. After applying Zorn's Lemma we may assume that $H_{\varepsilon_{n+1}}$ is the maximal subset of $S_{\varepsilon_{n+1}}$ where r_{n+1} can be extended as required. Then we need to prove that $S_{\varepsilon_{n+1}} = H_{\varepsilon_{n+1}}$. Suppose that there exits $v \in S_{\varepsilon_{n+1}} \setminus H_{\varepsilon_{n+1}}$ and let

$$P(v) = (\cap_{x\in A} B(x, d(x, v))) \cap (\cap_{u\in H_{\varepsilon_{n+1}}} B(r_{n+1}(u), d(u, v))) \\ \cap B\left(v, \frac{1}{2^{n+1}}\right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right) \cap A.$$

All we need to prove is that $P(v) \neq \emptyset$. Since A is weakly hyperconvex and only one of the above balls is centered outside A, it is enough to check that each two of such balls have nonempty intersection. In a case-by-case check it only remains to study those cases involving the ball centered at $r_n(v)$; other cases were already studied in [23]. For these cases it is enough to recall that $d(x, r_n(v)) \le d(x, v)$ for $x \in A$, now, since A is proximinal, let $p_v \in A$ such that $d(v, p_v) = \mathrm{dist}(v, A)$, so

$$d(v, r_n(v)) \le d(v, p_v) + d(p_v, r_n(v)) \\ \le 2\mathrm{dist}(v, A) = \frac{1}{2^n},$$

and finally, for $u \in H_{\varepsilon_{n+1}}$,

$$d(r_{n+1}(u), r_n(v)) \le d(r_{n+1}(u), r_n(u)) + d(r_n(u), r_n(v)) \\ \le \text{(by induction hypothesis)}\ \frac{1}{2^{n+1}} + d(u, v).$$

So we can consider r_{n+1} defined on the whole $S_{\varepsilon_{n+1}}$ as required. Next we show how to extend r_{n+1} to $A \cup S_{\varepsilon_{n+1}} \cup S_{2\varepsilon_{n+1}}$. Let $v \in S_{2\varepsilon_{n+1}} = S_{\varepsilon_n}$; then the set

$$P(v) = (\cap_{x\in A} B(x, d(x, v))) \cap (\cap_{u\in S_{\varepsilon_{n+1}}} B(r_{n+1}(u), d(u, v))) \\ \cap B\left(v, \frac{1}{2^n}\right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right) \cap A$$

is nonempty since $d(r_n(v), x) \le d(v, x)$ for $x \in A$,

$$d(r_{n+1}(u), r_n(v)) \le d(r_{n+1}(u), r_n(u)) + d(r_n(u), r_n(v)) \\ \text{(by induction)} \le \frac{1}{2^{n+1}} + d(u, v),$$

and, since the metric convexity of M implies that there exists $\hat{v} \in S_{\varepsilon_{n+1}}$ such that $d(v, \hat{v}) = \frac{1}{2^{n+1}}$, we have

$$d(v, r_n(v)) \le d(v, \hat{v}) + d(\hat{v}, r_n(\hat{v})) + d(r_n(\hat{v}), r_n(v)) \\ \le \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Now, by selecting a point in $P(v)$ it is possible to extend r_{n+1} as required from $S_{\varepsilon_{n+1}}$ to $S_{\varepsilon_{n+1}} \cup \{v\}$. This same argument shows how to extend r_{n+1} to $A \cup S_{\varepsilon_{n+1}} \cup S_{2\varepsilon_{n+1}}$ onto A as required.

Let $S = \cup_{i=1}^{\infty} S_{i\varepsilon_{n+1}}$. By proceeding as above but selecting $\hat{v} \in S_{(i-1)\varepsilon_{n+1}}$ for $v \in S_{i\varepsilon_{n+1}}$, and using induction it follows that there exists a nonexpansive retraction r_{n+1} of $A \cup S$ onto A as required. Let $v \in M\setminus(A \cup S)$; then we consider the set

$$P(v) = \left(\bigcap_{x\in A\cup S} B(r_{n+1}(x), d(x, v))\right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right).$$

$P(v)$ is nonempty from the hyperconvexity of A and the fact that, by induction hypothesis,

$$d(r_{n+1}(x), r_n(v)) \le d(r_{n+1}(x), r_n(x)) + d(r_n(x), r_n(v)) \\ \le d(x, v) + \frac{1}{2^{n+1}}.$$

Again, using induction it follows that r_{n+1} can defined on M as required. Hence

$$d(r_{n+1}(x), r_n(x)) \le \frac{1}{2^{n+1}}$$

for $x \in M$. Now let $\{r_n\}$ be the sequence of retractions given by the above procedure; then for $m > n$ and $x \in M$

$$\begin{aligned} d(r_m(x), r_n(x)) &\le \sum_{j=n+1}^{j=m} d(r_j(x), r_{j-1}(x)) \\ &\le \sum_{j=n+1}^{j=m} \frac{1}{2^j}. \end{aligned}$$

Hence the proof of the lemma is complete. □

The next result answers a question from [23] in the affirmative and basically characterizes proximinal nonexpansive retracts of hyperconvex spaces.

Theorem 4.8 *Let A be a complete weakly externally hyperconvex subset of a metrically convex metric space M. Then A is a proximinal nonexpansive retract of M.*

Proof Let $\{r_n\}$ be the sequence of retractions given by Lemma 4.7; then we define the mapping $r\colon M \to A$ as

$$r(x) = \lim_{n\to\infty} r_n(x).$$

From the convergence given in Lemma 4.7, r is a well-defined retraction on A. Moreover, since r_n is nonexpansive for each $n \in \mathbb{N}$, r is nonexpansive. Additionally we claim that $d(r(x), x) = \mathrm{dist}(x, A)$ for $x \in M$. For $x \in A$ there is nothing to prove, so let $x \in M \setminus A$. For each $n \in \mathbb{N}$ there exists $v_n \in M \setminus A$ such that

$$\begin{aligned} d(x, v_n) &\le \frac{1}{2^n} \quad \text{and} \\ d(v_n, r_n(v_n)) &= \mathrm{dist}(v, A). \end{aligned}$$

Hence we have

$$\begin{aligned} d(x, r_n(x)) &\le d(x, v_n) + d(v_n, r_n(v_n)) + d(r_n(v_n), r_n(x)) \\ &\le \frac{1}{2^n} + \mathrm{dist}(v_n, A) + \frac{1}{2^n} \\ &\le \frac{1}{2^n} + \mathrm{dist}(x, A) + d(v_n, x) + \frac{1}{2^n} \\ &= \frac{3}{2^n} + \mathrm{dist}(x, A). \end{aligned}$$

Taking limit as $n \to \infty$ the conclusion follows. □

Since Theorem 2.1 of [23] implies that proximinal nonexpansive retracts of hyperconvex spaces are weakly externally hyperconvex, the previous theorem can be re-written in the following way:

Theorem 4.9 *Let M be a hyperconvex metric space and $A \subseteq M$ nonempty. Then A is a proximinal nonexpansive retract of M if, and only if, A is a weakly externally hyperconvex subset of M.*

Now these results were applied, following the approach from [47], to the problem of best proximity pairs.

Lemma 4.10 *Let M be a hyperconvex metric space and let A and B be nonempty weakly externally hyperconvex subsets of M. Then A_0 and B_0 are nonempty and hyperconvex.*

Proof Since B is weakly externally hyperconvex, there exists a proximinal nonexpansive retraction $P_B\colon M \to B$. Let p_B denote the restriction of this retraction to A. Let $d = \text{dist}(A, B)$ and let $\varepsilon_n = d + 1/n$. Consider the set

$$A_n := \{x \in A\colon d(x, p_B(x)) \le \varepsilon_n\}.$$

Obviously A_n is nonempty. We claim that A_n is hyperconvex. To see this, let $\{x_\alpha\}$ be points of A_n and let $\{r_\alpha\} \subseteq \mathbb{R}^+$ be such that

$$d(x_\alpha, x_\beta) \le r_\alpha + r_\beta.$$

Let D be the weakly externally hyperconvex set

$$D := \cap B(x_\alpha, r_\alpha) \cap A.$$

Clearly D is nonempty since A is hyperconvex. By Lemma 4.4 there exists an ε_n-constant retraction π that maps $N_{\varepsilon_n}(D)$ onto D. If $x \in D$ then $p_B(x) \in N_{\varepsilon_n}(D)$ because

$$\begin{aligned} d(p_B(x), x) &\le d(p_B(x), p_B(x_\alpha)) + d(p_B(x_\alpha), x_\alpha) \\ &d(x, x_\alpha) + \varepsilon_n \le r_\alpha + \varepsilon_n. \end{aligned}$$

Now define $R\colon D \to D$ by setting $R = \pi \circ p_B$. Since R is nonexpansive, it has a fixed point $x_0 \in D$. But π is ε_n-constant, so

$$d(x_0, p_B(x_0)) = d(\pi \circ p_B(x_0), p_B(x_0)) \le \varepsilon_n.$$

Therefore, $x_0 \in \cap B(x_\alpha, r_\alpha) \cap A_n$, which proves that A_n is hyperconvex.

We now have a descending sequence $\{A_n\}$ of nonempty bounded hyperconvex subsets of A and so $\cap A_n$ is nonempty and hyperconvex. Clearly,

$$u \in \cap_{n=1}^{\infty} A_n \iff d(u, p_B(u)) = d = \text{diam}(A, B) \iff u \in A_0$$

and so $p_B(u) \in B_0$. □

Best proximity pair results follow from these results.

Theorem 4.11 *Let A and B be two weakly externally hyperconvex subsets of a hyperconvex metric space M with A bounded, and suppose $T^*\colon A \to 2^B$ is such that*

(i) *for each $x \in A$, $T^*(x)$ is a nonempty admissible (more generally, externally hyperconvex) subset of B;*
(ii) *$T^*\colon (A, d) \to (2^B, H)$ is nonexpansive (where H is the Hausdorff distance);*
(iii) *$T^*(A_0) \subseteq B_0$.*

Then there exists $x_0 \in A$ such that

$$\text{dist}(x_0, T^*(x_0)) = \text{dist}(A, B) = \inf\{\text{dist}(x, T^*(x))\colon x \in A\}.$$

Proof We know that A_0 is nonempty. By Theorem 3.8, T^* admits a nonexpansive selection T. Let r be a nonexpansive proximinal retraction of M onto A and $x \in A_0$. Since $r \circ T(x) \in A$ is a best approximation to $T(x)$ it follows that $T(x) \in B_0$. Therefore, there exists $a \in A$ such that

$$d(T(x), a) = \text{dist}(A, B).$$

On the other hand, $d(T(x), r \circ T(x)) \le d(T(x), a)$. Therefore,

$$d(T(x), r \circ T(x)) = \text{dist}(A, B).$$

Thus $r \circ T\colon A_0 \to A_0$ and so there exists $x_0 \in A_0$ such that $r \circ T(x_0) = x_0$. It follows then that $(x_0, T(x_0))$ is a best proximity pair for T and hence also for T^*. □

A topological version of this result can be proved in the similar way and reads as follows:

Theorem 4.12 *Let A and B be two weakly externally hyperconvex subsets of a hyperconvex metric space M with A compact, and suppose $T^*\colon A \to 2^B$ is such that*

(i) *for each* $x \in A$, $T^*(x)$ *is a nonempty admissible (more generally, externally hyperconvex) subset of* B*;*
(ii) $T^*\colon (A, d) \to (2^B, H)$ *is continuous;*
(iii) $T^*(A_0) \subseteq B_0$.

Then there exists $x_0 \in A$ *such that*

$$\mathrm{dist}(x_0, T^*(x_0)) = \mathrm{dist}(A, B) = \inf\{\mathrm{dist}(x, T^*(x))\colon x \in A\}.$$

The next theorem is a version of this result under milder topological conditions, see [54] for its proof.

Theorem 4.13 *Let* A *be an admissible subset and* B *a bounded externally hyperconvex subset of a hyperconvex metric space* M*. Assume* $T^*\colon A \to 2^B$ *is such that*

(i) *for each* $x \in A$, $T^*(x)$ *is an admissible subset of* B*;*
(ii) $T^*\colon (A, d) \to (2^B, H)$ *is continuous and condensing;*
(iii) $T^*(x) \cap B_0 \neq \emptyset$ *for each* $x \in A_0$.

Then there exists $x_0 \in A$ *such that*

$$\mathrm{dist}(x_0, T^*(x_0)) = \mathrm{dist}(A, B) = \inf\{\mathrm{dist}(x, T^*(x))\colon x \in A\}.$$

As a consequence of this kind of results we can obtain best approximation results in the spirit of Ky Fan lemma. The next is an example of these results. Many other Ky Fan results in hyperconvex spaces can be found, for instance, in [52].

Corollary 4.14 *Let* A *be a bounded weakly externally hyperconvex subset of a hyperconvex metric space* M *and suppose that* $T\colon A \to M$ *is a nonexpansive mapping. Then there exists* $x \in A$ *such that*

$$d(x, T(x)) = \inf\{d(y, T(x))\colon\ y \in A\}.$$

Another series of interesting related works on this kind of problems are [4,6,7]. In these works condensing conditions as well as lower and upper semicontinuous conditions from multivalued mappings are considered. Two results taken from these works are the following:

Theorem 4.15 *Let* M *be a hyperconvex metric space. Suppose that* $F\colon M \to 2^M$ *is an upper semicontinuous condensing multivalued mapping with nonempty closed acyclic values. Then* F *has a fixed point.*

Theorem 4.16 *Let* M *be a hyperconvex metric space and* A *a nonempty admissible subset of* M*. Let* $F\colon A \to 2^M$ *be a continuous condensing multivalued mapping with nonempty bounded externally hyperconvex values and* $G\colon A \to 2^A$ *an onto, quasiadmissible multivalued mappings for which* $G(B)$ *is closed for each closed set* $B \subseteq A$*. Assume that* $G^-\colon A \to A$ *is a* 1*-set contraction. Then there exists an* $x_0 \in A$ *such that*

$$\mathrm{dist}(G(x_0), F(x_0)) = \inf_{x\in X} \mathrm{dist}(x, F(x_0)).$$

5 $\mathbb{R}$-trees and fixed points

Complete $\mathbb{R}$-trees can be regarded as a subclass of hyperconvex metric spaces; this fact was made clear by Kirk in [40]. However, $\mathbb{R}$-trees are important elements on their own. In fact they have been extensively applied to graph discrete collections of data and very specifically have been applied in phylogenetics where they are one of the most relevant tools for modeling. Fixed point results for $\mathbb{R}$-tree metric spaces have existed since long ago, see for instance [56], but the interest of theorists of metric fixed point theory on these spaces did not begin until 2006 with [21]. That was the initial point for a burst on the systematic study of both topological and metric fixed point properties of $\mathbb{R}$-trees. The very peculiar structure of these spaces have allowed researchers to find results much more powerful than their counterparts for other kinds of spaces as normed spaces. It is also worth to recall that another burst in fixed point theory in the past decade has been given by the so-called CAT(0)-spaces and that $\mathbb{R}$-tree are CAT(0)-spaces too. We will not cover in this survey this fruitful and very interesting branch of metric fixed point theory on CAT(0)-spaces neither CAT(0)-spaces themselves; for a very extensive treatment on CAT(0)-spaces the interested reader may check [13].

In this section we take as our starting point results given in [21], where the first results for nonexpansive mappings on $\mathbb{R}$-trees were shown, to continue with the amazing sequence of results obtained by different authors after it. Results inspiring this section can be found in some of these references [1,5,12,17,40–42,44,45,57,58]. The fact that compact $\mathbb{R}$-trees have the fixed point property for continuous maps goes back to Young [66]. Some familiarity with geodesic spaces is assumed in this section, for details see [13].

We recall now the definition of $\mathbb{R}$-trees, also known as *metric trees*.

Definition 5.1 An $\mathbb{R}$-tree is a metric space T such that

(i) there is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in T$;
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.
From (i) and (ii) it is easy to deduce
(iii) If $p, q, r \in T$, then $[p, q] \cap [p, r] = [p, w]$ for some $w \in M$.

It is not hard to give examples of $\mathbb{R}$-trees; however, the following two examples are among the best well known:

Example 5.2 $\mathbb{R}^2$ with the metric "river", consider the following metric on $\mathbb{R}^2$:

$$d(v_1, v_2) = \begin{cases} | y_1 - y_2 |, & \text{if } x_1 = x_2, \\ | y_1 | + | y_2 | + | x_1 - x_2 |, & \text{if } x_1 \neq x_2, \end{cases}$$

where $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a complete $\mathbb{R}$-tree.

Example 5.3 $\mathbb{R}^2$ with the radial metric, consider the following metric on $\mathbb{R}^2$:

$$d(v_1, v_2) = \begin{cases} \rho(v_1, v_2), & \text{if } 0 = (0, 0), v_1, v_2 \text{ are colinear}, \\ \rho(v_1, 0) + \rho(v_2, 0), & \text{otherwise}, \end{cases}$$

where ρ denotes the usual Euclidean metric in $\mathbb{R}^2$ and $v_1, v_2 \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a complete $\mathbb{R}$-tree.

The following theorem can be found in [40]:

Theorem 5.4 *A metric space is a complete $\mathbb{R}$-tree if, and only if, it is hyperconvex and has unique metric segments joining its points.*

The main goal of [22] was to suggest a new metric approach to the classical fixed edge theorem of Nowakowski and Rival [56]. The first fact that is noticed in this work is the relation between $\mathbb{R}$-trees and gated sets.

Definition 5.5 Let M be a metric space and $A \subseteq M$. A is said to be gated if for any point $x \notin A$ there exists a unique point $x_A \in A$ (called the gate of x in A) such that for any $z \in A$,

$$d(x, z) = d(x, x_A) + d(x_A, z).$$

It is immediate to see that gated sets in a complete geodesic space are always closed and convex. (Remember that a *convex set* in a uniquely geodesic metric space is any set which contains any segment with endpoints in the same set.) Moreover, it was noticed in [18] that gated subsets of a complete geodesic space X are proximinal nonexpansive retracts of X. The next lemma is not hard to prove.

Lemma 5.6 *Gated subsets of an $\mathbb{R}$-tree are precisely its closed and convex subsets.*

Next we present a surprising fact from gated sets which is at the heart of many results obtained for $\mathbb{R}$-trees. Usually results asserting that a certain collection of descending closed sets has nonempty intersection require that the sets are bounded; that is not necessarily the case for gated sets.

Proposition 5.7 *Let M be a complete geodesic space, and let $\{H_\alpha\}_{\alpha\in\Lambda}$ be a collection of nonempty gated subsets of M which is directed downward by set inclusion. If M (or more generally, some H_α) does not contain a geodesic ray, then*

$$\bigcap_{\alpha\in\Lambda} H_\alpha \neq \emptyset.$$

Proof Let $H_0 \in \{H_\alpha\}_{\alpha\in\Lambda}$, select $x_0 \in H_0$ and let

$$r_0 = \sup\{\mathrm{dist}(x_0, H_0 \cap H_\alpha) : \alpha \in \Lambda\}.$$

If $x_0 \in \cap_{\alpha\in\Lambda} H_\alpha$ we are finished. Otherwise, choose $H_1 \in \{H_\alpha\}_{\alpha\in\Lambda}$ so that, $H_1 \subset H_0$, $x_0 \notin H_1$, and

$$\mathrm{dist}(x_0, H_1) \geq \begin{cases} r_0 - 1 & \text{if } r_0 < \infty; \\ 1 & \text{if } r_0 = \infty \end{cases}$$

Now take x_1 to be the gate of x_0 in H_1. Having defined x_n, let

$$r_n = \sup\{\mathrm{dist}(x_n, H_n \cap H_\alpha) : \alpha \in \Lambda\}.$$

Now choose $H_{n+1} \in \{H_\alpha\}_{\alpha\in\Lambda}$ so that $x_n \notin H_{n+1}$ (if possible), $H_{n+1} \subset H_n$, and

$$\mathrm{dist}(x_n, H_{n+1}) \geq \begin{cases} r_n - \dfrac{1}{n} & \text{if } r_n < \infty; \\ 1 & \text{if } r_n = \infty \end{cases}$$

Now take x_{n+1} to be the gate of x_n in H_{n+1}. Either this process terminates after a finite number of steps, yielding a point $x_n \in \cap_{\alpha\in\Lambda} H_\alpha$, or we have sequences $\{x_n\}$, $\{H_n\}$ for which $i < j \Rightarrow x_j$ is the gate of x_i in H_j. Since X does not contain a geodesic ray, it must be the case that $r_n < \infty$ for some n (and hence for all n). By transitivity of gated sets the sequence $\{x_n\}$ is linear and thus lies on a geodesic in X. Since X does not contain a geodesic ray, the sequence $\{x_n\}$ must in fact be Cauchy. Let $x_\infty = \lim_n x_n$. Since each of the sets H_n is closed, clearly $x_\infty \in \cap_{n=1}^{\infty} H_n$. Also $\sum_{n=1}^{\infty} r_n < \infty$, so $\lim_n r_n = 0$.

Now let P_α, $\alpha \in \Lambda$, be the nearest point projection of X onto H_α, and for each $n \in \mathbb{N}$, let $y_n = P_\alpha(x_n)$. Then $d(y_n, x_n) \leq r_n$, and since P_α is nonexpansive, for any $m, n \in \mathbb{N}$, $d(y_n, y_m) \leq d(x_n, x_m)$. It follows that $P_\alpha(x_\infty) = x_\infty$ for each $\alpha \in \Lambda$. Therefore, $x_\infty \in \cap_{\alpha\in\Lambda} H_\alpha$. □

Proposition 5.8 *Let M be a complete geodesic space, and let $\{H_n\}$ be a descending sequence of nonempty gated subsets of M. If $\{H_n\}$ has a bounded selection, then*

$$\bigcap_{n=1}^{\infty} H_n \neq \emptyset.$$

Proof Here we simply describe the step-by-step procedure. Let $\{z_n\}$ be a bounded selection for $\{H_n\}$. Let $x_0 = z_0$. Then let n_1 be the smallest integer such that $x_0 \notin H_{n_1}$. Let x_1 be the gate of x_0 in H_{n_1} and take $x_2 = z_{n_1}$. Now take n_2 to be the smallest integer such that $x_2 \notin H_{n_2}$ and take x_3 to be the gate of x_2 in H_{n_2}. Continuing this procedure inductively it is clear that one generates a sequence $\{x_n\}$ which is isometric to an increasing sequence of positive numbers on the real line. Since $\{x_{2n}\}$ is a subsequence of the bounded sequence $\{z_n\}$ it must be the case that $\{x_n\}$ is also bounded. Therefore, $\lim_n x_n$ exists and lies in $\cap_{n=1}^{\infty} H_n$. □

The next result also stands for gated sets and was noticed by Markin in [53].

Theorem 5.9 *Let M be a complete geodesic space with a convex metric (so, in particular, balls contain geodesic segments with endpoints in the given ball) and T a multivalued mapping with values that are bounded gated subsets of M. Then there is a mapping $f : M \to M$ such that $f(x) \in Tx$ for each $x \in M$ and $d(f(x), f(y)) \leq H(Tx, Ty)$ for each $x, y \in M$, where H stands for the Hausdorff distance.*

Proof For each $z \in M$ define the mapping $f : M \to M$ such that $f(x)$ is the unique closest point to z in Tx, that is, $f(x) = P_{Tx}z$. Take $\alpha = d(z, f(x))$ and $\beta = d(z, f(y))$ and assume that $\alpha \geq \beta$. Therefore $f(y) \in B(z, \alpha)$. Let p be the gate of $f(y)$ in Tx. Since $d(f(y), f(x)) = d(f(y), p) + d(p, f(x))$ the point p lies on the geodesic segment connecting $f(x)$ and $f(y)$. By convexity of the metric, this segment is contained in $B(z, \alpha)$. This implies that $p \in B(z, \alpha)$ and so it must be the case that $p = f(x)$. The conclusion trivially follows now. □

The main result in [21] requires the following lemma given in [43]:

Lemma 5.10 *Suppose M is uniquely geodesic with a convex metric, suppose $T: M \to M$ is nonexpansive, and suppose $x_0 \in M$ satisfies*

$$d(x_0, T(x_0)) = \inf\{d(x, T(x)) : x \in M\} > 0.$$

Then the sequence $\{T^n(x_0)\}$ is unbounded and lies on a geodesic ray.

Theorem 5.11 *Let M be a complete $\mathbb{R}$-tree, and suppose K is a closed convex subset of M which does not contain a geodesic ray. Then every commuting family $\mathfrak{F}$ of nonexpansive self-mappings on K has a nonempty common fixed point set.*

Proof Let $T \in \mathfrak{F}$. We first show that the fixed point set of T is nonempty. Let

$$d = \inf\{d(x, T(x)) : x \in K\}$$

and let

$$F_n := \left\{x \in K : d(x, T(x)) \leq d + \frac{1}{n}\right\}.$$

Since K is a closed convex subset of a complete $\mathbb{R}$-tree, K itself is hyperconvex and $\{F_n\}$ is a descending sequence of nonempty closed convex (hence gated) subsets of K. Since K does not contain a geodesic ray, Proposition 5.7 implies $F := \cap_{n=1}^{\infty} F_n \neq \emptyset$. Therefore, there exists $z \in K$ such that

$$d(z, T(z)) = d.$$

Since K does not contain a geodesic ray, in view of Lemma 5.10, $d = 0$.

Because $\mathbb{R}$-trees are uniquely geodesic, the fixed point set F of T is closed and convex, and hence again an $\mathbb{R}$-tree. Now suppose $G \in \mathfrak{F}$. Since G and T commute it follows that $G : F \to F$, and by applying the preceding argument to G and F we conclude that G has a nonempty fixed point set in F. In particular, the fixed point set of T and the fixed point set of G intersect. Since these are gated sets in X, by the Helly property of gated sets we conclude that every finite subcollection of $\mathfrak{F}$ has a nonempty common fixed point set (which is itself gated). Now let $\mathcal{A}$ be the collection of all finite subcollections of $\mathfrak{F}$, and for $\alpha \in \mathcal{A}$, let H_α be the common fixed point set of α. Then given $\alpha, \beta \in \mathcal{A}$, $H_{\alpha\cup\beta} \subseteq H_\alpha \cap H_\beta$, so clearly the family $\{H_\alpha\}_{\alpha\in\mathcal{A}}$ is directed downward by set inclusion. Since these are all gated sets, we again apply Proposition 5.7 to conclude that $\cap_{\alpha\in\mathcal{A}} H_\alpha \neq \emptyset$, and thus that $\mathfrak{F}$ has a nonempty common fixed point set. □

Remark 5.12 The significance of this result is the fact that K itself is not assumed to be bounded. This result might also be compared with Theorem 32.2 of [28] where it is shown that the complex Hilbert ball with a hyperbolic metric has the fixed point property for nonexpansive mappings if and only if it is geodesically bounded.

Surprisingly enough, Kirk noticed in [42] that when considered Theorem 5.11 on just one mapping the nonexpansiveness condition can be replaced by continuity.

Theorem 5.13 *Let M be a geodesically bounded complete $\mathbb{R}$-tree. Then every continuous mapping $T: X \to X$ has a fixed point.*

Proof For $u, v \in M$ we let $[u, v]$ denote the unique metric segment joining both points and let $[u, v) = [u, v]\backslash\{v\}$. To each $x \in M$ associate $\phi(x)$ as follows: For each $t \in [x, Tx]$, let $\xi(t)$ be the point in M for which

$$[x, Tx] \cap [x, Tt] = [x, \xi(t)].$$

Such a point always exists since M is an $\mathbb{R}$-tree. If $\xi(Tx) = Tx$ take $\phi(x) = Tx$. Otherwise, it must be the case that $\xi(Tx) \in [x, Tx)$. Let

$$A = \{t \in [x, Tx] : \xi(t) \in [x, t]\};$$
$$B = \{t \in [x, Tx] : \xi(t) \in [t, Tx]\}.$$

Now a connectedness reasoning yields that there exists $\phi(x) \in A \cap B$. If $\phi(x) = x$, then $Tx = x$ and we are fare. Otherwise, $x \neq \phi(x)$ and

$$[x, Tx] \cap [x, T\phi(x)] = [x, \phi(x)].$$

Let $x_0 \in M$ and let $x_n = \phi^n(x_0)$. Assuming that the process does not terminate upon reaching a fixed point of T, by construction, the points $\{x_0, x_1, x_2, \ldots\}$ are linear and thus lie on a subset of M which is isometric with a subset of the real line, that is, on a geodesic. Since M does not contain a geodesic of infinite length, it must be the case that $\{x_n\}$ is a Cauchy sequence. Let z be the limit of $\{x_n\}$; then, by continuity, Tz is the limit of $\{Tx_n\}$. Now, by construction,

$$d(Tx_n, Tx_{n+1}) = d(Tx_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1}).$$

From where it finally follows that $z = Tz$. □

The next result that we find in [42] is a non-self version of Theorem 5.13.

Theorem 5.14 *Let M be a complete $\mathbb{R}$-tree and K a closed convex subset of M which does not contain a geodesic ray, suppose $\operatorname{int}(K) \neq \emptyset$ and that $T: K \to M$ is a continuous mapping. Suppose there exists $p_0 \in \operatorname{int}(K)$ such that x is not in $[p_0, Tx]$ for any $x \in \partial K$. Then T has a fixed point in K.*

The technique developed to prove Theorem 5.13 would have a big impact and was applied by many other authors in different versions and extensions of this theorem. Among these new versions those ones dealing with multivalued mappings were to surprise us by the constantly and unexpected weakening of the required conditions.

Definition 5.15 Let $T: X \to 2^Y$ be a multivalued mapping with nonempty values; then

- T is said to be upper semicontinuous at $x_0 \in X$ if for each open $V \subseteq Y$ such that $T(x_0) \subseteq Y$ there exists an open set $U \subseteq X$ with contains x_0 such that

$$T(U) \subseteq V.$$

- T is said to be almost lower semicontinuous at $x_0 \in X$ if for each $\varepsilon > 0$ there is an open neighborhood $U \subseteq X$ of x_0 such that

$$\bigcap_{x \in U} N_\varepsilon(T(x)) \neq \emptyset.$$

- T is said to be ε-semicontinuous at $x_0 \in X$ if for each $\varepsilon > 0$ there is an open neighborhood U of x_0 such that

$$T(x) \cap N_\varepsilon(T(x_0)) \neq \emptyset$$

  for all $x \in U$.

A best approximation result was given by upper semicontinuous mappings by Kirk and Panyanak in [44] while a similar one for almost lower semicontinuous mappings was obtained by Markin in [53]. However, both were unified by Piątek in [57]. In fact the concept of ε-semicontinuity was introduced in [57] with the goal of unifying both results. The following proposition, not very hard to prove, was shown in [57]:

Proposition 5.16 *If a multivalued mapping is either almost lower semicontinuous or upper semicontinuous then it is ε-semicontinuous too.*

Then the following result, which contains those of [44,53] as particular cases, was proved in [57].

Theorem 5.17 *Let M be a complete $\mathbb{R}$-tree and let K be a nonempty convex closed and geodesically bounded subset of M. If $F: K \to 2^K$ is an ε-semicontinuous mapping with nonempty convex closed values then F has a fixed point.*

Proof Let $x \in X$ and let $r(x) = P_{F(x)}(x)$. If x is not a fixed point then $d(x, r(x)) > 0$. For each $t \in [x, r(x)]$ we define $\xi(t)$ as

$$[x, r(x)] \cap [x, r(t)] = [x, \xi(t)].$$

Let

$$A = \{t \in [x, r(x)] \mid \xi(t) \in [x, t]\},$$
$$B = \{t \in [x, r(x)] \mid \xi(t) \in [t, r(x)]\}.$$

Clearly $r(x) \in A$ and $x \in B$ and A and B are closed. Indeed, let (t_n) be a sequence of elements of B such that $t_n \to t$. Assume that $t \in A \setminus B$. Then $d(t, \xi(t)) = \varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $d(t, t_n) < \varepsilon/2$. For each $u \in F(t)$ and $v \in F(t_n)$ we obtain

$$r(t) \in [u, \xi(t)], \quad \xi(t) \in [r(t), \xi(t_n)], \quad \xi(t_n) \in [\xi(t), r(t_n)], \quad r(t_n) \in [\xi(t_n), v].$$

Then we have

$$[\xi(t), \xi(t_n)] \subset [u, v]$$

and finally $\inf_{z \in F(t_n)} d(z, F(t)) \geq d(\xi(t), \xi(t_n)) > \varepsilon/2$ for each $n \in \mathbb{N}$ sufficiently large which contradicts the ε-semicontinuity of F.

Since A is compact there is $\varphi(x) \in [x, r(x)]$ such that $d(x, \varphi(x)) = \inf_{t \in A} d(x, t)$. Moreover, $\varphi(x) \in A \cap B$ what implies that

$$[x, r(x)] \cap [x, r(\varphi(x))] = [x, \varphi(x)]. \tag{1}$$

Now suppose that F has not a fixed point in X. Therefore, we have

$$d(x, \varphi(x)) > 0, \quad x \in X. \tag{2}$$

Let us choose any $x_0 \in X$. We define a transfinite sequence $(x_\alpha)_{\alpha < \Omega}$ such that

$$d(x_0, x_\alpha) = \sum_{\beta < \alpha} d_\beta \tag{3}$$

and

$$d(x_0, \varphi(x_\alpha)) = d(x_0, x_\alpha) + d(x_\alpha, \varphi(x_\alpha)) \tag{4}$$

where Ω is the order type of the set $\{\alpha \mid \bar{\alpha} \leq \aleph_0\}$ and $d_\beta = d(x_\beta, \varphi(x_\beta))$.

Let α be a limit ordinal number. By the geodesically boundedness of X and (3) the countable sum $\sum_{\beta < \alpha} d_\beta$ is bounded. So there is a sequence of points x_{α_n} such that $\lim_{n \to \infty} \sum_{\beta < \alpha_n} = \sum_{\beta < \alpha}$ and $x_{\alpha_n} \to \bar{x} \in X$. Let us define $x_\alpha := \bar{x}$. Clearly (3) and (4) are satisfied. The proof of (4) is not different from the proof of the closedness of B.

If $\alpha = \beta + 1$ we define $x_\alpha := \varphi(x_\beta)$. By (1) with $x = x_\beta$, (3) and (4) we obtain $d(x_0, x_\alpha) = d(x_0, x_\beta) + d(x_\beta, x_\alpha) = \sum_{\gamma < \alpha} d_\gamma$ and $d(x_0, \varphi(x_\alpha)) = d(x_0, x_\alpha) + d_\alpha$.

Now let us define

$$m := \sup_{\alpha < \Omega} \sum_{\beta < \alpha} d_\beta. \tag{5}$$

If m were equal to infinity, points x_α would lie on the geodesic ray. Hence $m < \infty$ and one can find a sequence α_n for which $d(x_0, x_{\alpha_n}) \to m$. Clearly there is $\alpha < \Omega$ such that $\alpha_n < \alpha$ for each $n \in \mathbb{N}$. Moreover, $d(x_0, x_\alpha) = m$ what implies that $d(x_0, x_{\alpha+1}) = d(x_0, \varphi(x_\alpha)) > m$. This contradicts (5). □

The non-self version of this result is also given in [57].

Theorem 5.18 *Let M and K be as in the previous theorem with $F\colon K \to 2^M$ an ε-semicontinuous mapping with nonempty closed convex values. Then there exists a point $x_0 \in K$ for which*

$$dist(x_0, Fx_0) = \inf_{x \in K} dist(x, Fx_0).$$

Also invariant approximation theorems have been given for $\mathbb{R}$-trees. The next one has been taken from [54].

Theorem 5.19 *Let M be a complete $\mathbb{R}$-tree and K a closed bounded convex subset of M. Assume $t\colon K \to K$ and $T\colon K \to 2^M$ are nonexpansive mappings with Tx closed convex and $Tx \cap K \neq \emptyset$ for $x \in K$. If the mappings t and T commute then there is $z \in K$ such that $z = t(z) \in Tz$.*

Selections is also a very important feature of $\mathbb{R}$-trees and hyperconvex spaces [37,62]. This topic was already covered in [21, Section 9]. For the particular case of $\mathbb{R}$-trees check also Theorem 5.9 above or [1]. Very recently, in [25] we can find a result providing nice selections of a kind of generalized nonexpansive notion for multifunction mappings on $\mathbb{R}$-trees.

The nonexpansiveness condition (C), also known as Suzuki condition, was introduced in [65] to study mild nonexpansiveness conditions which still imply existence of fixed points.

Definition 5.20 Let M be a metric space, $K \subseteq M$ and $T\colon K \to X$. Then f satisfies condition (C) if

$$\frac{1}{2}d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le d(x, y),$$

for all $x, y \in K$.

This definition has been adapted for multivalued mappings in different ways. The next one was given in [59].

Definition 5.21 Let M be a metric space and $K \subseteq M$ and $T\colon K \to 2^M$. The mapping T satisfies condition (C) if for each $x, y \in K$ and $u_x \in Tx$ such that

$$\frac{1}{2}d(x, u_x) \le d(x, y),$$

there exists $u_y \in Ty$ such that

$$d(u_x, u_y) \le d(x, y).$$

This condition was extended to the multivalued case by different authors and studied, among other places, in [25]. It was in [25] that a new condition, weakening the multivalued version of (C) and called (C'), was introduced. This condition happened to work especially well for $\mathbb{R}$-trees.

Definition 5.22 Let M be a metric space, $K \subseteq M$ and $T\colon K \to 2^M$. The mapping T satisfies condition (C') if for each $x, y \in K$ and $u_x \in Tx$ with

$$d(x, u_x) = \operatorname{dist}(x, Tx) \quad \text{and} \quad \frac{1}{2}d(x, u_x) \le d(x, y),$$

there exists $u_y \in Ty$ such that

$$d(u_x, u_y) \le d(x, y).$$

We prove next a selection theorem in $\mathbb{R}$-trees for multivalued mappings satisfying condition (C').

Theorem 5.23 *Let M be an $\mathbb{R}$-tree, $K \subseteq M$ and $T : K \to 2^M$ a mapping with nonempty closed and convex values which satisfies (C'). Then the mapping $f : K \to M$ defined by $f(x) = P_{Tx}(x)$ for each $x \in K$ is a selection of T that satisfies condition (C).*

Proof Let $x, y \in K$ such that $f(x) \ne f(y)$ and $(1/2)d(x, f(x)) \le d(x, y)$. Consider $p(x) = P_{Ty}(f(x))$ and $p(y) = P_{Tx}(f(y))$.
First, suppose $p(x) \ne f(y)$ and $p(y) \ne f(x)$. Since $p(x)$ is the projection of $f(x)$ onto Ty it follows that

$$d(f(x), f(y)) = d(f(x), p(x)) + d(p(x), f(y)),$$

i.e., $p(x) \in [f(x), f(y)]$. Since Ty is convex, $[p(x), f(y)] \subseteq Ty$. This implies $[f(x), f(y)] \cap [f(y), y] = \{f(y)\}$ because otherwise the minimality of $f(y)$ would be contradicted. Thus, $f(y) \in [f(x), y]$. Similarly, $f(x) \in [f(y), x]$. Then $f(x), f(y) \in [x, y]$ (otherwise supposing for example that $z \in [x, f(y)] \cap [f(y), y]$ with $z \ne f(y)$ we have that $f(x) \in [z, f(y)]$ and $f(y) \in [z, f(x)]$ which is false). Therefore, $d(f(x), f(y)) \le d(x, y)$. In fact, $d(f(x), f(y)) = d(x, y) - \operatorname{dist}(x, Tx) - \operatorname{dist}(y, Ty)$.
Now assume $p(x) = f(y)$. Then $d(f(x), f(y)) = \operatorname{dist}(f(x), Ty)$ and so, by condition (C'),

$$d(f(x), f(y)) = \operatorname{dist}(f(x), T(y)) \le d(x, y).$$

Finally, suppose $p(x) \neq f(y)$ and $p(y) = f(x)$. As above, if $p(x) \neq f(y)$, we have that $f(y) \in [f(x), y]$. If $(1/2)d(y, f(y)) \leq d(x, y)$ then (C') yields that

$$d(f(x), f(y)) = \text{dist}(f(y), Tx) \leq d(x, y).$$

Otherwise, if $(1/2)d(y, f(y)) > d(x, y)$, then

$$\begin{aligned} d(f(x), f(y)) + 2d(x, y) < d(f(x), f(y)) + d(f(y), y) = d(f(x), y) \leq d(f(x), x) + d(x, y) \\ \leq 2d(x, y) + d(x, y). \end{aligned}$$

Consequently, $d(f(x), f(y)) \leq d(x, y)$. □

An immediate consequence of this result is its fixed point partner.

Corollary 5.24 *Let M be a bounded complete $\mathbb{R}$-tree. Suppose $T : M \to 2^M$ with nonempty closed and convex values satisfies condition (C'). Then* Fix(T) *is a nonempty complete $\mathbb{R}$-tree.*

More on fixed points on $\mathbb{R}$-trees has been done in the past year. Of particular interest are results on uniformly Lipschitzian mappings which can be found in [2,16].

Definition 5.25 Let M be a metric space; then a mapping $T : M \to M$ is said to be k-uniformly Lipschitzian if

$$d(T^n x, T^n y) \leq kd(x, y)$$

for any $x, y \in M$ and $n \in \mathbb{N}$.

The study of uniformly Lipschitzian mappings in metric fixed point theory was initiated by Goebel and Kirk in [27] (see also [26, Chapter 16]). The following result can be found in [2] where, surprisingly enough, it is shown that fixed points for uniformly Lipschitzian mappings in $\mathbb{R}$-trees are guaranteed under less restrictive conditions than, for instance, in Hilbert spaces.

Theorem 5.26 *Let M be a nonempty complete $\mathbb{R}$-tree and $T : M \to M$ a k-uniformly Lipschitzian mapping with bounded orbits and $k < 2$. Then T has a fixed point.*

6 Compact extensions of mappings

One of the main motivations for the seminal work of Aronszajn and Panitchpakdi [8] about extension of uniformly continuous mappings was to find a metric characterization of injectivity, that is, a metric property that characterizes those spaces which may play the role of the real line in Hahn–Banach's theorem. The solution to this problem was given by the intersecting condition of balls defining hyperconvex metric spaces. Therefore, hyperconvex metric spaces are intrinsically related to the notion of extension of operators. The next theorem was proved in [8] and was very generously explained in [21, Section 4].

Definition 6.1 A metric space M is said to be injective if it has the following extension property: whenever Y is a subspace of X and $T : Y \to M$ is nonexpansive, then T has a nonexpansive extension of the whole X.

Theorem 6.2 *Let M be a metric space. The following are equivalent:*

1. *M is hyperconvex*
2. *M is injective.*

Actually, Theorem 6.2 is presented in a more general form in [8]; instead of dealing with nonexpansive mappings, they work with uniformly continuous mappings with subadditive modulus of continuity and find extensions preserving the same modulus of continuity.

Definition 6.3 An extended valued nonnegative function $\delta : [0, +\infty) \to [0, +\infty]$ is said to be a modulus of continuity if it is nondecreasing and $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$. If T is a mapping from a metric space (X, d_1) into a metric space (Y, d_2), we say that $\delta(\varepsilon)$ is a *modulus of continuity of* T if

$$\delta(\varepsilon) \geq \sup\{d_2(T(x), T(y)) : \; x, y \in X \text{ and } d_1(x, y) \leq \varepsilon\}.$$

A modulus of continuity is called *subadditive* if

$$\delta(\varepsilon_1 + \varepsilon_2) \leq \delta(\varepsilon_1) + \delta(\varepsilon_2)$$

for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Extension of Lipschitz mappings has been a very important topic in linear spaces which also became very important in metric spaces after the work of Ball [10]. Lindenstrauss [51] studied compact extension of compact operators between linear spaces. The main object in this work turned out to be a certain class of linear spaces which satisfies a mild hyperconvexity condition, the so-called $\aleph_0$-hyperconvexity.

Definition 6.4 A metric space M is said to be $\aleph_0$-hyperconvex if given any family $\{x_\alpha \in \Lambda\}$ of points of M, with $|\Lambda| < \infty$, and any family $\{r_\alpha\}_{\alpha \in \Lambda}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that

$$\bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha) \neq \emptyset.$$

Under the light of [51] a characterization of $\aleph_0$-hyperconvex metric spaces in terms of compact extensions of compact uniformly continuous mappings was intended in [24]. These efforts led to the following characterization:

Theorem 6.5 *Let M be a metric space, then the following are equivalent:*

1. *M is an $\aleph_0$-hyperconvex complete metric space.*
2. *If Y is a metric space, $T : Y \to M$ is a uniformly continuous compact mapping with a subadditive modulus of continuity δ_T, and Z is a metric space containing Y metrically, then there exists a uniformly continuous compact extension $\tilde{T}$ of T to the whole of Z into M such that*

$$\delta_{\tilde{T}}(\varepsilon) = \max\{(1 + \eta)\varepsilon\delta_T(1), (1 + \eta)\delta_T(\varepsilon)\}$$

is a subadditive modulus of continuity of $\tilde{T}$. (Recall that $T : Y \to X$ is said to be compact if it is continuous and $T(A)$ is relatively compact in X for every bounded subset A of Y.)

In 2005 Lancien and Randrianantoanina [49] presented new results on extension of Hölder mappings on spaces of continuous functions. The fact that spaces of continuous functions treated in [49] are $\aleph_0$-hyperconvex, and that extensions found in [49] were mainly compact extensions, make it pertinent to try to revisit [49] from the approach given in [24]. What we offer next are some consequences of this approach.

Henceforth, uniformly continuous mappings are supposed to have a subadditive modulus of continuity.

Definition 6.6 For $K > 0$ and $\alpha \in (0, 1]$, a mapping T as above is said to be (K, α)-Hölder continuous if $\delta(\varepsilon) = K\varepsilon^\alpha$ is a modulus of continuity of T.

For $C \geq 1$, $\mathcal{B}_C(X, Y)$ denotes the set of all $\alpha \in (0, 1]$ such that any (K, α)-Hölder mapping T from a subset of X into Y can be extended to a (CK, α)-Hölder mapping from X to Y. In [49] the sets

$$\mathcal{A}(X, Y) = \mathcal{B}_1(X, Y) \quad \text{and} \quad \tilde{\mathcal{A}}(X, Y) = \bigcap_{C > 1} \mathcal{B}_C(X, Y)$$

are studied for Y a space of converging real sequences. In particular, it is obtained (Theorem 2.2 in [49]) that $\mathcal{A}(X, c_0) = (0, 1]$ for X a finite dimensional normed vector space. It is interesting to point out that the extensions given by this result share the same modulus of continuity than the extended mapping.

It is well known that spaces of continuous functions on compact Hausdorff domains are $\aleph_0$-hyperconvex; hence, in particular, the spaces of null real convergent sequences c_0 and real convergent sequences are $\aleph_0$-hyperconvex.

Notice that, under the terminology of Lancien–Randrianantoanina in [49], it follows from Theorem 6.5 that $\widetilde{\mathcal{A}}(Y, X) = (0, 1]$ for compact Hölder mappings whenever Y is a metric space and X is an $\aleph_0$-hyperconvex metric space.

For a metric space X, Isbell defined in [32] the set of extremal functions $\epsilon(X)$ of X as the set of all functions $f : X \to \mathbb{R}$ such that it satisfies $f(x) + f(y) \geq d(x, y)$ for all x and y in X and it is pointwise minimal (see [32] or [21, Section 8] for details). The following theorem shows that $\epsilon(X)$ can also be regarded as the hyperconvex hull of X:

Theorem 6.7 *Let X be a metric space and $\epsilon(X)$ the set of extremal functions on X. Then*

1. *$\epsilon(X)$ is a hyperconvex metric space with the metric $d_{\epsilon(X)}(f, g) = \sup_{x\in X} |f(x) - g(x)|$.*
2. *X is isometrically embedded into $\epsilon(X)$ by the mapping $I_X : X \to \epsilon(X)$ defined by $I_X(x)(\cdot) = d(x, \cdot)$.*
3. *If X is relatively compact, then $\epsilon(X)$ is compact.*

Next lemma adds some information on statement 3 from Theorem 6.7 above.

Lemma 6.8 *If X is a boundedly compact metric space then $\varepsilon(X)$ is boundedly compact too.*

Proof Let $\{f_n\} \subseteq \varepsilon(X)$ a bounded sequence of functions. Let $x_0 \in X$ fixed and f_n^m be the restriction of f_n to $B(x_0, m)$. Then, since $\{f_n^1\}$ is a bounded sequence of equicontinuous functions on a compact set, the Alaoglu theorem implies that it has a convergent subsequence $\{f_{n_k}^1\}$. If we apply the same reasoning to $\{f_{n_k}^2\}$, iterate the process and make use of the diagonalization technique at the end, the conclusion follows: □

In [24, Lemma 2.1] it was proved that if X is a complete $\aleph_0$-hyperconvex space and $A \subseteq X$ is relatively compact, then there exists a compact hyperconvex set $h(A)$ such that $A \subseteq h(A) \subseteq X$. After Lemma 6.8 and this remark it is tempting to prove whether the same remains true for boundedly compact subsets of $\aleph_0$-hyperconvex spaces. The following example, however, shows that this is not the case:

Example 6.9 Let c_0 be the space of null convergent real sequences with the maximum norm and consider the set $A = \{ne_n\}$ where $\{e_n\}$ is the standard basis of c_0. Trivially A is boundedly compact; however, there is not any hyperconvex set $h(A)$ such that $A \subseteq h(A) \subseteq c_0$. In fact, it is enough to consider the following intersection of balls

$$B(e_1, 1) \bigcap \left(\bigcap_{n\geq 2} B(ne_n, n-1) \right) = [0, 2] \times \{1\}^{\mathbb{N}}.$$

The next definition will help with the exposition.

Definition 6.10 A metric space X will be said boundedly compact hyperconvex (BCH for short) if for any boundedly compact subset A of X there exists a boundedly compact hyperconvex set $h(A)$ such that $A \subseteq h(A) \subseteq X$.

It trivially follows that if a metric space is BCH then it is $\aleph_0$-hyperconvex, and, by [21, Lemma 2.1], any bounded $\aleph_0$-hyperconvex space is BCH. The following proposition shows that hyperconvex metric spaces are BCH:

Proposition 6.11 *If a metric space X is hyperconvex then it is BCH.*

Proof Given $A \subseteq X$ nonempty it is known that for any hyperconvex set $h(A)$ such that $A \subseteq h(A)$ there exists another hyperconvex set $h'(A)$ such that $A \subseteq h'(A) \subseteq h(A)$ which can be isometrically embedded into X. Now Lemma 6.8 implies the proposition. □

There are, however, nonbounded $\aleph_0$-hyperconvex spaces which are BCH. An example of such spaces is the set of null convergent sequences with bounded tiles, i.e.,

$$\widetilde{c_0} = \mathbb{R}^n \times [-1, 1]^{\mathbb{N}},$$

endowed with the maximum norm.

Proposition 6.12 *The space $\widetilde{c}_0$ is BCH and $\aleph_0$-hyperconvex.*

Proof It is immediate to see that $\widetilde{c}_0$ is $\aleph_0$-hyperconvex. To see that $\widetilde{c}_0$ is additionally BCH it is enough to recall that $\mathbb{R}^n$ is hyperconvex and the set of null convergent sequences defined on $[-1,1]^{\mathbb{N}}$ with the maximum norm is bounded and $\aleph_0$-hyperconvex. Hence, if $A \subseteq \widetilde{c}_0$ boundedly compact, we can write $A \subseteq A_1 \times A_2$ where

$$A_1 = \{x \in \mathbb{R}^n : \text{ there exists } y \in A \text{ such that } x(i) = y(i) \text{ for } 1 \le i \le n\}, \tag{6}$$

$$A_2 = \{x \in [-1,1]^{\mathbb{N}} : \text{ there exists } y \in A \text{ such that } x(i) = y(i) \text{ for } i > n\} \tag{7}$$

are also boundedly compact. So there exist boundedly compact hyperconvex sets $h(A_1)$ and $h(A_2)$ such that $A_1 \subseteq h(A_1) \subseteq \mathbb{R}^n$ and $A_2 \subseteq h(A_2) \subseteq [-1,1]^{\mathbb{N}}$; hence $h(A_1) \times h(A_2)$ is boundedly compact hyperconvex and such that $A \subseteq h(A_1) \times h(A_2) \subseteq \widetilde{c}_0$. □

The following theorem states an extension property for BCH spaces:

Theorem 6.13 *Let X be a metric space and Y a BCH metric space. Let $M \subseteq X$ be a nonempty set. If $T : M \to Y$ is a uniformly continuous compact mapping with a subadditive modulus of continuity δ, then there exists a uniformly continuous compact extension $\tilde{T} : X \to Y$ of T for which δ is a modulus of continuity.*

Proof Let $x_0 \in M$ fixed. We first extend T from M to $M \cup B_X(x_0, 1)$. Let $y \in M$, $d_1 = \sup\{\delta(d(x, x_0)) : x \in B(x_0, 1)\}$ and $d_2(y) = \inf\{d(x, y) : x \in B(x_0, 1)\}$. Then we consider the nonempty set

$$\widetilde{M} = \{y \in M : d_1 + d(Tx_0, Ty) \ge \delta(d_2(y))\}.$$

Since T is a compact mapping, by construction, $T(\widetilde{M})$ is boundedly compact. Now let $h(T(\widetilde{M}))$ be the boundedly compact hyperconvex between $T(\widetilde{M})$ and Y. Hence we may apply Theorem 4 of [8] to obtain an extension of T to $\widetilde{M} \cup B(x_0, 1)$ into $h(T(\widetilde{M})) \subseteq Y$ such that $d(Ty, Tx) \le \delta(d(y, x))$ for any $y \in \widetilde{M}$ and $x \in B(x_0, 1)$. We check next that $T : M \cup B(x_0, 1) \to Y$ has the same modulus of continuity as T. It suffices to check that $d(Tx, Ty) \le \delta(d(x, y))$ for $y \in M \setminus \widetilde{M}$ and $x \in B(x_0, 1)$; in fact,

$$\begin{aligned} d(Tx, Ty) &\le d(Tx, Tx_0) + d(Tx_0, Ty) \le \delta(d(x, x_0)) + d(Tx_0, Ty) \\ &\le d_1 + d(Tx_0, Ty) < \delta(d_2(y)) \le \delta(d(x, y)). \end{aligned}$$

Now the result follows by induction on $B(x_0, n)$ as $n \in \mathbb{N}$. □

To prove the following corollary it is enough to recall Proposition 6.11. Notice that the next corollary guarantees compact extensions.

Corollary 6.14 *If in the previous theorem Y is hyperconvex then the same conclusion follows.*

To state our next result we need to introduce the following definition:

Definition 6.15 Let X be a metric space. We say that $M \subseteq X$ is a proximately centered set in X if for any $x \in X$ there exists $\Gamma(x) = \{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq M$ bounded such that for any $y \in M$ there exists $\alpha \in \mathcal{A}$ such that $d(y, x_\alpha) \le d(x, y)$.

We say that M is boundedly proximately centered in X if

$$\bigcup_{x \in B} \Gamma(x)$$

is bounded for any $B \subseteq X$ bounded.

The next example shows that the above definitions are not redundant.

Example 6.16 Let $\{e_n\}$ be as in Example *6.9* and, for $n \in \mathbb{N}$, $x_n \in c_0$ such that

$$x_n(i) = \begin{cases} 1, & \text{if } 1 \le i \le n, \\ 0, & \text{if } n < i. \end{cases}$$

Take $M = \{ne_n : n \in \mathbb{N}\}$, $B = \{x_n : n \in \mathbb{N}\}$ and $X = M \cup B \subseteq c_0$. Then, for $n \in \mathbb{N}$,

$$\|x_n - ie_i\| = \begin{cases} i - 1, & \text{if } 1 \le i \le n, \\ i, & \text{if } n < i, \end{cases}$$

and $\|ie_i - je_j\| = \max\{i, j\}$. Thus, M is proximately centered in M, but it is not boundedly proximately centered since $\Gamma(x_n) = \{ie_i : 1 \le i \le n\}$.

Our next result is an extension of [49, Theorem 2.2].

Theorem 6.17 *Let X be a metric space and $M \subseteq X$ boundedly proximately centered in X. If $T : M \to c_0$ is a compact uniformly continuous mapping, then there exists a compact extension $\widehat{T} : X \to c_0$ of T with the same modulus of continuity.*

Proof Let $x_0 \in X \setminus M$ and consider $B(x_0, 1) \setminus M$. We first find $\widehat{T} : M \cup B(x_0, 1) \to c_0$ as desired. Let $\Gamma = \{\Gamma(x) : x \in B(x_0, 1) \setminus M\}$; we take $x \in B(x_0, 1) \setminus M$ and pick $\varepsilon = \varepsilon(x)$ such that $\varepsilon < \frac{1}{2}\delta(\mathrm{dist}(x, M))$. Then, since Γ is bounded and T compact there exists $N \in \mathbb{N}$ such that $|T(y)(n)| < \varepsilon$ for $n \geq N$ and $y \in \Gamma$.

Now, since ℓ_∞ is hyperconvex, Corollary 6.14 implies that there exists a compact extension $G : M \cup B(x_0, 1) \to \ell_\infty$ of T with the same modulus of continuity. For x as above we make $G(x) = (\eta_n)$. Hence, for $n \leq N$ we fix $\widehat{T}(x)(n) = \eta_n$. For $n > N$, let $\delta_n \in \{-1, 1\}$ be the sign of η_n. Now we set

$$\widehat{T}(x)(n) = \delta_n \min\left\{|\eta_n|, \sup_{y\in\Gamma} |T(y)(n)|\right\}.$$

Since T is compact, $\widehat{T} : B(x_0, 1) \cup M \to c_0$. Next we prove that it admits the same modulus of continuity as T. We first show that $d(\widehat{T}x, Ty) = \sup_{n\in\mathbb{N}} |\widehat{T}(x)(n) - T(y)(n)| \leq \delta(d(x, y))$ when $x \in B(x_0, 1) \setminus M$ and $y \in M$. For $n \leq N$ it follows from the properties of G. For $n > N$ we have four cases:

1. If $|T(y)(n)| \leq |\widehat{T}(x)(n)|$, then
$$|T(y)(n) - \widehat{T}(x)(n)| \leq 2\varepsilon \leq \delta(d(x, y)).$$
2. If $|T(y)(n)| > |\widehat{T}(x)(n)|$, $\mathrm{sgn}(T(y)(n)) = \delta_n$, and $|\widehat{T}(x)(n)| = |\eta_n|$, then, by construction,
$$|T(y)(n) - \widehat{T}(x)(n)| \leq \delta(d(y, x)).$$
3. If $|T(y)(n)| > |\widehat{T}(x)(n)|$, $\mathrm{sgn}(T(y)(n)) = \delta_n$, and $|\widehat{T}(x)(n)| = \sup_{y\in\Gamma} |T(y)(n)|$, then let $y_0 \in \Gamma$ such that $d(y, y_0) \leq d(y, x)$ and so
$$\begin{aligned}|T(y)(n) - \widehat{T}(x)(n)| &= |T(y)(n)| - |\widehat{T}(x)(n)| \leq |T(y)(n) - T(y_0)(n)| \\ &\quad + |T(y_0)(n)| - |\widehat{T}(x)(n)| \leq \delta(d(y, y_0)) \leq \delta(d(y, x)).\end{aligned}$$
4. If $|T(y)(n)| > |\widehat{T}(x)(n)|$ and $\mathrm{sgn}(T(y)(n)) \neq \delta_n$, then
$$|T(y)(n) - \widehat{T}(x)(n)| = |T(y)(n)| + |\widehat{T}(x)(n)| \leq |T(y)(n)| + |\eta_n| = |T(y)(n) - \eta_n| \leq \delta(d(y, x)).$$

Now we take $x, y \in B(x_0, 1) \setminus M$ and let $G(x) = (\eta_n)$ and $G(y) = (\nu_n)$. We want to check that $d(\widehat{T}(x), \widehat{T}(y)) \leq \delta(d(x, y))$. Again, for $n \leq N$ it follows from the properties of G. For $n > N$ we have four cases:

1. If $\sup_{y\in\Gamma} |T(y)(n)| \leq \min\{|\eta_n|, |\nu_n|\}$ and $\mathrm{sgn}(\eta_n) = \mathrm{sgn}(\nu_n)$, then
$$|\widehat{T}(x)(n) - \widehat{T}(y)(n)| = 0.$$
2. If $\sup_{y\in\Gamma} |T(y)(n)| \leq \min\{|\eta_n|, |\nu_n|\}$ and $\mathrm{sgn}(\eta_n) \neq \mathrm{sgn}(\nu_n)$, then
$$|\widehat{T}(x)(n) - \widehat{T}(y)(n)| \leq |\eta_n - \nu_n| \leq \delta(d(x, y)).$$
3. If $|\eta_n| \leq \sup_{y\in\Gamma} |T(y)(n)| \leq |\nu_n|$, then
$$|\widehat{T}(x)(n) - \widehat{T}(y)(n)| = |\eta_n - \mathrm{sgn}(\nu_n) \sup_{y\in\Gamma} |T(y)(n)|| \leq |\eta_n - \nu_n|.$$
4. If $\sup_{y\in\Gamma} |T(y)(n)| \geq \max\{|\eta_n|, |\nu_n|\}$, then
$$|\widehat{T}(x)(n) - \widehat{T}(y)(n)| = |\eta_n - \nu_n|.$$

Hence $\widehat{T} : B(x_0, 1) \to c_0$ is uniformly continuous with modulus of continuity δ. Moreover $\widehat{T}$ is compact as a consequence of the fact that G is compact.

The proof is completed by induction on $\mathbb{N}$ as we extend successively from $B(x_0, n-1) \cup M$ to $B(x_0, n) \cup M$ by adding $B(x_0, n) \setminus (B(x_0, n-1) \cup M)$ to the domain of the new extension as above. □

The big question after this theorem is if c_0 can be replaced by a more general $\aleph_0$-hyperconvex space.

Remark 6.18 1. The condition of boundedness in the proximately centered character of M is required only for the compactness of the extension. The same result holds for noncompact extensions without this uniformity condition.
2. The boundedness condition is not required at all if X is supposed to be boundedly compact since the compactness of the extension follows from its continuity. This is the case in [49, Theorem 2.2].
3. The condition about M being centered cannot be dropped from the previous theorem. In fact, consider $A = \{ne_n\}$ where $\{e_n\}$ is the standard basis of c_0. It is readable to check that A is not proximately centered in $A \cup \mathbf{1}$, where $\mathbf{1}$ stands for the constant unit sequence, with the supremum norm. Now Example *6.9* shows that the identity map $id\colon A \to c_0$ cannot be extended as a nonexpansive map to $A \cup \mathbf{1}$.

Our next result requires the notion of λ-hyperconvexity (see [29,39] for more on this topic). Although the notion we present here is slightly different to that one given in [39], we still name it the same.

Definition 6.19 Let X be a metric space and $\lambda \geq 1$. We say that X is a λ-hyperconvex metric space if for every family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \mathcal{A}}$, each of radius r_α, centered at $x_\alpha \in X$ for $\alpha \in \mathcal{A}$, the condition $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for every $\alpha, \beta \in \mathcal{A}$, implies

$$\bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, \lambda r_\alpha) \neq \emptyset.$$

In [49] some results for the so-called $\mathcal{L}_\lambda^\infty$ normed spaces are obtained. Our next definition is an extension of this notion to the metric setting.

Definition 6.20 We say that a metric space X is compactly almost hyperconvex if for every $\lambda > 1$ and $M \subseteq X$ boundedly compact there exists $N \subseteq X$ boundedly compact, λ-hyperconvex and such that $M \subseteq N$.

It is easy to check that $\mathcal{L}_\lambda^\infty$-spaces are compactly almost hyperconvex. Our next result is related to [49, Proposition 3.1].

Theorem 6.21 *Let Y be a boundedly compact metric space, $M \subseteq Y$, X a compactly almost hyperconvex space, and $T\colon M \to X$ a compact uniformly continuous mapping with modulus of continuity δ. Then, for $\varepsilon > 0$, there exists $\widehat{T}\colon Y \to X$ uniformly continuous extension of T with modulus of continuity $(1+\varepsilon)\delta$.*

Proof Let $x_0 \in Y \setminus M$ and consider $B(x_0, 1)$; then we take $\widetilde{M}$ as in the proof of Theorem 6.13. Hence $T(\widetilde{M})$ is boundedly compact and so, for $\varepsilon > 0$, there exists a boundedly compact and $(1+\varepsilon)$-hyperconvex subset N_ε of X such that $T(\widetilde{M}) \subseteq N_\varepsilon$. Now notice that

$$\bigcap_{x \in \widetilde{M}} B(T(x), (1+\varepsilon)\delta(d(x, x_0))),$$

is nonempty. If we define $\widehat{T}(x_0)$ as any point in the above intersection $\widehat{T}\colon M \cup \{x_0\} \to X$ is an extension of T with modulus $(1+\varepsilon)\delta$. We complete the extension to the whole $B(x_0, 1)$ as follows: let (x_n) a dense sequence in $B(x_0, 1) \setminus M$. For a given $\varepsilon > 0$ we take (ε_n) so that $\prod_{n \geq 1}(1+\varepsilon_n) < 1+\varepsilon$. Iterating the above procedure on n and extending by continuity we finally have $\widetilde{T}\colon M \cup B(x_0, 1) \to X$.

The theorem follows by repeating the previous procedure in each $B(x_0, n) \setminus B(x_0, n-1)$ and picking an adequate sequence (ε_n). □

We finish this work with the following remarks:

Remark 6.22 1. Since Y is supposed to be boundedly compact all the mappings in the above theorem are compact.
2. Under the terminology of [49] the above theorem states that $\tilde{\mathcal{A}}(X, Y) = (0, 1]$ whenever X is boundedly compact and Y compactly almost hyperconvex.

3. If in Definition 6.20 we impose that for each N and λ we can choose N_λ so that $\cup_{\lambda\in\Lambda} N_\lambda$ is boundedly compact, then it is possible to replace Y boundedly compact by Y separable in the above theorem. We do not know, however, whether in this case being compactly almost hyperconvex implies BCH. Indeed, if Definition 6.20 is modified so that $\cup_{\lambda\in\Lambda} N_\lambda$ is boundedly compact for any choice of N_λ we make, then it is possible to prove that we have nothing else but BCH.

Acknowledgments The first author wishes to thank The University of Tabuk for its kind invitation to take part in *The International Mathematical Workshop (Fixed Point Theory and its Applications) May 26–27, 2012*. This work is a contribution to this event. This research is partially supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 and Grant P08-FQM-03543, and by MEC Grant MTM2009-10696-C02-01.

References

1. Aksoy, A.G.; Khamsi, M.A.: A selection theorem in metric trees. Proc. Am. Math. Soc. **134**, 2957–2966 (2006)
2. Aksoy, A.G.; Khamsi, M.A.: Fixed points of uniformly Lipschitzian mappings in metric trees. Sci. Math. Jpn. **65**, 31–41 (2007)
3. Ambrosio, L.; Tilli, P.: Topics on Analysis in Metric Spaces. Oxford University Press, Oxford (2004)
4. Amini-Harandi, A.; Farajzadeh, A.P.: A best approximation theorem in hyperconvex metric spaces. Nonlinear Anal. **70**, 2453–2456 (2009)
5. Amini-Harandi, A.; Farajzadeh, A.P.: Best approximation, coincidence and fixed point theorems for set-valued maps in R-trees. Nonlinear Anal. **71**, 1649–1653 (2009)
6. Amini-Harandi, A.; Farajzadeh, A.P.; O'Regan, D.; Agarwal, R.P.: Coincidence point, best approximation, and best proximity theorems for condensing set-valued maps in hyperconvex metric spaces. Fixed Point Theory and Appl. Article ID 543158 (2008)
7. Amini-Harandi, A.; Farajzadeh, A.P.: O'Regan, D.; Agarwal, R.P.: Fixed point theory for α-condensing set valued maps in hyperconvex metric spaces. Commun. Appl. Nonlinear Anal. **15**, 39–46 (2008)
8. Aronszajn, N.; Panitchpakdi, P.: Extensions of uniformly continuous transformations and hyperconvex metric spaces. Pac. J. Math. **6**, 405–439 (1956)
9. Baillon, J.B.: Nonexpansive mappings and hyperconvex spaces. Contemp. Math. **72**, 11–19 (1988)
10. Ball, K.: Markov chains, Riesz transforms and Lipschitz maps. Geom. Funct. Anal. **2**, 137–172 (1992)
11. Borkowski, M.; Bugajewski, D.: On fixed point theorems of Leray–Schauder type. Proc. Am. Math. Soc. **136**, 973–980 (2008)
12. Borkowski, M.; Bugajewski, D.; Phulara, D.: On some properties of hyperconvex spaces. Fixed Point Theory Appl. Article ID 213812 (2010)
13. Bridson, M.R.; Haefliger, A.: Metric Spaces of Non-Positive Curvature. Springer-Verlag, Berlin (1999)
14. Bugajewski, D.; Espínola, R.: Measure of nonhyperconvexity and fixed-point theorems. Abstr. Appl. Anal. **2003**, 111–119 (2003)
15. Chang, T.-H.; Chen, C.-M.; Peng, C.Y.: Generalized KKM theorems on hyperconvex metric spaces and some applications. Nonlinear Anal. **69**, 530–535 (2008)
16. Dhompongsa, S.; Kirk, W.A.; Sims, B.: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006)
17. Dhompongsa, S.; Kirk, W.A.; Panyanak, B.: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**, 35–45 (2007)
18. Dress, A.; Scharlau, R.: Gated sets in metric spaces. Aequationes Math. **34**, 112–120 (1987)
19. Espínola, R.: Darbo-Sadovski's theorem in hyperconvex metric spaces, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II **40**, 129–137 (1996)
20. Espínola, R.: On selections of the metric projection and best proximity pairs in hyperconvex spaces. Ann. Univ. Mariae Curie-Skłodowska Sect. A **59**, 9–17 (2005)
21. Espínola, R., Khamsi, M.A.: Introduction to hyperconvex spaces. In: Kirk, W.A., Sims, B. (eds.) Handbook of Metric Fixed Point Theory, pp. 391–435. Kluwer Academic Publishers, Dordrecht (2001)
22. Espínola, R.; Kirk, W.A.: Fixed point theorems in $\mathbb{R}$-trees with applications to graph theory. Topol. Appl. **153**, 1046–1055 (2006)
23. Espínola, R.; Kirk, W.A.; López, G.: Nonexpansive retracts in hyperconvex spaces. J. Math. Anal. Appl. **251**, 557–570 (2000)
24. Espínola, R.; López, G.: Extension of compact mappings and $\aleph_0$-hyperconvexity. Nonlinear Anal. **49**, 1127–1135 (2002)
25. Espínola, R.; Lorenzo, P.; Nicolae, A.: Fixed points, selections and common fixed points for nonexpansive-type mappings. J. Math. Anal. Appl. **382**, 503–515 (2011)
26. Goebel, K.; Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)
27. Goebel, K.; Kirk, W.A.: A fixed point theorem for transformations whose iterates have uniform Lipschitz constant. Studia Math. **47**, 135–140 (1973)
28. Goebel, K.; Reich, S.: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Marcel Dekker, New York (1984)

29. Grünbaum, B.: On some covering and intersection properties in Minkowski spaces. Pac. J. Math. **9**, 487–494 (1959)
30. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer, Berlin (2001)
31. Heinonen, J.: Nonsmooth calculus. Bull. Am. Math. Soc. (N.S.) **44**, 163–232 (2007)
32. Isbell, J.R.: Injective envelopes of of Banach spaces are rigidly attached. Bull. Am. Math. Soc. **70**, 727–729 (1964)
33. Khamsi, M.A.: KKM and Ky Fan Theorems in hyperconvex metric spaces. J. Math. Anal. Appl. **204**, 298–306 (1996)
34. Khamsi, M.A.: Sadovskii's fixed point theorem without convexity. Nonlinear Anal. **53**, 829–837 (2003)
35. Khamsi, M.A.; Lin, M.; Sine, R.: On the fixed points of commuting nonexpansive maps in hyperconvex spaces. J. Math. Anal. Appl. **168**, 372–380 (1992)
36. Khamsi, M.A.; Kirk, W.A.: An Introduction to Metric Spaces and Fixed Point Theory. Wiley Interscience, New York (2001)
37. Khamsi, M.A.; Kirk, W.A.; Martínez Yáñez, C.: Fixed point and selection theorems in hyperconvex spaces. Proc. Am. Math. Soc. **128**, 3275–3283 (2000)
38. Khan, A.R.; Hussain, N.; Thaheem, A.B.: Some generalizations of Ky Fan's best approximation theorem. Anal. Theory Appl. **20**, 189–198 (2004)
39. Khamsi, M.A.; Knaust, H.; Nguyen, N.T.; O'Neill, M.D.: Lambda-hyperconvexity in metric spaces. Nonlinear Anal. **43**, 21–31 (2001)
40. Kirk, W.A.: Hyperconvexity of $\mathbb{R}$-trees. Fundamenta Mathematicae **156**, 67–72 (1998)
41. Kirk, W.A.: A note on geodesically bounded $\mathbb{R}$-trees. Fixed Point Theory Appl. Article ID 393470 (2010)
42. Kirk, W.A.: Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees. Fixed Point Theory Appl. **4**, 309–316 (2004)
43. Kirk, W.A.: Krasnoselskii's iteration process in hyperbolic space. Numer. Funct. Anal. Optimiz. **4**, 371–381 (1981–1982)
44. Kirk, W.A.; Panyanak, B.: Best approximation in $\mathbb{R}$-trees. Numer. Funct. Anal. Optim. **28**, 681–690 (2007)
45. Kirk, W.A.; Panyanak, B.: Remarks on best approximation in $\mathbb{R}$-trees. Ann. Univ. Mariae Curie-Skłodowska Sect. A **63**, 133–138 (2009)
46. Kirk, W.A.; Shin, S.S.: Fixed point theorems in hyperconvex spaces. Houst. J. Math. **23**, 175–187 (1997)
47. Kirk, W.A.; Reich, S.; Veeramani, P.: Proximinal retracts and best proximity pair theorems. Numer. Funct. Anal. Opt. **24**, 851–862 (2003)
48. Lacey, H.E.: The Isometric Theory of Classical Banach Spaces. Springer-Verlag, New York (1974)
49. Lancien, G.; Randrianantoanina, B.: On the extension of Hölder maps with values in spaces of continuous functions. Isr. J. Math. **147**, 75–92 (2005)
50. Lin, M.; Sine, R.: On the fixed point set of order preserving maps. Math. Zeit. **203**, 227–234 (1990)
51. Lindenstrauss, J.: Extension of compact operators. Mem. Am. Math. Soc. AMS **48**, 1–112 (1964)
52. Markin, J.T.: A best approximation theorem for nonexpansive set-valued mappings in hyperconvex metric spaces. Rocky Mt. J. Math. **35**, 2435–2441 (2009)
53. Markin, J.T.: Fixed points, selections, and best approximation for multivalued mappings in $\mathbb{R}$-trees. Nonlinear Anal. **67**, 2712–2716 (2007)
54. Markin, J.T.; Shahzad, N.: Best approximation theorems for nonexpansive and condensing mappings in hyperconvex spaces. Nonlinear Anal. **70**, 2435–2441 (2009)
55. Nadler Jr., S.B.: Multi-valued contraction mappings. Pac. J. Math. **30**, 2059–2063 (2005)
56. Nowakowski, R.; Rival, I.: Fixed-edge theorem for graphs with loops. J. Graph Theory **3**, 339–350 (1979)
57. Piątek, B.: Best approximation of coincidence points in metric trees. Ann. Univ. Mariae Curie-Skłodowska Sect. A **62**, 113–121 (2008)
58. Piątek, B.; Espínola, R.: Fixed points and non-convex sets in CAT(0) spaces. Topol. Meth. Nonlinear Anal (to appear)
59. Razani, A.; Salahifard, H.: Invariant approximation for CAT(0) spaces. Nonlinear Anal. **72**, 2421–2425 (2010)
60. Sine, R.: On nonlinear contraction semigroups in sup norm spaces. Nonlinear Anal. **3**, 885–890 (1979)
61. Sine, R.: Hyperconvexity and approximate fixed points. Nonlinear Anal. **13**, 863–869 (1989)
62. Sine, R.: Hyperconvexity and nonexpansive multifunctions. Trans. Am. Math. Soc. **315**, 755–767 (1989)
63. Sine, R.: Hyperconvexity and approximate fixed points. Nonlinear Anal. **13**, 863–869 (1989)
64. Soardi, P.: Existence of fixed points of nonexpansive mappings in certain Banach lattices. Proc. Am. Math. Soc. **73**, 25–29 (1979)
65. Suzuki, T.: Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. **340**, 1088–1095 (2008)
66. Young, G.S.: The introduction of local connectivity by change of topology. Am. J. Math. **68**, 479–494 (1946)