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Algebraic determination of scattering matrices

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ABSTRACT. The algebraic approach to scattering allows the determination of S-matrices associated to a potential group describing the interaction region, through contraction and expansion mechanisms connecting the potential and asymptotic Lie algebras. We show that this procedure can be generalized to the $SO_q(2,1)$ algebra and extract the corresponding S-matrix. Possible applications of a three-dimensional generalization of our results are also discussed.

RESUMEN. El método algebraico permite determinar las matrices S asociadas a grupos de potencial que describen la región de interacción, mediante mecanismos de contracción y expansión que conectan las algebras de Lie del potencial con un algebra asintótica. Mostramos que este procedimiento puede ser generalizado a un algebra $SO_q(2,1)$ y determinamos la matriz S correspondiente. Posibles aplicaciones de una generalización tridimensional de nuestros resultados son también discutidas.

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1. INTRODUCTION

The use of group contractions has a long, albeit not widely known history, starting from the work of Inönu and Wigner, who used them as a means to determine representations of non-semisimple groups [1]. The opposite operation, that of group expansion [1], is even less familiar and has had few applications. Although most physicists have some degree of familiarity with the concepts and techniques of group theory and Lie algebras in connection with the use of symmetries and conserved quantities in physical systems, the contraction and expansion of these mathematical structures is by no means common knowledge. In the last years, however, these concepts were realized to be of central importance for the algebraic description of scattering processes [2,3], and the reason for it may be schematically understood from Fig. 1.

The interaction region in the algebraic approach is described by means of a group (or, more precisely, by its associated Lie algebra) which is refered to as the "potential group" and denoted by g in the figure. The asymptotic region is likewise described by the group g'. These correspond in the algebraic language to the potentials associated to these regions in the usual integro-differential framework. The algebras are related to each other through

$$\begin{array}{ccc} \longrightarrow & & \longrightarrow \\ g' & g & g' \\ \text{Asymptotic} & \text{Interaction} & \text{Asymptotic} \\ \text{Region} & \text{Region} & \text{Region} \end{array}$$

FIGURE 1. Schematic representation of the algebraic approach to scattering.

a contraction-expansion procedure

$$g \underset{\text{expansion}}{\overset{\text{contraction}}{\rightleftarrows}} g'$$

as will be explained below. In contrast to the case of bound systems, where the role of group theory is well understood, symmetry methods were very seldom useful for the description of continuous spectra, other than in the trivial form of angular momentum and energy conservation. The relevant information is in these cases contained in the S-matrix and the question is whether group theory can provide some information about it. In a series of papers [2–4] in the last years it was shown that the S-matrix of a scattering system may be evaluated by establishing a connection between the generators of the potential group, describing the interaction potential, and appropriate asymptotic generators, describing the long range behavior of the system. This connection formula was refered to as the "Euclidean connection" in Ref. [3]. It was shown in Ref. [4] that the connection formula is equivalent to the expansion [1] of the potential group generators in terms of asymptotic ones. This interpretation permits a fully algebraic determination of S-matrices for abstract potential groups of the general form $SO(n,m)$. The algebraic approach was later applied to heavy ion collisions [5], nuclear reactions [6,7], and to relativistic systems [8].

From a different perspective, quantum algebras have become a subject of great current interest [9–16]. Although they have up to now no direct physical interpretation, they have been shown to be a powerful tool to solve Yang-Baxter equations. The solutions are of interest both to integrable lattice models in statistical mechanics and to link and knot theories. Many properties of Lie algebras and groups and their representations have been extended to their quantum analogs. While Jimbo [10] has supplied the relations that define this extension for any classical Lie algebra, Celeghini *et al.* [11] have devised a contraction procedure to establish the representations of non-semisimple quantum groups, in the same spirit as in the work of Inönu and Wigner for Lie algebras. Many other q-generalizations have been carried out [13–16] and there is much interest in finding physical applications for these mathematical structures.

The purpose of the present paper is two-fold. On the one hand, since the algebraic approach to scattering can be viewed as an abstract procedure, the question arises as to whether it can be extended to the case where the potential region is described by a q-algebra. If the contraction-expansion procedure can indeed be generalized to cover these cases, we will then be able to extract the corresponding S-matrices and obtain, as a bonus, a new realization for the q-algebra generators in terms of asymptotic ones. We shall analyze the case of the contraction-expansion procedure for the $SO_q(2,1) \leftrightarrow E(2)$ algebras and carry out the above-mentioned steps. On the other hand, we study a three-dimensional version of this procedure, which corresponds to a q-deformation of Coulomb scattering and interpret the results in terms of a screened Coulomb potential, which may be useful for the study of electron-atom scattering.

2. Scattering from an $SO_q(2,1)$ Potential

We first define the $SO_q(2,1)$ representations, following a recent paper by Maekawa [17]. The quantum algebra $SU_q(1,1)$ (isomorphic to $SO_q(2,1)$) is defined by the operators $\hat{J}_+, \hat{J}_-$ and $\hat{J}_0$, satisfying

$$\left[\hat{J}_0, \hat{J}_\pm\right] = \pm \hat{J}_\pm; \qquad [\hat{J}_+, \hat{J}_-] = -\frac{q^{\hat{J}_0} - q^{-\hat{J}_0}}{q^{1/2} - q^{-1/2}}. \tag{1}$$

Introducing $\omega \equiv \ln q$, we find a different form for the second commutator

$$[\hat{J}_+, \hat{J}_-] = -\frac{\sinh(\omega \hat{J}_0)}{\sinh(\omega/2)}. \tag{2}$$

These relations reduce to the $SU(1,1)$ ones for $q \to 1 (\omega \to 0)$, and, except for the minus sign on the r.h.s. of (2), they coincide with the $SU_q(2)$ commutation relations. The Casimir invariant of $SU_q(1,1)$ takes the form [17]

$$\begin{aligned} \hat{C}_2(\omega) &= \cosh(\omega/2)\frac{\sinh^2(\omega \hat{J}_0/2)}{\sinh^2(\omega/2)} - \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) \\ &= \cosh(\omega/2)[\hat{J}_0]^2 - \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+), \end{aligned} \tag{3}$$

where we have introduced the notation

$$[X] = \frac{\sinh(\omega X/2)}{\sinh(\omega/2)}, \tag{4}$$

which reduces to X for $\omega \to 0$. The $SU_q(1,1)$ unitary irreps are discussed in Ref. [17]. We shall be interested only in the continuous (principal) series, defined by

$$\begin{aligned} \hat{C}_2(\omega)|j,m\rangle &= [j][j+1]|j,m\rangle, \\ \hat{J}_0|j,m\rangle &= m|j,m\rangle, \\ \hat{J}_\pm|j,m\rangle &= ([m \mp j][m \pm j \pm 1])^{1/2}|j, m \pm 1\rangle, \end{aligned} \tag{5}$$

where $j = -1/2 \pm i\sigma$, with real σ, and m takes all integral or half-integral values [17]. The quantities in square brackets are defined in (4).

We now carry out the contraction of (1) and (2) to the Euclidean group E(2), by first considering the change of scale transformation

$$\hat{P}_{\pm}^{\epsilon} = \epsilon \hat{J}_{\pm}; \qquad \hat{J}_0^{\epsilon} = \hat{J}_0 \tag{6}$$

and then taking the limit when $\epsilon \to 0$. We find

$$[\hat{J}_0^0, \hat{P}_{\pm}^0] = \pm \hat{P}_{\pm}^0; \qquad [\hat{P}_{+}^0, \hat{P}_{-}^0] = 0, \tag{7}$$

which are the E(2) commutation relations [1]. From now on we omit the superindex "0" in these operators. Can we now expand E(2) back to $SU_q(1,1)$?. In the usual expansion to $SU(1,1)$, we use the formula [3,5]

$$\hat{J}_{\pm} = \frac{1}{2ik}[\hat{J}_0^2, \hat{P}_{\pm}] + \frac{\alpha}{k}\hat{P}_{\pm}, \tag{8}$$

where α is a constant which depends on the representation label j and $k = \sqrt{\hat{P}_{+} \cdot \hat{P}_{-}}$. Using (7) we can verify that the operators in (8) satisfy the $SU(1,1)$ commutation relations. The expansion formula (8) can be understood in the following way [3,4]: the Casimir invariant J_0^2 of the compact subalgebra SO(2) (which is not modified by the contraction process) is not present in the contracted E(2) invariant $\hat{P}^2$. We should then use it to reconstruct the original scattering algebra. Because $\hat{J}_0^2$ is an SO(2) scalar and $P_{\pm}$ transform as an SO(2) vector, a new vector in SO(2) which is non linear in the E(2) generators can be constructed in the form (8), which therefore automatically satisfies the correct commutation relations with $\hat{J}_0$. The particular combination in (8) further guarantees that $[\hat{J}_{+}, \hat{J}_{-}] = -2\hat{J}_0$. We now attempt to repeat this argument for $SU_q(1,1)$. The form of the Casimir invariant (3) suggests that instead of $\hat{J}_0^2$, we may try the SO(2) invariant

$$\begin{aligned}\hat{\Im}_{\omega} &= \cosh(\omega/2)\frac{\sinh^2(\omega\hat{J}_0/2)}{\sinh^2(\omega/2)} \\ &= \cosh(\omega/2)[\hat{J}_0]^2. \end{aligned} \tag{9}$$

To test this idea we need the basic commutators

$$\begin{aligned}[\hat{P}_{\pm}, \sinh(\omega\hat{J}_0)] &= \hat{P}_{\pm}\Big(\sinh(\omega\hat{J}_0) - \sinh(\omega\hat{J}_0 \pm \omega)\Big), \\ [\hat{P}_{\pm}, \cosh(\omega\hat{J}_0)] &= \hat{P}_{\pm}\Big(\cosh(\omega\hat{J}_0) - \cosh(\omega\hat{J}_0 \pm \omega)\Big),\end{aligned} \tag{10}$$

which can be derived by using (7). We find that it is not precisely $\hat{\Im}_{\omega}$ of (9) which leads to the $SU_q(1,1)$ commutators (1), (2), but the slightly different form

$$\begin{aligned}\hat{J}_{\pm} &= \frac{4\cosh(\omega/4)}{2ik}\left[\frac{\sinh^2(\omega\hat{J}_0/4)}{\sinh^2(\omega/2)}, \hat{P}_{\pm}\right] + \frac{\alpha}{k}\hat{P}_{\pm}, \\ \hat{J}_0 &= \hat{J}_0.\end{aligned} \tag{11}$$

In the notation (4), formula (11) can be written in the alternative form

$$\begin{aligned} \hat{J}_\pm &= \frac{4\cosh(\omega/4)}{2ik}\left[[\hat{J}_0/2]^2, \hat{P}_\pm\right] + \frac{\alpha}{k}\hat{P}_\pm, \\ \hat{J}_0 &= \hat{J}_0, \end{aligned} \tag{12}$$

which clearly reduces to (8) for $\omega \to 0$. The $\hat{J}_\pm$ generators may be calculated from (11) and (10) to give the new $\mathrm{SU}_q(1,1)$ realization

$$\begin{aligned} \hat{J}_\pm &= \frac{1}{2ik}\hat{P}_\pm\left(\frac{\cosh(\omega\hat{J}_0/2)}{\cosh(\omega/4)} \pm \frac{\sinh(\omega\hat{J}_0/2)}{\sinh(\omega/4)} + 2i\alpha\right), \\ \hat{J}_0 &= \hat{J}_0 \end{aligned} \tag{13}$$

in terms of the E(2) generators. To define the constant α, we compute the Casimir operator (3) using (13). After some algebra, we find the simple relation

$$\hat{C}_2(\omega) = -\alpha^2 - \frac{1}{4\cosh^2(\omega/4)}. \tag{14}$$

Returning to equation (5),

$$\hat{C}_2(\omega)|j,m\rangle = [j][j+1]|j,m\rangle,$$

we find, for the principal series $j = -1/2 \pm i\sigma$, that

$$[j][j+1] = \frac{\sinh^2(i\sigma\omega/2)}{\sinh^2(\omega/2)} - \frac{1}{4\cosh^2(\omega/4)} = [i\sigma]^2 - \frac{1}{4\cosh^2(\omega/4)}. \tag{15}$$

By comparing with (14), we identify α as

$$\alpha = i[j+1/2] = \pm i[i\sigma]. \tag{16}$$

Equation (16) fixes α in terms of the $\mathrm{SU}_q(1,1)$ representation label j and gives its particular value for the principal series. We further define the appropriate sign in (16) below.

Inserting (16) into (13) and rearranging terms, we arrive at the final expression for the $\mathrm{SU}_q(1,1)$ generators in the continuous series representation

$$\begin{aligned} \hat{J}_+^\infty &= \frac{\hat{P}_+}{ik}\left([\hat{J}_0 + 1/2] \pm [i\sigma]\right), \\ \hat{J}_-^\infty &= \frac{\hat{P}_-}{ik}\left([\hat{J}_0 - 1/2] \pm [i\sigma]\right), \\ \hat{J}_0^\infty &= \hat{J}_0, \end{aligned} \tag{17}$$

where we attach the superindex "∞" to indicate that they arise from their expansion from the asymptotic E(2) generators. Again, these formulas reduce to the usual ones for $\omega \to 0$ [2–4]. Both signs in (17) are permissible in principle. We now calculate the S-matrix for scattering from an $\mathrm{SU}_q(1,1)$ potential group by following the usual procedure [2–4] and the explicit form of $\hat{J}_+^\infty$ in (17).

In the algebraic approach to scattering, the representation index σ becomes a real but otherwise arbitrary function of the momentum $k, i.e., \sigma(k)$. The asymptotic (contracted) form for the $\mathrm{SU}_q(1,1)$ wave functions is then given by

$$|j,m\rangle_\omega^\infty = A_m^\omega|-k,m\rangle + B_m^\omega|k,m\rangle, \tag{18}$$

where $|-k,m\rangle$ and $|k,m\rangle$ are identified with E(2) (*i.e.* free) incoming and outgoing waves, respectively. We now impose the equality

$$(\hat{J}_+|j,m\rangle_\omega)^\infty = \hat{J}_+^\infty|j,m\rangle_\omega^\infty, \tag{19}$$

which implies that the $\mathrm{SU}(1,1)_q$ relations are valid asymptotically [2–4] and use (5), (18) and the E(2) defining equations

$$\begin{aligned} \hat{P}_+|\pm k,m\rangle &= \pm k|\pm k,m+1\rangle, \\ \hat{J}_0|\pm k,m\rangle &= m|\pm k,m\rangle, \end{aligned} \tag{20}$$

to find recurrence relations for the S-matrix $S_m^\omega \equiv B_m^\omega/A_m^\omega$:

$$S_{m+1}^\omega = \frac{[m+1/2]+[i\sigma(k)]}{[m+1/2]-[i\sigma(k)]} S_m^\omega, \tag{21}$$

where the sign in (17) is fixed by whether the $\hat{J}_+$ generator acts on the $+k$ or $-k$ E(2) representation [2-4]. For half-integer m-values, we find

$$S_m^\omega = \prod_{n=1}^{m-1/2} \left(\frac{[n]+[i\sigma(k)]}{[n]-[i\sigma(k)]}\right) e^{i\varphi_\omega(k)}, \tag{22}$$

where $\varphi_\omega(k)$ is an arbitrary function. Equation (22) describes $\mathrm{SU}_q(1,1)$- like S-matrices and their m-dependence in terms of $\sigma(k)$ and the parameter $\omega = \ln q$. The form of $\sigma(k)$ is determined by the specific function of the Casimir invariant (14) which is taken as the Hamiltonian of the scattering system [3,5]. For example, for $\mathrm{SU}(1,1)$ scattering ($\omega = 0$) off a Poschl-Teller potential, the scattering Hamiltonian turns out to be given by [2]

$$\hat{H}\Psi(x) = \left(-\hat{C}_2(0) - 1/4\right)\Psi(x) = k^2\Psi(x), \tag{23}$$

for which (14) and (16) imply $\sigma(k) = \pm k$. The S-matrix (22) reduces to the ratio of two gamma functions for $\omega \to 0$. Note that a minor modification of Eq. (13) leads to a realization for the quantum group $\mathrm{SU}_q(2)$.

We have thus shown that the algebraic approach to scattering can be extended to q-algebras by means of an appropriate modification of the usual expansions formulas. In the next section we study a three-dimensional generalization of the procedure considered in this section and analyze its possible physical interpretation.

3. q-Coulomb Scattering and Screened Potentials

While the discussion of the previous section demonstrates that the contraction-expansion procedures can be successfuly applied to more complex potential-group structures, it does not shed much light into the nature of the new physical features which are in this way incorporated to the scattering processes. The usual approach, which deals with $SO(n,m)$ Lie algebras and their contraction, leads to S-matrices which are ratios of Γ functions [2–7]. In particular, for heavy ion scattering and reactions, both $SO(3,1)$ and $SO(3,2)$ [5-7] have been proposed as potential groups which incorporate both the Coulomb and short-range interactions, although the l- dependence is usually modified through an additional parametrization of the group labels [5–7]. In this section we shall show that a simple generalization of formula (21) for three-dimensional systems leads to an interesting modification of Coulomb scattering.

Before considering this generalization, however, we briefly indicate the results of the algebraic approach to pure Coulomb scattering [5]. The relevant contraction-expansion procedure is applied in this case to the algebras

$$\mathrm{SO}(3,1) \underset{\text{expansion}}{\overset{\text{contraction}}{\rightleftarrows}} \mathrm{E}(3),$$

as $SO(3,1)$ is well known to constitute a symmetry group for (positive energy) hydrogenic systems [18]. The analysis is fully analogous to the $SU(1,1) \approx SO(2,1) \leftrightarrow E(2)$ one discussed in the last section. For Coulomb potentials,

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{\beta}{r}, \tag{24}$$

the Hamiltonian may be written as

$$\hat{H} = -\mu\beta^2/2(\hat{C}_2 + 1), \tag{25}$$

where $\hat{C}_2$ is the $SO(3,1)$ second order Casimir invariant [5,18]. The corresponding recurrence relation for the S-matrix turns out to be given by [5]

$$S_{l+1}(k) = \frac{l+1+if(k)}{l+1-if(k)} S_l(k), \tag{26}$$

where $f(k)$ is fixed by (25) and given by

$$f(k) = \mu\beta/k. \tag{27}$$

Note the similarity with (21) (for the case $\omega \to 0$). The l plays the role of m and $f(k)$ that of $\sigma(k)$, while the $1/2$ is substituted by 1 due to the higher dimensionality of SO(3.1). Relation (26) leads to the well known result for Coulomb scattering

$$S_l(k) = \frac{\Gamma(l+1+i\mu\beta/k)}{\Gamma(l+1-i\mu\beta/k)} e^{i\varphi(k)}, \tag{28}$$

where $\varphi(k)$ is fixed by the s-wave amplitude (and cannot be determined by the algebraic procedure). We may now consider the q-deformation of the SO(3, 1) algebra [18] which is defined by the commutators

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k \tag{29.a}$$

$$[\hat{L}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}_k \tag{29.b}$$

$$[\hat{K}_i, \hat{K}_j] = -i\epsilon_{ijk}\frac{\sinh(\omega L_k)}{2\sinh(\omega/2)} = -i\epsilon_{ijk}\tfrac{1}{2}[2L_k], \tag{29.c}$$

where the square bracket on the r.h.s. of (29.c) was defined in (4). Note that these relations constitute a natural generalization of the commutators (1) corresponding to $SO_q(2,1)$. In fact, this kind of algebraic structures can be simply defined for $SO(n,1)$ with arbitrary n. The $SO(n,1)$ generators can be divided into compact and non-compact ones. The compact generators are those corresponding to the $SO(n)$ subalgebra ($n(n-1)/2$ of them) while the non-compact ones are the n remaining operators. Thus, in $SO(2,1)$ there is a single compact generator ($\hat{J}_0$) and two non-compact ones ($\hat{J}_\pm$), while in $SO(3,1)$ there are three compact ($\hat{L}_i$) and three non-compact ($\hat{K}_i$) generators. The q-deformation we are defining corresponds to leaving invariant all commutators except the ones among the non-compact generators, which are deformed as in (29.c). We now return to the S-matrix associated to the contraction-expansion of the algebras

$$SO_q(3,1) \underset{\text{expansion}}{\overset{\text{contraction}}{\rightleftarrows}} E(3),$$

where we denote by $SO_q(3,1)$ the mathematical structure defined by Eqs. (29). By following a procedure closely analogous to the one carried out in the last section, we find the recurrence relations for the $SO_q(3,1)$ S-matrix

$$S^{\omega}_{l+1(k)} = \frac{[l+1]+[if(k)]}{[l+1]-[if(k)]}\, S^{\omega}_l, \tag{30}$$

where $f(k)$ should be determined by the relation between the Hamiltonian and the $SO_q(3,1)$ Casimir invariant. We define the q-Coulomb Hamiltonian by generalizing (25) to

$$\hat{H}_w = -\mu\beta^2\left(2(\hat{C}_2(\omega) + \frac{1}{\cosh^2(\omega/4)})\right)^{-1}, \tag{31}$$

where $\hat{C}_2(\omega)$ is the Casimir invariant of the algebra (29) and w is related to q through $\omega = \ln q$, as before. This generalization follows from an analysis of the $\mathrm{SO}_q(3,1)$ representations. In the continuous series, $n = -1 \pm if(k)$, the eigenvalue equation for $\hat{C}_2(\omega)$ gives

$$\hat{C}_2(\omega)|nlm\rangle_\omega = [n][n+2]|nlm\rangle_\omega = \left([if(k)]^2 - \frac{1}{\cosh^2(\omega/4)}\right)|nlm\rangle_\omega. \tag{32}$$

Comparing with (31) and using $\hat{H}_\omega|nlm\rangle_\omega = \frac{k^2}{2\mu}|nlm\rangle_\omega$, leads to the identification of $f(k)$:

$$[if(k)] = i\frac{\mu\beta}{k}, \tag{33}$$

which reduces to (27) for $\omega \to 0$, as it should. Substitution into (30) gives

$$S_{l+1}^\omega(k) = \frac{[l+1] + i\mu\beta/k}{[l+1] - i\mu\beta/k}\, S_l^\omega(k) \tag{34}$$

for the S-matrix recurrence relation associated to q-Coulomb scattering. This form differs markedly in its l-dependence from relation (26). Writing $S_l^\omega = exp(2i\delta_l(\omega))$, we find

$$\delta_l(\omega) = \delta_{l-1}(\omega) + \arctan\left(\frac{\eta\sinh(\omega/2)}{\sinh(l\omega/2)}\right), \tag{35}$$

where $\eta = \mu\beta/k$ is the Sommerfeld parameter. The classical deflection function, that is, the angle of deviation as a function of l [19], is then given by

$$\Theta_\omega(l) = 2\frac{\partial\delta_l}{\partial l} \simeq 2\arctan\left(\frac{\eta\sinh(\omega/2)}{\sinh(l\omega/2)}\right), \tag{36}$$

to be compared with the Coulomb result ($\omega \to 0$)

$$\Theta_c(l) = 2\arctan(\eta/l). \tag{37}$$

Thus $\Theta_\omega(l)$ tends to zero exponentially with increasing l, which is typical of interaction potentials which fall exponentially. The determination of this potential involves either solving the inverse scattering problem or finding appropriate coordinate realizations for the algebraic relations (29) [20]. We shall follow a simpler route by carrying out a semiclassical analysis in search of an l-independent (but otherwise energy dependent) potential reproducing (34). We use the expression [20]

$$\Theta(l) = \pi - 2\int_{r_0}^{\infty}\frac{\hbar l\,dr}{r^2P(r)}; \qquad \frac{P^2(r)}{2\mu} = E - V - \frac{\hbar^2l^2}{2\mu r^2}, \tag{38}$$

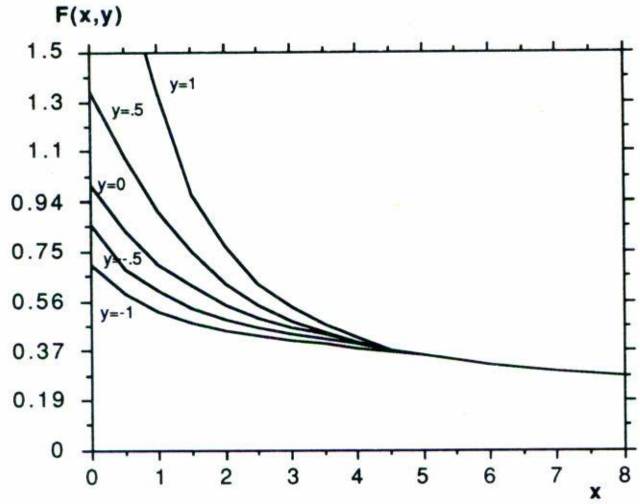

FIGURE 2. The function $F(x,y)$, with $x = \epsilon kr = r/\gamma$ and $y = \epsilon\eta = E_\gamma/2E$.

where r_0 is the classical turning point, $P(r_0) = 0$. If we propose a potential of the form

$$V(r,E) = \frac{2E\eta\sinh(\omega/2)}{\sinh(\frac{k\omega r}{2})} F\left(\frac{k\omega r}{2}, \eta\sinh\frac{\omega}{2}\right), \tag{39}$$

we can numerically determine the function $F(x,y)$ from (36) and (38). We obtain the results of Fig. 2. For $k\omega r/2$ large, $F(x,y) \simeq \sqrt{2/\pi x}$, while $F(0,0) = 1$. Also, we find the analytic approximation $F(x,0) \simeq \sqrt{\frac{2}{\pi}\arctan\left(\frac{1}{x}\right)}$. Note that for $\omega \to 0$ we recover $V(r,E) = 2E\eta/kr = \beta/r$, which is the Coulomb potential. To interpret our results, it is convenient to define the following parameter, with units of length

$$\gamma \equiv \frac{2}{k\omega}. \tag{40}$$

If we assume that γ is independent of the energy (which implies a k–dependence of the deformation parameter ω)

$$\eta\sinh(\omega/2) = \frac{\mu\beta}{k}\frac{\epsilon}{k\gamma} = \frac{E_\gamma}{2E}\epsilon, \tag{41}$$

where $\epsilon \equiv k\gamma$. Sinh $(1/k\gamma)$ and $E_\gamma \equiv \frac{\beta}{\gamma}$ has units of energy and is equal to the Coulomb potential at the distance γ. We may interpret the potential (39) as describing a screened Coulomb potential, where γ is a screening length. Substituting (40) and (41) into (38) we find

$$V(r,E) = \frac{E_\gamma\epsilon}{\sinh\left(\frac{r}{\gamma}\right)} F\left(\frac{r}{\gamma}, \frac{E_\gamma\epsilon}{2E}\right), \tag{42}$$

where $\epsilon \to 1$ for $k\gamma\rangle > 1$. Furthermore, for $k\gamma\rangle > \eta$, which implies $E\rangle > E_\gamma$ (the "sudden approximation") V becomes energy independent

$$V(r,E) = \frac{E_\gamma}{\sinh\left(\frac{r}{\gamma}\right)} F\left(\frac{r}{\gamma}, 0\right), \tag{43}$$

while for $E \lesssim E_\gamma$ the potential is strongly energy dependent, as illustrated by Fig. 2.

These characteristics are typical of potentials required to describe the scattering of charged particles by atoms: i) for projectile velocities large compared with the average velocity of the atomic electrons, the incident particle feels a static potential screened by the charge distribution, which makes it fall exponentially, and ii) for velocities less or equal to that of the electrons, there is enough time for them to reorganize, i.e., change their distribution. These dynamical effects give rise to an energy dependence of the "effective" potential.

We are currently investigating the possibility of using the q-Coulomb scattering S-matrix for the description of experimental charged particle-atom collision data [21].

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