A lower bound for the power of periodic solutions of the defocusing Discrete Nonlinear Schrödinger equation*

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Abstract

We derive a lower bound on the power of time periodic solutions of the *defocusing* Discrete Nonlinear Schrödinger Equation with power nonlinearity, supplemented with Dirichlet boundary conditions. The lower bound depends not only on the dimension of the lattice, the lattice spacing, and the frequency of the periodic solution, but also on the excitation threshold of time periodic and spatially localized solutions of the *focusing* DNLS, proved in M. Weinstein, Nonlinearity **12**, 673–691, 1999. The simple proof via a direct variational method, makes use of the interpolation inequality proved by M. Weinstein, and its optimal constant related to the excitation threshold. A numerical study is performed to test the efficiency of the lower bound.

1 Introduction

In [6], M. Weinstein considered the focusing Discrete Nonlinear Schrödinger Equation (DNLS) [2, 4]

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d\psi)_n + |\psi_n|^{2\sigma}\psi_n = 0,\ \ \sigma > 0,\ n \in \mathbb{Z}^N, \tag{1.1}$$

and resolved the hypothesis suggested by S. Flach, K. Kladko & R. MacKay [5] for this equation, on the existence of excitation thresholds for the existence of nonlinear localized modes for Hamiltonian dynamical systems defined on multidimensional lattices. More precisely, the numerical studies and heuristic arguments of [5], suggested that there is a lower bound on the energy of a breather (time periodic and spatially localized standing wave solutions), if the lattice dimension is greater than or equal to a certain critical value. For (1.1), where $(\Delta_d\psi)_n$, stands for the N-dimensional discrete Laplacian and ϵ is a discretization parameter $\epsilon \sim h^{-2}$ with h being the lattice spacing, the hypothesis of [5] was resolved by

Theorem 1.1 *(M. Weinstein [6, Theorem 3.1,pg. 678]). Let* $\sigma \geq \frac{2}{N}$*. Then there exists a ground state excitation threshold* $\mathcal{R}_{\mathrm{thresh}} > 0$.

A minimizer of the variational problem

$$\mathcal{I}_{\mathcal{R}} = \inf\left\{\mathcal{H}[\phi]\ :\ \mathcal{P}[\phi] = \mathcal{R}\right\}. \tag{1.2}$$

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is called a *a ground state* [6, Definition, pg. 676]. Here $\mathcal{H}[\phi]$ and $\mathcal{P}[\phi]$ are the fundamental conserved quantities

$$\mathcal{H}[\phi] = \epsilon(-\Delta_d\phi,\phi)_2 - \frac{1}{\sigma+1}\sum_{n\in\mathbb{Z}^N}|\phi_n|^{2\sigma+2}, \tag{1.3}$$

$$\mathcal{P}[\phi] = \sum_{n\in\mathbb{Z}^N}|\phi_n|^2, \tag{1.4}$$

the Hamiltonian and the power, respectively. By $(\cdot,\cdot)_2$ we denote the ℓ^2-scalar product.

Theorem 1.1, states that if $0<\sigma<\frac{2}{N}$, then $\mathcal{I}_{\mathcal{R}}<0$ for all $\mathcal{R}>0$. That is, the variational problem (1.2) has a solution for all $\mathcal{R}>0$ and there is no excitation threshold. However *when $\sigma\geq\frac{2}{N}$, there exists an excitation threshold* $\mathcal{R}_{\text{thresh}}$ such that *(a) if $\mathcal{R}>\mathcal{R}_{\text{thresh}}$ then $\mathcal{I}_{\mathcal{R}}<0$, and a ground state exists* and *(b) if $\mathcal{R}<\mathcal{R}_{\text{thresh}}$ then $\mathcal{I}_{\mathcal{R}}=0$, and there is no ground state minimizer of (1.2).*

Theorem 1.1, justifies the existence of an excitation threshold for spatially localized and time periodic solutions of the form

$$\begin{aligned}\psi_n(t) &= e^{\mathrm{i}\omega t}\phi_n,\ \omega>0,\ \ n\in\mathbb{Z}^N,\ \ t\in\mathbb{R},\\ \phi_n &\in \ell^2(\mathbb{Z}^N).\end{aligned} \tag{1.5}$$

The threshold value, $\mathcal{R}_{\text{thresh}}$, is related to the best constant of an interpolation inequality which is a discrete analogue of the Sobolev-Gagliardo-Nirenberg inequality.

Theorem 1.2 *(M. Weinstein [6, Theorem 4.1,pg. 682]) Assume that $\sigma\geq\frac{2}{N}$. Then there exists $C>0$, such that for all $\phi\in\ell^2$, the following interpolation inequality holds*

$$\sum_{n\in\mathbb{Z}^N}|\phi_n|^{2\sigma+2}\leq C\left(\sum_{n\in\mathbb{Z}^N}|\phi_n|^2\right)^{\sigma}(-\Delta_d\phi,\phi)_2. \tag{1.6}$$

If C_* is the infimum over all such constants for which inequality (1.6) holds, then the excitation threshold $\mathcal{R}_{\text{thresh}}$ is defined by [6, pg. 680, Eqn. (4.2)]

$$(\sigma+1)\epsilon\left(\mathcal{R}_{\text{thresh}}\right)^{-\sigma}=C_*, \tag{1.7}$$

and the optimal constant C_* has the variational characterization

$$\frac{1}{C_*}=\inf_{\substack{\phi\in\ell^2\\ \phi\neq 0}}\frac{\left(\sum_{n\in\mathbb{Z}^N}|\phi_n|^2\right)^{\sigma}(-\Delta_d\phi,\phi)_2}{\sum_{n\in\mathbb{Z}^N}|\phi_n|^{2\sigma+2}}.$$

In this short note we shall derive and perform a numerical study, on a lower bound on the power of *time-periodic solutions of the defocusing DNLS supplemented with Dirichlet boundary conditions*

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d\psi)_n-\Lambda|\psi_n|^{2\sigma}\psi_n=0,\ \Lambda>0,\ \ |n|\leq K, \tag{1.8}$$

$$\psi_n = 0,\ \ |n|>K. \tag{1.9}$$

Problem (1.8)-(1.9) has time periodic solutions of the form

$$\psi_n(t)=e^{-\mathrm{i}\Omega t}\phi_n,\ \ \Omega>0. \tag{1.10}$$

The existence of solutions (1.10) of prescribed frequency $\Omega>0$, and the lower bound on their power, is derived by a simple proof based also on a variational approach. It is shown that the lower bound exhibits an interesting relation between the parameters N,σ,Ω,ϵ *as well as on the excitation threshold for the periodic solutions of the focusing DNLS (1.1).* Although the study is *limited to the finite dimensional lattice, this case is of importance especially for numerical simulations*: since the infinite lattice cannot be modelled numerically, numerical investigations should consider finite lattices with Dirichlet or periodic boundary conditions. The choice of boundary conditions only matters, if the pulse is moving and collides with the boundary. We expect that similar bounds can be derived for the case of periodic boundary conditions, by considering appropriate variational problems, but the details have to be checked.

Let us mention at this point, that an analytical and numerical study, on various lower bounds of the power of time periodic solutions, of the DNLS equation with saturable and power nonlinearities in infinite and finite lattices, will be considered in [3].

2 A lower bound for time periodic solutions of the defocusing DNLS in a finite lattice: Relation to the excitation threshold of the focusing DNLS

Substitution of the solution (1.10) into (1.8)-(1.9) shows that ϕ_n satisfies the system of algebraic equations

$$-\epsilon(\Delta_d\phi)_n - \Omega\phi_n = -\Lambda|\phi_n|^{2\sigma}\phi_n, \;\; \Omega > 0, \;\Lambda > 0, \;\; |n| \leq K, \tag{2.1}$$

$$\phi_n = 0, \;|n| > K. \tag{2.2}$$

The finite dimensional problem (2.1)–(2.2) will be formulated in the finite dimensional subspaces of the sequence spaces ℓ^p, $1 \leq p < \infty$,

$$\ell^p(\mathbb{Z}_K^N) = \{\phi \in \ell^p \;:\; \phi_n = 0 \;\text{ for }\; |n| > K\}\,. \tag{2.3}$$

Clearly $\ell^p(\mathbb{Z}_K^N) \equiv \mathbb{C}^{(2K+1)^N}$ is endowed with the norm

$$||\phi||_p = \left(\sum_{|n|\leq K} |\phi_n|^p\right)^{\frac{1}{p}}.$$

Note that for any $1 \leq p \leq q < \infty$, there exist constants C_1, C_2 depending on K, such that

$$C_1||\psi||_p \leq ||\psi||_q \leq C_2||\psi||_p. \tag{2.4}$$

The principal eigenvalue of the operator $-\Delta_d$ denoted by $\lambda_1 > 0$, can be characterized as

$$\lambda_1 = \inf_{\substack{\phi \in \ell^2(\mathbb{Z}_K^N)\\ \phi \neq 0}} \frac{(-\Delta_d\phi,\phi)_2}{\sum_{|n|\leq K}|\phi_n|^2}, \tag{2.5}$$

Hence (2.5) implies the inequality

$$\epsilon\lambda_1 \sum_{|n|\leq K} |\phi_n|^2 \leq \epsilon(-\Delta_d\phi,\phi)_2 \leq 4\epsilon N \sum_{|n|\leq K} |\phi_n|^2. \tag{2.6}$$

Thus from (2.6), we obtain for λ_1 the bound

$$\lambda_1 \leq 4N. \tag{2.7}$$

In the case of an $1D$-lattice $n = 1, \ldots, K$, the eigenvalues of the discrete Dirichlet problem $-\Delta_d\phi = \lambda\phi$, with ϕ real, are given explicitly by

$$\lambda_n = 4\sin^2\left(\frac{n\pi}{4(K+1)}\right), \;\; n = 1, \ldots, K,$$

while for a N-dimensional problem, the eigenvalues are:

$$\lambda_{(n_1,n_2,\ldots,n_N)} = 4\left[\sin^2\left(\frac{n_1\pi}{4(K+1)}\right) + \sin^2\left(\frac{n_2\pi}{4(K+1)}\right) + \ldots + \sin^2\left(\frac{n_N\pi}{4(K+1)}\right)\right], \;\; n_j = 1, \ldots, K; \;\; j = 1, \ldots, N.$$

In consequence, the principal eigenvalue of the discrete Dirichlet problem $-\Delta_d\phi = \lambda\phi$, with ϕ real, is given by

$$\lambda_1 \equiv \lambda_{(1,1,\ldots,1)} = 4N\sin^2\left(\frac{\pi}{4(K+1)}\right).$$

We also mention that the inequality (1.6) holds for any element of the finite dimensional space $\phi \in \ell^2(\mathbb{Z}_K^N)$. The result of this note is stated in the following

Theorem 2.1 *We consider the functional*

$$\mathcal{E}[\phi] = \epsilon(-\Delta\phi, \phi)_2 - \Omega \sum_{n\in\mathbb{Z}^N} |\phi_n|^2, \quad \Omega > 0, \tag{2.8}$$

and the variational problem on $\ell^2(\mathbb{Z}_K^N)$

$$\inf\left\{\mathcal{E}[\phi] \ : \sum_{|n|\leq K} |\phi_n|^{2\sigma+2} = M > 0\right\}, \tag{2.9}$$

for some $\Omega > 0$*. Assume that*

$$\Omega > 4\epsilon N. \tag{2.10}$$

Then there exists a minimizer $\hat{\phi} \in \ell^2(\mathbb{Z}_K^N)$ *for the variational problem (2.9) and* $\Lambda(M) > 0$*, both satisfying the Euler-Lagrange equation (2.1), and* $\sum_{|n|\leq K} |\hat{\phi}_n|^{2\sigma+2} = M$*.*

Moreover if $\sigma \geq \frac{2}{N}$*, the power of the minimizer* $\mathcal{P}[\hat{\phi}]$ *satisfies the lower bound*

$$\mathcal{R}_{\text{thresh}} \cdot \left[\frac{\Omega - 4N\epsilon}{4N\epsilon(\sigma+1)^2\Lambda}\right]^{\frac{1}{\sigma}} \leq \mathcal{P}[\hat{\phi}], \tag{2.11}$$

where $\mathcal{R}_{\text{thresh}} \equiv \mathcal{R}_{\text{thresh}}(\sigma, N, \epsilon)$ *is the excitation threshold of solutions (1.5) of the focusing DNLS (1.1).*

Proof: We consider the set

$$B = \left\{\phi \in \ell^2(\mathbb{Z}_K^N) \ : \sum_{|n|\leq K} |\phi_n|^{2\sigma+2} = M\right\}. \tag{2.12}$$

Clearly $\mathcal{E} : B \to \mathbb{R}$ is a C^1-functional. Also, it is bounded from below: the equivalence of norms (2.4) implies the existence of a N-dependent constant C_2 such that

$$||\phi||_2^2 \leq C_2^2 ||\phi||_{2\sigma+2}^2, \quad \text{for all } \ \phi \in \ell^2(\mathbb{Z}_K^N). \tag{2.13}$$

By using (2.13), we derive the inequality

$$\begin{aligned} \mathcal{E}[\phi] \quad &\geq \quad -\Omega \sum_{|n|\leq K} |\phi_n|^2 && (2.14) \\ &\geq \quad -\Omega C_2^2 \left(\sum_{|n|\leq K} |\phi_n|^{2\sigma+2}\right)^{\frac{1}{\sigma+1}} && (2.15) \\ &\geq \quad -\Omega C_2^2 M^{\frac{1}{\sigma+1}}. && (2.16) \end{aligned}$$

We are restricted to the finite dimensional space $\ell^2(\mathbb{Z}_K^N)$, and it follows that any minimizing sequence associated with the variational problem (2.9) is precompact: any minimizing sequence has a subsequence, converging to a minimizer. Thus $\mathcal{E}$ attains its infimum at a point $\hat{\phi}$ in B. Now, for the C^1-functional

$$\mathcal{L}_M[\phi] = \sum_{|n|\leq K} |\phi_n|^{2\sigma+2} - M, \tag{2.17}$$

we get that for any $\phi \in B$

$$\langle \mathcal{L}'_M[\phi], \phi\rangle = 2(\sigma+1) \sum_{|n|\leq K} |\phi|^{2\sigma+2} > 0. \tag{2.18}$$

Thus the Regular Value Theorem ([1, Section 2.9], [7, Appendix A,pg. 556]) implies that the set $M = \mathcal{L}_M^{-1}(0)$ is a C^1-submanifold of $\ell^2(\mathbb{Z}_K^N)$. By applying the Lagrange multiplier rule, we obtain the existence of a parameter $\lambda = \lambda(M) \in \mathbb{R}$, such that

$$\begin{aligned} \left\langle \mathcal{E}'[\hat{\phi}] - \lambda \mathcal{L}'_M[\hat{\phi}], \psi \right\rangle \quad &= \quad 2\epsilon(-\Delta_d \hat{\phi}, \psi)_2 - 2\Omega \text{Re} \sum_{|n|\leq K} \hat{\phi}_n \overline{\psi}_n \\ &\quad -2(\sigma+1)\lambda \sum_{|n|\leq K} |\hat{\phi}_n|^{2\sigma} \hat{\phi}_n \overline{\psi_n} = 0, \quad \text{for all } \ \psi \in \ell^2(\mathbb{Z}_K^N). && (2.19) \end{aligned}$$

Setting $\psi = \hat{\phi}$ in (2.19), we find that

$$2\mathcal{E}[\hat{\phi}] = 2\epsilon(-\Delta_d\hat{\phi}, \hat{\phi})_2 - 2\Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2 = 2(\sigma+1)\lambda \sum_{|n|\leq K} |\hat{\phi}|^{2\sigma+2}. \tag{2.20}$$

By using inequality (2.6), we obtain that

$$\mathcal{E}[\hat{\phi}] \leq 4\epsilon N \sum_{|n|\leq K} |\hat{\phi}_n|^2 - \Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2. \tag{2.21}$$

Thus $\mathcal{E}[\hat{\phi}] < 0$ if (2.10) is satisfied. Due to the estimate (2.7), the condition (2.10) implies that

$$\Omega > \epsilon\lambda_1. \tag{2.22}$$

Then assuming (2.10), we find that $\lambda(M) < 0$. We set $\lambda = -\Lambda$, $\Lambda > 0$. Lastly, we assume that the power of the nontrivial minimizer $\hat{\phi}$ is $\sum_{|n|\leq K} |\hat{\phi}_n|^2 = R^2$. Equation (2.20) can be rewritten as

$$2\epsilon(-\Delta_d\hat{\phi}, \hat{\phi})_2 + 2(\sigma+1)\Lambda \sum_{|n|\leq K} |\hat{\phi}|^{2\sigma+2} = 2\Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2. \tag{2.23}$$

We shall use (1.6), with the optimal constant (1.7), to estimate the second term on the rhs of (2.23): we have

$$2\epsilon(-\Delta_d\hat{\phi}, \hat{\phi})_2 + 2(\sigma+1)\Lambda C_* \left(\sum_{n\in\mathbb{Z}^N} |\hat{\phi}_n|^2\right)^{\sigma} (-\Delta_d\hat{\phi}, \hat{\phi})_2 \geq 2\Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2. \tag{2.24}$$

Since from (2.6)

$$\sum_{|n|\leq K} |\hat{\phi}_n|^2 \geq \frac{1}{4N}(-\Delta_d\hat{\phi}, \hat{\phi})_2,$$

inequality (2.24) becomes

$$\begin{aligned} 2\epsilon(-\Delta_d\hat{\phi}, \hat{\phi})_2 \quad &+ \quad 2(\sigma+1)\Lambda C_* \left(\sum_{|n|\leq K} |\hat{\phi}_n|^2\right)^{\sigma} (-\Delta_d\hat{\phi}, \hat{\phi})_2 \\ &\geq \quad \frac{2\Omega}{4N}(-\Delta_d\hat{\phi}, \hat{\phi})_2. \end{aligned} \tag{2.25}$$

Thus, from (2.25), we get

$$\epsilon + (\sigma+1)\Lambda C_* R^{2\sigma} \geq \frac{\Omega}{4N}.$$

implying the lower bound

$$\left[\frac{\Omega - 4N\epsilon}{4N(\sigma+1)C_*\Lambda}\right]^{\frac{1}{\sigma}} < R^2. \tag{2.26}$$

Replacing the value C_*, given by (1.2), in inequality (2.26), we find

$$\mathcal{R}_{\mathrm{thresh}} \cdot \left[\frac{\Omega - 4N\epsilon}{4N\epsilon(\sigma+1)^2\Lambda}\right]^{\frac{1}{\sigma}} \leq R^2,$$

which is the lower bound (2.11). $\diamond$

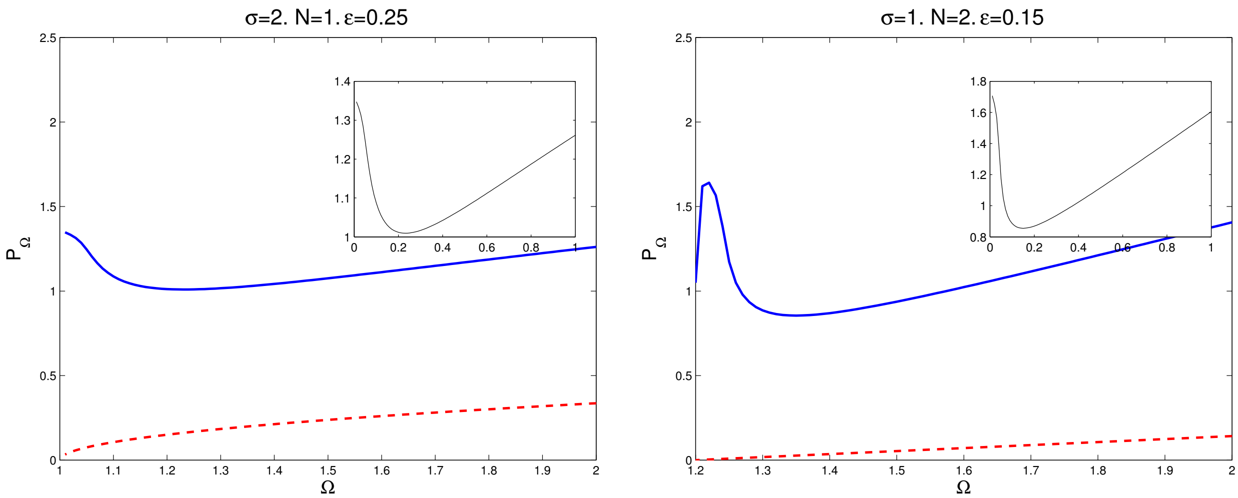

Figure 1: Numerical power for solutions (1.10), of the *defocusing* DNLS (1.8)-(1.9) (a) $\sigma = 2$, $N = 1$ ($\sigma = \frac{2}{N}$), (b) $\sigma = 1$, $N = 2$ ($\sigma = \frac{2}{N}$). The inset in each case, shows a magnification of the region where the power of periodic solutions (1.5) of the *focusing* DNLS (1.1), reaches its minimum value. In case (a), $\mathcal{R}_{\text{thresh}} = 1.009$ and in case (b), $\mathcal{R}_{\text{thresh}} = 0.855$.

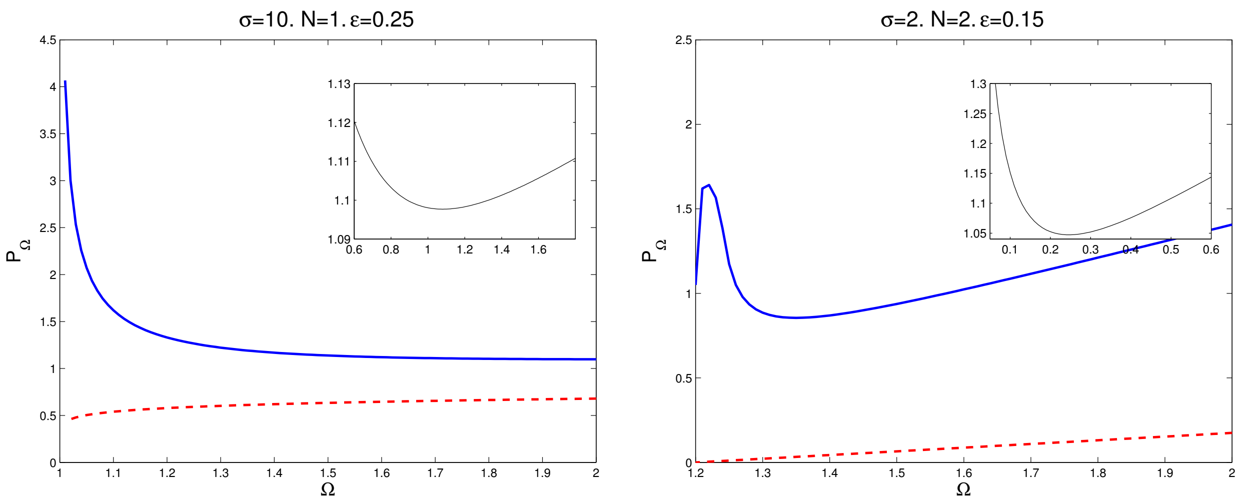

Figure 2: Numerical power for solutions (1.10), of the *defocusing* DNLS (1.8)-(1.9) (a) $\sigma = 10$, $N = 1$ ($\sigma > \frac{2}{N}$), (b) $\sigma = 2$, $N = 2$ ($\sigma > \frac{2}{N}$). The inset in each case, shows a magnification of the region where the power of periodic solutions (1.5) of the *focusing* DNLS (1.1), reaches its minimum value. In case (a), $\mathcal{R}_{\text{thresh}} = 1.098$ and in case (b), $\mathcal{R}_{\text{thresh}} = 1.047$.

Numerical study of the lower bound (2.11). We perform a numerical study to test the lower bound (2.11). In figures 1-2, the blue line corresponds to the numerically computed power of periodic solutions (1.10) for the *defocusing* DNLS (1.8)-(1.9). The inset in each figure shows the numericaly computed power of periodic solutions (1.5) for the *focusing* DNLS (1.1). The red line corresponds to the estimate (2.11).

Figure 1 refers to the cases (a) $\sigma = 2, N = 1, \epsilon = 0.25$ and (b) $\sigma = 1, N = 2, \epsilon = 0.15$, respectively. Both cases consider the critical value of Theorem 1.1, $\sigma = \frac{2}{N}$. The inset in each picture is a numerical verification of Theorem 1.1, demonstrating the region where the numerical power of periodic solutions (1.5) of the focusing DNLS (1.1) for the same values of σ, N, ϵ, reaches the minimum value $\mathcal{R}_{\text{thresh}}$. The numerical value for case (a) is $\mathcal{R}_{\text{thresh}} = 1.009$ and for case (b) is $\mathcal{R}_{\text{thresh}} = 0.855$. These numerical values have been inserted in the estimate (2.11). The numerical study shows that the numerical power of periodic solutions (1.10) of the defocusing DNLS (1.8)-(1.9), fullfils the estimate (2.11).

Figure 2 considers the cases (a) $\sigma = 10, N = 1, \epsilon = 0.25$ and (b) $\sigma = 2, N = 2, \epsilon = 0.15$, examples of the case $\sigma > \frac{2}{N}$. The inset in each picture, is again a numerical verification of Theorem 1.1. The numerical value of $\mathcal{R}_{\text{thresh}}$ for case (a) is $\mathcal{R}_{\text{thresh}} = 1.098$ and for case (b) is $\mathcal{R}_{\text{thresh}} = 1.047$. We observe that the lower bound (2.11) is fullfiled.

In conclusion, both figures show that the lower bound (2.11), is a quite satisfactory estimate of the power of time periodic solutions (1.10), of the defocusing DNLS (1.8)-(1.9).

References

[1] S. N. Chow, J. K. Hale *Methods of Bifurcation Theory*, Grundlehren der mathematischen Wissenschaften – A series of Comprehensive Studies in Mathematics 251, Springer-Verlag, New-York, 1982.

[2] J. C. Eilbeck, M. Johansson, *The Discrete Nonlinear Schrödinger Equation-20 Years on.* "Localization and Energy transfer in Nonlinear Systems, eds L. Vázquez, R.S. MacKay, M.P. Zorzano. World Scientific, Singapore, (2003), 44-67.

[3] J. Cuevas, J. C. Eilbeck, N. I. Karachalios, *Thresholds for time periodic solutions on the Discrete Nonlinear Schrödinger Equation with saturable and power nonlinearity*, submitted for publication.

[4] P. G. Kevrekidis, K. Ø. Rasmussen and A. R. Bishop, *The discrete nonlinear Schrödinger equation: A survey of recent results*, Int. Journal of Modern Physics B, **15** (2001), 2833-2900

[5] S. Flach, K. Kladko, R. MacKay, *Energy thresholds for discrete breathers in one-, two-, and three dimensional lattices*, Phys. Rev. Lett. 78, (1997), 1207-1210.

[6] M. Weinstein *Excitation thresholds for nonlinear localized modes on lattices*, Nonlinearity 12 (1999) 673-691.

[7] M. Haskins, J. M. Speight, *Breather initial profiles in chains of weakly coupled anharmonic oscillators* Phys. Letters A 299, (2002), 549-557.