Pullback permanence for non-autonomous partial differential equations

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Abstract

A system of differential equations is permanent if there exists a fixed bounded set of positive states strictly bounded away from zero to which, from a time on, any positive initial data enter and remain. However, this fact does not happen for a differential equation with general non-autonomous terms. In this work we introduce the concept of pullback permanence, defined as the existence of a time dependent set of positive states to which all solutions enter and remain for suitable initial time. We show this behaviour in the non-autonomous logistic equation $u_t - \Delta u = \lambda u - b(t)u^3$, with $b(t) > 0$, for all $t \in \mathbb{R}$, $\lim_{t \to \infty} b(t) = 0$. Moreover, a bifurcation scenario for the asymptotic behaviour of the equation is described in a neighbourhood of the first eigenvalue of the Laplacian. We claim that pullback permanence can be a suitable tool for the study of the asymptotic dynamics for general non-autonomous partial differential equations.

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1 Introduction

One of the main questions for a mathematical model from a natural phenomena is that on the long-time behaviour of its solutions. In particular, of special interest for an ecological

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model is to predict the existence of strictly positive solutions for all future times. In this sense, we can look for strictly positive globally attracting equilibria (or stationary solutions) for the corresponding partial differential equation associated to the model. But only in a restricted set of systems we can assure the existence of stationary solutions. However, the concept of *global attractor* comes to put some light in the understanding of the asymptotic behaviour of many dissipative systems (Hale [14], Temam [25]). Indeed, we can infer *uniform persistence or permanence* (Cantrell et al. [4], [5]) of solutions from the presence of a globally attracting positive set rather that a single attracting equilibrium. A system is said to have uniform persistence (Butler et al. [3]) if there exists a positive set which is bounded away from zero and globally attracting for all positive solutions. Note that this allows the systems to have a more complex dynamics, so that a wider set of (more natural) situations can be considered. On the other hand, we lose some information on the location and size of these new sets, so that any study on the *structure* of them are really helpful. There exists a substantial literature on this subject for autonomous differential equations (Hale and Waltman [15], Hess [17], Hutson and Schmitt [18]). The system is said to be *permanent* if it is also dissipative, i.e., the orbits enter into a compact set in a finite time. Afterwards, Cao and Gard [6] introduce the concept of *practical persistence*, defined as uniform persistence together with some information on the location of the positive attractor.

In this work we study problems on permanence of positive solutions for the following non-autonomous logistic equation

$$u_t - \Delta u = \lambda u - b(t)u^3,$$

with $b(t) > 0$, the interesting case being when we impose $\lim_{t \to +\infty} b(t) = 0$. Previous works for non-autonomous equations focus on the periodic or bounded by periodic functions in time cases (Cantrell and Cosner [4], Burton and Hutson [2], see also Nkashama [21] for the finite dimensional case with bounded and strictly positive non-autonomous terms). We treat a more general case, so that we allow the equation a very weak dissipation effect as time goes to infinite, and so previous works in the literature are not valid for our purposes.

The situation can be summarized as follows: when the parameter $\lambda < \lambda_1$, with λ_1 the first eigenvalue of the negative Laplacian, we get the existence of the zero solution as a globally attracting set. However, a drastic change in the asymptotic behaviour happens as the paramenter λ crosses the value λ_1. We describe in some detail this bifurcation scenario (Sections 3 and 4). Indeed, we firstly show that the equation leads to an order-preserving system and the method of sub and super solutions (Pao [22], Hess [17]) can be adapted to this case. Afterwards, we find a non bounded order interval depending on time in which

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all the asymptotic behaviour forward in time takes place (Section 4). That is, it does not exist any bounded absorbing set for the problem, and so any result on permanence in the sense of Cantrell and Cosner [4] can be expected.

However, very recently the theory of attractors for general non-autonomous differential equations has been introduced (Cheban et al. [8], Kloeden and Schamalfuss [19]; see also Crauel and Flandoli [12], Crauel et al. [13], for the same concept in a stochastic framework). In this case, the semigroup becomes a process, that is, a two-time dependent operator (Sell [24]), as the dependence of the initial times is so important as that on the final times. When the non-autonomous terms are periodic or quasi-periodic, the same concept of attractor in Temam [25] or Hale [14] can be used for these situations (Sell [24], Chepyzhov and Vishik [10]). But important changes in the concept must be introduced when we deal with general non-autonomous terms. Chepyzhov and Vishik [10] define *kernel* and *kernel sections.* This last concept is similar to that defined in Cheban et al. [8] as *cocycle or pullback attractor.* In our opinion, this is one of the right concepts to define the attractor for a general non-autonomous differential equation, as some results on the upper-semicontinuity of pullback attractors to the (autonomous) global attractor show (Caraballo and Langa [7]). The attractor in this situation is a time-dependent family of compact sets, invariant with respect to the cocycle and attracting from '$-\infty$' (see Definition 4).

We apply the theory of pullback attractors to our non-autonomous logistic equation. We also apply a result on the upper semicontinuity of this pullback attractor to the global attractor for the autonomous equation. This reinforces the choice of working with the pullback attractor to study the asymptotic behaviour of non-autonomous equations.

While forward in time we have not information on the stability of the equation when $\lambda > \lambda_1$, we describe a bifurcation scenario at the parameter value $\lambda = \lambda_1$ from the pullback procedure: the zero solution becomes unstable for $\lambda > \lambda_1$ and there exists a transfer of stability to the pullback attractor, which is a set strictly bigger than the zero solution, so that a result on permanence follows. We think this is the sensible concept for permanence for general non-autonomous partial differential equations (Definition 8). In Section 4.4 we are able to give more information on the structure of this pullback attractor and so on the bifurcation phenomena. Indeed, by introducing the concepts of sub-trajectories, super-trajectories and complete trajectories for non-autonomous systems in Section 3, as generalization of the theory of sub and super-equilibria in the sense of Hess [17], Arnold and Chueshov [1] and Chueshov [11], we describe the existence of a maximal complete trajectory on the attractor with some stability properties. We give a general theorem

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which can be applied to more general situations.

Finally, some conclusions and possible generalizations are given in the final Section.

2 Non-automous attractors

In this section, we introduce the general framework in which the theory of attractors for non-autonomous systems is going to be studied (see Cheban et al. [8] and Schmalfuss [23]). In a first step, we define processes as two-time dependent operators related with the solutions of non-autonomous differential equations. In this way, we are able to treat these equations as dynamical systems. Secondly, we write the general definitions of invariance, absorption and attraction and we finish with a general theorem on the existence of global attractors for these kind of equations. Finally, we give the definition of permanence for non-autonomous partial differential equations.

Let (X, d) be a complete metric space (with the metric d) with an order relation '$\leq$' and $\{S(t,s)\}_{t \geq s}$, $t, s \in \mathbb{R}$ be a family of mappings satisfying:

i) $S(t,s)S(s,\tau)u = S(t,\tau)u$, for all $\tau \leq s \leq t$, $u \in X$,

ii) $u \mapsto S(t,\tau)u$ is continuous in X.

This map is called a process. In general, we have to consider $S(t,\tau)u$ as the solution of a non-autonomous equation at time t with initial condition u at time τ.

Let $\mathcal{D}$ be a non-empty set of parameterized families of non-empty bounded sets $\{D(t)\}_{t \in \mathbb{R}}$. In particular, $D(t) \equiv B \in \mathcal{D}$, where $B \subset X$ is a bounded set. In what follows, we will consider fixed this *base of attraction* $\mathcal{D}$, so that the concepts of absorption and attraction in our analysis are always referred to it.

For $A, B \subset X$ define the Hausdorff semidistances as,

$$dist(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) \qquad Dist(A, B) = \inf_{a \in A} \inf_{b \in B} d(a, b).$$

Definition 1 *Given $t_0 \in \mathbb{R}$, we say that $K(t_0) \subset X$ is attracting at time t_0 if for every $\{D(t)\} \in \mathcal{D}$ we have that*

$$\lim_{\tau \to -\infty} dist(S(t_0, \tau)D(\tau), K(t_0)) = 0.$$

A family $\{K(t)\}_{t \in \mathbb{R}}$ is attracting if $K(t_0)$ is attracting at time t_0, for all $t_0 \in \mathbb{R}$.

The previous concept considers a fixed final time and moves the initial time to $-\infty$. Note that this does not mean that we are going backwards in time, but we consider the

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state of the system at time t_0 starting at $\tau \to -\infty$. This is called *pullback attraction* in the literature (cf. [19], [23]).

Definition 2 *Given $t_0 \in \mathbb{R}$, we say that $B(t_0) \subset X$ is absorbing at time t_0 if for every $\{D(t)\} \in \mathcal{D}$ there exists $T = T(t, D) \in \mathbb{R}$ such that*

$$S(t_0, \tau)D(\tau) \subset B(t_0), \ \text{for all } \tau \leq T.$$

A family $\{B(t)\}_{t \in \mathbb{R}}$ is absorbing if $B(t_0)$ is absorbing at time t_0, for all $t_0 \in \mathbb{R}$.

Note that every absorbing set at time t_0 is attracting.

Definition 3 *Let $\{B(t)\}_{t \in \mathbb{R}}$ be a family of subsets of X. This family is said to be invariant with respect to the process S if*

$$S(t, \tau)B(\tau) = B(t), \ \text{for all } (\tau, t) \in \mathbb{R}^2, \tau \leq t.$$

Note that this property is a generalization of the classical property of invariance for semigroups. However, in this case we have to define the invariance with respect to a family of sets depending on a parameter.

We define the *omega-limit set* at time t_0 of $D \equiv \{D(t)\} \in \mathcal{D}$ as

$$\Lambda(D, t_0) = \cap_{s \leq t_0} \overline{\cup_{\tau \leq s} S(t_0, \tau)D(\tau)}.$$

From now on, we assume that there exists a family $\{K(t)\}_{t \in \mathbb{R}}$ of compact absorbing sets, that is, $K(t) \subset X$ is non-empty, compact and absorbing for each $t \in \mathbb{R}$. Note that, in this case, $\Lambda(D, t_0) \subset K(t_0)$, for all $\{D(t)\} \in \mathcal{D}$, $t_0 \in \mathbb{R}$. As in the autonomous case, it is not difficult to prove that under these conditions $\Lambda(D, t_0)$ is non-empty, compact and attracts $\{D(t)\} \in \mathcal{D}$ at time t_0. The proof is similar to that in Crauel et al. [13], where the set $\mathcal{D}$ consists only of bounded sets.

Definition 4 *The family of compact sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be the global non-autonomous (or pullback) attractor associated to the process S if it is invariant, attracting every $\{D(t)\} \in \mathcal{D}$ (for all $t_0 \in \mathbb{R}$) and minimal in the sense that if $\{C(t)\}_{t \in \mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

Remark 5 *Chepyzhov and Vishik [9] define the concept of kernel sections for non-autonomous dynamical systems which corresponds to our definition of global non-autonomous attractor with $\{D(t)\} \equiv B \subset X$ bounded.*

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The general result on the existence of non-autonomous attractors is a generalization of the abstract theory for autonomous dynamical systems (Temam [25], Hale [14]):

Theorem 6 *(Crauel et al. [13], Schmalfuss [23]) Assume that there exists a family of compact absorbing sets. Then, the family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ defined by*

$$\mathcal{A}(t) = \overline{\cup_{D\in\mathcal{D}}\Lambda(D,t)}$$

is the global non-autonomous attractor.

Remark 7 *All the general theory of non-autonomous attractors can be written in the framework of cocycles (cf., among others, Cheban et al. [8], Crauel and Flandoli [12], Kloeden and Schmalfuss [19], Schmalfuss [23]). We could have also followed this notation here, but we think that, in this case, it is more clear to keep the explicit dependence on time of the attractor, which in addition, allows us to compare more straightforward with the concept of attractor in an autonomous framework.*

From the concept of non-autonomous attractor, we can now give the following definition of permanence, which will be suitable for non-autonomous partial differential equations.

Definition 8 *We say that a system has the property of pullback permanence (or that it is permanent in the pullback sense) if there exists a time-dependent family of sets $U : \mathbb{R} \longmapsto X$, satisfying*

1. *$U(t)$ is absorbing every bounded set $D \subset X$ (cf. Definition 2).*

2. *$Dist(U(t), \{0\}) > 0$ for all $t \in \mathbb{R}$.*

3   Order-preserving non-autonomous differential equations

We now introduce order-preserving systems and the concepts of sub, super and complete trajectories as a generalization of sub, super and equilibria in Hess [17], and of sub, super and equilibria in Arnold and Chueshov [1] in the stochastic case and Chueshov [11] in the non-autonomous case under stronger conditions.

Definition 9 *We say that the process $\{S(t,s) : X \to X\}_{t\geq s}$ is order-preserving if there exists an order relation '$\leq$' in X such that, if $u_0 \leq v_0$, then $S(t,s)u_0 \leq S(t,s)v_0$, for all $t \geq s$.*

Definition 10 *Let S be an order-preserving process. We call $\underline{u}$ $(\overline{u}) : \mathbb{R} \to X$ a subtrajectory (super-trajectory) of S if it satisfies*

$$S(t,s)\underline{u}(s) \geq \underline{u}(t), \text{ for all } t \geq s \quad (S(t,s)\overline{u}(s) \leq \overline{u}(t), \text{ for all } t \geq s \,).$$

Definition 11 *We call the continuous map $v : \mathbb{R} \to X$ a complete trajectory if, for all $s \in \mathbb{R}$, we have*

$$S(t,s)v(s) = v(t), \text{ for } t \geq s.$$

From a sub and super-trajectory $(\underline{u}, \overline{u})$ of a process such that $\underline{u}(t) \leq \overline{u}(t)$, for all $t \in \mathbb{R}$, we can define the "interval"

$$I_{\underline{u}}^{\overline{u}}(t) = \{u \in X : \ \underline{u}(t) \leq u \leq \overline{u}(t)\}.$$

Clearly, it is a closed forward invariant set, i.e. $S(t,s)I_{\underline{u}}^{\overline{u}}(s) \subset I_{\underline{u}}^{\overline{u}}(t)$, for all $t \geq s$.

The following result gives sufficient conditions for the existence of upper and lower asymptotically stable complete trajectories, giving some information on the structure of the non-autonomous attractor, adapting to our case the main results in Arnold and Chueshov [1] and Chueshov [11]. Note that we slightly generalize the results in [11] as we do not impose the set of parameters to be a compact set.

Suppose the pullback attractor attracts time-dependent families of sets in a base of attraction $\mathcal{D}$.

Theorem 12 *Let S be an order-preserving process and $\mathcal{A}(t)$ its associated pullback attractor. Let $\underline{u}, \overline{u}$ be sub and super-trajectories such that $\underline{u}(t) \leq \overline{u}(t)$, for all $t \in \mathbb{R}$, and $I_{\underline{u}}^{\overline{u}}(t)$ the corresponding associated interval, such that $\mathcal{A}(t) \subset I_{\underline{u}}^{\overline{u}}(t)$, for all $t \in \mathbb{R}$ and $\underline{u}, \overline{u} \in \mathcal{D}$. Suppose that there exists $\tau > 0$ such that $S(\tau + s, s)I_{\underline{u}}^{\overline{u}}(s)$ is relatively compact, for all $s \in \mathbb{R}$. Then, there exist complete trajectories $u_*(t)$, $u^*(t) \in \mathcal{A}(t)$ such that*

i)

$$\underline{u}(t) \leq u_*(t) \leq u^*(t) \leq \overline{u}(t), \text{ and } \mathcal{A}(t) \subset I_{u_*}^{u^*}(t), \text{ for all } t \in \mathbb{R}.$$

ii) *u_* (u^*) is minimal (maximal) in the sense that it does not exist any complete trajectory in the interval $I_{\underline{u}}^{u_*}(I_{u^*}^{\overline{u}})$.*

iii) *$u_*(t)$ is globally asymptotically stable from below, that is, for all $v \in \mathcal{D}$ with $\underline{u}(t) \leq v(t) \leq u_*(t)$, for all $t \in \mathbb{R}$, we have that*

$$\lim_{s \to +\infty} d(S(t,-s)v(-s), u_*(t)) = 0.$$

$u^*(t)$ is globally asymptotically stable from above, that is, for all $v \in \mathcal{D}$ with $u^*(t) \le v(t) \le \overline{u}(t)$, for all $t \in \mathbb{R}$, we have that

$$\lim_{s \to +\infty} d(S(t, -s)v(-s), u^*(t)) = 0.$$

Proof. Write $a_n(t) = S(t, -nt_0)\underline{u}(-nt_0), \quad b_n(t) = S(t, -nt_0)\overline{u}(-nt_0)$. Then, we have that

$$\underline{u}(t) \le a_n(t) \le a_m(t) \le b_m(t) \le b_n(t) \le \overline{u}(t), \quad \text{for all } m > n. \tag{1}$$

Indeed, $a_n(t) = S(t, -nt_0)\underline{u}(-nt_0) \ge \underline{u}(t)$, since $\underline{u}$ is a sub-trajectory. Moreover, for $s = \sigma + r$, $r > 0$,

$$
\begin{aligned}
a_s(t) &= S(t, -st_0)\underline{u}(-st_0) = S(t, -(\sigma + r)t_0)\underline{u}(-(\sigma + r)t_0) \\
&= S(t, -\sigma t_0)S(-\sigma t_0, -(\sigma + r)t_0)\underline{u}(-(\sigma + r)t_0) \ge S(t, -\sigma t_0)\underline{u}(-\sigma t_0) = a_\sigma(t).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
a_{n+1}(t) &= S(t, -(n+1)t_0)\underline{u}(-(n+1)t_0) \\
&= S(t, t - t_0)S(t - t_0, -(n+1)t_0)\underline{u}(-(n+1)t_0) = S(t, t - t_0)a_{n+1}(t - t_0),
\end{aligned}
$$

and so $a_{n+1}(t) \in S(t, t - t_0)I_{\underline{u}}^{\overline{u}}(t - t_0)$, for all $n \in \mathbb{N}$. Thus, from (1) and the relative compactness of $S(t, t - t_0)I_{\underline{u}}^{\overline{u}}(t - t_0)$, there exists the following limit

$$\lim_{n \to +\infty} a_n(t) \overset{\circ}{=} u_*(t).$$

Clearly, $u_* : \mathbb{R} \to X$ is a complete trajectory, as, by the continuity of the process $S(t, s)$,

$$
\begin{aligned}
S(t, s)u_*(s) &= S(t, s) \lim_{n \to +\infty} S(s, -nt_0)\underline{u}(-nt_0) \\
&= \lim_{n \to +\infty} S(t, s)S(s, -nt_0)\underline{u}(-nt_0) = \lim_{n \to +\infty} S(t, -nt_0)\underline{u}(-nt_0) = u_*(t).
\end{aligned}
$$

We now prove that $u_*(t)$, $u^*(t) \in \mathcal{A}(t)$. Indeed,

$$dist(S(t, s)u_*(s), \mathcal{A}(t)) \le d(S(t, s)u_*(s), S(t, s)\underline{u}(s)) + dist(S(t, s)\underline{u}(s), \mathcal{A}(t)),$$

and this right hand of the inequality tends to zero when $s \to -\infty$. As $S(t, s)u_*(s) = u_*(t)$, for all $s \in \mathbb{R}$, $u_*(t) \in \mathcal{A}(t)$.

Is is also straightforward that, for all $u(t) \in \mathcal{A}(t)$

$$\underline{u}(t) \le u_*(t) \le u(t) \le u^*(t) \le \overline{u}(t),$$

by the definition of u_* and u^*, the invariance of $\mathcal{A}(t)$ and the order in X.

On the other hand, for any complete trajectory $v(\cdot)$ such that $\underline{u}(t) \le v(t) \le u_*(t)$, for all $t \in \mathbb{R}$, and by the order in the process,

$$u_*(t) = \lim_{n \to +\infty} S(t, -nt_0)\underline{u}(-nt_0) \le \lim_{n \to +\infty} S(t, -nt_0)v(-nt_0) = v(t) \le u_*(t),$$

so that $v(t) = u_*(t)$, for all $t \in \mathbb{R}$. Note that this implies that u_* and u^* are uniquely defined by the order in X.

Finally, for iii), let be $v \in \mathcal{D}$ with $\underline{u}(t) \le v(t) \le u_*(t)$, for all $t \in \mathbb{R}$. Then, by the attraction property of $\mathcal{A}(t)$

$$u_*(t) = \lim_{s \to +\infty} S(t, -s)\underline{u}(-s) \le \lim_{s \to +\infty} S(t, -s)v(-s) \le \lim_{s \to +\infty} S(t, -s)u_*(-s) = u_*(t).$$

All these arguments also hold for u^*. ∎

Note that the same conclusions can be got under weaker hypotheses:

Corollary 13 *Let S be an order-preserving process and $\mathcal{A}(t)$ its associated pullback attractor. Let $\underline{u}, \overline{u} \in \mathcal{D}$ be such that $\underline{u}(t) \le \overline{u}(t)$, for all $t \in \mathbb{R}$, and assume that*

$$\mathcal{A}(t) \subset I_{\underline{u}}^{\overline{u}}(t), \quad \forall t \in \mathbb{R}.$$

Then there exists two trajectories $u_*(t), u^*(t) \in \mathcal{A}(t)$ such that

i) $u_*(t) \le u \le u^*(t)$, $\forall t \in \mathbb{R}$ and $\forall u \in \mathcal{A}(t)$.

ii) u_* (u^*) is minimal (maximal) in the sense that it does not exist any complete trajectory in the interval $I_{\underline{u}}^{u_*}(I_{u^*}^{\overline{u}})$.

iii) $u_*(t)$ is globally asymptotically stable from below and $u^*(t)$ is globally asymptotically stable from above.

Proof. Since $I_{\underline{u}}^{\overline{u}}(t) \subset \mathcal{D}$, the attractivity property of $\mathcal{A}(t)$ implies that

$$dist(S(t, -s)I_{\underline{u}}^{\overline{u}}(-s), \mathcal{A}(t)) \to 0, \text{ as } s \to +\infty.$$

Now, the compactness of $\mathcal{A}(t)$ and the order relation in X imply that there exist $u_*(t)$, $u^*(t) \in \mathcal{A}(t)$ with

$$\lim_{s \to +\infty} S(t, -s)\underline{u}(-s) = u_*(t) \text{ and } \lim_{s \to +\infty} S(t, -s)\overline{u}(-s) = u^*(t)$$

and the argument follows as in the previous theorem. ∎

4 Non-autonomous logistic equation

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 1$, with smooth boundary $\partial\Omega$. Consider the following non-autonomous logistic equation

$$\begin{cases} u_t - \Delta u = \lambda u - b(t)u^3 \\ \qquad u_{|\partial\Omega} = 0 \\ \qquad u(s) = u_0, \end{cases} \qquad (2)$$

with $\lambda \in \mathbb{R}$ and $b \in C^\nu(\mathbb{R})$, $\nu \in (0,1)$ verifying that there exists a positive constant B such that

$$0 < b(t) \leq B, \quad \text{for all } t \in \mathbb{R}. \qquad (Hb)$$

The next result provides us the existence, uniqueness and helpful estimates of the solution of (2). Consider $X = C_0^0(\overline{\Omega})$ with norm $|\cdot|_0$ and its positive cone

$$\mathcal{V}_+ = \{u \in X : u(x) \geq 0, \text{ a.a. } x \in \Omega\}.$$

For (2), we can define an order with respect to $\mathcal{V}_+$. That is, $u_0 \leq v_0$ if $v_0 - u_0 \in \mathcal{V}_+$.

Given a regular domain $D \subset \mathbb{R}^N$, λ_1^D and φ_1^D stand for the principal eigenvalue and the positive eigenfunction associated to $-\Delta$ under homogeneous Dirichlet condition, normalized such that $\max_{x \in \bar{D}} \varphi_1^D(x) = 1$. We write $\lambda_1 = \lambda_1^\Omega$ and $\varphi_1 = \varphi_1^\Omega$.

Theorem 14 *Assume (Hb) and $u_0 \in \mathcal{V}_+, u_0 \neq 0$. Then, there exists a unique solution $u(t) = u(t, s; u_0) \in X$ of (2), which is strictly positive for $t > s$.*

Proof. We use the sub-supersolution method, see for instance [22]. We take a domain D such that $\Omega \subset D$ and consider the pair

$$(\underline{u}, \overline{u}) := (0, \varepsilon e^{\gamma(t-s)}\varphi_1^D),$$

where ε and γ are constants to be chosen. The pair $(\underline{u}, \overline{u})$ is a sub-supersolution of (2) provided that

$$0 < \frac{\max_{\overline{\Omega}} u_0}{\min_{\overline{\Omega}} \varphi_1^D} \leq \varepsilon, \qquad (3)$$

and

$$0 \leq \gamma + \lambda_1^D - \lambda + b(t)\varepsilon^2 e^{2\gamma(t-s)}(\varphi_1^D)^2. \qquad (4)$$

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Now, it is clear that (3) and (4) are satisfied if ε is large enough and

$$0 \leq \gamma + \lambda_1^D - \lambda \qquad (5)$$

This shows the existence of a nonnegative and nontrivial solution u of (2) such that

$$\underline{u} \leq u \leq \overline{u}. \qquad (6)$$

Now, the strong maximum principle implies that u is strictly positive for $t > s$.

This completes the existence part. The uniqueness follows by a standard way (Pao [22], Chapter 2). ∎

So, we can define the following flow in X, for $t, s \in \mathbb{R}$, $t \geq s$:

$$S(t, s) \quad : \quad X \to X$$
$$S(t, s)u_0 \quad = \quad u(t, s; u_0),$$

with $u(t, s; u_0)$ the unique solution of (2). Furthermore, (2) can be written as the following differential equation in X

$$\begin{cases} \frac{du(t)}{dt} + Au = \lambda u(t) - b(t)u^3(t) \\ \qquad\qquad u_{|\partial\Omega} = 0 \\ \qquad\qquad u(s) = u_0 \end{cases} \qquad (7)$$

with $A = -\Delta$, the linear operator $A : D(A) \to X$ associated to the Laplacian. Moreover, it is clear that, by the maximum principle, $S(t, s)$ is an order-preserving system.

Remark 15 *Note that $v : \mathbb{R} \to X$ is a complete trajectory of problem (2) if*

$$u(t, s; v(s)) = v(t) \text{ in } X, \text{ for } t \geq s,$$

with $u(t, s; v(s))$ the unique solution of (2) with initial condition $u(s) = v(s)$.

4.1 Asymptotic behaviour forward in time

We are interested in the study of qualitative properties in the asymptotic behaviour of problem (2) when the parameter λ changes. The family of maps $\{S(t, s)\}_{t \geq s}$ will allow us to treat this problem from a dynamical system point of view.

If we fix the initial time s, and for $\lambda < \lambda_1$, note that the asymptotic behaviour of (2) is determined around the zero solution, that is, $\{0\}$ is globally asymptotically stable. The following result shows this fact as an easy consequence of Theorem 14.

Corollary 16 *Assume (Hb) and $\lambda < \lambda_1$. Then,*

$$|u(t, s; u_0)|_0 \to 0 \qquad as \; t \to +\infty.$$

Proof. From the monotonicity and continuity of the principal eigenvalue with respect to the domain, there exists a domain $D \supset \Omega$ such that $\lambda < \lambda_1^D < \lambda_1$. So, according to (5) we can take $\gamma < 0$ in Theorem 14. So, by (6)

$$0 < u(t, s; u_0) \le \varepsilon e^{\gamma(t-s)} \varphi_1^D, \tag{8}$$

whence the result follows. ∎

Denote by $\Theta_{[\lambda,c]}$ and Ψ_λ the unique positive solution respectively of

$$\begin{cases} u_t - \Delta u = \lambda u - c u^3 \\ u_{|\partial\Omega} = 0 \\ u(s) = u_0 \end{cases} \tag{9}$$

$$\begin{cases} u_t - \Delta u = \lambda u \\ u_{|\partial\Omega} = 0 \\ u(s) = u_0 \end{cases}$$

where c is a positive constant. Hence, by the maximum principle we get that

$$\Theta_{[\lambda,B]} \le u \le \Psi_\lambda \qquad \text{for } t \ge s.$$

When $\lambda > \lambda_1$, Ψ_λ goes to ∞ as $t \to +\infty$ and $\Theta_{[\lambda,B]}$ goes to $\theta_{[\lambda,B]}$, where $\theta_{[\lambda,B]}$ is the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u - B u^3 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Hence, when $\lambda > \lambda_1$ there exist $V \in C(\overline{\Omega})$ and $t_0(u_0) \in \mathbb{R}$ such that

$$0 < V(x) \le \Theta_{[\lambda,B]}(t, s; u_0) = \Theta_{[\lambda,B]}(t - s, 0; u_0) \le u(t, s; u_0) \qquad \text{for any } t - s \ge t_0(u_0), \tag{11}$$

so that a result on uniform persistence follows.

Remark 17 *If there exists a positive constant A such that $0 < A \le b(t)$, then there exists a positive funcion $W \in C(\overline{\Omega})$ such that*

$$u(t, s, u_0) \le W(x) \qquad for \; any \; t \ge s.$$

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Indeed, in this case if we take

$$0 < M \geq \max\{|u_0|_0, \sqrt{\frac{\lambda}{A}}\}, \tag{12}$$

then $(\underline{u}, \overline{u}) = (0, M)$ is a sub-supersolution of (2), whence the result follows. So, in this case we have that the equation is permanent.

In summary, when $\lambda > \lambda_1$ the behaviour of the positive solution of (2) changes drastically. In particular, the system is not permanent. The next result shows this fact.

Lemma 18 *Consider (2) with $\lambda > \lambda_1$, $u_0 \in \mathcal{V}_+$, $u_0 \neq 0$ and $\lim_{t \to +\infty} b(t) = 0$. Then, for all $M > 0$ and $t_0 \in \mathbb{R}$, there exists $t > t_0$ such that $|u(t, s; u_0)|_0 > M$.*

Proof. We argue by contradiction. Assume that there exist a positive constant $0 < K < +\infty$ and $t_0 \in \mathbb{R}$, such that for all $t \geq t_0$

$$|u(t, s; u_0)|_0 \leq K. \tag{13}$$

Since $\lambda > \lambda_1$, we can take $\varepsilon > 0$ such that

$$\lambda > \lambda_1 + \varepsilon K^2.$$

For this $\varepsilon > 0$, there exists $t_1 > 0$ such that $b(t) \leq \varepsilon$ for $t \geq t_1$. We define

$$q(t) = \int_\Omega u(t, s; u_0)\varphi_1(x)dx.$$

Multiplying the equation that satisfies u by φ_1, integrating over Ω and using the Green's formula, we obtain

$$q'(t) = (\lambda - \lambda_1 - \varepsilon K^2)q(t) + \int_\Omega (\varepsilon K^2 - b(t)u^2)u(t, s; u_0)\varphi_1(x)dx,$$

and so, by (13), we get for $t \geq \max\{t_0, t_1\}$

$$q'(t) \geq (\lambda - \lambda_1 - \varepsilon K^2)q(t) \quad \text{and} \quad q(s) > 0.$$

This is a contradiction with (13). ∎

The preceding result implies that there does not exist any bounded absorbing set for (2) in the sense of a bounded set $B \subset X$ such that, for any $D \subset X$ bounded, $S(t, s)D \subset B$, for t big enough (Chepyzhov and Vishik [9], Temam [25]). Thus, at the parameter value $\lambda = \lambda_1$ it occurs a qualitative change of the asymptotic behaviour of the equation, as a "dissapearance" of the dissipative effect in the equation. On the other hand, note that

the presence of the term $-b(t)u^3$ is also causing some dissipativeness in the problem. It is the possibility of being $b(t)$ as close to zero as time goes to ∞ which causes so big change in the asymptotic behaviour.

Recently, the theory of global attractors for general non-autonomous differential equations has been introduced (see Section 2). In the following section we apply this theory to our problem. Some new qualitative properties in the asymptotic behaviour of (2) will arise by using this theory. In particular, we will show a result on pullback permanence.

4.2 Existence of non-autonomous attractors for the logistic equation

4.2.1 Absorbing set in X

Consider the following non-autonomous differential equation

$$\begin{cases} \frac{dy(y)}{dt} = \lambda y(t) - b(t)y^3(y) \\ y(s) = y_s \end{cases}$$

whose solution satisfies

$$y^2(t, s; y_s) = \frac{e^{2\lambda t}}{\frac{e^{2\lambda s}}{y_s^2} + 2\int_s^t e^{2\lambda \tau} b(\tau)d\tau}.$$

Now, given $D \subset X$ bounded, i.e., $\sup_{d \in D} |d| \leq M$, for $M > 0$, and $u_0 \in D$, the pair $(0, y(t, s; M))$ is a sub-supersolution of (2) and so,

$$u(t, s; u_0) \leq y(t, s; M), \quad \text{for all } t \geq s.$$

Thus, there exists $T(t) \in \mathbb{R}$ such that

$$|u(t, s; u_0)|_0 \leq r_1(t) \quad \text{for} \quad s \leq T(t) \tag{14}$$

where

$$r_1^2(t) = \frac{e^{2\lambda t}}{\int_{-\infty}^t e^{2\lambda \tau} b(\tau)d\tau}.$$

Clearly, this means that the ball in X with radius $r_1(t)$, $B_X(0, r_1(t))$, is absorbing for the process $S(t, s)$.

4.2.2 Absorbing set in $C_0^1(\bar{\Omega})$

In order to obtain a family of absorbing sets in $C_0^1(\bar{\Omega})$ we need the following result which follows by [20], see also Lemma 3.1 in [5]. Here, for a Banach space Y, Y^β will denote the usual fractional power spaces with norm $|\cdot|_\beta$.

Lemma 19 *The operator A generates an analytic semigroup on $Y = C_0^k(\bar{\Omega})$ for $k = 0, 1$. Moreover, it holds*

$$Y^\beta \hookrightarrow C_0^{k+q}(\bar{\Omega}) \text{ for } q = 0, 1 \text{ and } 2\beta > q.$$

Given $D \subset X$ bounded, i.e., $\sup_{d \in D} |d| \leq M$, for $M > 0$, take $u_0 \in D$. We define

$$h(r, s) = \lambda u(r, s; u_0) - b(r)u^3(r, s; u_0) \text{ for } r \geq s.$$

Then, writing the equation from the variation of constants formula, we obtain

$$u(t, s; u_0) = e^{-A(t-s)}u_0 + \int_s^t e^{-A(t-r)}h(r, s)dr.$$

Hence, taking it between $t - 1$ and t, we get

$$u(t, s; u_0) = e^{-A}u(t - 1, s; u_0) + \int_{t-1}^t e^{-A(t-r)}h(r, s)dr.$$

Hence,

$$\begin{aligned}
|u(t, s; u_0)|_\beta &= \left|A^\beta u(t, s; u_0)\right|_0 \leq \left\|A^\beta e^{-A}\right\|_{op} |u(t - 1, s; u_0)|_0 + \\
&\quad \sup_{r \in [t-1, t]} |h(r, s)|_0 \int_{t-1}^t \left\|A^\beta e^{-A(t-r)}\right\|_{op} dr.
\end{aligned}$$

Now, using the estimate $\left\|A^\beta e^{-A(t-r)}\right\|_{op} \leq C_\beta(t-r)^{-\beta}e^{-\delta(t-r)}$ for some constants $C_\beta, \delta > 0$ (cf. Henry [16]) and (14), we obtain the existence of $M(t)$ and $T_0(t)$ such that

$$|u(t, s; u_0)|_\beta \leq M(t) \text{ for all } s \leq T_0(t)$$

with $\beta < 1 - \varepsilon$, and any $\varepsilon \in (0, 1)$. Applying now Lemma 19 with $q = 1$ and $\beta > 1/2$, we obtain

$$|u(t, s; u_0)|_{C^1} \leq R_1(D, t) \text{ for all } s \leq T_0(t),$$

and then the ball in $C_0^1(\bar{\Omega})$, $B(0, R_1(t))$ is absorbing in $C_0^1(\bar{\Omega})$.

We can repeat the argument taking now $Y = C_0^1(\bar{\Omega})$ and D a bounded set in Y. In this case, using again Lemma 19, we obtain that

$$|u(t, s; u_0)|_{C^2} \leq R_2(D, t) \text{ for all } s \leq T_1(t),$$

and hence, the existence of an absorbing set in $C_0^2(\bar{\Omega})$, and so compact in X.

Remark 20 *From (14) we conclude that the non-autonomous attractor $\mathcal{A}(t)$ attracts not only the "pullback pseudotrajectories" $\cup_{s\leq t}S(t,s)u_0$, but we have a stronger attraction property: Consider the base of attraction*

$$\mathcal{D} = \{v : \mathbb{R} \to X \text{ continuous, such that, } \lim_{s\to-\infty} \frac{e^{2\lambda s}}{|v(s)|_0^2} = 0\}$$

Then, we have that, given $v \in \mathcal{D}$,

$$\lim_{s\to-\infty} dist(S(t,s)v(s), \mathcal{A}(t)) = 0. \tag{15}$$

Indeed, we have that for s small enough

$$|u(t,s;v(s))|_0^2 \leq \frac{e^{2\lambda t}}{\frac{e^{2\lambda s}}{|v(s)|_s^2} + 2\int_s^t e^{2\lambda\tau}b(\tau)d\tau} \leq r_1^2(t).$$

Note that every map v, with $v(t) \equiv v_0$, for all t, is in $\mathcal{D}$.

4.3 Upper semicontinuity of non-autonomous attractors to the global attractor

Let b^σ be a family of functions satistying (Hb) and $S_\sigma(t,s)$ be the non-autonomous dynamical system associated to

$$u_t - \Delta u = \lambda u - b^\sigma(t)u^3, \qquad \lim_{\sigma\searrow 0} b^\sigma(t) = \alpha > 0 \tag{16}$$

uniformly on bounded sets of $t \in \mathbb{R}$, $\lambda > \lambda_1$, defined as a small perturbation of the given semigroup S_0 associated to the autonomous equation

$$u_t - \Delta u = \lambda u - \alpha u^3. \tag{17a}$$

Remark 21 *Note that this holds, for example, for $0 < b^\sigma(t) = \alpha e^{-\sigma|t|}$.*

The asymptotic behaviour of this autonomous logistic equation is very well known (cf., for example, Hale [14]). Indeed, it can be proved that (17) has a global attractor $\mathcal{A}$, that is, a compact invariant set ($S_0(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$) attracting every bounded set in X forward in time, i.e. $\lim_{t\to+\infty} dist(S(t)D, \mathcal{A}) = 0$, for all $D \subset X$ bounded. Moreover, we can get some information about the *structure* of this set. Indeed, at the parameter value $\lambda = \lambda_1$ we find a *pitchfork bifurcation*, that is, it bifurcates from the zero solution a globally asymptotically stationary solution $u_+ \in X$, i.e. u_+ satisfies

$$\lim_{t\to+\infty} |u(t;u_0) - u_+|_0 = 0, \text{ for all } u_0 \in \mathcal{V}_+.$$

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Note that the asymptotic behaviour of (16) is rather different. However, we can apply a result on the upper semicontinuity of attractors associated to (16) and (17). Indeed, suppose, for all $t > s$,

$$\lim_{\sigma \searrow 0} d(S_\sigma(t,s)u_0, S_0(t)u_0) = 0 \tag{h1}$$

uniformly on bounded sets of X.

On the other hand, suppose that there exist the pullback attractors $\mathcal{A}_\sigma(t)$ and $\mathcal{A}$, associated to S_σ and S_0 respectively, such that $\mathcal{A}_\sigma(t) \subset K_\sigma(t)$, $\mathcal{A} \subset K$, where $K_\sigma(t)$ and K are compact absorbing sets associated to the corresponding flows, and satisfying

$$\lim_{\sigma \searrow 0} dist(K_\sigma(t), K) = 0, \quad \text{for every } t \in \mathbb{R}. \tag{h2}$$

Then we have (Caraballo and Langa [7])

Theorem 22 *Under the preceding assumptions (h1), (h2), it follows that, for all $t \in \mathbb{R}$,*

$$\lim_{\sigma \searrow 0} dist(\mathcal{A}_\sigma(t), \mathcal{A}) = 0.$$

It remains to prove that conditions (h1) and (h2) are satisfied in our case. Indeed, (h2) is a consequence of the expression for $r_1^\sigma(t)$, with $r_1^\sigma(t)$ the corresponding radius of the absorbing ball associated to (2) with $b^\sigma(t)$. From (14),

$$\lim_{\sigma \searrow 0} r_1^\sigma(t)^2 = \lim_{\sigma \searrow 0} \frac{e^{2\lambda t}}{\int_{-\infty}^t e^{2\lambda \tau} b^\sigma(\tau) d\tau} = 2\frac{\lambda}{\alpha},$$

which is independent of $t \in \mathbb{R}$, so that the same is true for $R_1^\sigma(t)$ and (h2) holds.

On the other hand, (h1) is the content of the following lemma

Lemma 23 *Given $u_\sigma(t,s;u_0), u(t;u_0)$ solutions of (16) and (17) respectively with initial data $u_\sigma(s) = u(s) = u_0$, it holds that, for all $t > s$,*

$$\lim_{\sigma \searrow 0} |u_\sigma(t,s;u_0) - u(t;u_0)|_0 = 0. \tag{18}$$

Proof. Since $\lim_{\sigma \searrow 0} b^\sigma(t) = \alpha > 0$, for σ sufficiently small, positive constants are supersolutions of (16), see (12), and so

$$|u_\sigma(t,s;u_0)|_0 \leq M \text{ (independent of } \sigma.) \tag{19}$$

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If we call $v_\sigma(t, s; u_0) = u_\sigma(t, s; u_0) - u(t; u_0)$, and using the variation of constants formula, we have

$$v_\sigma(t, s; u_0) = \int_s^t e^{-A(t-r)}(\lambda v_\sigma(r, s; u_0) + (\alpha - b^\sigma(r))u_\sigma^3(r, s; u_0) +$$
$$\alpha(u_\sigma^3(r, s; u_0) - u^3(r; u_0)))dr.$$

Since $\left\| e^{-A(t-r)} \right\|_{op} \le e^{-\delta(t-r)} \le 1$, we get

$$|v_\sigma(t, s; u_0)|_0 \le \lambda \int_s^t |v_\sigma(r, s; u_0)|_0 \, dr + \int_s^t |\alpha - b^\sigma(r)| \sup_\sigma |u_\sigma^3(r, s; u_0)|_0 \, dr$$
$$+3\alpha \int_s^t \eta_\sigma^2(\xi_r) \, |v_\sigma(r, s; u_0)|_0 \, dr.$$

Using now (19) and Gronwall's lemma, we get (18). ∎

4.4 Bifurcation scenario for positive solutions

In this section we describe the changes in the asymptotic behaviour of the equation as the parameter value crosses λ_1. For $\lambda < \lambda_1$, note that the zero solution is globally asymptotically stable, in both the forward and the pullback sense. Indeed, from (8) we have that

$$\lim_{t \to +\infty} |u(t, s; u_0)|_0^2 = \lim_{s \to -\infty} |u(t, s; u_0)|_0^2 = 0.$$

This means that, in this case, the non-autonomous attractor reduces to a fixed (not depending on time) point, i.e. $\mathcal{A}(t) \equiv \{0\}$, for all $t \in \mathbb{R}$.

On the other hand, a nontrivial attractor exists for values of the parameter bigger than λ_1. In particular, we prove that the attractor is bigger than the zero solution, i.e. $\{0\} \subsetneq \mathcal{A}(t)$.

Proposition 24 *Given $u_0 \in C_0^1(\bar{\Omega})$ strictly positive, $\lambda > \lambda_1$ and $t \in \mathbb{R}$, there exists $\varepsilon > 0$ such that, for all $s \le t$*

$$|S(t, s)u_0|_0 > \varepsilon.$$

Proof. Since $\lambda > \lambda_1$ and $u_0 \in C_0^1(\bar{\Omega})$ is strictly positive, it is not hard to prove that $\underline{u} = \varepsilon\varphi_1$ is a subsolution of (2) provided that ε verifies

$$0 < \varepsilon \le \min_{x \in \bar{\Omega}} \frac{u_0(x)}{\varphi_1(x)} \quad and \quad \varepsilon^2 B^2 \le \lambda - \lambda_1.$$

So, taking ε sufficiently small, we get

$$\varepsilon\varphi_1 \le u(t, s; u_0)$$

whence the result follows. ∎

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Remark 25 *Note that this implies that $\{0\} \subsetneq \mathcal{A}(t)$. In fact, there exists a subset of $\mathcal{A}(t)$ bounded away from zero "attracting" every $u_0 \in C_0^1(\bar{\Omega})$ strictly positive.*

4.4.1 On the structure of the pullback attractor

In this Section we apply the results of Section 3 (Corollary 13) to our equation (2).

We take $(\underline{u}(t), \overline{u}(t)) = (0, r_1(t))$. Observe that $r_1(t) \in \mathcal{D}$, because

$$\lim_{s \to -\infty} \frac{e^{2\lambda s}}{r_1^2(s)} = 0.$$

Moreover, $\mathcal{A}(t) \subset I_0^{r_1}(t)$, by (14). So, applying Corollary 13, there exists a complete trajectory $u^*(\cdot)$ in the pullback attractor. Since $\mathcal{A}(t) \neq \{0\}$, we have that $u^*(t) \neq \{0\}$, for all $t \in \mathbb{R}$.

Remark 26 *Note that $u^*(\cdot) \in \mathcal{A}(\cdot)$ is a very special complete trajectory. On the one hand, it is a maximal trajectory, an upper bound on the pullback attractor in the positive cone, that is, for all $u(t) \in \mathcal{A}(t)$*

$$u(t) \leq u^*(t).$$

On the other hand, it is globally asymptotically stable from above (cf. Theorem 12).

Finally, we conclude

Theorem 27 *Assume (Hb) and $\lambda > \lambda_1$. Then, (2) is permanent in a pullback sense.*

Proof. Given a bounded set $D \subset X$ and $u_0 \in D$, by (11) we have that

$$0 < V(x) \leq u(t, s; u_0) \text{ for any } s \leq T(t, D).$$

Thus, the permanence follows for

$$U(t) = \{u \in X : V(x) \leq u \leq r_1(t)\}.$$

■

5 Conclusions

We have studied permanence for a general non-autonomous logistic equation. We allow the non-autonomous term to tend to zero, so that we have obtained some information on the transfer of stability at the parameter value $\lambda = \lambda_1$ from the zero solution to the pullback attractor.

On the one hand, we have introduced in a general way the concepts of sub, super and trajectories to get some information on the structure of the pullback attractor of order-preserving systems, finding $u^*(t)$ as a maximal complete trajectory in the attractor with some properties of stability. We think that all this general framework could be appropiate to study bifurcation phenomena and permanence for other non-autonomous partial differential equations. But note that we do not have that u^* it is globally asymptotically stable, that is, for all $u_0 \in \mathcal{V}_+$, $\lim_{s \to +\infty} |S(t, -s)u_0 - u^*(t)|_0 = 0$. Observe that this being true would lead to a pitchfork bifurcation scenario, just as in the autonomous case for the logistic equation.

But, on the other hand, it should be possible to define and apply to particular non-autonomous examples a general theory of stable and unstable manifolds, so that problems on the structure of attractors can be completed as it is known in the autonomous case (Hale [14]). Also the definition of different kinds of bifurcations has to be stated, as well as the application of these ideas to systems of partial differential equations. But we think that what it is done in this work put some light in these challenging problems, which we think to follow studying in future.

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