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A new method for classifying complex filiform Lie algebras

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Abstract

In this paper, we describe a new method to classify complex filiform Lie algebras based on the concept of isomorphism between Lie algebras. This method, which has the advantage of being applied to any dimension, gives the families of algebras in each dimension in an explicit way. In order to apply, only the corresponding structure theorem of complex filiform Lie algebras in each dimension is needed. As a consequence of our study, we also predict that the increase (in terms of quotiens) in the number of algebras families when passing from even dimension to odd dimension tends to 1 whereas it grows in a no finite way if passing from odd dimension to immediate even dimension.

Keywords: Lie algebras; Filiform; Classification; Isomorphism

1. Introduction

Classifying Lie algebras is actually an open problem, which has been treated by several authors by using different methods. In fact, only the classifications of solvable, nilpotent and filiform complex Lie algebras of dimension n, with $n \leqslant 5$, $n \leqslant 7$ and $n \leqslant 12$, respectively, are known.

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Dealing with the problem of classifying complex filiform Lie algebras only, which is the aim of our study, the first author who defined the filiform Lie algebras was M. Vergne, in her thesis in 1966, which was later published in [10]. She firstly classified these algebras in the cases of low dimensions and she also showed that they are a subset of the nilpotent Lie algebras (in fact, the most structured subset of nilpotent Lie algebras).

In 1988, Goze and Ancochea [1], through the introduction of a new invariant which they call the *characteristic sequence*, obtained the classification of these algebras in dimension 8. This classification was later corrected by the same authors [2] and independently same time, by Seeley [9]. Later, Echarte and Gómez [6] classified using this method filiform Lie algebras of dimension 9.

Echarte and two of the authors of this paper classified in 1996 these algebras in dimension 10 by the introduction of another invariant, which they call *the pair* (i, h) [4].

Recently, Gómez et al. [7] by the method of the *elementary changes of basis* gave a correct classification of complex filiform Lie algebras of dimension n with $n \leqslant 11$.

Finally, at the end of 1997, the last classification of complex filiform Lie algebras known until now appeared. The authors of this paper [5] classified complex filiform Lie algebras of dimension 12 by using this method which we explain in this paper, based on the concept of *isomorphism* between Lie algebras. However, this list, in which 496 families of these algebras appeared, had some errors which were checked by Gómez et al. [7]. The authors thank them for their constructive suggestions in the application of this method.

It is convenient to note that by classifying a set, we mean to find a certain property of it which let us define an equivalence relation among its elements. The equivalence relation which we will use to classify complex filiform Lie algebras will be isomorphic. So, we will explicitly compute a representative of each class of complex filiform Lie algebras in each dimension. However, this process requires hard and complicated calculations, which are impossible without the use of a computer. In our case, most of the calculations needed were made by using the *Mathematica* program, although another symbolic computation program could be used.

2. Definitions and notations

In this paper, all the Lie algebras which appear will be considered over the complex field $\mathbb{C}$.

Let $\mathbf{g} = (\mathbb{C}^n, \mu)$ be a Lie algebra of dimension n, with μ the associated law. We consider the lower central series of $\mathbf{g}$ defined by $C^1\mathbf{g} = \mathbf{g}$, $C^i\mathbf{g} = \mu(\mathbf{g}, C^{i-1}\mathbf{g})$. The Lie algebra $\mathbf{g}$ is *filiform* if $\dim_{\mathbb{C}} C^i\mathbf{g} = n - i$ for $2 \leqslant i \leqslant n$. If $x \in \mathbf{g}$ we denote by $\mathrm{ad}(x)$ the adjoint mapping associated to x (i.e., the map $y \mapsto \mu(x, y)$).

Let $\mathbf{g}$ be a filiform Lie algebra of dimension n. Then there exists a basis $\mathscr{B} = \{e_1, \ldots, e_n\}$ of $\mathbf{g}$ such that $e_1 \in \mathbf{g} \setminus C^2\mathbf{g}$, the matrix of $\mathrm{ad}(e_1)$ with respect to $\mathscr{B}$ has a Jordan block of order $n-1$ and $C^i\mathbf{g}$ is the vector space generated by $\{e_2, \ldots, e_{n-(i-1)}\}$ with $2 \leqslant i \leqslant n-1$. Such a basis is called an *adapted basis*.

Sometimes, for the sake of simplicity, we will use $[x, y]$ instead of $\mu(x, y)$ for the Lie bracket in a Lie algebra.

Let $\mathbf{g}$ and $\mathbf{g}'$ be two Lie algebras. A map $\Phi : \mathbf{g}' \mapsto \mathbf{g}$ is said to be a *homomorphism* between Lie algebras if Φ is a linear application such that $\Phi : [X, Y] \mapsto [\Phi(X), \Phi(Y)]\ \forall X, Y \in \mathbf{g}'$. In the case of being Φ a bijection, it is called *isomorphism*.

3. A new method for classifying filiform Lie algebras

In order to classify complex filiform Lie algebras of any dimension, we use the structure theorem of these algebras. So, let $\mathbf{g}$ be a complex filiform Lie algebra of dimension n with an adapted basis $\{e_1, \ldots, e_n\}$. Then, $[e_1, e_h] = e_{h-1}$, $3 \leqslant h \leqslant n$. A structure theorem for $\mathbf{g}$ consists of giving explicitly the rest of the non-null brackets. These brackets are expressed as function of some structure constants a_{ij}. These constants, which define the law of the algebra, are related themselves by polynomial relations coming from the Jacobi identities. These theorems are already known for dimension n, with $n \leqslant 14$ (see [3,4,8], for instance).

To make this idea more understandable, we now present, for the sake of example, the structure theorem corresponding to the case of the dimension 8.

If $\mathbf{g} = (\mathbb{C}^8, \mu)$ *is a filiform Lie algebra, then there exists an adapted basis* $\mathscr{B} = \{e_1, e_2, \ldots, e_8\}$ *of* $\mathbf{g}$ *such that*

- $[e_1, e_h] = e_{h-1}$ $(3 \leqslant h \leqslant 8)$,
- $[e_4, e_7] = a_{47}e_2$,
- $[e_5, e_6] = -a_{47}e_2$,
- $[e_4, e_8] = a_{47}e_3 + a_{48}e_2$,
- $[e_5, e_7] = a_{57}e_2$,
- $[e_5, e_8] = a_{47}e_4 + (a_{48} + a_{57})e_3 + a_{58}e_2$,
- $[e_6, e_7] = a_{57}e_3 + a_{67}e_2$,
- $[e_6, e_8] = a_{47}e_5 + (a_{48} + 2a_{57})e_4 + (a_{58} + a_{67})e_3 + a_{68}e_2$,
- $[e_7, e_8] = a_{47}e_6 + (a_{48} + 2a_{57})e_5 + (a_{58} + a_{67})e_4 + a_{68}e_3 + a_{78}e_2$,

where the rest of the brackets are null and the coefficients $a_{47}, a_{48}, \ldots, a_{78} \in \mathbb{C}$ *verify the following equation*: $a_{47}(2a_{48} + 5a_{57}) = 0$.

From now on, we say that a complex filiform Lie algebra $\mathbf{g}$, *with basis* $\{e_1, e_2, \ldots, e_8\}$, *verifying the above condition, has the law* $\mu(a_{47}, a_{48}, a_{57}, a_{58}, a_{67}, a_{68}, a_{78})$, *where the constants define the law* μ *of this algebra.*

As it can be seen in this example, the subindexes of these structure constants have been denoted according to the *first* bracket in which they appear. In order to decide which is the *first* bracket, we have set a certain order relation $\prec$ in the set of the subindexes pairs (i,j) with $i<j$. This order is defined by

$$(i,j) \prec (k,l) \Longleftrightarrow [i+j<k+l \ \vee \ (i+j=k+l \wedge i<k)].$$

In dimension n, the *first* bracket we consider is $[e_4, e_n] = a_{4n}e_2$ and we also use $\prec$ to order the constants in the law of the algebra.

The main idea to classify these algebras is defining two Lie algebras and demanding them to be isomorphic. In this way, we get the necessary conditions for both to be isomorphic. In fact, sufficient conditions are deduced when we classify each particular case, due to the isomorphisms between them. We obtain the classification required from the different possibilities of some of the last conditions not to be verified, that is, after grouping all the complex filiform Lie algebras of each dimension in isomorphic classes.

So, first of all, we consider two bases over a same vector space verifying the thesis of the structure theorem of each dimension. Secondly, we set the existence of an isomorphism which maps the basis of the second algebra in the basis of the first algebra. According to the definition of the isomorphism and to these theorems, we will get some restrictions, from which we can deduce the different families of algebras.

In short, we give the following steps:

Step 1. Let $\mathbf{g}$ and $\mathbf{g}'$ be two complex filiform Lie algebras of dimension n, with bases $\{u_1,\ldots,u_n\}$ and $\{v_1,\ldots,v_n\}$, respectively. From now on we denote the coefficients defining their laws by a_{ij} and a'_{ij}.

Let consider the isomorphism $\Phi : \mathbf{g}' \to \mathbf{g}$, this is, the bases change given by

$$v_1 = \sum_{i=1}^{n} s_i u_i, \quad v_n = \sum_{i=1}^{n} r_i u_i$$

and by the property of filiformity of the algebra.

Step 2. By straightforward computations in the previous step, we can deduce the following general expression:

$$v_2 = s_1(s_1 r_n - s_n r_1)(s_1 - s_n a_{4,n-1})^{n-4} u_2,$$

where $a_{4,n-1}$ is the coefficient of u_{n-2} in the bracket $[u_{n-1}, u_n]$. In fact, it can be easily proved that $a_{4,n-1} = 0$ if n is odd, whereas $a_{4,n-1}$ is the first structure constant with respect to the order previously defined, if n is even.

Then, as v_2 belongs to a basis, necessarily $v_2 \neq 0$ and thus, we have $s_1 \neq 0$, $s_1 r_n - s_n r_1 \neq 0$ and $s_1 - s_n a_{4,n-1} \neq 0$.

Step 3. Now, to make next the computations easier, by using again the property of filiformity, we have $0 = [v_3, v_n] = -r_1(s_1 r_n - s_n r_1)(s_1 - s_n a_{4,n-1})^{n-4}$ u_2. Then, by taking into consideration the previous step, we have $r_1 = 0$. So,

$v_1 = \sum_{i=1}^{n} s_i u_i,\ v_n = \sum_{i=2}^{n} r_i u_i$. We also deduce, in this step, the following conditions between the elements of the law of the algebra: $s_1 \neq 0,\ r_n \neq 0$ and $s_1 - s_n a_{4,n-1} \neq 0$.

Step 4. As the bases change Φ is an isomorphism between Lie algebras, then $[\Phi(v_k), \Phi(v_l)] - \Phi([v_k, v_l]) = 0\ \forall k, l = 1, \ldots, n$. We obtain some vector equations by using these last expressions.

Step 5. It is easy to prove in each dimension that we do not lose any piece of information although we operate with second components only. So, from now on, we will take into consideration these components only.

Step 6. In this step, we already have new equations from the previous vector equations (which we denote $c_{ij} = 0$) and the equations consisting on $P_k = 0$ (these last ones are the restrictions coming from the Jacobi identities in the corresponding structure theorem). So, we have a set of equations involving the coefficients which appear in the laws of each algebra and the coefficients r_i and s_j from the isomorphisms. Now, we wish to solve the equation system constituted by the $P_k = 0$. It obliges us to distinguish several cases (for example, 16 of them are considered in dimension 12). By using the first $c_{ij} = 0$ we can easily prove that each case corresponds to families of algebras which are non-isomorphic to the ones of other cases.

Step 7. Finally, we wish that, in all of the cases of the previous step, the equations $c_{ij} = 0$ give specific values (that is, constants) for the coefficients of the law or of the isomorphisms or, in other case, relations between them.

In this way, for instance, if we find an expression like the following: $a_1 = s_1 a'_1$, we would distinguish two non-isomorphic cases: $a_1 = 0 = a'_1$ and $a_1 \neq 0 \neq a'_1$ (where a_1 and a'_1 represent any coefficient a_{ij} or a'_{ij} of the laws of $\mathbf{g}$ and $\mathbf{g}'$, respectively).

In this method, we use the following particular kind of sets to describe algebras having the same starting point in our method, in a non-redundant way and by using a unique family:

$$\mathbb{C}_m(d) = \left\{ d + r\mathrm{e}^{\mathrm{i}\varphi}\colon r > 0,\ \varphi \in \left[0, \frac{2\pi}{m}\right) \right\} \quad \text{and we denote } \mathbb{C}_m(0) = \mathbb{C}_m.$$

Now, we will explain something more about these sets. The set $\mathbb{C}_m$ has been already used in earlier classifications to solve some situations as the following.

As we described in the general explanation, it is sometimes convenient to distinguish two cases depending on $a_2 = 0$ or not. We now consider the second case. Then, let us suppose that we find an expression like $a_2 - s_1^m a'_2 = 0$. Then, if we consider an isomorphism with $s_1 = 1/(a'_2)^{1/m}$, any algebra of the family which we are considering is isomorphic to one with $a_2 = 1$. So, from then on, we study the subfamily with $a_2 = 1$. Then, we must suppose $a_2 = 1 = a'_2$ and we must also suppose that s_1 is a mth root of the unit. We can do it because this last condition will be verified for any isomorphism between algebras belonging to that subfamily.

If, when continuing the study of this same case, an equation of the kind $a_3 - s_1 a'_3 = 0$ appears, we would have to distinguish two new cases:

- $a_3 = 0 = a'_3$ and s_1 is a mth root of the unit.
- $a_3 = s_1 a'_3 \neq 0$. In this case, the sector $\mathbb{C}_m$ is sufficient to represent in a unique way all possible values of a_3 of the algebras of that family, as a convenient value of s_1 relates both a_3 and a'_3 by one isomorphism.

So, from then on, we suppose $a_3 = a'_3 \in \mathbb{C}_m$ and $s_1 = 1$.

Something similar occurs with the set $\mathbb{C}_m(d)$. There is no need for using these kinds of sets in dimensions less than 12, but it appears on some occasions for greater dimensions.

For instance, let us suppose that we are using some expressions of the kind $Aa'_4 + Ba_4 + C = 0$, where A, B, C are expressions depending on r_{12} (with r_{12} a square root of the unit), such that: if $r_{12} = 1$, it is deduced that $a_4 = a'_4$ and if $r_{12} = -1$, then $a_4 + a'_4 = d \in \mathbb{C}$.

That is, it occurs that either $a_4 = a'_4$ or $d/2$ is the middle point of a_4 and a'_4 in the complex plane. Then, any possible value of a_4 in an algebra of the family which we are studying can be represented by one and only one element belonging to the set $\mathbb{C}_2(d/2)$.

In this way, by applying our method we have obtained the classification of complex filiform Lie algebras of dimension n with $n \leqslant 12$ and we have observed that our results coincide with the ones previously obtained by other authors for dimension $\leqslant 11$, although they were reached by using different methods. So, we note that there is only a finite number of complex filiform Lie algebras of dimension 6 pairwise non-isomorphic. In dimension 7 a no finite family of complex filiform Lie algebras appears for the first time. In dimension 8 some families have parameters not belonging to $\mathbb{C}$, but either to $\mathbb{C}_2$ or to another subset of $\mathbb{C}$ defined by an easy algebraic restriction. In dimension 9 our method groups the algebras by using two two-parameter families and in the case of dimension 10, nine three-parameter families and the use of the set $\mathbb{C}_3$ is needed. New difficulties appear when applying our method to dimension 11. In this way, we find four- and five-parameter families of algebras. We also find the appearance of some parameters belonging to $\mathbb{C}_4$ and others appearing as solutions of third-grade equations. Finally, in dimension 12, the greatest dimension in which our method has been applied, we found the sets $\mathbb{C}_5, \mathbb{C}_2(d)$ and $\mathbb{C}_3(d)$. Some equations with grades up to 6 containing parameters of exponent 4 were also used.

With respect to the number of families obtained in each dimension when classifying complex filiform Lie algebras, we have observed that this number (1 family of algebras in dimension 4, 2 in dimension 5, 5 in 6, 8 in 7, 22 in 8, 34 in 9, ...) allows us to predict that the increase (in terms of quotiens) in the number of algebras families when passing from even dimension to odd dimension $(2/1, 8/5, 34/22, \ldots)$ is going to tend to 1 whereas it could grow in a no finite way if passing from odd dimension to immediate even dimension (5/2, 22/8, 104/34, ...).

This conjecture allows us to suppose that the classification of complex filiform Lie algebras of dimension 13 is not going to be very complicated with respect to that already obtained in dimension 12, whereas to get the classification in dimension 14 must result in more difficult and it must present quite a great complexity, not only in computations but in the number of families of algebras finally obtained.

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