A Correction of the Friction Term in Depth-Averaged Granular Flow Models Related to the Motion/Stop Criterion

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Abstract In the pioneer work of Savage and Hutter [7] the first depth-averaged model for granular landslides was proposed written in local coordinates over a reference plane. From then, there has been proposed a hugh amount of variations, corrections and generalizations. In particular, for their impact in real applications, we mention the work developed in [1] to include the topography curvature effects and the new friction law proposed in [6] to better reproduce the spread of the granular mass through a velocity dependent friction coefficient. In [3] a correction of the friction law has been proposed in order to improve the motion/stop criterion in terms of the repose angle of the material. This correction have any influence for uniform flows, nevertheless, it plays an important role in case of a free surface not parallel to the reference plane. In this work we show two new applications of this model to relevant situations. In the first one we show the influence of the correction on the spread area of the avalanche over variable topography. In the second one a comparison with experimental data of a submarine avalanche is presented.

Keywords Depth-averaged · Friction term · Motion criteria · Granular landslides

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C. Parés et al. (eds.), *Hyperbolic Problems: Theory, Numerics, Applications. Volume II*,
SEMA SIMAI Springer Series 35, https://doi.org/10.1007/978-3-031-55264-9_24

1 Introduction and Notation

In this work we study the motion/stop criteria of depth-averaged models for aerial and submarine avalanches. We consider as a base the Savage-Hutter's model, which can be written in the following form:

$$\begin{cases} \partial_t H + \cos\theta \partial_X (HU) = 0, \\ \partial_t (HU) + \cos\theta \partial_X (HU^2 + g\frac{H^2}{2}\cos\theta) = \\ \qquad\qquad -gH\cos\theta\partial_X(\hat{b}\cos\theta) - g\,\mathrm{sgn}(U)H\cos\theta\mu, \end{cases} \tag{1}$$

where the unknowns are H, the thickness of the avalanche measured perpendicularly to a reference plane with slope $\tan\theta$ (see Fig. 1) and U, the averaged velocity tangential to the reference plane. g is the constant of gravity and μ is the coefficient of the friction law. μ can be defined as a constant value, for the case of a Coulomb law, $\mu = \mu_C = \tan\delta_0$, being δ_0 the angle of repose of the material. Another possibility is to consider a more complex law, as for example the Pouliquen's law proposed in [6]. In this case, the friction coefficient depends on the velocity and thickness of the flow through the Froude number Fr. Finally, $\hat{b}$ describes the topography, taking into account the reference plane, it is described in what follows and defined by (4).

In model (1) local coordinates type are considered over an inclined plane. Denoting by $\mathbf{x} = (x, z)$ the Cartesian coordinates, we define the reference plane determined by a fixed point $(\bar{x}, \bar{z})$ and the slope $\tan\theta$. For every point P in the granular layer, we introduce the coordinates $\mathbf{X} = (X, Z)$, X being the abscissa of the orthogonal projection of P onto the reference plane and Z being the distance from P to the reference plane (see Fig. 1). The relation between the Cartesian coordinates $\mathbf{x}$ and the $\mathbf{X}$ coordinates is

$$\mathbf{x} = (X - Z\sin\theta, b(X) + Z\cos\theta), \tag{2}$$

where $b(X)$ is the level of the reference plane:

$$b(X) = \bar{z} + \tan\theta(X - \bar{x}). \tag{3}$$

The fixed topography measured perpendicularly to the reference plane is defined by $\tilde{b}(X)$. In order to describe the interfaces in a more suitable way, we also introduce the distance $\hat{b}$ defined by

$$\hat{b}(X)\cos\theta = b(X) + \tilde{b}(X)\cos\theta, \tag{4}$$

that is, $\hat{b}(X)\cos\theta$ designs the vertical distance (with sign) of the bottom to the abscissa axis (see Fig. 1).

A generalization of model (1) taking into account curvature terms of the topography is presented in [1]. In the present work we only consider the model written

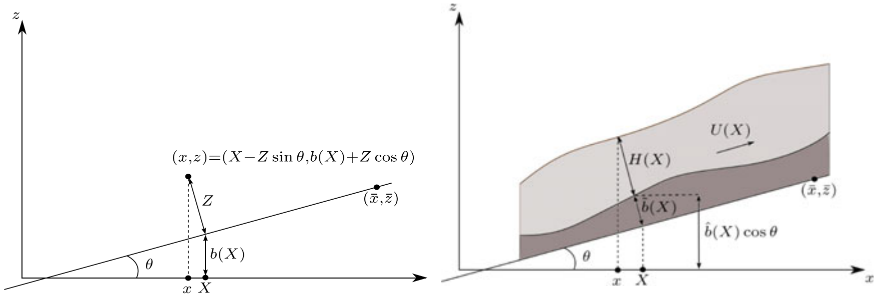

Fig. 1 Left: Cartesian and local coordinates notation. Right: Sketch of the domain and notation

in previous introduced local coordinate system from a reference plane. We refer the reader to the reference [3] for its generalization to the case of general topography.

For the case of submarine avalanches a complete extension of the Savage-Hutter model for this case is introduced in [4]. Here we consider a simplification of this model, consisting of considering the system (1) with a reduced gravity instead, that is, g is replaced with $g(1-\rho_f/\rho_g)$, for ρ_f, ρ_g being the density of the fluid and granular material, respectively.

2 Proposed Correction

In depth-averaged models, the interaction between the granular material and the bottom surface is considered through a term depending on a friction coefficient μ multiplied by the normal stress $gH\cos\theta$. For instance, for the Coulomb friction law, $\mu = \tan\delta_0$, δ_0 being the angle of repose of the material. Thus, if the granular layer is initially at rest and $z = \eta(x)$ defines the avalanche free surface, the equality $|\partial_x \eta| = \mu$ marks the inflection between motion or stillness of the avalanche. In local coordinates, that inflection should be taken into account in the modeling of the friction law. If the momentum equation of system (1) is rewritten as follows

$$\partial_t(HU) + \cos\theta\,\partial_X(HU^2) = -gH\cos\theta\Big(\partial_X(\hat{b}+H)\,\cos\theta + \mathrm{sgn}(U)\mu\Big), \qquad (5)$$

a glance to the right hand side member reveals that the inflection between the motion or stillness of the avalanche is related now to the equality $|\partial_X(\hat{b}+H)\,\cos\theta| = \mu$.

The following basic example explains the problem with the globally friction law used in the literature. Figure 2 illustrates an avalanche on a leaning plane (for simplicity, we take $\tilde{b} = 0$). Consider two points X_1 and X_2 on the abscissa axis as in the figure, and let $x_i = X_i - H(X_i)\,\sin\theta$, $i = 1, 2$. Note that the slope of the blue line

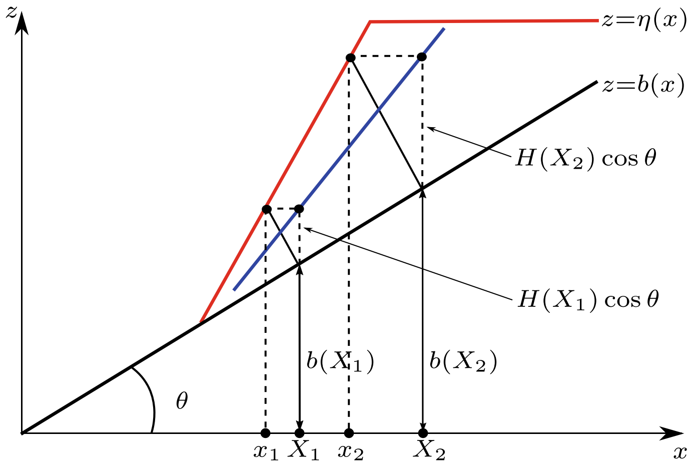

Fig. 2 A basic example showing the problem with the globally friction coefficient

is different to the slope of the free surface of the avalanche (red line), so

$$\frac{(\hat{b}(X_2) + H(X_2))\ \cos\theta - (\hat{b}(X_1) + H(X_1))\ \cos\theta}{X_2 - X_1} \neq \frac{\eta(x_2) - \eta(x_1)}{x_2 - x_1}.$$

Hence, $\partial_X(\hat{b} + H)\ \cos\theta \neq \partial_x \eta$ in general, although we have the equality for uniform flows. This suggests that the statement of the coefficient of friction law μ should be adapted if the model is written in local coordinates in order to reflect effectively the inflection between motion or stillness of the avalanche. Now then, from the relation between Cartesian and local coordinates given by $x = X - (\tilde{b}(X) + H(X))\ \sin\theta$, we have

$$\partial_X(\hat{b}(X) + H(X))\ \cos\theta = \mathcal{J}\ \partial_x(\eta(x)), \tag{6}$$

where $\mathcal{J}$ is the Jacobian of the change of variables related to the local coordinates:

$$\mathcal{J} = 1 - \partial_X(\tilde{b}(X) + H(X))\ \sin\theta.$$

This leads us to propose the following correction of the friction coefficient,

$$\mu_{\mathcal{J}} = \mathcal{J}\mu;$$

that is, we propose to replace μ with $\mu_{\mathcal{J}}$ in the friction term coefficient in model (1). Note that $\mu_{\mathcal{J}} = \mu$ when H is constant as, for example, in the case of uniform flows. From now on, model (1) will be referred as *local-JCF* (respectively, *local*) when $\mu_{\mathcal{J}}$ (respectively, μ) is the friction coefficient.

Here we presented the main lines of this friction correction. A complete deduction has been developed in [3], firstly taking into account the definition of the friction term in the depth-averaged procedure, and secondly, as a second order correction of the bed pressure.

3 Numerical Tests

In this section two tests are presented, the first one is a qualitative test that compare three models. In the second one a comparison with experimental data for a submarine avalanche is presented. The three models are denoted as follows: the first one is named *Cartesian* model, for which the heigh is measured in the vertical direction, obtained by setting $\theta = 0$. *Local* and Local-JCF correspond to the models where the height is measured perpendicularly to the reference bottom, without or with the proposed correction, respectively. For the numerical test a finite volume method with a hydrostatic reconstruction is considered, by applying an apparent bottom topography to discretize the friction term (see [2]). Note that the same numerical scheme can be considered for the case with the proposed correction, by replacing μ by $\mu_{\mathcal{J}}$.

3.1 Test 1: Comparison of the Three Models for a Variable Topography

This test explores the evolution of the avalanche, firstly, over a smooth bottom and, subsequently, by adding a bump. A comparison of the results of the three models are presented in what follows. No curvature for the bottom is considered; the angle θ of the reference plane has been set to 20° when local coordinates are used.

The domain is $x \in [-7, 2]$ discretized with 3600 points. Let us consider a bottom defined as

$$z_b(x) = \frac{3}{2}\arctan(x)$$

and the initial condition for the granular free surface is

$$\eta(x) = \max\left(-\frac{1}{2}\left(x - \frac{3}{2}\right)^4 + \frac{3}{2}\arctan\left(\frac{3}{2}\right), z_b(x)\right)$$

(see Fig. 3). In Fig. 4 right we show the comparison of the solution at rest obtained for the three models when setting the repose friction angle $\delta_0 = 30°$. The model written in local coordinates without the proposed correction predicts a shorter runout than the other two models, that gives quite similar solutions with some differences in the

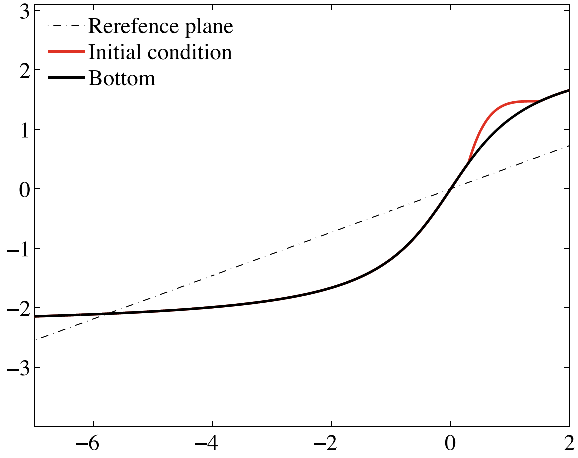

Fig. 3 Test 1: Initial condition. Reference plane, landslide free surface and bottom

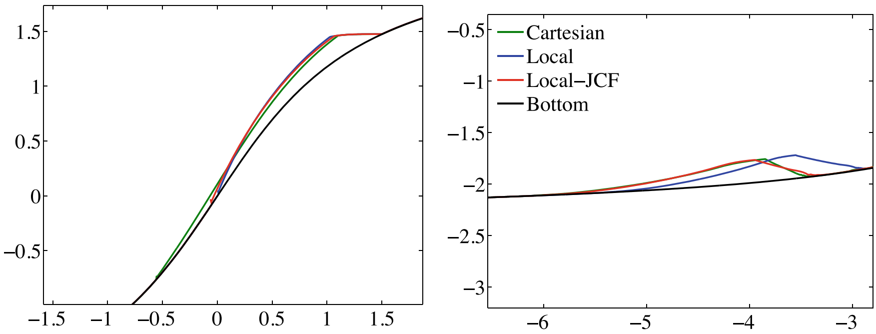

Fig. 4 Test 1: Comparison between the models *Cartesian*, *Local* and *Local-JCF*. Left: $t = 0.3$ s. Right: solution at rest

rear part of the avalanche. Nevertheless, an opposed behavior is found during the flow of the avalanche in the first stages (see Fig. 4 left related to time $t = 0.3$ s). At this moment, contrary to the final state, the faster solution is given by the *Cartesian* model, while the solutions corresponding to *Local* and *Local-JCF* also keep quite similar and remain backward.

Although the solution at rest predicted by the *Cartesian* model is quite similar to the *Local-JCF* one, the evolution of the corresponding avalanches at the first stages reveals different behaviors through the most sloping part of the bottom (Fig. 4). To further explore this situation, we have repeated the test by including a bump in the bottom in order to show how the direction of the velocity of the profile of the landslide (related to the angle of reference plane) affects the predictions. So, consider now the new bottom defined by (see Fig. 5)

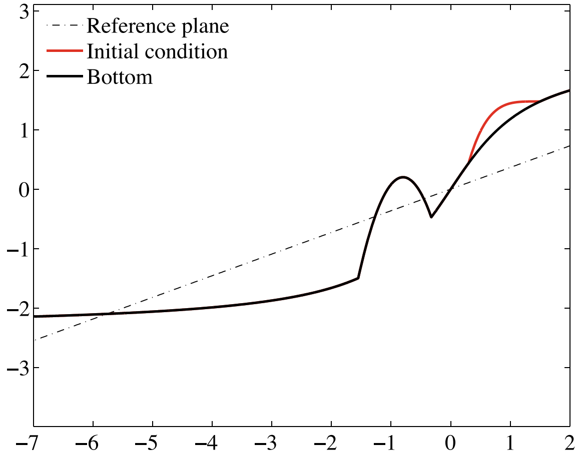

Fig. 5 Test 1 (bottom with bump): Initial condition. Reference plane, landslide free surface and bottom

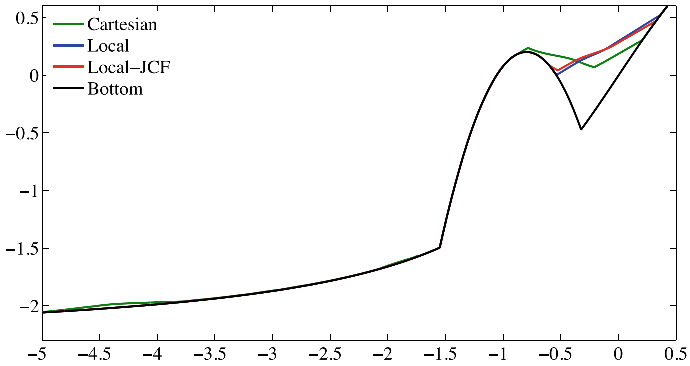

Fig. 6 Test 1 (bottom with bump): Solution at rest. The prediction of the *Cartesian* model exceeded the bump, but *Local* and *Local-JCF* have not

$$z_b(x) = \max\left(\frac{3}{2}\arctan(x), -3\left(x+\frac{4}{5}\right)^2+\frac{1}{5}\right).$$

Figure 6 shows the comparison of the solution at rest for the three models. Notice that the avalanches modeled using *Local* and *Local-JCF* stop when they achieve the bump. On the contrary the avalanche *Cartesian* model is able to pass over it. Its is due to the fact that the unknown of this model is the horizontal velocity and assumes a small vertical velocity, instead of a small velocity perpendicular to the bottom, as in *Local* models.We can also observe the mass resting after the bump around $x = -4.5$. That is, it predicts an erroneous spread of the avalanche.

3.2 Test 2: Comparison with Experimental Data for Submarine Avalanches

In this test we apply the correction of the friction coefficient to a submarine avalanche model and we show the comparison of the numerical results with experimental data. We focus on the evolution of the granular layer, so system (1) is adopted by considering the reduced gravity instead, $g' = (1 - r)g$, where r is the density ratio, $r = \frac{\rho_f}{\rho_b} < 1$ for ρ_b the bulk density of the mixture granular/fluid mass.

The laboratory experiments have been performed by Viroulet [9] and also presented in [5]. The experiment consists of a release of a granular mass confined behind a removable gate in the upper part of a plane with slope α (see Fig. 7 for the sketch and notation). The granular material is made of glass beads of diameter $d_s = 4$ mm and density $\rho_g = 2500$ kg/m^3. A thin layer of these beads are sticked all along the bottom surface. The reservoir is filled with 2 kg of those beads after being compacted to a compaction index of $\varphi = 0.6$. Notice that we use the bulk density to calculate the density ratio r, that is, $\rho_b = \varphi\rho_g + (1 - \varphi)\rho_f$ with $\rho_f = 1000$ kg/m^3, what gives $r = 0.5263$.

The tank is 2.20 m long, 0.20 m large and 0.40 m high. Two experiments have been carried out for two different slopes, $\alpha = 35°$ (Case A) and $\alpha = 45°$ (Case B). The initial level water is placed at the same depth h_{w0} for both cases. The initial height of the granular layer is denoted by h_{s0} that is maximum at point x_0 (given by the location of the lifting gate). In Table 1, we specify these data of the experimental tests. In our case, a longitudinal section is considered to perform the 1D simulation and we take the slope of the reference plane equal to the inclination α in each case.

The results have been obtained by using 400 points for the discretization of the domain [0, 2.20 m], then $\Delta x = 0.0055$ m and a CFL condition of 0.5.

In Figs. 8 and 9, we show the final results for all studied options together with the experimental data when the granular layer stops. The curves for experiments are

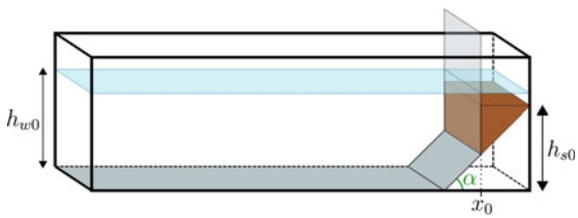

Fig. 7 Test 2. Sketch of the experiment and notation

Table 1 Experiment parameters for Test 2

Case	α (°)	h_{w0} (m)	h_{s0} (m)	x_0 (m)
A	35	0.28	0.2448	0.2071
B	45	0.28	0.271	0.145

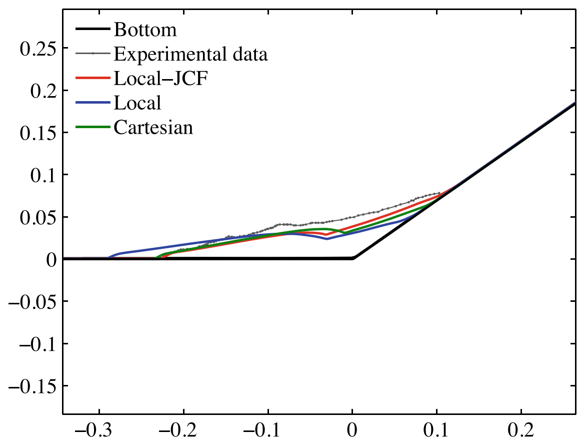

Fig. 8 Test 2: Comparison with experimental data, plane slope of 35°

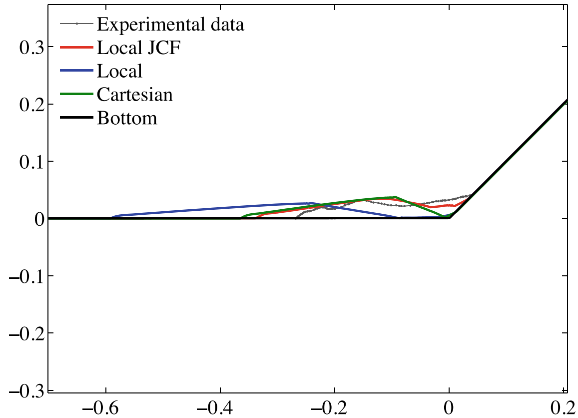

Fig. 9 Test 2: Comparison with experimental data, plane slope of 45°

obtained from digital images and kindly provided by Poulain [5]. As in Test 1, the *Local* solution goes further than the other computations followed by *Cartesian* and *Local-JCF* solutions. Looking first at the Case A (Fig. 8), we can see that even the *Cartesian* option gives a good result of the runout, the local-JFC solution improves it in both, front and rear part of the granular mass. Notice that the apparent loss of mass is due to the 1D approximation of the real problem. In Case B (Fig. 9) the *Local* solution is very far from the experimental data, but also *Cartesian* and local-JFC are not so close, unless in the front part. Nevertheless we observe the same relative behavior, obtaining an improved approximation of local-JFC coordinates mainly in the rear position, where part of the mass is retained on the inclined plane at stationary state. In order to improve this result it would be necessary to consider the model taking into account curvature effects. In [3] it has been shown the crucial effects of curvature terms in comparison with experimental data for the case of aereal avalanches.

Conclusions

A correction of the friction coefficient for models written in local coordinates proposed in [3] has been applied to two numerical tests. As we have shown in the numerical results, its influence cannot be predicted a priori with respect to a classical definition of the friction term because it depends on the gradient of the free surface. In the two numerical tests, we have shown that the correction has a great influence on the runout and the evolution of the avalanche. Moreover, it improves the results for the submarine avalanches in comparison with experimental data. A generalization of this correction for 2D models can be found in [3].

Acknowledgements This research has been partially supported by the Spanish Government and FEDER through the research project RTI2018-096064-B-C22, and by the ERC contract ERC-CG-2013-PE10-617472 SLIDE-QUAKES.

References

1. Bouchut, F., Mangeney, A., Perthame, B., Vilotte, J.P.: A new model of Saint Venant and Savage-Hutter type for gravity driven shallow water flows. Comptes Rendus Mathematique **336**, 531–536 (2003)
2. Bouchut, F.: Nonlinear stability of finite volume methods for hyperbolic conservation laws, and well-balanced schemes for sources. Frontiers in Mathematics Series, Birkhäuser (2004)
3. Bouchut, F., Delgado-Sánchez, J.M., Fernández-Nieto, E.D., Mangeney, A., Narbona-Reina, G.: A bed pressure correction of the friction term for depth-averaged granular flow models. Appl. Math. Model. **106**, 627–658 (2022)
4. Fernández-Nieto, E.D., Bouchut, F., Bresch, D., Castro Díaz, M.J., Mangeney, A.: A new Savage-Hutter type model for submarine avalanches and generated tsunami. J. Comput. Phys. **227**, 7720–7754 (2008)
5. Poulain, P.: Granular flows simulation and application to volcanic flank collapse. Master's thesis, Master GEO3: Risques Naturels Telluriques (2017)
6. Pouliquen, O., Forterre, Y.: Friction law for dense granular flows: application to the motion of a mass down a rough inclined plane. J. Fluid Mech. **453**, 133–151 (2002)
7. Savage, S.B., Hutter, K.: The motion of a finite mass of granular material down a rough incline. J. Fluid Mech. **199**, 177–215 (1989)
8. Savage, S.B., Hutter, K.: The dynamics of avalanches of granular materials from initiation to runout. Part I: analysis. Acta Mechanica **86**(1), 201–223 (1991)
9. Viroulet, S., Sauret, A., Kimmoun, O., Kharif, C.: Granular collapse into water: toward tsunami landslides. J. Visu. **16**, 189–191 (2013)