# On irreducible triangulations of Punctured Torus and Punctured Klein Bottle[S]

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**Abstract.** We extend Lawrencenko and Negami's results in [4, 5] by establishing lower and upper bounds for the number of irreducible triangulations of the punctured torus and the punctured Klein bottle.

*Keywords. Closed 2-manifold; punctured 2-manifold; irreducible triangulation; contraction of edges; vertex splitting.*

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## 1 Introduction and Terminology

In this work we will deal with 2-manifolds with boundary. By a *2-manifold with boundary* we mean the polyhedron underlying a pure, strongly connected simplicial 2-complex which has no singular vertices, each edge of $K$ belongs to two triangles at most and whose boundary is also a 1-manifold (and hence, a disjoint union of circles).

The simplicial 2-complex $K$ is said to be a *triangulation* of a 2-manifold $P$ (in fact, the simplicial complex $K$ is often called an abstract 2-manifold). For our purpose, when there is no ambiguity, we will replace a triangulation $K$ of $P$ by the edge graph $G(K)$ of $K$ and triangles of $K$ will be called faces of $G(K)$.

Let $K$ be a triangulation of a 2-manifold $P$ and $G = G(K)$ be the edge graph of $K$, we define a *contraction* of an edge $e = v_1v_2$ of $G$ to be the process of shrinking $e$ to a vertex $v_1 = v_2$ and then removing multiple edges; we denote by $G/e$ the graph obtained by contracting $e$ in $G$. We say that an edge $e$ of $K$ is *contractible* if $G/e$ is the edge graph of a triangulation $K'$ of $P$. If $G$ is a graph not isomorphic to $K_4$, it is readily checked that $e$ is not contractible

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if and only if $e$ belongs to a cycle of length 3 which is not a triangle of the triangulation $K$. We say that a triangulation is *irreducible* if it has no contractible edge. The contractible edge $e$ is said to be *$k$-contractible* if $\delta(G/e) \geq k$. Here $\delta(G(K)) = min\{deg(v) : v \in K \ dim(v) = 0\}$ denotes the minimum degree of $G(K)$. Notice that the minimum degree condition $\delta(G) \geq 4$ implies that no triangle in $G$ has more than one edge lying in $\partial(P)$.

*Splitting of* a vertex $v_1 = v_2$ of $G$ is the inverse operation of contracting the edge $e = v_1v_2$. When $deg(v_i) \geq k$ for $i = 1, 2$ after the splitting, this is called *$k$-splitting.* From the definition, it is easy to deduce that any triangulation of $P$ comes from an irreducible one by a sequence of 3-splittings. It can be also proved that an edge $e$ is 4-contractible if and only if $e$ belongs to triangles $t_1t_2t_3$ and $t_1t_2t_4$ such that $e = t_1t_2$, and $\deg(t_i) \geq 5$ for $i = 3, 4$.

A cycle in $G$ is *critical* if it consists of three edges which do not bound a face of $G$. From the definition, it is easy to deduce that an edge is contractible if it is not in a critical cycle of $G$.

Beside splitting and contracting, we also consider the operation of *adding an octahedron* introduced by Nakamoto and Negami ([6]), which is a particular case of face subdivision defined as follows. Given a face $f = v_1v_2v_3$ of $G$, we place a new triangle $t_1t_2t_3$ inside $f$ and add the new edges $v_it_j$, for $i \neq j$.

Irreducible triangulations of the sphere, the Klein bottle, the projective plane and the torus have been studied ([7, 5, 1, 4]). All of them, except those of the sphere, have minimum degree greater than or equal to 4.

In [6] Nakamoto and Negami proved that every triangulation on a closed surface with minimum degree at least 4 can be obtained from an irreducible triangulation by a finite sequence of 4-splittings of vertices and addition of octahedra. In a previous work, we present analogous results for the class of 2-pseudomanifolds: *Every triangulation of a 2-pseudomanifold with (possibly non-empty) boundary and minimum degree at least four can be constructed from an irreducible triangulation by a finite sequence of vertices splittings and additions of octahedra ([3]). In the same paper, we gave the set of irreducible triangulations of the disk, the cylinder, the Möbius strip, the pinched sphere and the pinched projective plane. Recall that a pinched 2-manifold is the result of identifying two points in a 2-manifold.*

In this paper we provide lower and upper bounds for the number of irreducible triangulations of the punctured torus and the punctured Klein bottle. As usual, by a *punctured 2-manifold* we mean a 2-manifold with a disk removed.

The method presented here can be considered as a "surface simplification" that preserves the topology of the surface, a very useful research topic in computer graphics.

## 2 Main Results

Now, following similar techniques as in [3], we tackle the problem of finding the set of irreducible triangulations of the punctured Torus and the punctured Klein bottle. For this we need the following lemmas:

**Lemma 1.** *Let $K$ be an irreducible triangulation of a 2-manifold without boundary $P$, let $G(K) = (V(K), A(K))$ denote the graph of $K$ and let $D \subset P$ be a disk of $P$. Let $K - v$ denote the triangulation obtained by removing $v$ and its adjacent edges from $G(K)$. Let $K - a$ denote the triangulation obtained by removing the edge $a$ from $G(K)$; The following triangulations are irreducible for $P - D$:*

1. *$K - v$ for any $v \in V(K)$, if $K$ is minimal in the vertex set,*
2. *$K - a$ for any $a \in A(K)$, if $K$ is minimal in the vertex set and*
3. *$K$ minus one triangle $t$.*

**Lemma 2.** *Let $K$ be an irreducible triangulation of a 2-manifold without boundary $P$, let $G(K) = (V(K), A(K))$ denotes the graph of $K$. The following statements hold:*

1. *For any $v \in V(K)$, the edges in the boundary of $K - v$ are not contractible in $K - v$.*
2. *For any $a \in A(K)$, the edges in the boundary of $K - a$ are not contractible in $K - a$.*
3. *Let $e \in A(K)$ such that $e$ belongs to exactly one critical cycle $C$ in $G$, let $a \in A(C) - e$, then $e$ is contractible in $K - a$.*

By using the previous lemmas and the set of 21 irreducible triangulations of the torus given by Lawrencenko in [4], we can establish:

**Theorem 1.** *The number of irreducible triangulations of the punctured torus is at least 98 and at most 663.*

Analogously, by considering the set of 25 irreducible triangulations of the Klein bottle given by Lawrencenko and Negami in [5], we prove:

**Theorem 2.** *The number of irreducible triangulations of the punctured Klein bottle is at least 50 and at most 1344.*

We conjecture that both the lower and upper bounds in Theorems 3 and 4 can be improved.

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