NORMING SETS ON A COMPACT COMPLEX MANIFOLD

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Abstract. We describe the norming sets for the space of global holomorphic sections to a k-power of a positive holomorphic line bundle on a compact complex manifold X. We characterize in metric terms the sequence of measurable subsets $\{G_k\}_k$ of X such that there is a constant $C > 0$ where

$$\|s\|^2 \leq C \int_{G_k} |s(z)|^2 \, dV(z)$$

for every $s \in H^0(X, \mathcal{O}(L^{\otimes k}))$ and for all $k \in \mathbb{N}$.

1. Introduction

Let X be a compact complex manifold which is endowed with a smooth hermitian metric ω. As we know, this metric induces a distance function $d(x, y)$ on X, that will be used to define the balls, and a volume form V, which will be used to integrate over the manifold.

We assume a holomorphic line bundle L on a compact complex manifold X endowed with a smooth hermitian metric ϕ. Moreover, we consider that the line bundle (L, ϕ) is positive, that is, $i\partial\overline{\partial}\phi$ is a positive form.

We denote by $H^0(X, \mathcal{O}(L^{\otimes k}))$ the space of global holomorphic sections to a k-power $L^{\otimes k}$ of a positive line bundle L, where $L^{\otimes k}$ is endowed with the product metric $k\phi$.

The statement of the main result is the following:

Theorem 1.1. *For a sequence of measurable subsets $\{G_k\}$ in X the following are equivalent:*

(1) *There is a constant $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\int_X |s|^2 \, dV \leq C \int_{G_k} |s|^2 \, dV$$

for every $s \in H^0(X, \mathcal{O}(L^{\otimes k}))$. We will say that $\{G_k\}$ is a norming sequence for the section ring

$$R_X(L) := \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{O}(L^{\otimes k})).$$

(2) *There is a constant $\delta > 0$ and a radius R such that*

$$Vol\left(G_k \cap B\left(a, k^{-1/2}R\right)\right) \geq \delta \; Vol\left(B\left(a, k^{-1/2}R\right)\right)$$

for all $a \in X$ and $k \in \mathbb{N}$. We will say that G_k is relatively dense in X.

As we will see below this problem has already proved for some space of functions. In [LS74] it appears the proof of the Logvinenko-Sereda theorem for functions of the Paley-Wiener space PW_K for a fixed K:

Theorem 1.2. *For a measurable subset $G \subset \mathbb{R}^n$ the following are equivalent:*

(1) *There is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |f|^2 \, dm \leq C \int_G |f|^2 \, dm$$

for every $f \in PW_K$. We will say that G is a norming set for the Paley-Wiener space*.*

(2) *There is a cube $Q \subset \mathbb{R}^n$ ans a constant $\gamma > 0$ such that*

$$m((Q + x) \cap G) \geq \gamma$$

for all $x \in \mathbb{R}^n$. We will say that G is relatively dense in $\mathbb{R}^n$*.*

See also [HJ94, pag 112-115].

Theorem 1.2 has an analogue for functions in the Bergman space A^p in the unit disk $\mathbb{D}$ in [Lue81]:

Theorem 1.3. *For a measurable subset $G \subset \mathbb{D}$ the following are equivalent:*

(1) *There is a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f|^p \, dm \leq C \int_G |f|^p \, dm$$

for every $f \in A^p$. We will say that G is a norming set for the Bergman space*.*

(2) *There is a constant $\delta > 0$ and radius $R \in (0, 1)$ such that*

$$m(G \cap D(a, (1 - |a|)R)) \geq \delta m(D(a, (1 - |a|)R))$$

for all $a \in \mathbb{D}$. We will say that G is relatively dense in the disc $\mathbb{D}$*.*

It should be noted that using similar arguments to those used in [Lue81] one can prove the Fock space version of D.H. Luecking:

Theorem 1.4. *For a a measurable subset $G \subset \mathbb{C}^n$ the following are equivalent:*

(1) *There is a constant $C > 0$ such that*

$$\int_{\mathbb{C}^n} |f|^2 e^{-2|z|^2} \, dm(z) \leq C \int_G |f|^2 e^{-2|z|^2} \, dm(z)$$

for every $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$. We will say that G is a norming set for the classical Fock space*.*

(2) *There is a constant $\delta > 0$ and a radius $R > 0$ such that*

$$m(G \cap B(z, R)) \geq \delta \; m(B(a, R))$$

for all $z \in \mathbb{C}^n$. We will say that G is relatively dense in $\mathbb{C}^n$*.*

2. Peak Sections

2.1. **The Bergman Kernel.** The space $H^0(X, \mathcal{O}(L^{\otimes k}))$ admits a Hilbert space structure endowed with the scalar product

$$\langle u, v\rangle = \int_X \langle u(x), v(x)\rangle, \quad u, v \in H^0(X, \mathcal{O}(L^{\otimes k}))$$

where the integration is taken with respect to the volume form V.

The Bergman kernel $\Pi_k(x, y)$ associated to this space is a section to the line bundle $L^{\otimes k} \boxtimes \overline{L^{\otimes k}}$ over the manifold $X \times X$, defined by

$$\Pi_k(x, y) = \sum_{j=1}^{N} s_j(x) \otimes \overline{s_j(y)}$$

where $s_1, \ldots, s_N$ is an orthonormal basis for $H^0(X, \mathcal{O}(L^{\otimes k}))$. Moreover, this definition does not depend on the choice of the orthonormal basis. Notice that $L^{\otimes k} \boxtimes \overline{L^{\otimes k}} = \pi_1^*(L^{\otimes k}) \otimes \pi_2^*(\overline{L^{\otimes k}})$ where $\pi_i = X \times X \to X$ is the projection onto the i-factor.

The Bergman kernel $\Pi_k(x, y)$ is in a sense the reproducing kernel for the space $H^0(X, \mathcal{O}(L^{\otimes k}))$, satisfying the reproducing formula

$$s(x) = \int_X \langle s(y), \Pi_k(x, y)\rangle \; dV(y)$$

for $s(y) \in H^0(X, \mathcal{O}(L^{\otimes k}))$.

The pointwise norm of the Bergman kernel in symmetric, $|\Pi_k(x, y)| = |\Pi_k(y, x)|$. Moreover, it satisfies

$$|\Pi_k(x, x)| = \int_X |\Pi_k(x, y)|^2 \; dV(y).$$

Lemma 2.1. *Let (L, ϕ) be a positive line bundle. We have the off-diagonal estimate*

$$|\Pi_k(x, y)| \lesssim k^n e^{-c\sqrt{k}\; d(x,y)}$$

where c is an appropriate positive constant and the diagonal estimate

$$|\Pi_k(x, x)| \asymp k^n.$$

Moreover, from this we obtain the estimate of the dimension of $H^0(X, \mathcal{O}(L^{\otimes k}))$

$$\dim H^0(X, \mathcal{O}(L^{\otimes k})) \asymp k^n.$$

See [Ber03].

2.2. **Construction of Peak Sections.** The main goal of the lemma is to prove for each ball the existence of a normalized peak-section, that is, a section such that most of the mass is in such ball. For this we will use the reproducing kernel of $H^0(X, \mathcal{O}(L^{\otimes k}))$.

Lemma 2.2. *Given $\varepsilon > 0$, there exist $R > 0$ and $k_0 > 0$ such that for all $k \geq k_0$ and all $y \in X$ there exists a section $s = s_{y,k} \in H^0(X, \mathcal{O}(L^{\otimes k}))$ such that*

- $\|s\|^2 = \int_X |s|^2 \; dV = 1$, *and*

- $\int_{X\setminus B\left(y,k^{-1/2}R\right)} |s|^2\, dV < \varepsilon.$

Proof. Given $\varepsilon > 0$. First of all, we take any $k > k_1 = \max\{k \in \mathbb{N} \ : \ H^0(X, \mathcal{O}(L^{\otimes k})) = 0\}$, since there are ample line bundles L without non-zero sections.

We consider the reproducing kernel of the Hilbert space $H^0(X, \mathcal{O}(L^{\otimes k}))$

$$\Pi_k(x,y) = \sum_{j=1}^{N} s_j(x) \otimes \overline{s_j(y)}$$

where $s_1, \dots, s_N$ is an orthonormal basis for $H^0(X, \mathcal{O}(L^{\otimes k}))$.

If we fix $y \in X$, there is a section $\Phi_{k,y} \in H^0(X, \mathcal{O}(L^{\otimes k}))$ such that

$$|\Phi_{k,y}(x)| = |\Pi_k(x,y)|, \quad x \in X,$$

by the lemma in [LOC12, pag. 431]. So, we assume the section

$$s_{k,y}(x) = \frac{\Phi_{k,y}(x)}{\sqrt{|\Pi_k(y,y)|}}, \quad x \in X,$$

where Π_k denotes the Bergman kernel for the k'th power $L^{\otimes k}$ of the line bundle L.

First,

$$\int_X |s_{k,y}(x)|^2\, dV(x) = \frac{1}{|\Pi_k(y,y)|} \int_X |\Pi_k(x,y)|^2\, dV(x) = \frac{|\Pi_k(y,y)|}{|\Pi_k(y,y)|} = 1.$$

Next,

$$\int_{X\setminus B\left(y,k^{-1/2}R\right)} |s_{k,y}(x)|^2\, dV(x) = \frac{1}{|\Pi_k(y,y)|} \int_{X\setminus B\left(y,k^{-1/2}R\right)} |\Pi_k(x,y)|^2\, dV(x)$$

$$= \frac{1}{|\Pi_k(y,y)|} \int_0^\infty Vol\left(\{x : |\Pi_k(x,y)| > \lambda\} \setminus B\left(y, k^{-1/2}R\right)\right) 2\lambda\, d\lambda$$

$$\leq \frac{1}{|\Pi_k(y,y)|} \int_0^{k^n D} Vol\left(\left\{x : De^{-c\sqrt{k}\ d(x,y)} \geq k^{-n}\lambda\right\} \setminus B\left(y, k^{-1/2}R\right)\right) 2\lambda\, d\lambda.$$

where we use that $|\Pi_k(x,y)| \leq k^n De^{-c\sqrt{k}\ d(x,y)}$ for some constants $c, D > 0$, as we see in Lemma 2.1. Applying the change of variable $\lambda = k^n De^{-cu}$.

$$\frac{1}{|\Pi_k(y,y)|} \int_0^\infty Vol\left(B\left(y, k^{-1/2}u\right) \setminus B\left(y, k^{-1/2}R\right)\right) (2k^n De^{-cu})| - ck^n De^{-cu}|\, du$$

$$\lesssim \frac{1}{k^n} \int_R^\infty \left(\frac{u}{\sqrt{k}}\right)^{2n} (2k^n De^{-cu})| - ck^n De^{-cu}|\, du$$

$$\lesssim \int_R^\infty u^{2n} e^{-2cu}\, du = e^{-2cR} \int_0^\infty (\mu + R)^{2n} e^{-2c\mu}\, d\mu$$

$$= e^{-2cR} \sum_{j=0}^{2n} \binom{2n}{k} R^{2n-j} \int_0^\infty \mu^j e^{-2c\mu}\, d\mu$$

$$\lesssim e^{-2cR} \sum_{j=0}^{2n} \binom{2n}{k} \frac{R^{2n-j}}{(2c)^j} j! \lesssim e^{-2cR} \sum_{j=0}^{2n} \binom{2n}{k} (2cR)^{2n-j} = e^{-2cR}(1 + 2cR)^{2n}.$$

Finally, for R large enough we obtain that

$$\int_{X\setminus B\left(y,k^{-1/2}R\right)} |s_{k,y}(x)|^2 \, dV(x) < \varepsilon.$$

In addition, we choose $k_0 >> R^2$ such that $k_0 > k_1$ and for all $k \geq k_0$ the balls $B(y, k^{-1/2}R)$ are not the whole space. So, we conclude the proof of the lemma. □

Remark 2.3. Notice that taking the same section of the previous lemma and using the Cauchy-Schwarz inequality we obtain that

$$|s(x)|^2 = |\langle s(y), \Pi_k(x,y)\rangle|^2 \leq \|s\|^2 |\Pi(x,x)| \lesssim k^n.$$

3. Proof of the Main Theorem

We are ready to prove Theorem 1.1.

The proof that (1) implies (2) is the easiest. In this proof, we consider a section with the properties of Lemma 2.2.

So, given $\varepsilon \leq 1/2C$ and applying the Lemma 2.2 there is a radius R and $k_0 > 0$ such that for all balls $B\left(a, k^{-1/2}R\right)$ and $k \geq k_0$ there is a section $s^* = s^*_{a,k}$ verifying the properties of the lemma. Hence, applying the Remark 2.3 we obtain

$$\frac{Vol\left(G_k \cap B\left(a, k^{-1/2}R\right)\right)}{Vol\left(B\left(a, k^{-1/2}R\right)\right)} \gtrsim \frac{1}{R^{2n}} \int_{G_k \cap B\left(a,k^{-1/2}R\right)} |s^*|^2 \, dV$$

and using the other properties with (1) we have for $k \geq k_0$ that

$$\begin{aligned} \frac{Vol\left(G_k \cap B\left(a, k^{-1/2}R\right)\right)}{Vol\left(B\left(a, k^{-1/2}R\right)\right)} &\gtrsim \frac{1}{R^{2n}} \left(\int_{G_k} |s^*|^2 \, dV - \int_{X\setminus B\left(a,k^{-1/2}R\right)} |s^*|^2 \, dV \right) \\ &\geq \frac{1}{R^{2n}} \left(\frac{1}{C} - \varepsilon \right) \geq \frac{1}{2CR^{2n}}. \end{aligned}$$

Therefore, we have proved that (1) implies (2).

The proof that (2) implies (1) is the difficult one. We will use a similar argument as in [MOC08, Chapter 4]. We will partition X in two pieces. The first piece, denoted by $\mathcal{A}_s$, is the set of points where the section is much smaller that its average and the second is the complementary $X \setminus \mathcal{A}_s$. The following lemma proves that the integral over $\mathcal{A}_s$ is irrelevant as most of the mass is carried by $X \setminus \mathcal{A}_s$.

Lemma 3.1. *Let $s \in H^0(X, \mathcal{O}(L^{\otimes k}))$. Define the set*

$$\mathcal{A}_s = \left\{ a \in X \mid |s(a)|^2 < \frac{1}{2Vol(B(a, R/\sqrt{k}))} \int_{B(a,k^{-1/2}R)} |s|^2 \, dV \right\}.$$

Then

$$\int_{\mathcal{A}_s} |s|^2 \, dV < \frac{1}{2} \int_X |s|^2 \, dV.$$

Proof. For $a \in \mathcal{A}_s$ we have

$$|s(a)|^2 < \frac{1}{2Vol(B(a,k^{-1/2}R))} \int_{B(a,k^{-1/2}R)} |s(z)|^2 \, dV(z) = \frac{1}{2}\int_X |s(z)|^2 \frac{\chi_{B(a,k^{-1/2}R)}(z)}{Vol(B(a,k^{-1/2}R))} \, dV(z).$$

Integrating with respect to a and applying Fubini's theorem

$$\begin{aligned}\int_{\mathcal{A}_s} |s|^2 \, dV &< \frac{1}{2}\int_X |s(z)|^2 \left(\int_{\mathcal{A}_s} \frac{\chi_{B(a,k^{-1/2}R)}(z)}{Vol(B(a,k^{-1/2}R))} \, dV(a)\right) dV(z) \\ &\leq \frac{Vol(B(a,k^{-1/2}R))}{Vol(B(a,k^{-1/2}R))} \frac{1}{2}\int_X |s(z)|^2 \, dV(z) = \frac{1}{2}\int_X |s|^2 \, dV.\end{aligned}$$

Therefore, we obtain

$$\int_{\mathcal{A}_s} |s|^2 \, dV < \frac{1}{2}\int_X |s|^2 \, dV.$$

□

Let $\mathcal{F}_s = X \setminus \mathcal{A}_s = \left\{a \in X \ : \ |s(a)|^2 \geq \frac{1}{2Vol(B(a,k^{-1/2}R))} \int_{B(a,k^{-1/2}R)} |s|^2 \, dV\right\}$. We have

$$\int_X |s|^2 \, dV < 2\int_{\mathcal{F}_s} |s|^2 \, dV$$

since

$$\int_X |s|^2 \, dV = \int_{\mathcal{F}_s} |s|^2 \, dV + \int_{\mathcal{A}_s} |s|^2 \, dV < \int_{\mathcal{F}_s} |s|^2 \, dV + \frac{1}{2}\int_X |s|^2 \, dV.$$

Thus it is enough to show that

$$\int_{\mathcal{F}_s} |s|^2 \, dV \lesssim \int_{G_k} |s|^2 \, dV.$$

All we need to prove is the existence of a constant $C > 0$ such that for all $k >> 0$, all $s \in H^0(X, \mathcal{O}(L^{\otimes k}))$ and all $w \in \mathcal{F}_s$

$$|s(w)|^2 \leq \frac{C}{Vol\left(B\left(w,k^{-1/2}R\right)\right)} \int_{B\left(w,k^{-1/2}R\right)\cap G_k} |s|^2 \, dV. \tag{3.1}$$

Indeed, if this is the case then

$$\begin{aligned}\int_{\mathcal{F}_s} |s|^2 \, dV &\leq C\int_{G_k} |s(z)|^2 \left(\int_{\mathcal{F}_s} \frac{\chi_{B\left(w,k^{-1/2}R\right)}(z)}{Vol\left(B\left(w,k^{-1/2}R\right)\right)} \, dV(w)\right) dV(z) \\ &\leq C\int_{G_k} |s|^2 \, dV.\end{aligned}$$

To prove (3.1) we argue by contradiction. If (3.1) is false then for any $q \in \mathbb{N}$ there are sections $s_q \in H^0(X, \mathcal{O}(L^{\otimes k_q}))$ and $w_q \in \mathcal{F}_s$ such that

$$|s_q(w_q)|^2 > \frac{q}{Vol\left(B\left(w_q, k_q^{-1/2}R\right)\right)} \int_{B(w_q,k_q^{-1/2}R)\cap G_{k_q}} |s_q(z)|^2 \, dV(z) \tag{3.2}$$

By compactness of X we can choose a convergent subsequence $\{w_q^*\}$ of w_q to some $w^* \in X$ such that there is a local chart (U, φ) where

$$B\left(w_q^*, k_q^{-1/2}R\right) \subset U, \ \forall q \in \mathbb{N}$$

and $\varphi(w^*) = 0$.

As ϕ is given by the negative logarithm of the norm of some frame ξ of L, that is, $\phi = -\log|\xi|^2$. We can choose a frame properly such that

$$\phi(\varphi^{-1}(z)) = |z|^2 + q(z)$$

in the local ball $B(0, \tau)$ where $q(z) = o(|z|^2)$. Hence, we have that

$$|2\phi(\varphi^{-1}(z)) - 2|z|^2| \leq c(\tau)\tau^2$$

in $B(0, \tau)$ for nondecreasing continuous function $c(\tau)$ such that $c(0) = 0$. Notice that we can use the local chart $(\varphi^{-1}(B(0, \tau*)), \varphi)$ instead since there exists $\nu \in \mathbb{N}$ such that for all $q \geq \nu$

$$B\left(w_q^*, k_q^{-1/2}R\right) \subset \varphi^{-1}(B(0, \tau^*)) \quad \text{and} \quad \tau^* \leq \tau.$$

Now, we consider the sequence of holomorphic functions

$$F_q(z) = f_q(\varphi^{-1}(z)), \quad q \in \mathbb{N}$$

where $s_q = f_q \ \xi^{\otimes k}$ with $|s_q|^2 = |f_q|^2 e^{-k\phi}$ and we need to see that

$$|s_q(w_q^*)|^2 \lesssim |F_q(\eta_q)|^2 \tag{3.3}$$

where $\varphi(w_q^*) = \eta_q$,

$$|s_q(\varphi^{-1}(z))|^2 \gtrsim |F_q(z)|^2 \tag{3.4}$$

for a certain ball and

$$|F_q(\lambda_q)|^2 \gtrsim \frac{q}{m(B(\eta_q, \delta_q))} \int_{B(\eta_q,\delta_q)\cap\varphi(G_{k_q})} |F_q(z)|^2 \, dm(z). \tag{3.5}$$

First, for the sequence $\{w_q^*\}$ we have

$$|s_q(w_q^*)|^2 = |f_q(\varphi^{-1}(\eta_q))|^2 e^{-k\phi(\varphi^{-1}(\eta_q))} \leq |f_q(\varphi^{-1}(\eta_q))|^2 e^{\frac{k}{2}c(\tau^*)\tau^{*2}} \lesssim |F_q(\eta_q)|^2.$$

Furthermore for $z \in B(0, \tau^*)$ we obtain

$$|F_q(z)|^2 = |f_q(\varphi^{-1}(z))|^2 \lesssim |s_q(\varphi^{-1}(z))|^2, \quad z \in B(0, \tau^*).$$

Finally, using (3.2), (3.3), (3.4) and taking the local chart $(\varphi^{-1}(B(0, \tau^*)), \varphi)$, it follows that

$$\begin{aligned}
|F_q(\eta_q)|^2 \gtrsim |s_q(w_q^*)|^2 &\gtrsim \frac{q}{Vol\left(B\left(w_q^*, \ k_q^{-1/2}R\right)\right)} \int_{B\left(w_q^*,k_q^{-1/2}R\right)\cap G_{k_q}} |s_q|^2 \, dV \\
&\gtrsim \frac{q}{Vol\left(B\left(w_q^*, k_q^{-1/2}R\right)\right)} \int_{\varphi\left(B\left(w_q^*,k_q^{-1/2}R\right)\cap G_{k_q}\right)} |f_q(\varphi^{-1}(z))|^2 \ e^{-k\phi(\varphi^{-1}(z))} \, dm(z) \\
&\gtrsim \frac{q}{m(B(\eta_q, \delta_{k_q}))} \int_{B\left(\eta_q,\delta_{k_q}\right)\cap G_{k_q}^*} |F_q|^2 \, dm
\end{aligned}$$

where $G^*_{k_q} = \varphi(G_{k_q})$ and there is a ball $B(\eta_{k_q}, \delta_{k_q}) \subset \varphi(B(w^*_q, k_q^{-1/2}))$ such that $m(B(\eta_q, \delta_{k_q})) \asymp Vol\left(B\left(w^*_q, k_q^{-1/2}R\right)\right)$, because as $\varphi, \varphi^{-1} \in \mathcal{C}^1$ we have that φ is bi-Lipschitz.

Next, by means of a dilation and a translation we send η_q to the origin of $\mathbb{C}^n$, the ball $B(\eta_q, \delta_{k_n})$ to $B(0,1) \subset \mathbb{C}^n$ and the set $G^*_{k_q}$ to G^*_q. We will denote the set $\overline{B(0,1)} \cap G^*_q$ by H_q. Moreover, we apply the above transformations to the function F_q and multiply by a constant such that the resulting function, denoted by $\tilde{F}_q$, verifies

$$\int_{B(0,1)} |\tilde{F}_q(z)|^2 \, dm = 1.$$

The subharmoniticity of $|\tilde{F}_q|^2$ and the fact that $w^*_q \in \mathcal{F}_s$ tells us that

$$\frac{1}{2} \lesssim |\tilde{F}_q(0)|^2 \lesssim 1. \tag{3.6}$$

Notice that we can modify the inequality of the definition of $\mathcal{F}_s$ in the same way as the expression (3.5). This property (3.6) together with (3.5) yields

$$\frac{1}{q} \gtrsim \int_{H_q} |\tilde{F}_q(u)|^2 \, dm(u). \tag{3.7}$$

Now, we have that $\{\tilde{F}_q\}$ is a locally bounded sequence of holomorphic functions defined in $B(0,1)$ and therefore using Montel theorem there exist a subsequence converging locally uniformly on $B(0,1)$ to some holomorphic function G.

We observe that the relative dense hypothesis yields

$$\inf_q m(H_q) > 0.$$

The Banach-Alaoglu theorem [Rud06, pag. 68] guarantees the existence of a weak-$*$limit μ of a subsequence of $\mu_q = \chi_{H_q} m$ such that $\sigma \not\equiv 0$. Condition (3.7) implies that $\tilde{F}_q = 0$ μ-a.e. and therefore $supp\mu \subset \{\tilde{F}_q = 0\}$.

We want to show that $supp\mu$ cannot lie on a complex $(n-1)$-dimensional submanifold $S \subset \mathbb{C}^n$. We argue by contradiction.

By definition of μ_q we have $\mu_q(B(x,r)) \leq m(B(x,r)) \lesssim r^{2n}$. As this inequality is true for all μ_q, the same holds for μ. But, using the Frostman lemma [Mat99, pag. 112] we obtain that $\dim_{Haus}(supp\tau) \geq 2n > 0$.

Therefore, we have proved that (2) implies (1).

□

4. Sampling Sequences

Definition 4.1. Let $\{\mu_k\}_k \subset \mathfrak{M}(X)$ be a sequence of measures. We will say that it defines *uniformly standard norms* for the section ring $R_X(L)$ if there is a constant $C > 0$ such that

$$\int_X |s|^2 \, d\mu_k \leq C \int_X |s|^2 \, dV$$

for each $k \in \mathbb{N}$ and all $s \in H^0(X, \mathcal{O}_X(L^{\otimes k}))$. Moreover, we will say that it is a *sampling sequence* for the section ring $R_X(L)$ if it is norming and it defines uniformly standard norms, that is, there is a constant $C > 0$ such that

$$\frac{1}{C}\int_X |s|^2 \, dV \leq \int_X |s|^2 \, d\mu_k \leq C \int_X |s|^2 \, dV$$

for each $k \in \mathbb{N}$ and all $s \in H^0(X, \mathcal{O}_X(L^{\otimes k}))$.

Now, we will give a characterization of the sequences of measures that define uniformly standard norms. However, the characterization for norming sequences of measures continues being an open problem, as does the characterization of the sampling sequences of measures.

Theorem 4.2. *For a sequence of measures $\{\mu_k\}$ the following are equivalent:*

(1) *The sequence $\{\mu_k\}$ defines uniformly standard norms for the section ring $R_X(L)$.*

(2) *There exists a constant $C_2 > 0$ such that for all $k \in \mathbb{N}$,*

$$T[\mu_k](z) = \int_X \left| \frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 d\mu_k(w) \leq C_2$$

for every $z \in X$, that is, the sequence of Berezin transforms $\{T[\mu_k]\}_k$ are uniformly bounded in X.

(3) *There is a constant $C_3 > 0$ such that*

$$\mu_k(B(z, 1/\sqrt{k})) \leq \frac{C_2}{k^n}, \qquad z \in X.$$

Proof. The proof that (1) implies (2) is trivial because $\Pi_k(z,w) \in H^0(L^k)$ for each $w \in X$ and $k \in \mathbb{N}$.

To prove that (2) implies (3) it is enough to establish the following lemma.

Lemma 4.3. *There are constants $\varepsilon > 0$, $M > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\left| \frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 \geq M k^n > 0 \tag{4.1}$$

for every $w \in B(z, \varepsilon/\sqrt{k})$ and $k > k_0$.

Proof. To prove (4.1) we use that

$$\left| \frac{\Pi_k(z,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 - \left| \frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 = \int_0^1 \frac{d}{dt} \left(\left| \frac{\Pi_k(w-(z-w)t, z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 \right) dt$$

and by the bound of the derivatives in [LOC12, pag. 434]

$$\left| D_w \left| \frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2 \right| \leq C\sqrt{k} \sup_{\gamma \in U_k(w)} \left| \frac{\Pi_k(\gamma,z)}{\sqrt{|\Pi_k(z,z)|}} \right|^2$$

where $U_k(w)$ is the polydisk with polyradius $1/\sqrt{k}$, together with the sub-mean value property in [LOC12, pag. 432], we obtain that

$$\left|\frac{\Pi_k(z,z)}{\sqrt{|\Pi_k(z,z)|}}\right|^2 - \left|\frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}}\right|^2 \leq M' k^{\frac{2n+1}{2}} \sqrt{n}|z-w|.$$

By the diagonal estimate of Lemma 2.1 we have

$$\left|\frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}}\right|^2 \geq Dk^n - M' k^{\frac{2n+1}{2}} \sqrt{n}|z-w|,$$

where $D > 0$ is a certain constant.

Then, if we consider the ball $B(z, \varepsilon/\sqrt{k})$ where $\varepsilon < \frac{D}{2M'\sqrt{n}}$ we have

$$\left|\frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}}\right|^2 \geq (D - M'\varepsilon\sqrt{n})k^n > \frac{Dk^n}{2} > 0$$

for $w \in B(z, \varepsilon/\sqrt{k})$. □

Applying Lemma 4.3 we have that there are constants $\varepsilon > 0$, $M > 0$ and $k_0 \in \mathbb{N}$ such that

$$Mk^n \mu_k(B(z, \varepsilon/\sqrt{k})) \leq \int_{B(z,\varepsilon/\sqrt{k})} \left|\frac{\Pi_k(w,z)}{\sqrt{|\Pi_k(z,z)|}}\right|^2 d\mu_k(w) \leq T[\mu_k](z) \leq C_2$$

for every $z \in X$ and $k > k_0$.

We take a covering of the ball $B(z, 1/\sqrt{k})$ by a collection of balls $B(z_j, \varepsilon/5\sqrt{k})$, $z_j \in B(z, 1/\sqrt{k})$ and applying the Vitali covering lemma we obtain a sub-collection of these balls which are disjoint such that

$$\cup B(z_j, \varepsilon/5\sqrt{k}) \subset \cup B(z_{j_q}, \varepsilon/\sqrt{k}).$$

Moreover, as

$$B(z_{j_q}, \varepsilon/\sqrt{k}) \subset B(z, 2/\sqrt{k})$$

by means of the volume we have that

$$N \cdot \left(\frac{\varepsilon}{\sqrt{k}}\right)^{2n} \lesssim \left(\frac{2}{\sqrt{k}}\right)^{2n},$$

that is, $N \lesssim \left(\frac{2}{\varepsilon}\right)^{2n}$ where N is the number of balls of the sub-collection. Therefore, we conclude

$$\mu_k(B(z, 1/\sqrt{k})) \lesssim \left(\frac{2}{\varepsilon}\right)^{2n} m_k(B(z_{j_q}), \varepsilon/\sqrt{k}) \lesssim \frac{1}{k^n}.$$

Notice that for $k \leq k_0$ (3) holds immediately since $\mu_k(B(z, 1/\sqrt{k}))$ are bounded by $\mu_k(X)$.

All we need to prove that (3) implies (1) is the existence of a constant $Q > 0$ such that for all $w \in X$

$$|s(w)|^2 \leq Qk^n \int_{B(w,1/\sqrt{k})} |s(z)|^2 \, dV(z), \quad w \in X. \tag{4.2}$$

This is proved by the sub-mean value property in [LOC12, pag. 432]. Indeed, if this is the case then

$$\int_X |s|^2 \, d\mu_k \leq Qk^n \int_X |s(z)|^2 \left(\int_X \chi_{B(w,1/\sqrt{k})}(z) \, d\mu_k(w) \right) \, dV(z)$$
$$\lesssim \int_X |s|^2 \, dV$$

for every $s \in H^0(X, \mathcal{O}(L^{\otimes k}))$.

□

A sequence of measures μ_k whose Radon-Nykodym derivative with respect the Lebesgue measure in X is the characteristic function of a set G_k that is relatively dense in X always define uniformly standard norms and it is norming by Theorem 1.1. So, this sequence of measures is sampling.

Example 4.4 (Hyperplane line bundle)**.** We will apply the Theorem 1.1 to the complex projective space $\mathbb{CP}^n$ with the hyperplane bundle $\mathcal{O}(1)$, endowed with the Fubini-Study metric. Notice that the holomorphic sections to $\mathcal{O}(k)$, the k'th power of $\mathcal{O}(1)$, can be identified with the homogeneous polynomials of degree k in homogeneous coordinates.

Using Theorem 1.1 we obtain that for a given sequence of measurable subsets $\{\Omega_k\}_k$ in $\mathbb{C}^n$, we have that the following statements are equivalent:

(1) For all $k \in \mathbb{N}$, it is verified that

$$\int_{\Omega_k} \frac{|p_k(z)|^2}{(1+|z|^2)^k} \frac{dV(z)}{(1+|z|^2)^{n+1}} \asymp \int_{\mathbb{C}^n} \frac{|p_k(z)|^2}{(1+|z|^2)^k} \frac{dV(z)}{(1+|z|^2)^{n+1}}$$

for every polynomial p_k of degree less or equal than k in n variables.

(2) There is a radius R such that

$$\int_{\Omega_k} \frac{\chi_{H(z,R/\sqrt{k})}(w)}{(1+|w|^2)^{n+1}} \, dV(w) \asymp \left(\frac{R}{\sqrt{k}} \right)^{2n}$$

for all $z \in \mathbb{C}^n$ and $k \in \mathbb{N}$, where

$$H(z, R/\sqrt{k}) = \left\{ w \in \mathbb{C}^n \ : \ |z - w| < \left| \tan \left(\frac{R}{\sqrt{k}} \right) \right| \cdot |1 + z\overline{w}| \right\}.$$

Here the measure V is the Lebesgue measure on $\mathbb{C}^n$.

Next, we will see an application of the Theorem 4.2 to an example that appears in the section 4 of [Ber03].

Example 4.5. Let L be a positive line bundle and we consider the space of sections $H^0(X, \mathcal{O}_X(L^{\otimes k}))$. For $k > 0$, we consider a discrete set D_k of points in X such that any two points have a distance bounded from below by a positive constant C times $k^{-1/2}$.

Now, we consider the sequence of measures $\{\nu_k\}$ such that

$$d\nu_k = k^{-n} \sum_{x \in D_k} \delta_x,$$

We can see that there is a positive constant C_0 independent of k such that

$$\int_{D\left(z,\frac{1}{\sqrt{k}}\right)} d\nu_k = k^{-n} \# \left\{ x \in D_k \ : \ x \in D(z, k^{-1/2}) \right\} \leq \left(\frac{C+2}{C} \right)^{2n} k^{-n} = C_0 k^{-n}$$

for all $z \in X$.

Therefore, using Theorem 4.2 it follows that the sequence of measures $\{\nu_k\}$ defines uniformly standard norms for the section ring $R_X(L)$, that is, there is a positive constant C_1 such that

$$k^{-n} \sum_{x \in D_k} |s(x)|^2 \leq C_1 \int_X |s|^2 \, dV$$

for each $k \in N$ and all $s \in H^0(X, \mathcal{O}_X(L^{\otimes k}))$.

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