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# Traffic flow continuum modeling by hypersingular boundary integral equations

Luis M. Romero, Francisco G. Benitez

Transportation Engineering, University of Seville, Spain

***Summary***

The quantity of data necessary in order to study traffic in dense urban areas through a traffic network, and the large volume of information that is provided as a result, causes managerial difficulties for the said model. A study of this kind is expensive and complex, with many sources of error connected to each step carried out. A simplification like the continuous medium is a reasonable approximation and, for certain dimensions of the actual problem, may be an alternative to be kept in mind. The hypotheses of the continuous model introduce errors comparable to those associated with geometric inaccuracies in the transport network, with the grouping of hundreds of streets in one same type of link and therefore having the same functional characteristics, with the centralization of all journey departure points and destinations in discrete centroids and with the uncertainty produced by a huge origin/destination matrix that is quickly phased out, etc. In the course of this work, a new model for characterizing traffic in dense network cities as a continuous medium, the diffusion–advection model, is put forward. The model is approached by means of the boundary element method, which has the fundamental characteristic of only requiring the contour of the problem to be discretized, thereby reducing the complexity and need for information into one order versus other more widespread methods, such as finite differences and the finite element method. On the other hand, the boundary elements method tends to give a more complex mathematical formulation. In order to validate the proposed technique, three examples in their fullest form are resolved with a known analytic solution.

**KEYWORDS.** Traffic flow, Continuum theory, Two-dimensional modeling.

## INTRODUCTION

The modeling of regional or zonal traffic currently requires the zone studied to be discretized by means of a discrete network, concentrating on the journeys generated and attracted by each of the transport zones into points or nodes, and the road sections to be modeled as links between the nodes. Each of the links has some characteristics, which specify its functional behavior, assigned to it – the time or generally the cost of travelling the said link, which will be dependent on the intensity of the circulating traffic flow. This network is given a departure-point/destination matrix that constitutes the network demand, the journeys running through the network that are distributed between the links in keeping with the principle of user equilibrium, thereby resolving the problem of assignment. The number of parameters involved in this model is as high as the process needs in order to calibrate a network, perhaps over a period of several years, and even then the results will never be one hundred percent reliable and hence the process will require continual revision. In a large number of cases, the obtained results yield a pattern of flows with an excessive level of detail, which, at times, makes it difficult to interpret the results qualitatively. Consequently, if the actual network of lines of communication is very complex, the methods involved in the network theory may be an inadequate tool. The errors themselves made in a network model (routes not taken into account, lost interconnections, undefined crossovers, inappropriate parameters ...) will introduce a large level of inaccuracy into the results obtained in resolving problems in accordance with the given model. However, increasing the level of accuracy of the model of the network may render the said model unpractical from the point of view of calculation. In such cases, a different form of analysis, such as one based on the continuum theory of traffic flow, will be more convenient, at least in the initial stage of planning where general patterns of the distribution of journeys and changes to them in response to different transport policies are looked at.

Over the last 25 years, various researchers have put forward different methods of working with traffic distribution in dense urban areas. Most of these methods could only be applied to simple geometries and were far from being practical when used, although they provided a basis for future research (see [1] and [2]). It is also worth mentioning the pioneering work by Taguchi and Iri [3] of using the finite element method for solving a real transportation problem, and the published work of Yang, Yagar and Iida [4] on the problem of assigning traffic in a congested city comprising many departure points and a single journey destination. The model used shows a typical discrete network for modeling the system of main routes and a continuum model for the network of secondary streets. In the aforementioned work, the continuous model is resolved by applying the finite elements method. Wong [5] took this line, studying a more complex case with multi-commodity traffic in the network. In this paper, a new continuous model is put forward and the finite elements method is replaced by the boundary elements method.

In traffic theory, a dense network is defined as a route network that is sufficiently compressed for it to be presumed that the space between the route sections and their longitude is small by comparison with a dimension characterizing the region under consideration. In this case, the course of the roads can be considered as a continuum; the vehicles can circulate at any point $\mathbf{x} = (x, y)$ of the area to be studied that occupies an area $E^2$, where $E^2$ is a Euclidian, two-dimensional space. In reality, there will be a discrete number $I$ of feasible directions at each point $\mathbf{x}$ with direction cosines $(u_x^i, u_y^i)$. Thus, if the discrete set contains practically any direction, the said zone can be considered as a zone with an isotropic layout, while if the roads are preferably orientated in two directions, where $I = 2$, it can be considered as having an orthotropic layout. Generally speaking, any spatial area can be divided up, in accordance with the orientation and density of its streets or roads, into different zones with different isotropic, orthotropic, or anisotropic zones. Here, only isotropic zones will be considered.

It will be normal, during the course of this paper, to distinguish between different types of vehicle. Sometimes, this distinction can be made on the basis of the actual type of the vehicle (light or heavy vehicles, motorbikes ...), although, in most cases, this distinction will be made in accordance with the point of departure and the destination. Here, only those vehicles having the same point of departure and traveling to the same destination are considered to be vehicles of the same kind, herein referred to as a specific commodity. The methodology put forward in this paper is based on the hypothesis that there is just one commodity circulating for the continuous medium. Romero [6] develops the method generalization put forward here into multiple commodity traffic, thereby enabling the modeling of situations that are more realistic than the case of a single item.

# BASIC TRAFFIC FLOW CONCEPTS. DIFFUSION–ADVECTION TRAFFIC MODEL

The models dealt with in this section are framed in macroscopic flow dynamics theory, considering traffic flow to be similar to that of a fluid, without taking the behavior of separate vehicles into account; they are also deterministic in that they do not consider random variables in order to describe a phenomenon as complex as traffic, very dependent, as it is, on human behavior.

The two-dimensional model comes from a traffic conservation equation, the simple continuous model, introducing a change, which has been criticized in the one-dimensional case because it may require a violation of the anisotropic nature of the traffic in specific traffic conditions, but which will be duly justified for the two-dimensional case. Another basic difference of the model put forward, with regard to the simple, continuous model, is that, in the proposed model, it is not necessary to know any traffic intensity–density relationship in advance in order to close it, as may be required in the simple continuous model.

## One-Dimensional Models

Lighthill and Whitham [7] formulated a continuous model for traffic flow starting only with a conservation equation, which, in a one-dimensional space, will have the form:

$$\frac{\partial \phi(x,t)}{\partial t} + \frac{\partial F(x,t)}{\partial x} = \rho(x,t) \tag{1}$$

where $\phi(x,t)$ is the traffic density or concentration, $F(x,t)$ is the net vehicle flow and $\rho(x,t)$ the vehicle generation function.

This model, known as the simple model, is clearly very limited since it does not allow variations in speed in relation to equilibrium values, and allows the appearance of shock waves that mean instantaneous changes in speed.

The rising of shock waves threatens the continuity and unicity of solutions. Characteristics' intersections imply multi-evaluated points, corresponding to shock waves. There are different approaches to dealing mathematically with problems of shock waves. One of them makes use of the weak solution concept, which admits solutions with discontinuity in the shock wave variable; in order to guarantee the unicity of the solution, the Ansorge entropy condition [8] has to be imposed. A second method to ensure unicity, proposed in [9], is to consider an artificial diffusion term, $q = Q(k) - \mu\, \partial k/\partial x$, knowing a priori the existence of a unique solution, and later setting the limit $\mu \to 0$ to eliminate the effect of this term.

Since then, and up until the present time, a large number of models have been put forward, although the one most frequently used has been Payne's model [10], which introduces a new equation in which speed is one more unknown magnitude:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{V_e - V}{T} - \frac{\nu}{T}\frac{1}{\phi}\frac{\partial \phi}{\partial x} \qquad (2)$$

where $V$ and $V_e$ stand for speed and equilibrium speed, respectively, T is the reaction time and ν is an anticipation coefficient.

Whitham [9] subsequently introduced a diffusion term into the traffic-flow expression, such that:

$$f = F(\phi) - \mu\frac{\partial \phi}{\partial x}, \qquad v = V(\phi) - \mu\frac{1}{\phi}\frac{\partial \phi}{\partial x} \qquad (3)$$

accompanied by the reaction time, giving rise to a model that is very similar to that of Payne, and hence many references are to what is termed the PW (Payne–Whitham) model.

These models, known in the literature as second-order methods, were postulated in order to try to improve the simple continuous model, but even like this they do not resolve its problems and they introduce new errors, postulated in [11] and [12], such as allowing for the possibility of negative speeds in the face of high-density gradients and for wave velocities that are greater than vehicle speeds, implying that impact waves would speed vehicles up. Such vehicle behavior is a breach of the anisotropic property of traffic circulation, established by [12], which, in essence, states that drivers basically only react to frontal stimuli. In general, a fast vehicle cannot force a slow vehicle to speed up, particularly in dense, one-lane traffic where the slower driver does not have enough front spacing to speed up. The fast vehicle has to slow down when it catches up with a slow vehicle. However, it is shown in [13] how non-anisotropic waves can arise in multi-lane traffic even when the flow in each lane behaves anisotropically.

The publication of a couple of works, [11] and [12], jeopardized the use of second-order approaches. Finally, these concerns only supposed a short interruption in their use to take a step back and define a set of properties to be fulfilled for any acceptable model. New refined second-order models appeared, verifying the consistency properties ([13] - [17]).

**Two-Dimensional Models**

Generalization of the expression (1) to two dimensions is immediate:

$$\frac{\partial \phi(x,t)}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{x},t) = \boldsymbol{\rho}(\mathbf{x},t) \qquad (4)$$

where $\mathbf{f}(\mathbf{x},t)$ is the traffic flow vector, $\phi(\mathbf{x},t)$ the traffic density and $\boldsymbol{\rho}(\mathbf{x},t)$ the traffic generation function (notation is particularized for the two-dimensional case).

Allowing that the traffic circulation at point $x$ at instant $t$ not only depends on traffic density and speed at this point and at this instant, but also depends on the density gradients as observed physically, it can be written that:

$$\mathbf{f}(\mathbf{x},t) = -\mu \nabla \phi(x,t) + u(x,t)\phi(x,t) \qquad (5)$$

where, in this case, speed is a vector of components $u_x$ and $u_y$ (velocity).

Incorporating a diffusive component into expression (5) is equivalent to the two-dimensional version of the variation of the simple continuous model put forward by Whitham [9], expression (3), rejected by Daganzo [12] because it allowed for negative speeds in the face of high-density gradients. In a one-dimensional case, where the vehicles always circulate in one lane, it is inadmissible that a driver should react to a traffic jam ahead, given that that driver has no other option than the given lane, for making his journey. However, in the two-dimensional case, the driver can react in the face of a high concentration of vehicles, changing his route towards zones in which there are fewer vehicles and, hence, in this case, negative speed components are no breach of the actual behavior of the physical problem.

This dependence on the gradient looks like a violation of the anisotropic property of the traffic flow, an undisputed characteristic of one-lane traffic but questionable in other types of flows. It has been proved, i.e. [13], that the said property is not valid for multi-lane traffic flow, even being valid for each lane independently. Drivers know roughly the zone traffic conditions, either through any intelligent transport system (ITS) that informs of the real-time situation or by reckoning the usual traffic patterns. Due to these factors, it is feasible that the driver not only takes into account the space available in front of him in the lane he is circulating along, but that he also considers an additional, more global, variable related to the zone he is located in as the difference from vehicle densities between zones closer to his location.

Expression (3) supposes that the decrements/increments of circulation are given not only as a result of decrements/increments in velocity and/or density, but also in response to increments/decrements in the density gradient. The level of variation of traffic flow in relation to the density gradient is defined in the model by the scalar parameter μ, which is dependent on all the usual urban traffic factors (the town planning policy, the design of routes, atmospheric conditions, drivers' behavior, etc.). The magnitude of parameter $\mu$ plays an important role in the modeling. Its weight in traffic behavior has not been clarified as no conclusive experimental studies have been carried out.

Given that the spatial variation of velocity in relation to an equilibrium velocity is small, we can write:

$$u(x,t) = \bar{u}(t) + \tilde{u}(x,t)\,. \tag{6}$$

Substituting expressions (5) and (6) in (4), we obtain:

$$\mu\nabla^2\phi(x,t) - \bar{u}(t)\cdot\nabla\phi(x,t) = \frac{\partial\phi(x,t)}{\partial t} - \rho(x,t) + \nabla\left(\tilde{u}(x,t)\phi(x,t)\right) \tag{7}$$

an equation that defines the diffusion–advection model already known to describe other fluid mechanics problems.

## THE BOUNDARY ELEMENTS METHOD APPLIED TO THE DIFFUSION–ADVECTION TRAFFIC MODEL

The strategy followed to solve a problem ruled by an equation simpler than (7), the convection–diffusion equation, with neither the traffic generation nor the divergence terms, has already been published [18]. In the following, the methodology needed to apply the BEM to equation (7) is presented.

### Diffusion–Advection Integral Equation

Multiplying equation (7) by $\psi(\mathbf{x},\boldsymbol{\xi},t)$ and integrating within the domain $\Re(\eta)\equiv\Re-\mathrm{B}(\eta)$, where $\mathrm{B}(\eta)$ is a ball centered on $\boldsymbol{\xi}$ and having a radius $\eta$, with $\eta\to 0$, we obtain

$$\int_{\Re(\eta)}\left[\mu\nabla^2\phi(x,t)-\bar{u}(t)\cdot\nabla\phi(x,t)\right]\psi(x,\xi,t)d\Re_x = \int_{\Re(\eta)}\left[\frac{\partial\phi(x,t)}{\partial t}-\rho(x,t)+\nabla\left(\tilde{u}(x,t)\phi(x,t)\right)\right]\psi(x,\xi,t)d\Re_x \qquad (8)$$

where $\psi(\mathbf{x},\boldsymbol{\xi},t)$ corresponds to the fundamental solution defined by the equation:

$$\mu\nabla^2\psi(x,\xi,t)+\bar{u}(t)\cdot\nabla\psi(x,\xi,t)=-\delta(\xi). \qquad (9)$$

The main advantage of the boundary elements method by comparison with other methods is its capacity to provide a complete solution to the problem solely using information on the boundary. For this, the domain integrals of equation (8) must be transformed into integrals concerned with the boundary. To do this with the integrals of the first member, it suffices to integrate by parts, and apply the Gauss theorem and expression (9), obtaining:

$$\int_{\Re(\eta)}\left[\mu\nabla^2\phi(x,t)-\bar{u}(t)\cdot\nabla\phi(x,t)\right]\psi(x,\xi,t)d\Re_x = c(\xi)\phi(\xi,t)+ + \int_{\partial\Re_\eta}\phi(x,t)\left[\mu\nabla^2\frac{\partial\psi(x,\xi,t)}{\partial n}+\psi(x,\xi,t)\right]d\Gamma_x-\mu\int_{\partial\Re_\eta}\psi(x,\xi,t)\frac{\partial\phi(x,t)}{\partial n}d\Gamma_x \qquad (10)$$

An initial restriction of the boundary elements method is that the fundamental solution to the original, differential equation is required in order thereby to obtain the boundary integral equation. Another is the inclusion of domain integrals in the problem formulation. The dual reciprocity method deals with this problem by using a simpler fundamental solution, expanding the remaining terms of the original equation as a series using some global approximate functions. The following approximation is used for each of the terms of the second member of equation (8) that are represented generically as $\chi(x,t)$:

$$\chi(x,t) \approx \sum_{k=1}^{M} \alpha_k(t) \cdot f_k(x) \tag{11}$$

where the approximate functions, $f_k(x)$, the same for each one of the terms of the second member of equation (8), verify that:

$$\mu \nabla^2 \varsigma_k(x,\xi,t) + \overline{u}(t) \cdot \nabla \varsigma_k(x,\xi,t) = -f_k(\xi) \tag{12}$$

while the coefficients $\alpha_k$ are given by the solution to the equations system:

$$\chi_i(x,t) = \sum_{k=1}^{M} \alpha_k(t) \cdot f_{ik}(x), \quad \text{with } i = 1,\dots,M$$

which means demanding that the approximation (11) is exactly verified at the *M* points used for it. There is abundant literature on choosing the aforementioned approximate functions ([18] and [19]).

With all of this, equation (8) is transformed into the following equation, now with integrals only within the boundary:

$$\begin{aligned} & c(\xi)\phi(\xi,t) + \int_{\partial\Re_\eta} \phi(x,t)\left[\mu \frac{\partial \psi(x,\xi,t)}{\partial n} + \psi(x,\xi,t) \cdot u_n\right] d\Gamma_x - \\ & -\mu \int_{\partial\Re_\eta} \psi(x,\xi,t) \frac{\partial \phi(x,t)}{\partial n} d\Gamma_x = \sum_{k=1}^{M} (\alpha_k(t) + \beta_k(t) + \gamma_k(t)) \cdot \left[c(\xi)\varsigma_k(\xi,t) + \right. \\ & \left. + \int_{\partial\Re_\eta} \phi(x,t)\left[\mu \frac{\partial \varsigma_k(x,\xi,t)}{\partial n} + \varsigma_k(x,\xi,t) \cdot u_n\right] d\Gamma_x - \mu \int_{\partial\Re_\eta} \varsigma_k(x,\xi,t) \frac{\partial \phi(x,t)}{\partial n} d\Gamma_x \right] \end{aligned} \tag{13}$$

The coefficient of the free term in the above expression, c(ξ), is only a function of the internal angle $\theta$, generally specified by $c(\xi) = \theta/2\pi$, such that expression (13) is valid when the collocation point ξ is defined both in the boundary $\partial\Re$ and within the domain $\Re$. The expressions relative to the fundamental solution can be found in [18].

The boundary integrals are solved by the discretization of the boundary in which we define some natural coordinates. Within the elements, the geometric values like those of any other type of function are approximated by means of the nodal values of the said functions. With this discretization, the expression (13), in vectorial form, becomes:

$$c(\theta)\phi(\boldsymbol{\xi})+\{h(\boldsymbol{\xi})\}^T\cdot\{\phi\}-\{g(\boldsymbol{\xi})\}^T\cdot\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}=$$
$$=\left(c(\boldsymbol{\xi})\zeta_k(\boldsymbol{\xi})+\{h(\boldsymbol{\xi})\}^T[\zeta]-\{g(\boldsymbol{\xi})\}^T\left[\frac{\partial\zeta}{\partial\mathbf{n}}\right]\right)\cdot \tag{14}$$
$$\cdot\left[F^{-1}\right]\left(\left\{\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\}-\{\boldsymbol{\rho}(\mathbf{x},t)\}+\left\{\nabla\cdot\left[\tilde{\mathbf{u}}\phi\right]\right\}\right)+\sum_{p=1}^{NP}\boldsymbol{\rho}_p(t)\psi(\mathbf{x}_p,\boldsymbol{\xi})$$

where $\boldsymbol{\rho}_p$ represents $NP$ discrete traffic generation sources and means that, in order to simplify formulation, the same $M$ points and the same approximate functions have been used for the dual reciprocity approximation of the terms $\{\partial\phi(\mathbf{x},t)/\partial t\}$, $\{\boldsymbol{\rho}(\mathbf{x},t)\}$ and $\{\nabla\cdot[\tilde{\mathbf{u}}\phi]\}$.

Defining the vector $\{\tau\}$ as

$$\{\tau\}^T=\left(c(\theta)\{\zeta(\boldsymbol{\xi})\}^T+\{h(\boldsymbol{\xi})\}^T\cdot[\zeta]-\{g(\boldsymbol{\xi})\}^T\cdot\left[\frac{\partial\zeta}{\partial\mathbf{n}}\right]\right)[F]^{-1} \tag{15}$$

equation (14) remains

$$c(\theta)\phi(\boldsymbol{\xi})+\{h(\boldsymbol{\xi})\}^T\cdot\{\phi\}-\{g(\boldsymbol{\xi})\}^T\cdot\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}=$$
$$=\{\tau\}^T\left(\left\{\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\}-\{\boldsymbol{\rho}(\mathbf{x},t)\}+\left\{\nabla\cdot\left[\tilde{\mathbf{u}}\phi\right]\right\}\right)+\sum_{p=1}^{NP}\boldsymbol{\rho}_p(t)\psi(\mathbf{x}_p,\boldsymbol{\xi}) \tag{16}$$

**Hypersingular Equation of the Diffusion–Advection Gradient**

Multiplying equation (7) by $\nabla_\xi\psi(x,\xi,t)$ and integrating the domain $\Re(\eta)$, carrying out transformations similar to those performed for equation (10), we obtain the following expression in which the symbol $⨏$ represents a hypersingular integral:

$$\mathbf{a}(\xi)\phi(\xi,t)+\mathbf{E}(\xi)\nabla\phi(\xi,t)-\int_{\partial\Re_\eta}\mu\cdot\nabla_\xi\phi^*(\mathbf{x},\xi)\cdot\frac{\partial\phi(\mathbf{x},t)}{\partial\mathbf{n}}d\Gamma+$$

$$+\quad ⨍\phi(\mathbf{x},t)\left[\mu\cdot\nabla_\xi\left(\frac{\partial\phi^*(\mathbf{x},\xi)}{\partial\mathbf{n}}\right)+\nabla_\xi\phi^*(\mathbf{x},\xi)\cdot\bar{\mathbf{v}}_\mathbf{n}\right]d\Gamma=$$

$$\sum_{k=1}^{M}(\alpha_k(t)+\beta_k(t)+\gamma_k(t))\cdot\left[\mathbf{a}(\xi)\phi(\xi,t)+\mathbf{E}(\xi)\nabla\phi(\xi,t)-\int_{\partial\Re_\eta}\mu\cdot\nabla_\xi\phi^*(\mathbf{x},\xi)\cdot\frac{\partial\phi(\mathbf{x},t)}{\partial\mathbf{n}}d\Gamma+\right.$$

$$\left.+\quad ⨍\phi(\mathbf{x},t)\left[\mu\cdot\nabla_\xi\left(\frac{\partial\phi^*(\mathbf{x},\xi)}{\partial\mathbf{n}}\right)+\nabla_\xi\phi^*(\mathbf{x},\xi)\cdot\bar{\mathbf{v}}_\mathbf{n}\right]d\Gamma\right] \tag{17}$$

where $\partial\Re_\eta$ stands for the region domain excluding the collocation point $\xi$. In (17), the coefficients of the free terms, the vector a(ξ) and the matrix E(ξ), are given by the expressions

$$\mathbf{a}(\xi)=-\frac{M\cdot\mathbf{v}}{4\pi\mu}$$

$$\mathbf{E}(\xi)=\frac{1}{2\pi}\left([I]\theta+M\right) \tag{18}$$

with

$$M=-\frac{1}{2}\begin{pmatrix}-\sin(2\gamma) & \cos(2\gamma)\\ \cos(2\gamma) & \sin(2\gamma)\end{pmatrix}\Bigg|_{\gamma_1}^{\gamma_2}$$

Discretizing the boundary into elements and expressing the integrals as the sum of the integrals of the elements, on the basis of the nodal values, the vector form is obtained:

$$\mathbf{a}(\theta)\phi(\xi)+\left[D(\theta)\right]\nabla\phi(\xi)+\left[h_\nabla(\xi)\right]\cdot\{\phi\}-\left[g_\nabla(\xi)\right]\cdot\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}=$$

$$\left[\tau_\nabla(\xi)\right]\left(\left\{\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\}-\{\boldsymbol{\rho}(\mathbf{x},t)\}+\{\nabla\cdot[\mathbf{u}\phi]\}\right)+\sum_{p=1}^{NP}\boldsymbol{\rho}_p(t)\nabla\psi(\mathbf{x}_p,\xi) \tag{19}$$

which is an equation of a hypersingular nature, the solution to which requires special treatment, presented in [18], where, similarly, the expressions relative to the gradient of the fundamental solution can be found. A new unknown $\nabla\phi(\xi,t)$ appears in each equation of this kind.

## SOLVING THE DIFFUSION–ADVECTION TRAFFIC MODEL

### Equations

The equations system obtained agrees with the *integral density equation*, expression (16), and the *integral equation of the density gradient*, expression (17).

By the boundary conditions, either the traffic density $\phi$ or its flow $\partial\phi/\partial\mathbf{n}$ is known for each point of the boundary or district, and therefore each point brings with it an unknown concerning the primary variables of the problem. If the point is an interior one, no boundary condition is imposed on it and its unknown will always be the traffic density $\phi$. Furthermore, each point, whether internal or boundary, gives the traffic density gradient as an additional unknown, hence the need to state simultaneously the equation systems resulting from applying equations (16) and (17) to each point used. However, there are terms in the equation systems requiring special treatment in order for the problem to be closed, as discussed in the following paragraphs.

### Temporal Derivative

An appropriate integration scheme for relating variables between two consecutive time steps, *m* and *m+1*, is used for dealing with the temporal derivative, as follows:

$$\begin{aligned} \{\phi\} &= \left(1-\theta_\phi\right)\{\phi\}^m + \theta_\phi\{\phi\}^{m+1} \\ \left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\} &= \left(1-\theta_q\right)\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}^m + \theta_q\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}^{m+1} \\ \left\{\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right\} &= \frac{\{\phi\}^{m+1}-\{\phi\}^m}{\Delta t} \end{aligned} \tag{20}$$

In order to simplify the way the equations are shown, it will be assumed that $\theta_\phi = \theta_q = \theta$, and the same weight $\theta$ is used for all the variables.

Introducing the outline expressed by (18) into equation (16), and dividing by the weight $\theta$, with $\theta \neq 0$, we obtain:

$$
\begin{aligned}
& c(\theta)\phi^{m+1}(\xi)+\{h(\xi)\}^T\cdot\{\phi\}^{m+1}-\{g(\xi)\}^T\cdot\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}^{m+1}= \\
& =\{\tau\}^T\left(\frac{1}{\theta\cdot\Delta t}\{\phi\}^{m+1}-\{\boldsymbol{\rho}(\mathbf{x})\}^{m+1}+\{\nabla\cdot[\tilde{\mathbf{u}}\phi]\}^{m+1}\right)+d^m+\sum_{p=1}^{NP}\boldsymbol{\rho}_p^{m+1}\psi(\mathbf{x}_p,\xi)
\end{aligned}
\tag{21}
$$

where the vector $d^m$ has been defined as an independent term that includes all the values associated with previous time steps, $t^m$.

Introducing the outline expressed in (20) and dividing by the weight $\theta$ in equation (19), we obtain:

$$
\begin{aligned}
& \mathbf{a}(\theta)\phi^{m+1}(\xi)+[D(\theta)]\boldsymbol{\nabla}\phi^{m+1}(\xi)+[h_\nabla(\xi)]\cdot\{\phi\}^{m+1}-[g_\nabla(\xi)]\cdot\left\{\frac{\partial\phi}{\partial\mathbf{n}}\right\}^{m+1}= \\
& =[\tau_\nabla(\xi)]\left(\frac{1}{\theta\cdot\Delta t}\{\phi\}^{m+1}-\{\boldsymbol{\rho}(\mathbf{x},t)\}^{m+1}+\{\nabla\cdot[\mathbf{u}\phi]\}^{m+1}\right)+\mathbf{d}_\nabla{}^m+\sum_{p=1}^{NP}\boldsymbol{\rho}_p^{m+1}(t)\boldsymbol{\nabla}\psi(\mathbf{x}_p,\xi)
\end{aligned}
\tag{22}
$$

where the vector $\mathbf{d}_\nabla{}^m$ has again been defined as including the values corresponding to previous time steps, $t^m$.

**Dealing with Divergence**

This section will deal with the term $\{\boldsymbol{\nabla}\cdot[\tilde{\mathbf{u}}\phi]\}$ of equations (21) and (22). Applying the product rule, the term can be broken down in accordance with the following expression:

$$
\{\boldsymbol{\nabla}\cdot[\tilde{\mathbf{u}}\phi]\}=\{\phi\boldsymbol{\nabla}\cdot[\tilde{\mathbf{u}}]\}+\{\tilde{\mathbf{u}}\cdot\boldsymbol{\nabla}[\phi]\} \tag{23}
$$

In expression (23), the second member has two terms in which the density $\phi$ and the density gradient $\boldsymbol{\nabla}\phi$ appear, given that they already appear in the problem equations and therefore do not give new unknowns. It is necessary to deal with $\tilde{\mathbf{u}}$ and $\boldsymbol{\nabla}\cdot[\tilde{\mathbf{u}}]$ sufficiently to process the term $\{\boldsymbol{\nabla}\cdot[\tilde{\mathbf{u}}\phi]\}$ without including new unknowns in order to close the problem.

The actual speed is a quantity known at certain points of the network through the corresponding measurements used for this purpose. The available information on velocities will be used in order to carry out an interpolation by means of dual reciprocity for the velocity at the remaining points. This approximation makes it possible to extract the velocity of the set of problem variables, without the need to determine a hypothesis of the behavior of the problem by way of a fundamental $\mathbf{u} = \mathbf{u}(\phi)$ diagram. The approximation expression has the form:

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \overline{\mathbf{u}}(t) \simeq \sum_{k=1}^{M} \boldsymbol{\alpha}_k(t) s_k(\mathbf{x}) \tag{24}$$

As is usual in the dual reciprocity method, the coefficients $\boldsymbol{\alpha}_k(t)$ are obtained by requiring that the approximation be verified exactly at the M points used:

$$\left\{\boldsymbol{\alpha}^T\right\} = [S]^{-1} \cdot \left\{\tilde{\mathbf{u}}^T\right\} \tag{25}$$

Calculating the expression (24), we obtain:

$$\nabla \cdot \tilde{\mathbf{u}}(\mathbf{x},t) = \sum_{k=1}^{M} \boldsymbol{\alpha}_k \cdot \nabla s_k(\mathbf{x}) \tag{26}$$

That, in matrix form, and using (25), is

$$\left\{\nabla \cdot \tilde{\mathbf{u}}\right\} = [S]^{-1}[\nabla S] \cdot \left\{\tilde{\mathbf{u}}^T\right\} \tag{27}$$

Once the approximate functions have been chosen by means of the expressions (24) and (26), we obtain the expressions of $\tilde{\mathbf{u}}(\mathbf{x},t)$ and $\nabla \cdot \tilde{\mathbf{u}}(\mathbf{x},t)$ on the basis of values known at some specified points. In short, it is necessary to know the said magnitudes at the *M* points used in the dual reciprocity approximation of the overall problem. $\nabla \cdot \tilde{\mathbf{u}}$ is obtained at the *M* dual reciprocity points by means of (25).

Substituting (27) in expression (23), we have:

$$\left\{\nabla \cdot \left[\tilde{\mathbf{u}}\phi\right]\right\} = [S]^{-1} \cdot [\nabla S]\left\{\tilde{\mathbf{u}}^T \phi\right\} + \left\{\tilde{\mathbf{u}} \cdot \nabla\left[\phi\right]\right\} \tag{28}$$

**Method**

Resolution of the diffusion–advection model in a traffic problem consists solely of simultaneously stating the equations systems (19) and (20), substituting the term $\left\{\nabla \cdot \left[\tilde{\mathbf{u}}\phi\right]\right\}$ of both equations by means of expression (28). The resulting equations system is a linearly independent algebraic equations system with the same number of equations as unknowns and is easily solved using any standard method. The accuracy and convergence of the method are related to the choice of the dual reciprocity approximating functions (see [20]).

The difficulty in applying the model may appear when compiling the data needed to apply the presented methodology. A brief summary of the data concerned is given below.

***Data Needed in Order to Apply the Model***

Next, the data needed to solve the problem are summarized.

– *Geometric discretization* of the urban zone to be examined. The coordinates of the different nodes to be used to discretize the boundary can be obtained by means of adequate cartography and a Geographic Information System (*GIS*). The nodes concerned will form the elements that discretize the real boundary.

– *Boundary conditions* at the points used in discretizing the boundary and its corresponding evolution with time. Either the traffic density $\phi$ or its flow $\partial\phi/\partial\mathbf{n}$ needs to be known for each point. This information is needed for all of the time period in question.

– *Traffic generation* data $\rho$, defined at the generation nodes to model either the entries to and exits from a ring-road motorway network or a higher road hierarchy network that is not modeled by means of the continuous medium hypothesis, and the continuous generation of traffic to model, for example, the departure of vehicles from a residential area in the morning rush hour or the attraction of employment centers.

– *Measured actual velocity* data, at a series of points in the region (domain). By means of this information, the velocity and velocity gradient will be determined, by means of the dual reciprocity approximation, at any point in the area studied.

– *Definition of the approximate functions*, both for the basic unknowns of the problem and for velocity.

– *Specification of an average velocity*, present in the fundamental solution.

– *Specification of parameters* that characterize the time integration:

- The time interval of the study ($t_0$, $t_f$).
- Integration weights, generally $\theta_\phi$ and $\theta_q$.

- The integration step or time increment $\Delta t$ .

## Empirical Application

***The One-Dimensional Stationary Case***

A first example of an analytic solution put forward for the case of diffusion convection is outlined. Here, it is a one-dimensional problem in which the velocity is functionally dependent on the *x* coordinate that is involved.

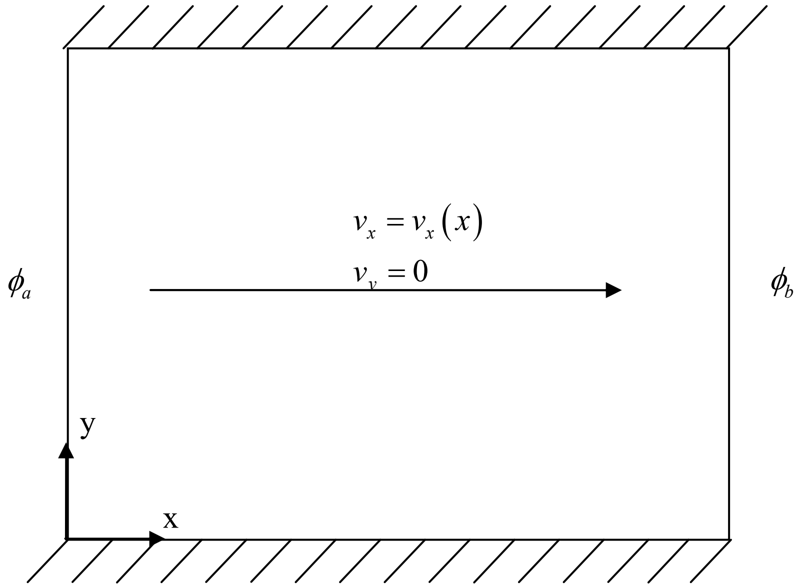



**FIGURE 1 Boundary geometry and conditions**

Its geometry is that of a square of unit-length sides, with zero flow on the horizontal sides of the square, and $q = 0$ when $y = 0$ and when $y = 1$ and $\phi$ is constant on the vertical sides; $\phi_a$ is at $x = 0$ and $\phi_b$ is at $x = 1$; with $\phi_a > \phi_b$, as shown diagrammatically in Figure 1.

The resulting analytic density expression is as follows:

$$\phi = \phi_a \cdot \left(\frac{\phi_b}{\phi_a}\right)^{x^2} e^{-k_1 \cdot x \cdot (1-x)} \tag{29}$$

while velocity meets the expression:

$$v_x = 2 \cdot x \cdot \left(\ln\left(\frac{\phi_b}{\phi_a}\right) - k_1\right) - k_1 + \frac{k_2}{\phi_a} \cdot \left(\frac{\phi_b}{\phi_a}\right)^{-x^2} e^{k_1 \cdot x \cdot (1-x)} \tag{30}$$

$$v_y = 0$$

The specific values of the constants taken for this example were $k_1 = 2$ and $k_2 = 100$, with $\phi_a = 30$ and $\phi_b = 10$, values with which the magnitudes obtained behave compatibly with an example of traffic. The said behavior, in the range within which the problem is defined, can be summarized by the graph shown in Figure 2. This problem has a moderate value of the Peclet number, $Pe \approx 10$.

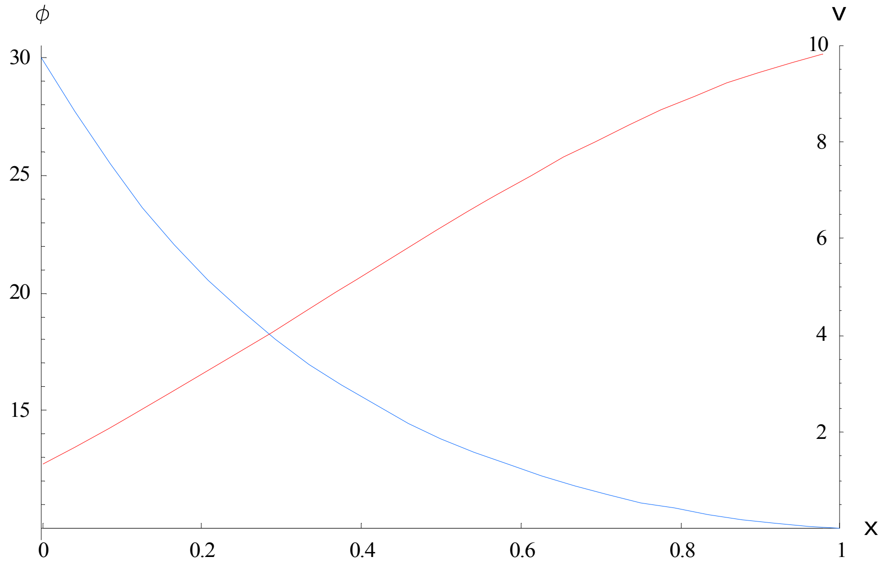
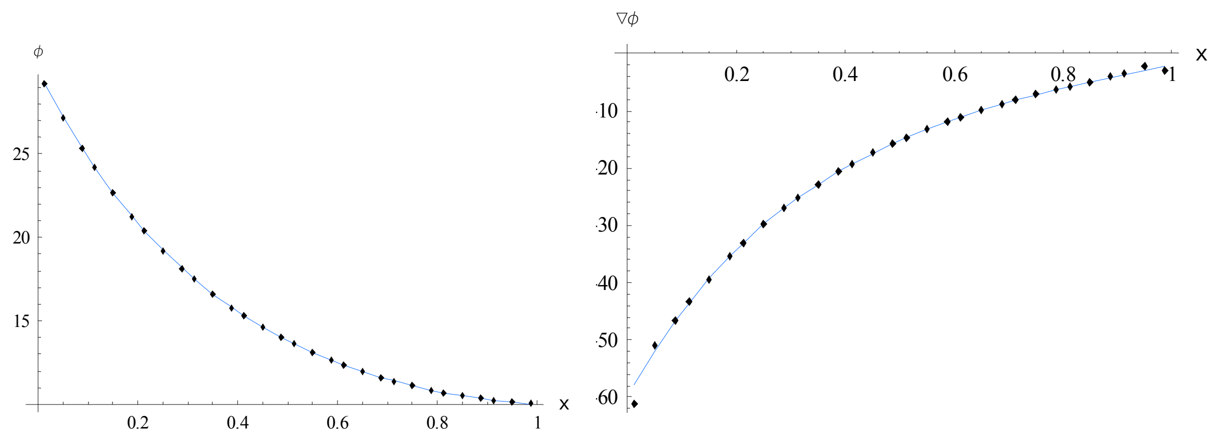



**FIGURE 2 Development of density and velocity along the x-axis**

Calculating equation (29), we obtain the analytic expression of the gradient:

$$\frac{\partial \phi}{\partial x} = \left( 2 \cdot x \cdot (k_1 + \mathrm{Log}\left( \frac{\phi_b}{\phi_a} \right)) - k_1 \right) \cdot \phi_a \cdot \left( \frac{\phi_b}{\phi_a} \right)^{x^2} e^{-k_1 \cdot x \cdot (1-x)} \tag{31}$$

In order to solve the problem using the boundary elements method, 10 discontinuous parabolic elements have been used on each side of the square, using the nodes on the lower side ($y = 0,\ 0 < x < 1$) as dual reciprocity points.




(a) (b)

**FIGURE 3 Density and gradient results along the x-axis**

Figure 3 shows the result obtained for density, by applying the stated technique, which, as can be seen, coincides with the analytic solution. Figure 3b also shows the density gradient results to be sufficiently in agreement with the analytic solution. The density gradient results are shown in order to highlight that it is necessary to obtain the gradient as a whole in order to calculate the traffic density.

*The Two-Dimensional Stationary Case*

The example that follows is a two-dimensional case. The velocity depends on the *x* and *y* coordinates, and its analytical expression is:

$$v(x,y)=\sqrt{\frac{1+4y^2}{4\left(-4+x^2\right)}-\frac{\left(-4+x^2\right)\mathrm{ArcSinh}\,\frac{2y}{4-x^2}}{4y}}\left\{\frac{1}{2\sqrt{1+\frac{4x^2y^2}{\left(-4+x^2\right)}}},\frac{\frac{xy}{4-x^2}}{\sqrt{1+\frac{4x^2y^2}{\left(-4+x^2\right)}}}\right\} \tag{32}$$

The domain is an ellipse centered on the origin, of size 1 and 0.5 for the semimajor and semiminor axes, respectively. Figure 4 shows the velocity field in the region of study.

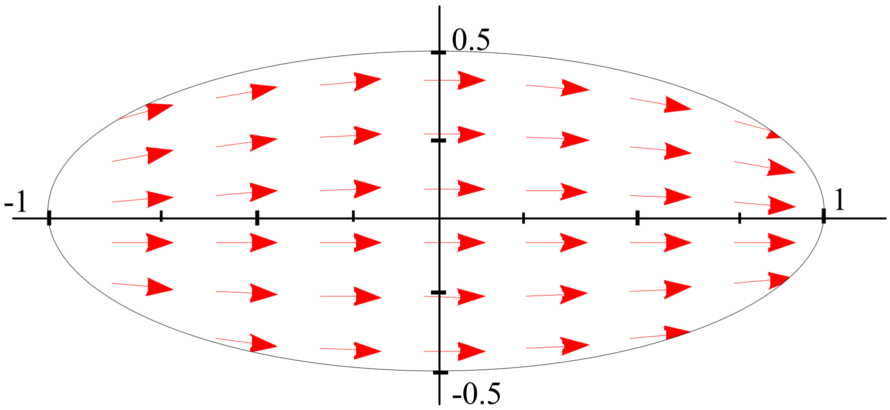



**FIGURE 4 Velocity field of a two-dimensional stationary case**

The density scalar field is also functionally dependent on the *x* and *y* coordinates:

$$\phi=\phi_a\cdot\left(\left(\frac{\phi_b}{\phi_a}\right)^{x^2}e^{-k_1\cdot(x\cdot(1-x)+(y-1)y)}\right) \tag{33}$$

It simulates a continuous generation of traffic responding to the expression:

$$\rho(x,t) = -\mu\nabla^2\phi(x,y) + \nabla\left(\mathbf{v}(x,y)\phi(x,y)\right) \tag{34}$$

Figure 5 depicts density and traffic generation in the domain of study. The case has a Peclet number of the order 2.

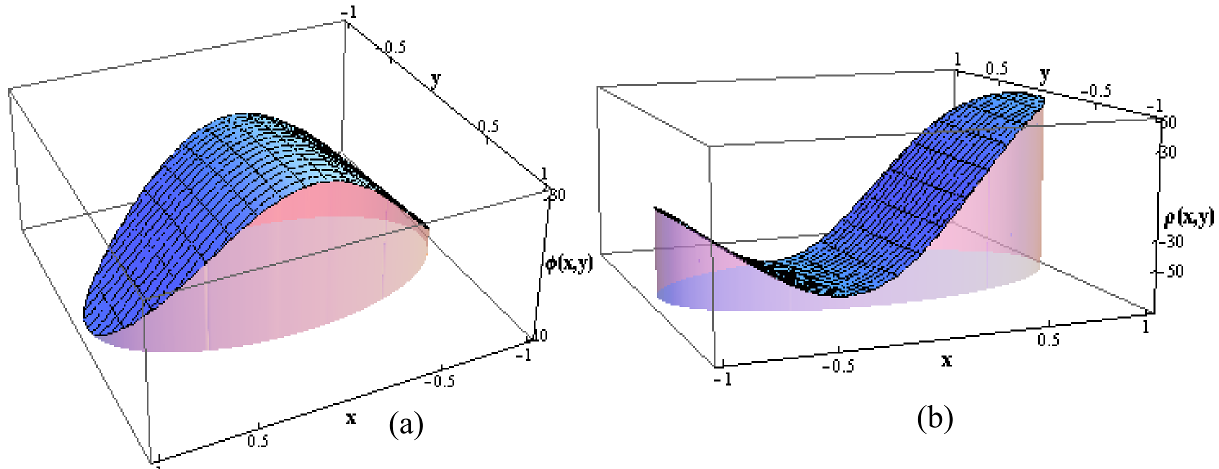



**FIGURE 5 (a) Density and (b) generation of traffic in the domain of the two-dimensional case**

The contour geometry is discretized using 34 discontinuous elements. The computation time is of the order of 30 minutes to evaluate numerical integrals, and seconds for the resolution of the equation systems. The convergence of the method depends on the approximate functions used in the dual reciprocity scheme. In this example, radial basis functions were used. Figure 6 shows the numerical results obtained in the lower side of the ellipse.

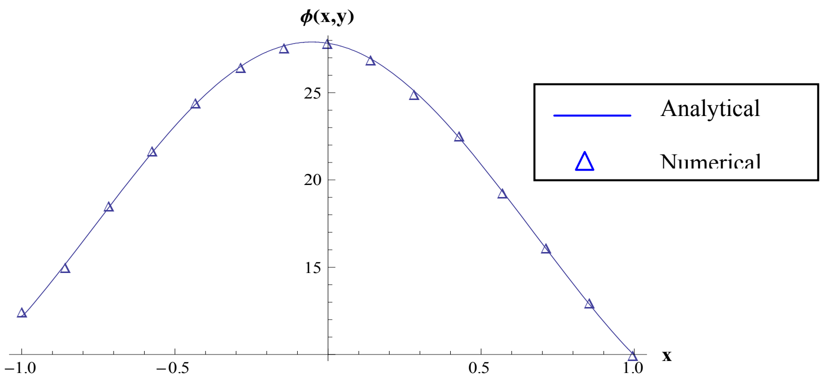



**FIGURE 6 Numerical results versus analytical density values**

### *The Transitory Case*

Romero [6] outlines various examples of a known analytic solution in order to validate the proposed method. A one-dimensional example with the density and the velocity varying with time and the x coordinate is shown below. The geometry of the problem is defined in Figure 7, while the analytic solution and the boundary conditions are determined by the expression

$$\phi = \phi_a \cdot \left(\frac{\phi_b}{\phi_a}\right)^{x^2} e^{-k_1 \cdot x \cdot (1-x) - t^2/k_3} \tag{35}$$

In this case, the velocity is given by:

$$\begin{aligned} v_x &= 2 \cdot x \cdot \left(\ln\left(\frac{\phi_b}{\phi_a}\right) - k_1\right) - k_1 + \frac{k_2}{\phi_a} \cdot \left(\frac{\phi_b}{\phi_a}\right)^{-x^2} e^{k_1 \cdot x \cdot (1-x) + t^2/k_3} \\ v_y &= 0 \end{aligned} \tag{36}$$

Moreover, a continuous generation of traffic was modeled, meeting the expression:

$$\rho(x,t) = -\frac{2 \cdot t}{k_3} \phi_a \cdot \left(\frac{\phi_b}{\phi_a}\right)^{x^2} e^{-k_1 \cdot x \cdot (1-x) - t^2/k_3} \tag{37}$$

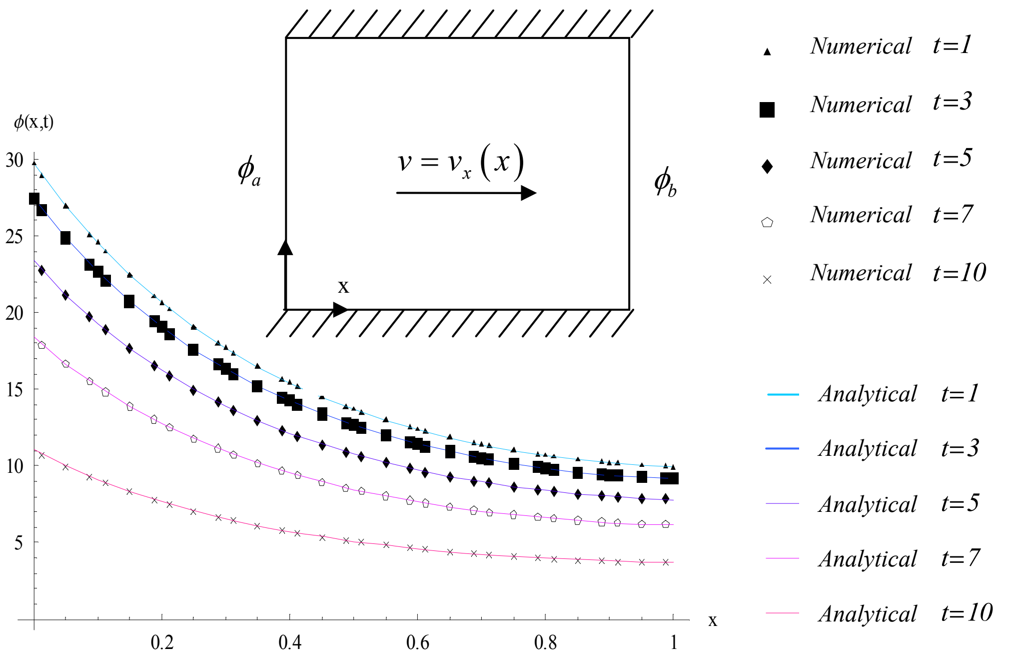



**FIGURE 7 Density value in a transitory example**

The particular values of the constants that were taken for this example were $k_1 = 2$ and $k_2 = k_3 = 100$, with $\phi_a = 30$ and $\phi_b = 10$, values with which one obtains magnitudes that behave compatibly with those expected in a traffic example, with velocities that are positive and rise as the density of the traffic is reduced. For this case the Peclet number is also moderated ($Pe \approx 10$) as expected in traffic flow situations.

In order to solve the problem using the boundary elements method, a discretization of the geometry comprising 10 discontinuous, parabolic elements has been used on each side of the square, using the nodes on the lower side ($y = 0,\ 0 < x < 1$) as dual reciprocity points. The time interval studied was from $t = 0$ to $t = 10$ with integration weights $\theta_\phi = \theta_q = 0.9$ and an integration step of time increment $\Delta t = 5/100$.

Figure 7 shows the results of the density along the *x*-axis, within the range in which the problem is defined. The curves corresponding to different time instants are shown, indicating the perfect matching of theoretical data with those obtained numerically.

The examples given in this document serve to confirm the technical accuracy of the proposed solution. It remains to validate the methodology in real scenarios. The model's complexity lies above all in the theoretical formulation developed over the course of this investigation and brought together in this document. Applying it requires a lot of information, though mainly within the boundary of the zone to be studied. Collecting this quantity of information in a zone that is sufficiently wide to be dealt with by means of this technique may be a difficult task. To avoid this, it is possible to make use of the latest technological advances in mobile-telephone user data-capture and infer from it the parameters that define the mobility for the zone [21]. In this way, the density and flow data necessary for applying the proposed method can be obtained.

## CONCLUSION

This investigation has developed new modeling of traffic flow in two-dimensional areas. Mathematical modeling introduces many simplifications with regard to the actual physical problem, the most obvious being the hypothesis of the continuous medium that, in short, means the elimination of the road from the modeling, treating traffic as a fluid, and hence eliminating the individual vehicle aspect. As a result, the proposed method represents a complementary alternative to other numerical schemes such as finite differences and the finite element method. In this work, the technique has been tested with three cases that, though simplistic, validate the proposed scheme. All cases tested have a low-moderate Peclet number, in accordance to the nature of the problems dealt with. For high-value Peclet numbers, of scarce interest in the traffic application, numerical refinements techniques (i.e. finer mesh [22, 23]) should be investigated.

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