On Multivariate Extensions of the Conditional Value-at-Risk measure

Di Bernardino, E.[a], Fernández-Ponce, J.M.[b,*], Palacios-Rodríguez, F.[b], Rodríguez-Griñolo, M.R.[c]

[a] *CNAM, Paris, Département IMATH, Laboratoire Cédric EA4629, 292 Rue Saint-Martin, Paris 75003, France*

[b] *Universidad de Sevilla, Departamento de Estadística e Investigación Operativa, Calle Tarfia sin número, 41012 Sevilla, España*

[c] *Universidad Pablo de Olavide, Departamento de Economía, Métodos Cuantitativos e Historia Económica, Carretera Utrera, km 1, 41013 Sevilla, España*

Abstract

CoVaR is a systemic risk measure proposed by Adrian and Brunnermeier [1] able to measure a financial institution's contribution to systemic risk and its contribution to the risk of other financial institutions. CoVaR stands for conditional Value-at-Risk, i.e. it indicates the Value at Risk for a financial institution that is conditional on a certain scenario. In this paper, two alternative extensions of the classic univariate Conditional Value-at-Risk are introduced in a multivariate setting. The two proposed multivariate CoVaRs are constructed from level sets of multivariate distribution functions (*resp.* of multivariate survival distribution functions). These vector-valued measures have the same dimension as the underlying risk portfolio. Several characterizations of these new risk measures are provided in terms of the copula structure and stochastic orderings of the marginal distributions. Interestingly, these results are consistent with existing properties on univariate risk measures. Furthermore, comparisons between existent risk measures and the proposed multivariate CoVaR are developed. Illustrations are given in the class of Archimedean copulas. Estimation procedure for the multivariate proposed CoVaRs is illustrated in simulated studies and insurance real data.

Keywords: Copulas and dependence, Level sets of distribution functions, Multivariate risk measures, Stochastic orders, Value-at-Risk.

Introduction

A risk-based approach for supervision and regulation of the financial sector is gaining ground in both emerging and industrialized countries. As part of this approach, regulators need to measure, monitor, and manage market risk. Value-at-Risk (VaR) is one measure being explored

*Corresponding author: ferpon@us.es

for this purpose. One of the most important sectors in which this practice has been adopted is the pension fund industry. As the recent financial crisis has shown, risks are generally difficult to measure and to manage. This becomes crucial in the case of pensions, where people rely on their savings to finance their old age. Risk is a complex notion and can take on varied forms with diverse applications. In the context of trading firms, managing risk has been traditionally achieved by the introduction of Value-at-Risk (VaR) thresholds on the portfolio risk accumulated by traders. Over recent decades, this problem has been handled mostly in a univariate version. Moreover, the risk allocation problem only involves internal risks associated with businesses in the subsidiaries. However, the solvability of financial institutions could also be affected by external risks whose sources cannot be controlled. These risks may also be strongly heterogeneous in nature and difficult to diversify away. One can think, for instance, of systemic risk or contagion effects in a strongly interconnected system of financial companies.

In the last decade, much research has been devoted to risk measures and many multidimensional extensions have been investigated. On theoretical grounds, Jouini et al. [23] propose a class of set-value coherent risk measures. Unsurprisingly, the main difficulty regarding multivariate generalizations of risk measures is the fact that vector preorders are, in general, partial preorders. In order to generalize the Value-at-Risk measure, Embrechts and Puccetti [15], Nappo and Spizzichino [30], and Prékopa [32] use the notion of quantile curve which is defined as the boundary of the upper-level set of a distribution function or the lower-level set of a survival function. Cousin and Di Bernardino [6] introduce two alternative extensions of the classic univariate Value-at-Risk in a multivariate setting. The proposed measures, which are based on the definitions of multivariate quantiles in Embrechts and Puccetti [15], are real-valued vectors with the same dimension as the considered portfolio of risks. This feature can be considered relevant from an operational point of view. Both measures satisfy the positivite homogeneity and translation invariance property. Cousin and Di Bernardino [7] propose two extensions of the classic univariate Conditional-Tail-Expectation (CTE) in a multivariate setting. These multivariate extensions in Cousin and Di Bernardino [6] and Cousin and Di Bernardino [7] are constructed from level sets of multivariate distribution functions and multivariate survival distribution functions, respectively. This level sets approach is also used in this paper.

Another recent interesting risk measure is that of the CoVaR, which stands for Conditional Value-at-Risk. CoVaR is a systemic risk measure proposed by Adrian and Brunnermeier [1] that measures a financial institution's contribution to systemic risk and its contribution to the risk of other financial institutions. In the original unidimensional model, the CoVaR (of a particular bank, portfolio of asset, etc.) indicates the Value-at-Risk for a financial institution which is conditional on a certain (stress) scenario.

Assume now that X_j represents asset returns of the financial system (or bank j) and X_i represents the asset returns of bank i. The $\mathrm{CoVaR}_\alpha^{j|i}$ can then be defined by:

$$P[X_j \leq \mathrm{CoVaR}_\alpha^{j|i} \,|\, X_i = \mathrm{VaR}_q(X_i)] = \alpha, \qquad \text{for } \alpha \in (0,1), \tag{1}$$

where $\mathrm{VaR}_\alpha(X_i)$ is the quantile function of the random variable X_i at risk-level α, i.e., $\mathrm{VaR}_\alpha(X_i) =$

$\inf\{x \in \mathbb{R} : F_{X_i}(x) \geq \alpha\}$. Equation (1) implicitly defines the CoVaR of the bank j which is conditional on bank i being at its α%-VaR level (see Adrian and Brunnermeier [1]).

CoVaR in (1) represents one of the major threads in the current regulatory and scientific discussion of systemic risks. In the literature, several alternative definitions of CoVaR can be found (see Girardi and Ergün [19] and Goodhart and Segoviano [21]). Starting from (1), we can also consider the CoVaR given by

$$\mathrm{CoVaR}_{\alpha}^{j}(\mathbf{X}) = \mathrm{VaR}_{\alpha}(L|X_j \geq \mathrm{VaR}_{\alpha}(X_j)),$$

where the financial system is represented via the total risk $L = X_1 + \ldots + X_d$, i.e., the aggregated total risk of the firm network and the component j of the vector $\mathbf{X} = (X_1, \ldots, X_d)$ represents the risk exposure of the company j.

In this paper, two new multivariate generalizations of CoVaR based on the multivariate quantile settings of Embrechts and Puccetti [15], Cousin and Di Bernardino [6], and Cousin and Di Bernardino [7] are introduced. These proposed CoVaR measures can be useful in the analysis of multiple financial institutions all together in the systemic context.

Several properties have been obtained. In particular, the positive homogeneity and translation property are shown. The behaviour of the components of the proposed CoVaR vectors with respect to the univariate VaR of margins and to the multivariate VaR in Cousin and Di Bernardino [6] is also analysed. We also study how these measures are influenced by a change in marginal distributions, by a change in dependence structure, and by a change in risk level.

Adrian and Brunnermeier [1] defined a systemic risk measure, called ΔCoVaR, as the difference between the VaR of the institution j (or financial system) conditional on the distress of a particular financial institution i (see (1)) and the VaR of the institution j. ΔCoVaR and other interesting systemic risk measures are introduced and gathered in Mainik and Schaanning [24]. The in-depth study of ΔCoVaR systemic risk measures using the multivariate CoVaR proposed in this paper goes beyond the scope of the present work. A more practical analysis on systemic risks using multivariate ΔCoVaR measures is currently in preparation.

The paper is organized as follows. In Section 1, the piecewise-linear weighted loss function which, can be used to generalize several risk measures, is introduced. Moreover, some notations, tools, and technical assumptions are given. In Section 2, properties of invariance for the proposed multivariate CoVaR are shown. Furthermore, we analyse how these multivariate measures behave when the marginal risks or the copula structures increase with respect to stochastic orders (see Section 3). Illustrations and properties for the Archimedean copula class are presented in Section 4. In Section 5, estimation procedure for the multivariate proposed CoVaRs is illustrated in simulated studies and insurance real data. Conclusion discusses open problems and possible directions for future work.

1. Preliminaries and Definitions

Let X be a non-negative random variable with distribution function F_X and quantile function at level ω in $[0,1]$ given by $Q_X(\omega) = \inf\{x : F_X(x) \geq \omega\}$. Note that the quantile function is also defined as a *Value-at-Risk* in the economics literature and denoted as $\mathrm{VaR}_\omega(X)$ (see also (1)). Let $\mathcal{L}_1(\Omega, \mathcal{A}, P)$ be the set of all random variables with finite expectations. Assuming that X is a random variable of $\mathcal{L}_1$, the Weighted Loss function (WL) is defined by

$$L_X(x;\omega) = \omega\,\mathbb{E}[(X-x)^+] + (1-\omega)\,\mathbb{E}[(X-x)^-] \quad \text{for all } x \in \mathbb{R} \text{ and } \omega \in [0,1], \tag{2}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.
Note that if X is a non-negative random variable, then $L_X(x;\omega) = \omega\mathbb{E}[X]$ for all $x < 0$. This function has a key role in an actuarial context. Indeed, it represents the expected cost for the reinsurance company, called *net premium*, where X denotes the risk for the insurance company. If the insurance company prefers not to bear all the risk, passes on parts of the risk to a reinsurance company. The part retained by the original insurance company is usually called the *retention.* A *stop-loss* contract establishes a fixed retention x (see Section 8.3 in Müller and Stoyan [29]). This means that the maximum risk for the insurance company is x. Thus, if $X > x$ then, the reinsurance company will take over $X - x$. This class of contracts is useful to protect companies from insolvency due to excessive claims. In an actuarial context, the threshold x is often called the *deductible* or *priority* (see Section 1.7.1 in Denuit et al. [11]).

Certain interesting properties of the WL function in (2) are now recalled. The properties (P1)-(P6) are trivially obtained by the same arguments as those used by Muñoz Pérez and Sánchez-Gómez [27] to prove the properties of the dispersion function.

(P1) It holds that
$$L_X(x;\omega) = \omega \int_x^{+\infty} \bar{F}(t)\,\mathrm{d}t + (1-\omega) \int_{-\infty}^{x} F(t)\,\mathrm{d}t.$$

(P2) Let C_F denote the set of continuity points of F_X and $X \in \mathcal{L}_1$. Then
$$F_X(x) = L_X^{'}(x;\omega) + \omega, \quad \forall\, x \in C_F \text{ and } x \geq 0$$
where L'_X is the derivative of L_X with respect to x.

(P3) The WL function is differentiable and its derivative has, at most, a countable number of discontinuity points.

(P4) $L_X(x;\omega)$ is a convex function on $\mathbb{R}^+$.

(P5) $\lim_{x\to+\infty} L_X^{'}(x;\omega) = 1-\omega$; and $\lim_{x\to-\infty} L_X^{'}(x;\omega) = 0$.

(P6) $\lim_{x\to+\infty}[L_X(x;\omega) - (1-\omega)x] = -(1-\omega)\,\mathbb{E}[X]$.

(P7) Finally,

$$\mathrm{VaR}_\omega(X) = \arg\min_{x\in\mathbb{R}^+} L_X(x;\omega), \text{ for } w \in [0,1],$$

with $\mathrm{VaR}_0(X) = x_{F^-}$ and $\mathrm{VaR}_1(X) = x_{F^+}$, where x_{F^+} and x_{F^-} are, respectively, the right and left endpoints of F, such that $x_{F^+} = \sup\{x \in \mathbb{R} : F(x) < 1\}$ and $x_{F^-} = \inf\{x \in \mathbb{R} : F(x) > 0\}$.

It is easy to see that Properties (P1)-(P7) uniquely characterize a WL function, i.e., if $L_X(x;\omega)$ is a function that satisfies Properties (P1)-(P7) above, then there exits a unique distribution function which has $L_X(x;\omega)$ as its WL function. Therefore, it uniquely determines a probability measure P_F on $\mathcal{B}$ (the σ-field of Borel set on $\mathbb{R}$).

An interesting interpretation of the WL function is that $2\,L_X(x;1/2)$ is the L_1-distance between F_X and F_x, where F_x is the distribution function of the degenerate random variable at the point $x \in \mathbb{R}$ (Muñoz Pérez and Sánchez-Gómez [27]). It is also interesting to remark that $L_X(x;1)$ is the well-known *stop-loss* function of X, and that $L_X(x;0)$ could be interpreted as the *stop-gain* function of X. Consequently, the WL function is a weighting of both functions in terms of x. Now, let $\mathbf{X} = (X_1,\ldots,X_d)$ be a non-negative d-dimensional random vector[1]. Cousin and Di Bernardino [6] defined, under certain regularity conditions, the multivariate Lower-Orthant Value-at-Risk at probability level α as the d-dimensional vector

$$\underline{\mathrm{VaR}}_\alpha(\mathbf{X}) = \mathbb{E}[\mathbf{X}\,|\,F(\mathbf{X}) = \alpha], \;\text{ for }\; \alpha \in (0,1),$$

where F is the distribution function of $\mathbf{X}$. Particularly, the i-th component of this vector trivially verifies

$$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = L_{X_i|F(\mathbf{X})=\alpha}(0;1). \tag{3}$$

Using Property (P7), our purpose is now to give a new multivariate approach of the classic Conditional Value-at-Risk model (see CoVaR in (1)) which, as introduced previously, is defined as the VaR of a financial institution, conditional on a certain scenario (see Adrian and Brunnermeier [1]). In this case, the approach is based on the conditional scenario being a restriction for both financial institutions. Thus, in general, no relationship exists between the two CoVaRs.

From now on, assume that $\mathbf{X} = (X_1,\ldots,X_d)$ is a non-negative absolutely-continuous random vector (with respect to Lebesgue measure λ on $\mathbb{R}^d$) with distribution function F and survival function $\overline{F}$. Furthermore, the multivariate distribution function F is assumed to be partially strictly-increasing[2] such that $E(X_i) < \infty$ for $i = 1,\ldots,d$. Such F is said to verify the *regularity*

[1] We restrict ourselves to $\mathbb{R}^d_+$ because, in our applications, components of $d-$dimensional vectors correspond to random losses and are then valued in $\mathbb{R}_+$.

[2] A function $F(x_1,\ldots,x_n)$ is partially strictly-increasing on $\mathbb{R}^d_+\backslash\mathbf{0}$ if the function of one variable $g(\cdot) = F(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_d)$ are strictly-increasing.

conditions. Note that if $\overline{F}$ is the survival function of $\mathbf{X}$, and F verifies the regularity conditions, then $\overline{F}$ is a partially strictly-decreasing function. Unless stated otherwise, the dimension of the vectors is d, and the null vector of dimension d will be denoted by $\mathbf{0}$, and the unity vector of dimension d by $\mathbf{1}$. Therefore, the order $\leq$ between vectors will be considered component-wise. Throughout the paper, given a random variable or a vector $\mathbf{X}$ and any event A, $\mathbf{X}|A$ is denoted as the random variable or vector whose distribution is the conditional distribution of $\mathbf{X}$ given A. Eventually, the equality in law is given by $\stackrel{d}{=}$.

Several useful definitions of stochastic orders are now recalled. Further details, equivalent definitions and applications may be found in Shaked and Shanthikumar [37], Müller [28], and Joe [22].

Definition 1.1. *Let X and Y be two random variables with distribution functions F_X and F_Y respectively. X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if*

$$F_X(x) \geq F_Y(x), \text{ for all } x \in \mathbb{R}.$$

Definition 1.2 (Supermodular function)**.** *A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if, for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$, it satisfies*

$$f(\mathbf{x}) + f(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}),$$

where the operators $\wedge$ and $\vee$ denote coordinate-wise minimum and maximum respectively.

Definition 1.3 (Supermodular Order)**.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two $d-$dimensional random vectors. $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ with respect to the supermodular order (denoted by $\mathbf{X} \leq_{sm} \mathbf{Y}$) iff*

$$\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y})),$$

for all supermodular functions $f : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist.

In Definition 1, from the discussion above, a multivariate generalization of the CoVaR measure is now introduced.

Definition 1 (Multivariate Lower-Orthant CoVaR)**.** *Consider a random vector $\mathbf{X}$ which satisfies the regularity conditions. For $\alpha \in (0,1)$, we define the multivariate lower-orthant* CoVaR *at probability level α by*

$$\underline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \text{VaR}_{\boldsymbol{\omega}}(\mathbf{X}|\mathbf{X} \in \partial \underline{L}(\alpha)) = \begin{pmatrix} \text{VaR}_{\omega_1}(X_1|\mathbf{X} \in \partial \underline{L}(\alpha)) \\ \vdots \\ \text{VaR}_{\omega_d}(X_d|\mathbf{X} \in \partial \underline{L}(\alpha)) \end{pmatrix}, \quad (4)$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$, *for* $i = 1, \ldots, d$, *and* $\partial \underline{L}(\alpha)$ *is the boundary of the set* $\underline{L}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : F(\boldsymbol{x}) \geq \alpha\}$. *Therefore,*

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|F(\mathbf{X}) = \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|F(\mathbf{X}) = \alpha) \end{pmatrix}. \tag{5}$$

In a similar way, the multivariate upper-orthant CoVaR can be defined.

Definition 2 (Multivariate Upper-Orthant CoVaR)**.** *Consider a random vector* $\mathbf{X}$ *which satisfies the regularity conditions. For* $\alpha \in (0,1)$, *we define the multivariate upper-orthant* CoVaR *at probability level* α *by*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\boldsymbol{\omega}}(\mathbf{X}|\mathbf{X} \in \partial \overline{L}(\alpha)) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|\mathbf{X} \in \partial \overline{L}(\alpha)) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|\mathbf{X} \in \partial \overline{L}(\alpha)) \end{pmatrix}, \tag{6}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$, *for* $i = 1, \ldots, d$, *and* $\partial \overline{L}(\alpha)$ *is the boundary of the set* $\overline{L}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : \overline{F}(\boldsymbol{x}) \leq 1 - \alpha\}$. *Therefore,*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|\overline{F}(\mathbf{X}) = 1 - \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|\overline{F}(\mathbf{X}) = 1 - \alpha) \end{pmatrix}. \tag{7}$$

Remark 1.1. *Using the same notation and framework of Definitions 1 and 2, we can also consider a modified version of the multivariate upper and lower CoVaR proposed in Equations (4) and (6). Indeed, consider a financial institution* X_i *and the firm network without* X_i, *i.e.,* $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d) := \mathbf{X}_{d-1}$. *The following modified version of the lower CoVaR in Definition 1 can therefore be proposed:*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}(X_i|F(\mathbf{X}_{d-1}) = \alpha),$$

where F_{d-1} *is the* $(d-1)$*-dimensional distribution function associated to the vector* $\mathbf{X}_{d-1}$. *Analogously, a modified version of the upper CoVaR in Definition 2 can be :*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}(X_i|\overline{F}(\mathbf{X}_{d-1}) = 1 - \alpha),$$

where $\overline{F}_{d-1}$ *is the survival* $(d-1)$*-dimensional distribution function associated to the vector* $\mathbf{X}_{d-1}$. *It should be borne in mind that, using this modified versions, when* $d = 2$ *and* $\omega_i = \alpha$, $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ *and* $\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ *become the classic* CoVaR *in* (1).

The following interpretation of our measures can be considered. The ith component of multivariate lower-orthant CoVaR of $\mathbf{X}$ (*resp.* multivariate upper-orthant CoVaR of $\mathbf{X}$) corresponds to the point x^* that minimizes the WL function of the associated ith marginal given that $\mathbf{X}$ stands in the $\alpha-$level curve of its multivariate distribution function (*resp.* multivariate survival distribution function).

It is worth mentioning that under regularity conditions, $\partial \underline{L}(\alpha)$ (*resp.* $\partial \overline{L}(\alpha)$) is the α-level curve (*resp.* $(1-\alpha)$-level curve) of F (*resp.* $\overline{F}$) (see for instance Di Bernardino et al. [12], Cuevas et al. [8]). This means that there is no plateau in the graph of F for each level α. Therefore, *regularity conditions* guarantee that the minimizer x^* is unique for each component $i = 1, \ldots, d$.

Trivially, given that our CoVaRs are the minimizers of suitable expected losses (see (P7)), they therefore verify the *elicitability* property. This property was studied by Gneiting [20], while Bellini and Bignozzi [4] suggested a slightly more restrictive definition. Recently, Embrechts and Hofert [13] stated that elicitability is a very important property of a risk measure since it provides a natural methodology to perform backtesting. Ziegel [41] has also studied the connections between elicitability and coherence properties of risk measures.

Moreover, the solvency of an insurance company depends on the frequency of large claims. One of the advantages of working with the quantile function is that this function is more robust to extreme values than other central tendency measures.

2. Properties of the multivariate CoVaR

In this section, the aim is to analyse the lower-orthant and upper-orthant CoVaR introduced in Definitions 1 and 2 in terms of classic suitable properties of risk measures (see, for instance, Artzner et al. [2], Denuit et al. [11]).

We focus on invariance properties (see Section 2.1). Furthermore, in Section 2.2, the relationships between our CoVaR, the univariate VaR, and the multivariate VaR introduced by Cousin and Di Bernardino [6] are analysed. In Section 2.3, some comonotonic dependence properties for our measures are investigated.

2.1. Invariance properties

The following results (Proposition 2.1 and Corollary 2.1) are now introduced, which will be central in proving invariance properties of our risk measures.

Proposition 2.1. *Let the function h be such that $h(x_1, \ldots, x_d) = (h_1(x_1), \ldots, h_d(x_d))$. Let $\boldsymbol{\omega}$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$.*

(1) If $h_1, \ldots, h_d$ are non-decreasing functions, then, for $i = 1, \ldots, d$,

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = \mathrm{VaR}_{\omega_i}(h_i(X_i)|F(\mathbf{X}) = \alpha).$$

(2) If $h_1, \ldots, h_d$ are non-increasing functions, then, for $i = 1, \ldots, d$,

$$\underline{\text{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = \text{VaR}_{\omega_i}(h_i(X_i)|\overline{F}(\mathbf{X}) = \alpha).$$

Proof. By Definition 1,

$$\begin{aligned}
\underline{\text{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) &= \text{VaR}_{\omega_i}(h_i(T_i)) \\
&= \underset{x \in [h_i(\text{VaR}_\alpha(X_i)), +\infty)}{\arg\min} \left\{ \omega_i \, \mathbb{E}[(h_i(T_i) - x)^+] + (1 - \omega_i) \, \mathbb{E}[(h_i(T_i) - x)^-] \right\},
\end{aligned}$$

where $h_i(T_i) = [h_i(X_i)|F_{h(\mathbf{X})}(h(\mathbf{X})) = \alpha]$, for $i = 1, \ldots, d$.
Since

$$F_{h(\mathbf{X})}(y_1, \ldots, y_d) = \begin{cases} F(h_1^{-1}(y_1), \ldots, h_d^{-1}(y_d)) \text{ if } h_1, \ldots, h_d \text{ are non-decreasing functions,} \\ \overline{F}(h_1^{-1}(y_1), \ldots, h_d^{-1}(y_d)) \text{ if } h_1, \ldots, h_d \text{ are non-increasing functions,} \end{cases}$$

then

$$\underline{\text{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = \begin{cases} \text{VaR}_{\omega_i}(h_i(X_i)|F(\mathbf{X}) = \alpha) \text{ if } h_1, \ldots, h_d \text{ are non-decreasing functions,} \\ \text{VaR}_{\omega_i}(h_i(X_i)|\overline{F}(\mathbf{X}) = \alpha) \text{ if } h_1, \ldots, h_d \text{ are non-increasing functions.} \end{cases}$$

□

As in Proposition 2.1, a similar result can also be obtained for the multivariate upper-orthant CoVaR, by interchanging F with $\overline{F}$. From Proposition 2.1, one can trivially obtain the following property which links the multivariate upper-orthant CoVaR and lower-orthant CoVaR.

Corollary 2.1. *Let h be a linear function such that $h(x_1, \ldots, x_d) = (h_1(x_1), \ldots, h_d(x_d))$. Let $\boldsymbol{\omega}$ be a vector in $[0, 1]^d$ and $\alpha \in (0, 1)$.*

(1) If $h_1, \ldots, h_d$ are non-decreasing functions, then

$$\underline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = h(\underline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})) \text{ and } \overline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = h(\overline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})).$$

(2) If $h_1, \ldots, h_d$ are non-increasing functions, then

$$\underline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = h(\overline{\text{CoVaR}}_{1-\alpha,1-\boldsymbol{\omega}}(\mathbf{X})) \text{ and } \overline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) = h(\underline{\text{CoVaR}}_{1-\alpha,1-\boldsymbol{\omega}}(\mathbf{X})).$$

The following result proves the positive homogeneity and invariance translation properties for risk measures in Definitions 1 and 2.

Proposition 2.2. *Consider a random vector* $\mathbf{X}$ *with a distribution function, which satisfies the regularity conditions. Let* $\boldsymbol{\omega}$ *be a vector in* $[0,1]^d$ *and* $\alpha \in (0,1)$. *The multivariate lower-orthant and upper-orthant* CoVaR *satisfy the following properties:*
Positive Homogeneity: $\forall\, \boldsymbol{c} = (c_1, \ldots, c_d) \in \mathbb{R}_+^d$,

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}\,\mathbf{X}) = \boldsymbol{c}\,\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \text{ and } \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}\,\mathbf{X}) = \boldsymbol{c}\,\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}),$$

where $\mathbf{c}\,\mathbf{X} = (c_1\, X_1, \ldots, c_d\, X_d)$.
Translation Invariance: $\forall \boldsymbol{c} \in \mathbb{R}_+^d$,

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c} + \mathbf{X}) = \boldsymbol{c} + \underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \text{ and } \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c} + \mathbf{X}) = \boldsymbol{c} + \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}).$$

The proof is trivially obtained from Corollary 2.1.

2.2. Relationships between our CoVaR and other risk measures

The relationships between the marginal components of multivariate lower-orthant CoVaR(resp. multivariate upper-orthant CoVaR) and the univariate VaR are given in Proposition 2.3. Furthermore, Proposition 2.4 provides a comparison between the multivariate VaR introduced by Cousin and Di Bernardino [6] and our corresponding multivariate CoVaR.

Proposition 2.3. *Consider a random vector* $\mathbf{X}$ *with distribution function* F, *which satisfies the regularity conditions. Let* $\boldsymbol{\omega}$ *be a vector in* $[0,1]^d$ *and* $\alpha \in (0,1)$. *Therefore,*

$$\overline{\mathrm{CoVaR}}^{\,i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \mathrm{VaR}_\alpha(X_i) \leq \underline{\mathrm{CoVaR}}^{\,i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}), \text{ for } i = 1, \ldots, d.$$

Proof. From Definitions 1 and 2,

$$\begin{aligned}
\underline{\mathrm{CoVaR}}^{\,i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) &= \mathrm{VaR}_{\omega_i}(T_i) \\
&= \underset{x \in [\mathrm{VaR}_\alpha(X_i),\, +\infty)}{\arg\min} \left\{ \omega_i\, \mathbb{E}[(T_i - x)^+] + (1-\omega_i)\, \mathbb{E}[(T_i - x)^-] \right\},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathrm{CoVaR}}^{\,i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) &= \mathrm{VaR}_{\omega_i}(\overline{T}_i) \\
&= \underset{x \in (-\infty,\, \mathrm{VaR}_\alpha(X_i)]}{\arg\min} \left\{ \omega_i\, \mathbb{E}[(\overline{T}_i - x)^+] + (1-\omega_i)\, \mathbb{E}[(\overline{T}_i - x)^-] \right\},
\end{aligned}$$

where $T_i = [X_i \,|\, F(\mathbf{X}) = \alpha]$ and $\overline{T}_i = [X_i \,|\, \overline{F}(\mathbf{X}) = 1 - \alpha]$, for $i = 1, \ldots, d$. Hence, the result is trivially verified since $\mathrm{VaR}_\alpha(X_i)$ is the lower and upper bound of the domain for the corresponding WL function, respectively. □

Proposition 2.4. *Let* α *be a fixed risk level in* $(0,1)$. *Let us denote by* $\underline{\mathrm{VaR}}^{\,i}_\alpha(\mathbf{X})$ *and* $\overline{\mathrm{VaR}}^{\,i}_\alpha(\mathbf{X})$ *the multivariate lower and upper VaR defined by Cousin and Di Bernardino [6]. Given a level* $\boldsymbol{\omega}_* \in [0,1]^d$ *such that* $\underline{\mathrm{CoVaR}}^{\,i}_{\alpha,\boldsymbol{\omega}_*(\mathbf{X})} = \underline{\mathrm{VaR}}^{\,i}_\alpha(\mathbf{X})$, *for any* $i \in \{1, \ldots, d\}$, *then*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \geq \underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}), \textit{ for all } \boldsymbol{\omega} \geq \boldsymbol{\omega}_*.$$

Given a level $\boldsymbol{\omega}^* \in [0,1]^d$ *such that* $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}^*}(\mathbf{X}) = \underline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$ *for any* $i \in \{1,\ldots,d\}$*, then*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \geq \overline{\mathrm{VaR}}^i_\alpha(\mathbf{X}), \textit{ for all } \boldsymbol{\omega} \geq \boldsymbol{\omega}^*.$$

The proof is based on the increasing property of the quantile function. An illustration of Proposition 2.4 in the Clayton copula case is given in Example 4.4.

2.3. Comonotonic dependence properties

Recall that a non-negative random vector $\mathbf{X}$ is said to be a comonotonic random vector if there exists a random variable Z and d increasing functions $g_1,\ldots,g_d$ such that $\mathbf{X} \stackrel{d}{=} (g_1(Z),\ldots,g_d(Z))$ (see Proposition 5.16 in McNeil et al. [25]). The following property of the multivariate CoVaR of a comonotonic random vector can be shown.

Proposition 2.5. *Consider a comonotonic random vector* $\mathbf{X}$ *with distribution function* F*, which satisfies the regularity conditions. Let* $\boldsymbol{\omega}$ *be a vector in* $[0,1]^d$ *and* $\alpha \in (0,1)$*. Therefore,*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i) = \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \textit{ for } i = 1,\ldots,d.$$

Proof. Let $\alpha \in (0,1)$. Therefore

$$\begin{aligned}
\mathbb{E}[(X_i - x)^+ | F(\mathbf{X}) = \alpha] &= \mathbb{E}[(X_i - x)^+ | \min\{g_1^{-1}(x_1),\ldots,g_d^{-1}(x_d)\} = \mathrm{VaR}_\alpha(Z)] \\
&= \mathbb{E}[(X_i - x)^+ | g_i^{-1}(x_i) = \mathrm{VaR}_\alpha(Z)] \\
&= \mathbb{E}[(\mathrm{VaR}_\alpha(X_i) - x)^+], \text{ for all } x \text{ in the support of } X_i.
\end{aligned}$$

In the same way, $\mathbb{E}[(X_i - x)^- | F(\mathbf{X}) = \alpha] = \mathbb{E}[(\mathrm{VaR}_\alpha(X_i) - x)^-]$, for all x in the support of X_i.

In addition,

$$\begin{aligned}
\mathrm{VaR}_{\omega_i}(X_i | F(\mathbf{X}) = \alpha) &= \underset{x \in [\mathrm{VaR}_\alpha(X_i), +\infty)}{\arg\min} \left\{\omega_i\, \mathbb{E}[(\mathrm{VaR}_\alpha(X_i) - x)^+] + (1-\omega_i)\, \mathbb{E}[(\mathrm{VaR}_\alpha(X_i) - x)^-]\right\} \\
&= \underset{x \in [\mathrm{VaR}_\alpha(X_i), +\infty)}{\arg\min} (1-\omega_i)\{x - \mathrm{VaR}_\alpha(X_i)\} \\
&= \mathrm{VaR}_\alpha(X_i), \text{ for } i = 1,\ldots,d.
\end{aligned}$$

By using similar arguments to the lower CoVaR and taking into account that $\overline{F}_{\mathbf{Z}}(u_1,\ldots,u_d) = \overline{F}_{\mathbf{Z}}(\max_{i=1,\ldots,d} u_i)$, the result for the upper CoVaR is obtained. □

The additivity of the multivariate CoVaR for π-comonotonic couple of random vectors is now proposed. From Puccetti and Scarsini [33], a couple $(\mathbf{X},\mathbf{Y})$ of d-dimensional random vectors is a π-comonotonic random vector if there exists a d-dimensional random vector $Z = (Z_1,\ldots,Z_d)$ and non-decreasing functions $f_1,\ldots,f_d,g_1,\ldots,g_d$ such that

$$(\mathbf{X},\mathbf{Y}) \stackrel{d}{=} ((f_1(Z_1),\dots,f_d(Z_d)),(g_1(Z_1),\dots,g_d(Z_d))).$$

Thus, one can prove the following result.

Proposition 2.6. *Let $(\mathbf{X},\mathbf{Y})$ be a π-comonotonic couple of random vectors. Therefore, for $\boldsymbol{\omega}\in[0,1]^d$ and $\alpha\in(0,1)$,*

$$\begin{aligned}
\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}+\mathbf{Y}) &= \underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})+\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}),\\
\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}+\mathbf{Y}) &= \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})+\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}).
\end{aligned}$$

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be two π-comonotonic random vectors. There exists a random vector $\mathbf{Z}$ such that, for any $i=1,\dots,d$, $X_i=f_i(Z_i)$ and $Y_i=g_i(Z_i)$, where f_i and g_i are non-decreasing functions. Let f be the function defined by $f(x_1,\dots,x_d)=(f_1(x_1),\dots,f_d(x_d))$, g be the function defined by $g(x_1,\dots,x_d)=(g_1(x_1),\dots,g_d(x_d))$, and h be the function defined by $h(x_1,\dots,x_d)=(h_1(x_1),\dots,h_d(x_d))$, where $h_i:=f_i+g_i, i=1,\dots,d$. Since the function h_i, $i=1,\dots,d$ is a sum of non-decreasing functions, h_i is a non-decreasing function for $i=1,\dots,d$. Furthermore, $\mathbf{X}+\mathbf{Y}=h(\mathbf{Z})$. From Proposition 2.1, it follows that

$$\begin{aligned}
\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}+\mathbf{Y}) &= \mathrm{VaR}_{\omega_i}(h_i(Z_i)|F_{\mathbf{Z}}(\mathbf{Z})=\alpha)\\
&= \mathrm{VaR}_{\omega_i}(f_i(Z_i)|F_{\mathbf{Z}}(\mathbf{Z})=\alpha)+\mathrm{VaR}_{\omega_i}(g_i(Z_i)|F_{\mathbf{Z}}(\mathbf{Z})=\alpha),
\end{aligned}$$

where $F_{\mathbf{Z}}$ denotes the distribution function of $\mathbf{Z}$. Consequently,

$$\mathrm{VaR}_{\omega_i}(f_i(Z_i)|F_{\mathbf{Z}}(\mathbf{Z})=\alpha)=\mathrm{VaR}_{\omega_i}(f_i(Z_i)|F_{f(\mathbf{Z})}(f(\mathbf{Z}))=\alpha)=\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}),$$

and

$$\mathrm{VaR}_{\omega_i}(g_i(Z_i)|F_{\mathbf{Z}}(\mathbf{Z})=\alpha)=\mathrm{VaR}_{\omega_i}(g_i(Z_i)|F_{g(\mathbf{Z})}(g(\mathbf{Z}))=\alpha)=\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}),$$

which concludes the proof for the lower-orthant CoVaR. Similar arguments can be used for the upper-orthant CoVaR . ☐

2.4. Behaviour of multivariate CoVaR in terms of risk levels

Trivially, due to the increasing property of the quantile function, the components of the multivariate risk measures $\underline{\mathrm{CoVaR}}$ and $\overline{\mathrm{CoVaR}}$ are increasing functions of the risk levels $\omega_i\in[0,1]$.

A property of the monotony of the CoVaR for the risk level α is now given. The increasing behaviour of CoVaR in terms of level α means that the measures increase with the dangerousness of the stress scenarios considered.

This monotony is based on the concept of *positive regression dependence*. Recall that a bivariate random vector (X,Y) is said to admit positive dependence with respect to X, $PRD(Y|X)$, if $[Y|X=x_1]\leq_{st}[Y|X=x_2]$, $\forall x_1\leq x_2$, where $\leq_{st}$ denotes the usual stochastic order (see Shaked and Shanthikumar [37]).

From now on, we denote $U_i=F_{X_i}(X_i)$, $\mathbf{U}=(U_1,\dots,U_d)$, $V_i=\overline{F}_{X_i}(X_i)$, and $\mathbf{V}=(V_1,\dots,V_d)$.

Proposition 2.7. *Consider a d-dimensional random vector $\mathbf{X}$, which satisfies the regularity conditions, with marginal distributions F_{X_i}, for $i = 1, \ldots, d$, copula C and survival copula $\overline{C}$.*

(1) If $(U_i, C(\mathbf{U}))$ is $PRD(U_i|C(\mathbf{U}))$ then, for $\boldsymbol{\omega} \in [0,1]^d$, $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ is a non-decreasing function of α.

(2) If $(V_i, \overline{C}(\mathbf{V}))$ is $PRD(V_i|\overline{C}(\mathbf{V}))$ then, for $\boldsymbol{\omega} \in [0,1]^d$, $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ is a non-decreasing function of α.

Proof. If $\alpha_1 \leq \alpha_2$, then $[U_i|C(\mathbf{U}) = \alpha_1] \leq_{st} [U_i|C(\mathbf{U}) = \alpha_2]$ and $[V_i|\overline{C}(\mathbf{V}) = 1-\alpha_2] \leq_{st} [V_i|\overline{C}(\mathbf{V}) = 1-\alpha_1]$ hold. By using Theorem 1.A.3.a from Shaked and Shanthikumar [37], it is verified that

$$[F^{-1}_{X_i}(U_i)|C(\mathbf{U}) = \alpha_1] \leq_{st} [F^{-1}_{X_i}(U_i)|C(\mathbf{U}) = \alpha_2],$$

and

$$[\overline{F}^{-1}_{X_i}(V_i)|\overline{C}(\mathbf{V}) = 1-\alpha_2] \geq_{st} [\overline{F}^{-1}_{X_i}(V_i)|\overline{C}(\mathbf{V}) = 1-\alpha_1].$$

Thus, $\underline{\mathrm{CoVaR}}^i_{\alpha_1,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha_2,\boldsymbol{\omega}}(\mathbf{X})$ and $\overline{\mathrm{CoVaR}}^i_{\alpha_1,\boldsymbol{\omega}}(\mathbf{X}) \leq \overline{\mathrm{CoVaR}}^i_{\alpha_2,\boldsymbol{\omega}}(\mathbf{X})$, for any $\alpha_1 \leq \alpha_2$ which proves that $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ and $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ are non-decreasing functions of α. □

Assumptions of Proposition 2.7 are automatically satisfied by the large class of Archimedean copulas. This result will be proved in Corollary 4.3.

3. Comparing CoVaR using stochastic orders

The comparison of risks is an important topic of actuarial sciences, especially in insurance business. The behaviour of multivariate CoVaR risk measures is studied under different stochastic ordering conditions. The results below compare the multivariate CoVaR risk measures for random vectors with the same copula by assuming that margins change in the sense of some particular stochastic order.

Proposition 3.1. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two d-dimensional random vectors, which satisfy the regularity conditions and with the same copula C. If $X_i \leq_{st} Y_i$, then*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}),$$

and

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}),$$

for $\alpha \in (0,1)$ and $\boldsymbol{\omega} \in [0,1]^d$.

Proof. Let us denote the i-margins of $\mathbf{X}$ and $\mathbf{Y}$ by F_{X_i} and F_{Y_i} respectively. Since $X_i \leq_{st} Y_i$, then $F_{X_i}^{-1}(u) \leq F_{Y_i}^{-1}(u)$, $\forall\, u \in [0,1]$. Using Sklar's Theorem (see Theorem 2.3.3 in Nelsen [31]), the random variables $U_i \stackrel{d}{=} F_{X_i}(X_i)$, for $i = 1, \dots, d$, are uniformly distributed and their joint distribution is equal to C. Similarly, the random variables $U_i' \stackrel{d}{=} F_{Y_i}(Y_i)$, for $i = 1, \dots, d$. Therefore,

$$[X_i \,|\, C(\mathbf{U}) = \alpha] \stackrel{d}{=} [F_{X_i}^{-1}(U_i) \,|\, C(\mathbf{U}) = \alpha], \text{ and } [Y_i \,|\, C(\mathbf{U}') = \alpha] \stackrel{d}{=} [F_{Y_i}^{-1}(U_i') | C(\mathbf{U}') = \alpha],$$

for $i = 1, \dots, d$. Observe that $[U_i \,|\, C(\mathbf{U}) = \alpha] \stackrel{d}{=} [U_i' \,|\, C(\mathbf{U}') = \alpha]$. From Theorem 1.A.2 in Shaked and Shanthikumar [37], $[X_i \,|\, C(\mathbf{U}) = \alpha] \leq_{st} [Y_i \,|\, C(\mathbf{U}') = \alpha]$ holds. Hence, the statement for the lower-orthant CoVaR is verified. The proof of the second statement is also verified using the same arguments. □

The result in Proposition 3.1 will be illustrated in the Archimedean case in Example 4.5.

Corollary 3.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* d*-dimensional random vectors satisfying the regularity conditions and with the same copula* C*. If* $X_i \stackrel{d}{=} Y_i$*, then, for* $\alpha \in (0,1)$ *and* $\boldsymbol{\omega} \in [0,1]^d$*,*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}), \quad \textit{and} \quad \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}).$$

Finally, some results are provided for the behaviour of our CoVaR measures with respect to a variation of the copula structure, with unchanged marginal distributions.

Proposition 3.2. *Let* $\mathbf{X}$ *and* $\boldsymbol{X}^*$ *be two* d*-dimensional continuous random vectors, which satisfy the regularity conditions with joint distribution functions* F *and* G*, and with the same margins* F_{X_i} *and* $F_{X_i^*}$*, for* $i = 1, \dots, d$*. Let* C *(resp.* C^**) be the copula function associated with* $\mathbf{X}$ *(resp.* $\boldsymbol{X}^*$*) and* $\overline{C}$ *(resp.* $\overline{C}^*$*) the survival copula function associated with* $\mathbf{X}$ *(resp.* $\boldsymbol{X}^*$*).*

(1) *Let* $U_i = F_{X_i}(X_i)$*,* $U_i^* = F_{X_i{}^*}(X_i^*)$*,* $\boldsymbol{U} = (U_1, \dots, U_d)$ *and* $\boldsymbol{U}^* = (U_1^*, \dots, U_d^*)$*.*
If $[U_i \,|\, C(\mathbf{U}) = \alpha] \leq_{st} [U_i^* \,|\, C^*(\mathbf{U}^*) = \alpha]$*, then*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*) \textit{ for } \alpha \in (0,1), \omega_i \in [0,1], i = 1, \dots, d.$$

(2) *Let* $V_i = \overline{F}_{X_i}(X_i)$*,* $V_i^* = \overline{F}_{X_i{}^*}(X_i^*)$*,* $\boldsymbol{V} = (V_1, \dots, V_d)$ *and* $\boldsymbol{V}^* = (V_1^*, \dots, V_d^*)$*.*
If $[V_i \,|\, \overline{C}(\mathbf{V}) = 1-\alpha] \leq_{st} [V_i^* \,|\, \overline{C}^*(\mathbf{V}^*) = 1-\alpha]$*, then*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \geq \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*) \textit{ for } \alpha \in (0,1), \omega_i \in [0,1], i = 1, \dots, d.$$

Proof. By using (P2) and (P7), trivially it holds that

$$\omega_i = F_{X_i^*|F_{\mathbf{X}^*}(\mathbf{X}^*)=\alpha}(\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*)) = F_{X_i|F_{\mathbf{X}}(\mathbf{X})=\alpha}(\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})), \text{ for } i = 1, \dots, d. \tag{8}$$

On the other hand, since $F^{-1}_{X_i}(u)$ for $u \in [0,1]$ is a non-decreasing function, and since X_i and X_i^* have the same distribution, then from Theorem 1.A.3.a in Shaked and Shanthikumar [37], it is verified that

$$F_{F^{-1}_{X_i}(U_i^*)\,|\,C^*(\mathbf{U}^*)=\alpha}(u) \leq F_{F^{-1}_{X_i}(U_i)\,|\,C(\mathbf{U})=\alpha}(u),\ \forall u \in [0,1]. \tag{9}$$

Therefore, from (8) and (9), $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*)$.

Following the above development for $X_i\,|\,\overline{F}_{\mathbf{X}}(\mathbf{X}) = 1-\alpha$ and $X_i^*\,|\,\overline{F}_{\mathbf{X}^*}(\mathbf{X}^*) = 1-\alpha$, and by using the survival quantile function $\overline{F}^{-1}_{X_i}(u)$ for $u \in [0,1]$, the result for upper-orthant CoVaR holds. □

An application of Proposition 3.2 in the case of Archimedean copulas is given in Corollary 4.4.

4. Multivariate CoVaR for Archimedean copulas

Interestingly, one can readily show that when the random vector $\mathbf{X}$ follows an Archimedean copula then the analytical expression for the CoVaR can be easily computed, in a similar way to that used in Cousin and Di Bernardino [6] to compute their multivariate Value-at-Risk. Indeed, Archimedean copulas have useful relationships between their generator and the probability associated to their level curves $\overline{L}(\alpha)$ and $\underline{L}(\alpha)$ (see the notion of multivariate probability integral transformation in Genest and Rivest [18], Barbe et al. [3] and references therein). Furthermore, the results and properties, which were previously proved in this paper, can easily be applied in the large class of Archimedean copulas.

Note that a d-dimensional Archimedean copula with generator ϕ and its inverse ϕ^{-1} is defined by

$$C(u_1,\ldots,u_d) = \phi^{-1}(\phi(u_1)+\ldots+\phi(u_d)), \text{ for all } (u_1,\ldots,u_d) \in [0,1]^d.$$

McNeil and Nešlehová [26] obtained an important stochastic representation of Archimedean copulas, recalled in Proposition 4.1 below.

Proposition 4.1 (McNeil and Nešlehová [26])**.** *Let $\mathbf{U} = (U_1,\ldots,U_d)$ be distributed according to a d-dimensional Archimedean copula with generator ϕ, then*

$$(\phi(U_1),\ldots,\phi(U_d)) \stackrel{d}{=} R\boldsymbol{S},$$

where $\boldsymbol{S} = (S_1,\ldots,S_d)$ is uniformly distributed on the unit simplex $\left\{\boldsymbol{x} \geq 0 | \sum_{k=1}^d x_k = 1\right\}$ and R is an independent non-negative scalar random variable which can be interpreted as the radial part of $(\phi(U_1),\ldots,\ \phi(U_d))$ since $\sum_{k=1}^d S_k = 1$. The random vector $\boldsymbol{S}$ follows a symmetric Dirichlet distribution, whereas the distribution of $R \stackrel{d}{=} \sum_{k=1}^d \phi(U_k)$ is directly related to the generator ϕ through the inverse Williamson transform of ϕ^{-1}.

As a result, any random vector $\mathbf{U} = (U_1, \ldots, U_d)$ which follows an Archimedean copula with generator ϕ can be represented as a deterministic function of $C(\mathbf{U})$ and an independent random vector $\mathbf{S} = (S_1, \ldots, S_d)$ uniformly distributed on the unit simplex, i.e.,

$$(U_1, \ldots, U_d) \stackrel{d}{=} (\phi^{-1}(S_1\phi(C(\mathbf{U}))), \ldots, \phi^{-1}(S_d\phi(C(\mathbf{U})))). \tag{10}$$

Corollary 4.1. *Let* $\mathbf{X}$ *be a* d*-dimensional random vector with an Archimedean copula with generator* ϕ *and* $\alpha \in (0, 1)$*. Therefore,*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}\left[F^{-1}_{X_i}(\phi^{-1}(S_i\phi(\alpha)))\right], \ \textit{for } i = 1, \ldots, d, \tag{11}$$

where $\boldsymbol{\omega} \in [0, 1]^d$ *and* S_i *is a random variable with* $Beta(1, d-1)$ *distribution.*

Proof. Note that $\mathbf{X}$ is distributed as $(F^{-1}_{X_1}(U_1), \ldots, F^{-1}_{X_d}(U_d))$, where $\mathbf{U} = (U_1, \ldots, U_d)$ follows an Archimedean copula C with generator ϕ. Consequently, each component $i = 1, \ldots, d$ of the multivariate risk measure introduced in Definition 1 can be expressed as

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \underset{x\in[\mathrm{VaR}_\alpha(X_i),+\infty)}{\arg\min} \left\{\omega_i\, \mathbb{E}[(T_i - x)^+] + (1-\omega_i)\, \mathbb{E}[(T_i - x)^-]\right\},$$

where $T_i = [F^{-1}_{X_i}(U_i)|C(\mathbf{U}) = \alpha]$. Moreover, from representation (10), the following relation is verified

$$[\mathbf{U}|C(\mathbf{U}) = \alpha] \stackrel{d}{=} (\phi^{-1}(S_1\phi(\alpha)), \ldots, \phi^{-1}(S_d\phi(\alpha))), \tag{12}$$

since $\mathbf{S}$ and $C(\mathbf{U})$ are stochastically independent. The result comes from the fact that the random vector $\mathbf{S}$ follows a symmetric Dirichlet distribution. □

Note that, by using (12), the marginal distributions of $\mathbf{U}$ given $C(\mathbf{U}) = \alpha$ can be expressed in a very simple way, that is,

$$P(U_k \le u \,|\, C(\mathbf{U}) = \alpha) = \left(1 - \frac{\phi(u)}{\phi(\alpha)}\right)^{d-1} \ \text{for } 0 < \alpha < u < 1, \text{ and any } k = 1, \ldots, d. \tag{13}$$

Corollary 4.2. *Let* $\mathbf{X}$ *be a* d*-dimensional random vector with an Archimedean survival copula with generator* ϕ *and* $\alpha \in (0, 1)$*. Therefore,*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}\left[\overline{F}^{-1}_{X_i}(\phi^{-1}(S_i\phi(1-\alpha)))\right] \ \textit{for } i = 1, \ldots, d, \tag{14}$$

where $\boldsymbol{\omega} \in [0, 1]^d$ *and* S_i *is a random variable with* $Beta(1, d-1)$ *distribution.*

The proof is similar to Corollary 4.1 and is therefore omitted here.

From (11) and (14), analytical expressions of the lower-orthant and the upper-orthant CoVaR for a vector $\mathbf{X} = (X_1, \ldots, X_d)$ with a particular Archimedean copula are now derived. Assume that X_i is uniformly-distributed on $[0,1]$, for $i = 1,\dots,d$. Since Archimedean copulas are exchangeable, the components of $\underline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ (*resp.* $\overline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$) are equal in the case where $\omega_1 = \dots = \omega_d$. Furthermore, it is also possible to obtain expressions for the upper-orthant $\overline{\text{CoVaR}}_{\alpha,\boldsymbol{\omega}}$ for $\tilde{\mathbf{X}} = (1-X_1,\dots,1-X_d)$ since, by using Corollary 2.1:

$$\overline{\text{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\tilde{\mathbf{X}}) = 1 - \underline{\text{CoVaR}}^i_{1-\alpha,1-\boldsymbol{\omega}}(\mathbf{X}).$$

4.1. Analytical expressions of CoVaR measures for Archimedean copulas

In the following, Corollary 4.1 is illustrated for some commonly used Archimedean copula families (see Example 4.1, 4.2, 4.3).

Example 4.1 (Bivariate Clayton family)**.** *In Table 1 (left), the bivariate random vector (X,Y) is considered with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$ is considered. One can readily show that*

$$\frac{\partial \underline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}}{\partial\theta} \leq 0 \;\; \textit{and} \;\; \frac{\partial \overline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}}{\partial\theta} \geq 0, \;\; \textit{for} \;\; \theta \geq -1, \;\; \alpha \in (0,1) \;\; \textit{and} \;\; \omega \in [0,1].$$

Hence, the components of the multivariate $\underline{\text{CoVaR}}$ *(resp.* $\overline{\text{CoVaR}}$*) are decreasing (resp. increasing) functions of the dependence parameter θ. Interestingly, in the comonotonic case, both multivariate risk measures* $\underline{\text{CoVaR}}$ *and* $\overline{\text{CoVaR}}$ *correspond to the vector composed of the univariate* VaR *at level α associated with each component. These properties are illustrated in Figure 1 where upper and lower* CoVaR *are plotted as functions of the risk level $\boldsymbol{\omega}$ for different values of dependence parameter θ and for a fixed level α. Note that, when the parameter θ increases, the lower* CoVaR *tends to decrease. Conversely, the upper bound for the upper* CoVaR *is represented by the perfect positive dependence case. The latter empirical behaviours will be formally confirmed in the following (see Corollary 4.4).*

θ	$\underline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X,Y)$
$(-1,\infty)$	$\left(1+\left(\frac{1}{\alpha^\theta}-1\right)(1-\omega_1)\right)^{-1/\theta}$
-1	$1-(1-\omega_1)(1-\alpha)$
0	$\alpha^{1-\omega_1}$
1	$\frac{\alpha}{(1-\alpha)(1-\omega_1)+\alpha}$
∞	α

θ	$\underline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X,Y)$
$[-1,1)$	$\frac{1-\theta}{\left(\frac{1-\theta(1-\alpha)}{\alpha}\right)^{(1-\omega_1)}-\theta}$
0	$\alpha^{1-\omega_1}$

Table 1: $\underline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$, for a bivariate Clayton copula (left) and a bivariate Ali-Mikhail-Haq copula (right).

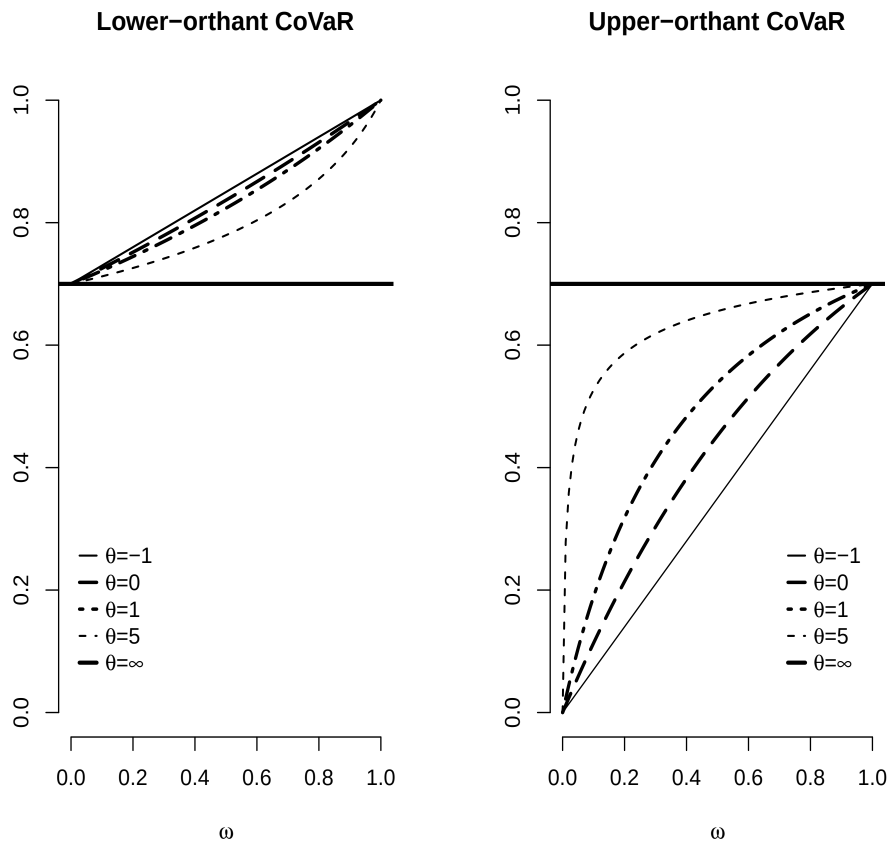

Figure 1: Behaviour of $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$ (left) and $\overline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(1-X,1-Y)$ (right) with respect to the risk level ω for different values of dependence parameter θ and for $\alpha = 0.7$. Here, (X,Y) is a bivariate random vector with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$.

Example 4.2 (Bivariate Ali-Mikhail-Haq family). *Table 1 (right) illustrates the analytical expressions of* $\underline{\mathrm{CoVaR}}$ *for the first component of a bivariate random vector with uniform marginal distributions and a Ali-Mikhail-Haq copula, for* $\theta \in [-1,1)$.

Recall that bivariate Archimedean copulas can be extended to $d-$dimensional copulas, with $d > 2$, on the condition that the generator ϕ is a $d-$monotone function in $[0,\infty)$ (see McNeil and Nešlehová [26]). The bivariate Gumbel family can be generalized in dimension d, for $\theta \geq 1$ (see Example 4.25 in Nelsen [31]).

Example 4.3 (3−dimensional Gumbel family). *In this case, analytical expressions of the first component of lower* CoVaR *of a* $3-$*dimensional random vector* (X_1, X_2, X_3) *with uniform marginal distributions and a Gumbel copula, for* $\theta \geq 1$ *are provided in Table 2.*

θ	$\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X_1,X_2,X_3)$
$[1,\infty)$	$\alpha^{(1-\sqrt{\omega_1})^{1/\theta}}$
1	$\alpha^{(1-\sqrt{\omega_1})}$
∞	α

Table 2: $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X_1,X_2,X_3)$ for a 3−dimensional Gumbel copula.

4.2. Illustrations of some properties for Archimedean copulas

In the following, some theoretical properties presented in Section 2 are illustrated in the large class of $d-$dimensional Archimedean copula. Firstly, using Corollary 4.1, an illustration of Proposition 2.4 in the Clayton copula case is provided.

Example 4.4. *Assume that* $\mathbf{X}$ *is a bivariate random vector with uniform marginal distributions and Clayton copula. The distribution function of* $\mathbf{X}$ *is therefore given by:*

$$F(x_1,x_2)=\Big[\max\{x_1^{-\theta}+x_2^{-\theta}-1,0\}\Big]^{-1/\theta},\qquad \textit{for } \theta\in[-1,\infty)\backslash\{0\} \textit{ and } (x_1,x_2)\in[0,1]^2.$$

Then, by straightforward computation, one can obtain, for $\alpha\in(0,1)$ *and* $\omega_1\in[0,1]$,

$$\underline{\mathrm{VaR}}^1_{\alpha}(\mathbf{X})=\tfrac{\theta}{\theta-1}\tfrac{\alpha^{\theta}-\alpha}{\alpha^{\theta}-1},\quad \textit{and}\quad \underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X})=\left[1+\left(\tfrac{1}{\alpha^{\theta}}-1\right)(1-\omega_1)\right]^{-1/\theta},$$

where $\underline{\mathrm{VaR}}^1_{\alpha}(\mathbf{X})$ *is the first-component lower* VaR *proposed by Cousin and Di Bernardino [6]. Consequently, both measures coincide in*

$$\omega_*=\left(\alpha^{-\theta}-\left(\tfrac{\theta}{\theta-1}\tfrac{\alpha^{\theta}-\alpha}{\alpha^{\theta}-1}\right)^{-\theta}\right)[\alpha^{-\theta}-1]^{-1}.$$

For a fixed $\alpha=0.6$ *we obtain the results gathered in Figure 2.* $\underline{\mathrm{VaR}}_{\alpha}(\mathbf{X})$ *represents the case that the complete risk of the insurance company is reinsured by another company (*$x=0$*) (see Cousin and Di Bernardino [6]). The insurance company gives the total weight to the expected cost of the reinsurance company, that is, establishes* $\omega=1$*. By contrast,* CoVaR *defines the minimum retention of the insurance company given a weight* $\omega\in[0,1]$ *for the expected cost of the reinsurance company. For instance, for* $\theta=2$*, it can be observed in Figure 2 that* $\underline{\mathrm{VaR}}^1_{0.6}(\mathbf{X})=0.75$ *and the cut-off point is* $\omega_*=0.56$*. Similarly, analytical expressions for multivariate upper* CoVaR *and comparisons with the associated* $\overline{\mathrm{VaR}}_{\alpha}(\mathbf{X})$ *(see Cousin and Di Bernardino [6]) can be obtained.*

Corollary 4.3 proves that assumptions of Proposition 2.7 are automatically satisfied in the large class of d-dimensional Archimedean copulas.

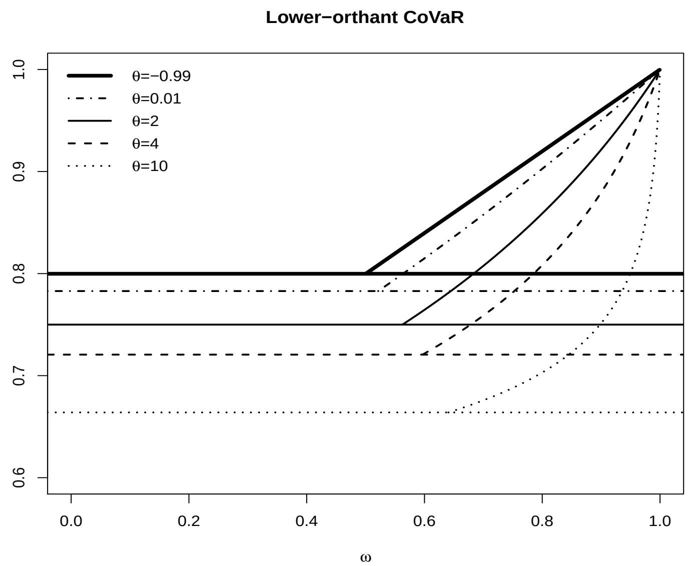

Figure 2: $\underline{\mathrm{VaR}}^1_\alpha(\mathbf{X})$ and $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$. Here, (X,Y) is a bivariate random vector with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$, and $\alpha = 0.6$.

Corollary 4.3. *Consider a d-dimensional random vector $\mathbf{X}$, which satisfies the regularity conditions, with marginal distributions F_{X_i}, for $i = 1, \ldots, d$, copula C and survival copula $\overline{C}$.*

(1) If C is a d-dimensional Archimedean copula, then $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ is a non-decreasing function of α with $\boldsymbol{\omega} \in [0,1]^d$.

(2) If $\overline{C}$ is a d-dimensional Archimedean copula, then $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ is a non-decreasing function of α with $\boldsymbol{\omega} \in [0,1]^d$.

Proof. Let $U_i = F_{X_i}(X_i)$, $\mathbf{U} = (U_1, \ldots, U_n)$, $V_i = \overline{F}_{X_i}(X_i)$ and $\mathbf{V} = (V_1, \ldots, V_n)$. Since C is the copula of $\mathbf{X}$, then $\mathbf{U}$ is distributed as C. If C is an Archimedean copula, from (13), $P(U_i > u | C(\mathbf{U}) = \alpha)$ is a non-decreasing function of α. Similarly, $P(V_i > u | \overline{C}(\mathbf{V}) = 1 - \alpha)$ is a non-decreasing function of α. The results are therefore trivially derived from Proposition 2.7. □

In the following, an illustration of Proposition 3.1 is provided in the Archimedean case.

Example 4.5. *Three different random vectors (X, Y_i), for $i = 1, \ldots, 3$ are considered with the same bivariate Clayton copula with dependence parameter 2, such that*

$$X \sim Exp(1), \;\; Y_1 \sim Exp(2), \;\; Y_2 \sim Burr(5,1), \;\; Y_3 \sim Fréchet(4).$$

Since $Y_1 \leq_{st} Y_2 \leq_{st} Y_3$, from Proposition 3.1, then

$$\underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_1) \leq \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_2) \leq \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_3),$$

for any $\boldsymbol{\omega} \in [0,1]^2$ *and* $\alpha \in (0,1)$*. The results are gathered in Figure 3. It should also be emphasised that, by Corollary 3.1, the first components of the multivariate lower-orhant* CoVaR *and upper-orthant* CoVaR *for the four vectors coincide.*

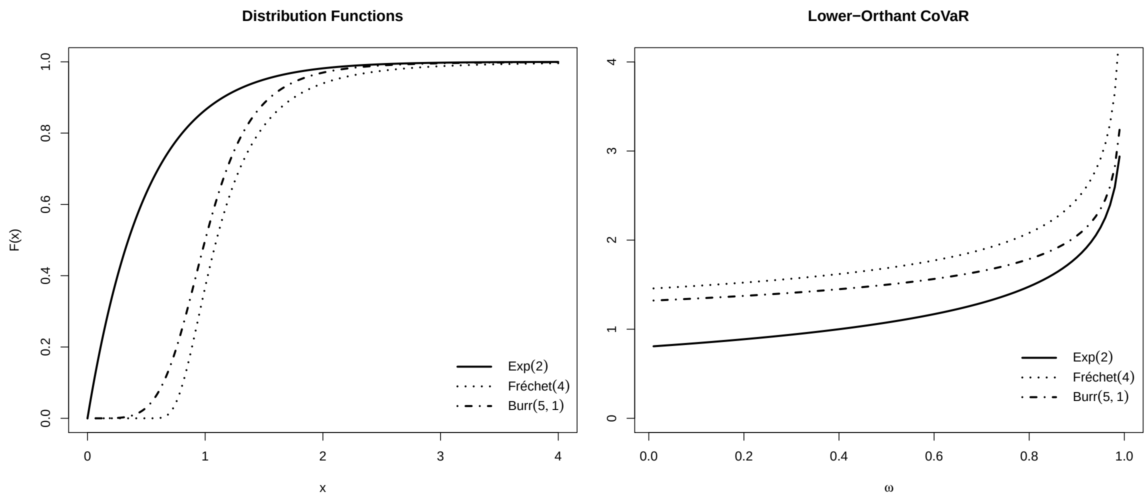

Figure 3: Left: Distribution functions of random variables Y_i, for $i = 1, \ldots, 3$, with $Y_1 \sim Exp(2)$, $Y_2 \sim Burr(5,1)$ and $Y_3 \sim Fr\acute{e}chet(4)$. Right: $\underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_i)$ for $i = 1, \ldots, 3$, with the same copula Clayton with parameter 2, $X \sim Exp(1)$, $Y_1 \sim Exp(2)$, $Y_2 \sim Burr(5,1)$, $Y_3 \sim Fr\acute{e}chet(4)$ and $\alpha = 0.8$.

The following remark will be useful in Corollary 4.4.

Remark 4.1. *Let* $\boldsymbol{U}$ *and* $\boldsymbol{U}^*$ *be two random vectors with copula* C *and* C^**, respectively, and with uniform marginal distributions. It is easy to prove that* $\boldsymbol{U} \leq_{sm} \boldsymbol{U}^*$ *implies* $C(\boldsymbol{u}) \leq C^*(\boldsymbol{u})$*, for* $\boldsymbol{u} \in [0,1]^d$ *(Section 6.3.3 in Denuit et al. [11]). In addition, for Gumbel, Frank, Clayton, and Ali-Mikhail-Haq families, it can be shown that an increase of* θ *yields an increase of dependence in the sense of the supermodular order (see examples in Wei and Hu [39], Joe [22]). As a consequence, in these cases,*

$$\theta \leq \theta^* \Rightarrow C(\boldsymbol{u}) \leq C^*(\boldsymbol{u}), \quad for \quad \boldsymbol{u} \in [0,1]^d. \tag{15}$$

Corollary 4.4. *Let* $\mathbf{X}$ *be a* $d-$*dimensional random vector satisfying the regularity conditions with copula* C *and survival copula* $\overline{C}$*.*

If C *is a* $d-$*dimensional Archimedean copula that satisfies Property* (15) *in Remark* (4.1)*, each component of* $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ *is a decreasing function of* θ*, with* $\alpha \in (0,1)$ *and* $\boldsymbol{\omega} \in [0,1]^d$*.*

If $\overline{C}$ *is a* $d-$*dimensional Archimedean copula that satisfies Property* (15) *in Remark* (4.1)*, each component of* $\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ *is a increasing function of* θ*, with* $\alpha \in (0,1)$ *and* $\boldsymbol{\omega} \in [0,1]^d$*.*

It should be noted that, for instance for Gumbel, Frank, Clayton and Ali-Mikhail-Haq families, assumptions of Corollary 4.4 are satisfied. The reader is referred, for instance, to the behaviour of the lower and upper CoVaR with respect to the copula parameter θ presented in Figure 1.

Proof. We consider two Archimedean copulas of the same family, C_θ (associated to vector $\mathbf{U}$) and C_{θ^*} (associated to vector $\mathbf{U}^*$) with generator ϕ_θ and ϕ_{θ^*} such that $\theta \leq \theta^*$. By Proposition 3.2, we have to prove that $[U_i^*|C_{\theta^*}(\mathbf{U}^*) = \alpha] \leq_{st} [U_i|C_\theta(\mathbf{U}) = \alpha]$ holds for $i = 1, \ldots, d$. On the other hand, from Eq. (13), it is readily obtained that

$$[U_i^*|C_{\theta^*}(\mathbf{U}^*) = \alpha] \leq_{st} [U_i|C_\theta(\mathbf{U}) = \alpha] \text{ for any } \alpha \in (0,1) \Leftrightarrow \frac{\phi_{\theta^*}}{\phi_\theta} \text{ is a decreasing function.}$$

Finally, by taking into account Remark 4.1 for Clayton, Frank, Gumbel and Ali-Mikhail-Haq families, the function $\frac{\phi_{\theta^*}}{\phi_\theta}$ is decreasing when $\theta \leq \theta^*$. Therefore, from Proposition 3.2, an increase of the parameter θ yields a decrease in each component of $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$. The second statement is obtained trivially using the same arguments. □

4.3. A weak subadditivity tail property in the Archimedean cases

The additivity of our CoVaR is provided in Section 2.3 in a comonotonic dependence vectorial case (see Proposition 2.6 for π-comonotonic vectors). In the following, the aim is to study the condition for a copula to obtain subadditivity inequalities for our lower CoVaR .

To this end, as in the univariate case (see Daníelsson et al. [9]), we focus on the *tails* of the considered multivariate distribution.

In the following, two notions of regular variation are applied. A measurable function $U : \mathbb{R} \to \mathbb{R}$ is regularly varying at ∞ with index ρ (denoted by $U \in RV_\rho$), if it holds that $\lim_{t\to\infty} \dfrac{U(tx)}{U(t)} = x^\rho$, for any real number $x > 0$. Also, a random vector $\mathbf{X}$ with joint distribution function F is said to be multivariate regularly varying ($\mathbf{X} \in MRV$) if there exists a Radon measure ν on $[\mathbf{0}, \infty]\backslash\{\mathbf{0}\}$, such that

$$\lim_{t\to\infty} \frac{1 - F(t\mathbf{x})}{1 - F(t\mathbf{1})} = \nu([\mathbf{0}, \mathbf{x}]^c),$$

for all points $\mathbf{x} \in [\mathbf{0}, \infty)\backslash\{\mathbf{0}\}$, which are continuity points of the function $\nu([\mathbf{0}, \cdot]^c)$. Observe also that for any non-negative MRV random vector $\mathbf{X}$, its non-degenerate univariate margins X_i have regularly varying right tails, that is,

$$\overline{F}_i(t) := t^{-\beta} L(t),\ t \geq 0,$$

where $\beta > 0$ is the marginal heavy-tail index and $L(t)$ is a slowly varying function, i.e. $L(x\,t)/L(t) \to 1$ as $t \to \infty$ for any $x > 0$. Further details about regular variation can be found in Resnick [34], Resnick [35] and Embrechts et al. [14]. Therefore in this setting, the following result can be obtained.

From now on, the following notation is considered. Let $\mathbf{X}$ be a bivariate random vector with distribution function F, Archimedean copula C and with same margins F_{X_i}, $i = 1, 2$. Let us denote $T_i = [X_i|F(\mathbf{X}) = \alpha]$, for $\alpha \in (0, 1)$, $i = 1, 2$.

Theorem 4.1. *Assume that ϕ is twice differentiable and that $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 0$. Then $\boldsymbol{T} := (T_1, T_2) \in MRV$.*

Proof. Firstly, the copula of random vector $\mathbf{T}$ is computed. Note that

$$F(x_1, x_2) = \phi^{-1}(\phi(F_{X_1}(x_1)) + \phi(F_{X_2}(x_2))).$$

For simplicity, the univariate random variable $F(X_1, X_2)$ is denoted by V. Similarly to Theorem 1 in Wang and Oakes [38], we obtain

$$\mathbb{P}[V \le \alpha, X_1 \le x_1, X_2 \le x_2] = \begin{cases} \alpha - \frac{\phi(\alpha)}{\phi'(\alpha)} + \frac{\phi(F(x_1,x_2))}{\phi'(\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2). \end{cases} \tag{16}$$

By straightforward calculation, it can be shown that the distribution function of $\mathbf{T}$ is defined as

$$\begin{aligned} F_{\mathbf{T}}(x_1, x_2) &= \begin{cases} \frac{\mathbb{P}[V=\alpha, X_1 \le x_1, X_2 \le x_2]}{P(V=\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2), \end{cases} \\ &= \begin{cases} 1 - \frac{\phi(F(x_1,x_2))}{\phi(\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2), \end{cases} \end{aligned} \tag{17}$$

where $P(V = \alpha)$ is the density in α of random variable V.

On the other hand, for $i = 1, 2$,

$$F_{T_i}(x_i) = \begin{cases} 1 - \frac{\phi(F_{X_i}(x_i))}{\phi(\alpha)}, & \text{if } \alpha \le F_{X_i}(x_i); \\ 0, & \text{if } \alpha > F_{X_i}(x_i), \end{cases}$$

and

$$F_{T_i}^{-1}(w_i) = \begin{cases} (\phi \circ F_{X_i})^{-1}(\phi(\alpha)(1 - w_i)), & \text{if } 0 < w_i \le 1; \\ 0, & \text{if } w_i = 0. \end{cases}$$

Therefore, the copula of the random vector $\mathbf{T}$ is

$$C_{\mathbf{T}}(u_1, u_2) = F_{\mathbf{T}}(F_{T_1}^{-1}(u_1), F_{T_2}^{-1}(u_2)) = \begin{cases} u_1 + u_2 - 1, & \text{if } u_1 + u_2 \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

It is now shown that $\mathbf{T} \in MRV$ by Theorem 3.2 in Weng and Zhang [40]. Therefore, conditions (C1) and (C2) of Theorem 3.2 in Weng and Zhang [40] are proved. As a result of that $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 0$, we trivially obtain $\overline{F}_{T_1} \in RV_{-\beta}$, $\beta > 0$ (C1).

In addition, since $\mathbf{X}$ has the same margins then,

$$\lim_{t \to \infty} \frac{\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)} = 1,$$

that is, $\overline{F}_{T_1}$ and $\overline{F}_{T_2}$ have equivalent tails. (C2)

Finally, the lower tail dependence function of the survival copula of $\mathbf{T}$,

$$\lambda_2(u_1, u_2) = \lim_{t \to 0^+} \frac{\overline{C}_{\mathbf{T}}(tu_1, tu_2)}{t},$$

is equal to 0. Due to that and considering (C1) and (C2), by Theorem 3.2 in Weng and Zhang [40], $\mathbf{T} \in MRV$.

□

Remark 4.2. *Note that, if $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 1$, by applying Theorem 4.1 and Proposition 1 in Daníelsson et al. [9] for $\boldsymbol{T}$, then the* VaR *of $\boldsymbol{T}$ is subadditive sufficiently deep in the tail regions. In this case, a weak subadditivity of the proposed multivariate lower* CoVaR *is obtained, that is, since* $\mathrm{VaR}_\omega(T_i) = \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$, *then*

$$\mathrm{VaR}_\omega(T_1 + T_2) < \underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) + \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \tag{18}$$

sufficiently deep in tail regions.

Now, an illustration of Remark 4.2 is presented (see Figure 4 and Example 4.6 below).

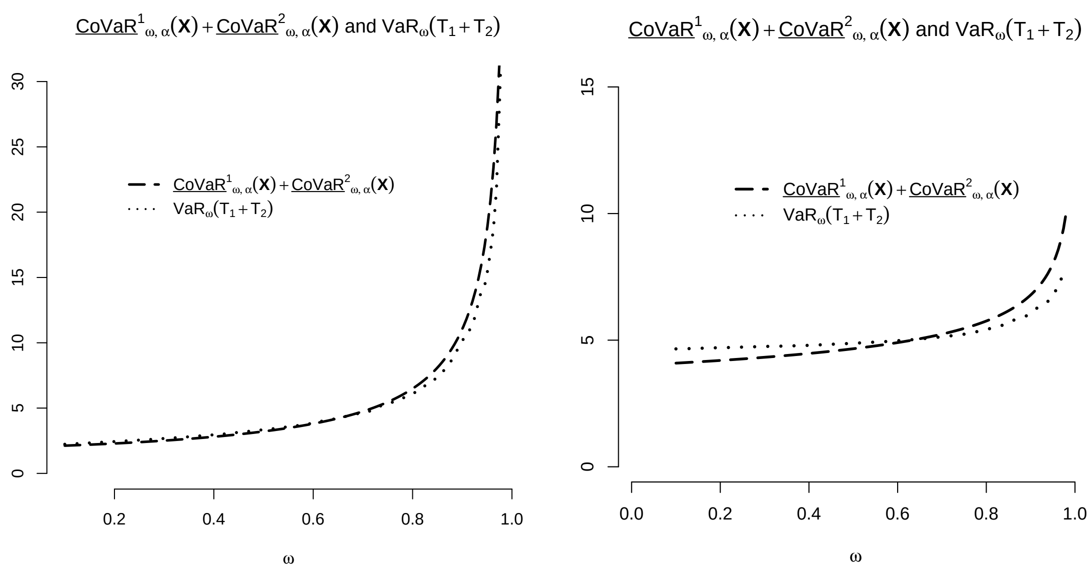

Figure 4: $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) + \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ and $\mathrm{VaR}_\omega(T_1 + T_2)$ for $\mathbf{X}$ with $X_1 \sim X_2 \sim Pareto(2)$ and a Gumbel copula with $\theta = 2$, as in Example 4.6, for $\alpha = \omega$ (left panel) and for $\alpha = 0.75$ (right panel).

Example 4.6. *In this example, a bivariate random vector, $\mathbf{X}$, with $X_1 \sim X_2 \sim Pareto(2)$ and a Gumbel copula, $\theta = 2$, is considered. Analytical expressions of* $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$, *$i = 1, 2$ are* *obtained. In addition,* $\mathrm{VaR}_\omega(T_1+T_2)$ *is calculated by numeric approximation. The obtained results are gathered in Figure 4: for* $\omega=\alpha\in(0,1)$ *(see Figure 4, left) and for* $\alpha=0.75$, $\omega\in(0,1)$ *(see Figure 4, right). It can be easily observed that (18) is verified for large* ω.

5. Estimation

Semiparametric estimators by assuming Archimedean copula for the proposed multivariate CoVaRs are given in this section. Moreover, illustrations with simulated and insurance real data are provided.

Firstly, let assume that $\mathbf{X}$ has an Archimedean copula structure. The generator of an Archimedean copula depends on the dependence parameter θ of the copula (see, e.g., Table 4.1. in Nelsen [31]). Consequently, a semiparametric estimator of the generator is obtained by considering a maximum pseudo-likelihood estimator of the dependence parameter θ associated with this generator. Following these considerations and using Equation (11), we introduce a semiparametric estimator for the multivariate lower CoVaR (see Definition 5.1) by using a semiparametric estimation for θ and the empirical quantile estimation.

Definition 5.1. *Let* $\boldsymbol{X}$ *be a* $d-$*dimensional random vector with Archimedean copula with generator* ϕ_θ *and* $\alpha\in(0,1)$. *A semiparametric estimator of the* $i-$*component of the multivariate lower* CoVaR *is defined as*

$$\underline{\widehat{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})=\widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{F}^{-1}_{X_i}(\phi^{-1}_{\hat{\theta}_n}(S_i\phi_{\hat{\theta}_n}(\alpha)))\right],\quad \text{for } i=1,\dots,d, \tag{19}$$

where $\boldsymbol{\omega}\in[0,1]^d$, S_i *is a random variable with* $Beta(1,d-1)$ *distribution,* $\widehat{\mathrm{VaR}}_\omega(X)$ *is the empirical estimator of* $\mathrm{VaR}_\omega(X)$, $\phi_{\hat{\theta}_n}$ *is the semiparametric estimator of* ϕ_θ *and* $\hat{F}^{-1}_{X_i}$ *is the empirical estimator of* $F^{-1}_{X_i}$ *for* $i=1,\dots,d$.

Secondly, let assume that $\mathbf{X}$ has an Archimedean survival copula structure. From Equation (14), we introduce a semiparametric estimation of multivariate upper CoVaR (see Definition 5.2) using the semiparametric estimation of the generator of the Archimedean survival copula and the empirical estimation of the quantile functions.

Definition 5.2. *Let* $\boldsymbol{X}$ *be a* $d-$*dimensional random vector with Archimedean survival copula with generator* ϕ_θ *and* $\alpha\in(0,1)$. *A semiparametric estimator of the* $i-$*component of the multivariate upper* CoVaR *is defined as*

$$\overline{\widehat{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})=\widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{\overline{F}}^{-1}_{X_i}(\phi^{-1}_{\hat{\theta}_n}(S_i\phi_{\hat{\theta}_n}(1-\alpha)))\right],\quad \text{for } i=1,\dots,d, \tag{20}$$

where $\boldsymbol{\omega}\in[0,1]^d$, S_i *is a random variable with* $Beta(1,d-1)$ *distribution,* $\widehat{\mathrm{VaR}}_\omega(X)$ *is the empirical estimator of* $\mathrm{VaR}_\omega(X)$, $\phi_{\hat{\theta}_n}$ *is the semiparametric estimator of* ϕ_θ *and* $\hat{\overline{F}}^{-1}_{X_i}$ *the empirical estimator of* $\overline{F}^{-1}_{X_i}$ *for* $i=1,\dots,d$.

The estimator of the dependence parameter θ considered in Definitions 5.1 and 5.2 is obtained by a pseudo-likelihood estimation procedure. Genest et al. [17] investigate the properties of the semiparametric estimator for θ and study the efficiency, consistency and asymptotic normality of $\hat{\theta}_n$. Proposition 2.1 in Genest et al. [17] shows that, under regularity conditions, $\hat{\theta}_n$ is consistent and $n^{1/2}(\hat{\theta}_n - \theta)$ converges in distribution to a normal distribution with known variance. The regularity conditions of Proposition 2.1 in Genest et al. [17] are satisfied, among others, by Archimedean copulas. Therefore, since ϕ_θ is a continuous function, $\phi_{\hat{\theta}_n}$ is consistent from Proposition 2.1 in Genest et al. [17]. On the other hand, empirical quantile estimator $\hat{F}_{X_i}^{-1}(p)$ is consistent if quantile $F_{X_i}^{-1}(p)$ is unique (see Serfling [36] page 75). The empirical quantile estimator $\hat{F}_{X_i}^{-1}(p)$ is also asymptotically normal if F_{X_i} possesses left- or right-hand derivative in the point $F_{X_i}^{-1}(p)$ (see Serfling [36] page 77). However, due to Definitions 5.1 and 5.2, CoVaRs estimators are the quantiles of non-independent observations. Consequently, consistency and asymptotic normal properties of these estimators need a supplementary study, by using the above results in Genest et al. [17] and in Serfling [36], which is beyond the scope of this paper. Actually, a new research in this line is now being developed by the authors.

5.1. Simulated data

The aim of this section is to evaluate the performance of the estimators introduced before. In particular, we focus on Definition 5.1 (the multivariate upper CoVaR estimator could similarly be studied). For this purpose, several simulated cases of the bivariate lower CoVaR estimator are studied. Although we restrict ourselves to the bivariate case, these illustrations could be adaptable in any dimension.

In the following, the ratio $\widehat{\underline{\text{CoVaR}}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)/\underline{\text{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$ is considered for different values of α and ω and two different sizes of the sample: $n = 600$ (Figures 5 and 7) and $n = 1000$ (Figures 6 and 8). We generate our simulated data from the following two models: Ali-Mikhail-Haq copula with $\theta = 0.5$ and uniform marginals (Figures 5 and 6), and Gumbel copula with $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2 (Figures 7 and 8, Tables 3, 4 and 5).

We analyse misspecification model error, in order to study the bias and the variance of the estimation when the parametric form of the copula is not appropriate to the data. To this aim we use Clayton, Gumbel and Frank copula in Figures 5 and 6; Joe, Clayton and Frank copula in Figures 7 and 8. Obviously, the true model is included in the boxplot analysis, i.e., Ali-Mikhail-Haq copula in Figures 5 and 6 and Gumbel copula in Figures 7 and 8.

Remark in Figures 5 and 6 that boxplots associated to Ali-Mikhail-Haq, Clayton and Frank copulas are similar in terms of bias and variance. Conversely, the Gumbel boxplot is obviously the worst one. This is clearly related to the domain of attraction (in the upper tails) of these copula structures (asymptotically dependence structure for Gumbel copula, asymptotically independence structure for Ali-Mikhail-Haq, Clayton and Frank copulas, see Remark 5.1).

In Figures 7 and 8 the Gumbel copula is the true (best) model. Joe copula behaves asymptotically similar to Gumbel one. Conversely, Frank and Clayton copulas are clearly different.

Remark 5.1. *Recall that a copula has upper tail dependence if the upper tail dependence parameter λ_U for this copula is in $(0,1]$. If $\lambda_U = 0$, the copula is no upper tail dependence, that is, independent in the tail. Clayton, Frank and Ali-Mikhail-Haq copulas are independent in the tail (i.e., $\lambda_U^{Clayton} = \lambda_U^{Frank} = \lambda_U^{AMH} = 0$). Gumbel and Joe copulas are upper tail dependence copulas (i.e., $\lambda_U^{Gumbel} = \lambda_U^{Joe} = 2 - 2^{1/\theta}$). For more details see Nelsen [31].*

Finally, for both Ali-Mikhail-Haq with uniform marginals and Gumbel copula with Pareto marginals, the larger sample size n is, the better the estimation is.

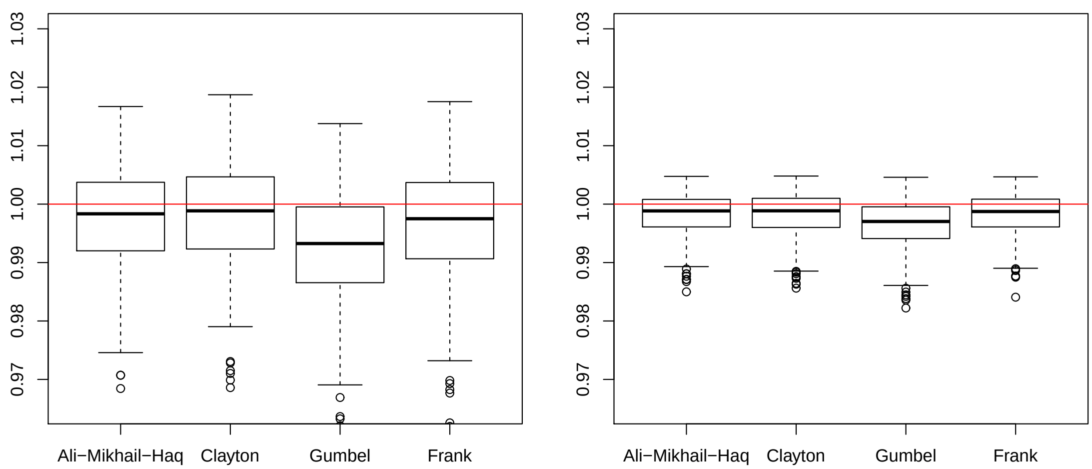

Figure 5: (X,Y) follows a bivariate Ali-Mikhail-Haq copula with parameter $\theta = 0.5$ and uniform marginals. Box plot for the ratio $\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega} / \underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$ for $n = 600$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$, and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 0.9698$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 0.9946$. We take $M = 500$ Montecarlo simulations.

In the following we denote $\overline{\widehat{\underline{\mathrm{CoVaR}}}}_{\alpha,\omega}(X,Y) = \left(\overline{\widehat{\underline{\mathrm{CoVaR}}}}^1_{\alpha,\omega}(X,Y), \overline{\widehat{\underline{\mathrm{CoVaR}}}}^2_{\alpha,\omega}(X,Y)\right)$ the mean (coordinate by coordinate) of $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ on M Montecarlo simulations ($M = 500$ in this section).

From now on, the empirical standard deviation (coordinate by coordinate) is defined as $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$ with

$$\hat{\sigma}_1 = \sqrt{\frac{1}{M-1} \sum_{j=1}^{M} \left(\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega}(X,Y)_j - \overline{\widehat{\underline{\mathrm{CoVaR}}}}^1_{\alpha,\omega}(X,Y) \right)^2}.$$

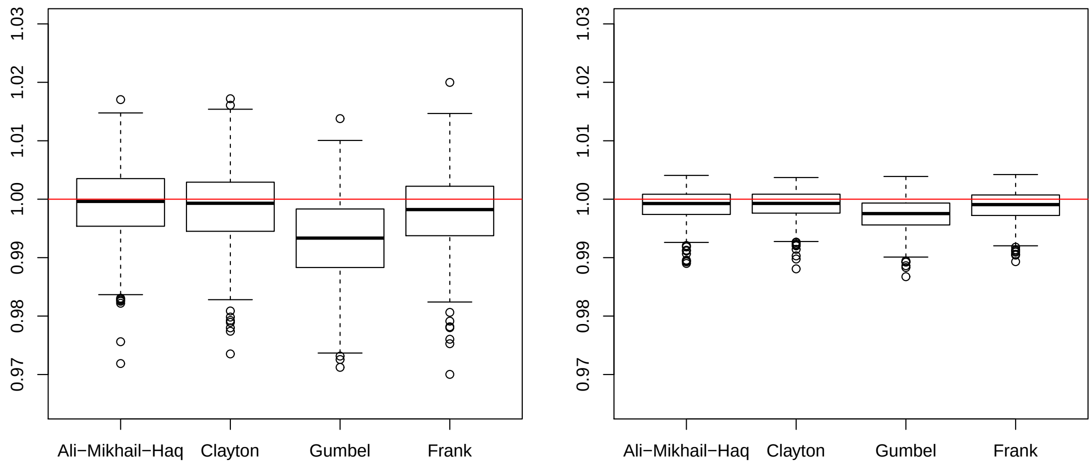

Figure 6: (X,Y) follows a bivariate Ali-Mikhail-Haq copula with parameter $\theta = 0.5$ and uniform marginals. Box plot for the ratio $\underline{\widehat{\text{CoVaR}}}^1_{\alpha,\omega}/\underline{\text{CoVaR}}^1_{\alpha,\omega}$, for $n = 1000$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\text{CoVaR}}^1_{0.75,0.9} = 0.9698$ and $\underline{\text{CoVaR}}^1_{0.9,0.95} = 0.9946$. We take $M = 500$ Montecarlo simulations.

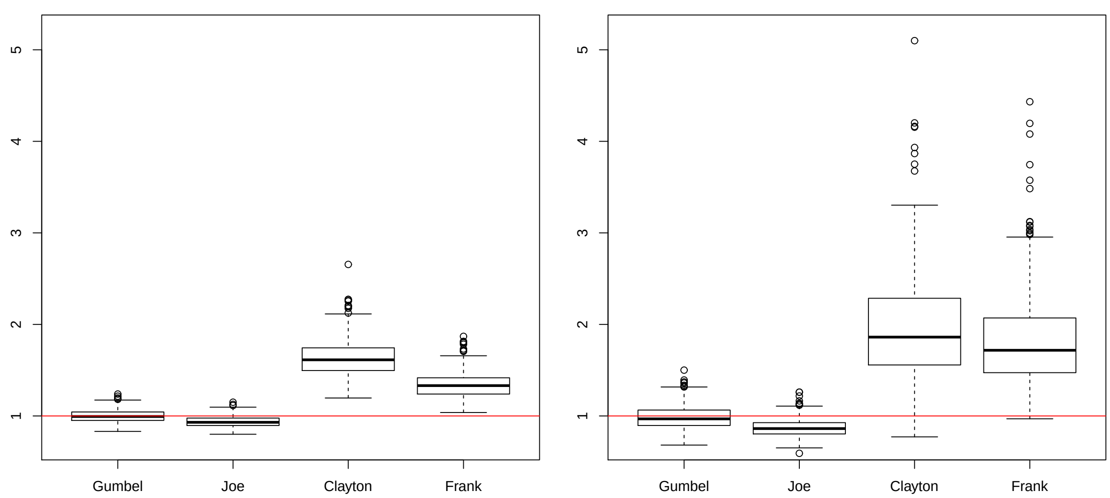

Figure 7: (X,Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Box plot for the ratio $\underline{\widehat{\text{CoVaR}}}^1_{\alpha,\omega}/\underline{\text{CoVaR}}^1_{\alpha,\omega}$ for $n = 600$ with $\alpha = 0.75$ $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\text{CoVaR}}^1_{0.75,0.9} = 3.3911$ and $\underline{\text{CoVaR}}^1_{0.9,0.95} = 6.5535$. We take $M = 500$ Montecarlo simulations.

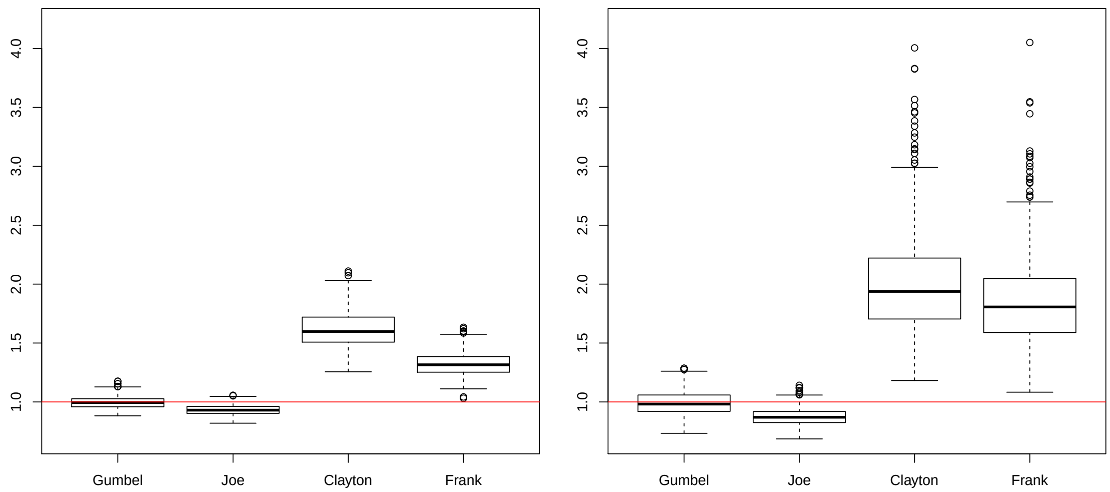

Figure 8: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Box plot for the ratio $\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega}/\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$ for $n = 1000$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 3.3911$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 6.5535$. We take $M = 500$ Montecarlo simulations.

$RMSE = (RMSE_1, RMSE_2)$ corresponds to the relative mean square error (coordinate by coordinate) with

$$RMSE_1 = \sqrt{\frac{1}{M}\sum_{j=1}^{M}\left(\frac{\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega}(X,Y)_j - \underline{\mathrm{CoVaR}}^1_{\alpha,\omega}(X,Y)}{\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}(X,Y)}\right)^2},$$

where M is the number of Montecarlo simulations. Similarly, $RMSE_2$ and $\hat{\sigma}_2$ are defined.

$RMES_1$ and $\hat{\sigma}_1$ in terms of ω (*resp.* α) with $\alpha = 0.7$ fixed (*resp.* $\omega = 0.75$ fixed) for Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2 are shown in Table 3 (*resp.* Table 4). It can be observed that, the more α and ω increase, the more $RMES_1$ and $\hat{\sigma}_1$ increase. Similarly, for Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2, $RMES_1$ and $\hat{\sigma}_1$ in terms of the sample size n for $\alpha = 0.9$ and $\omega = 0.98$ fixed are given in Table 5. As expected, $RMES_1$ and $\hat{\sigma}_1$ decrease when the sample size increases.

5.2. Insurance real data

The estimators of the multivariate CoVaRs measures proposed in Definitions 1 and 2 are now calculated in an insurance real case: **Loss-ALAE data** (in the log scale). Considered data set

ω	Gumbel	Joe	Clayton	Frank
0.70	0.034 (0.081)	0.039 (0.078)	0.253 (0.145)	0.095 (0.098)
0.75	0.037 (0.091)	0.044 (0.084)	0.304 (0.175)	0.116 (0.122)
0.80	0.037 (0.096)	0.049 (0.088)	0.369 (0.210)	0.144 (0.134)
0.86	0.043 (0.122)	0.060 (0.111)	0.491 (0.333)	0.211 (0.207)
0.90	0.046 (0.140)	0.069 (0.122)	0.618 (0.437)	0.280 (0.264)
0.95	0.059 (0.212)	0.097 (0.173)	0.914 (0.864)	0.464 (0.503)
0.98	0.080 (0.360)	0.135 (0.281)	1.394 (2.040)	0.812 (1.248)
0.99	0.106 (0.559)	0.169 (0.415)	1.911 (3.900)	1.188 (2.831)

Table 3: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parenthesis) in terms of ω for $\alpha = 0.7$ fixed. We take 500 Montecarlo simulations.

α	Gumbel	Joe	Clayton	Frank
0.75	0.038 (0.105)	0.049 (0.094)	0.324 (0.219)	0.153 (0.154)
0.80	0.045 (0.138)	0.059 (0.126)	0.338 (0.286)	0.190 (0.212)
0.85	0.051 (0.183)	0.065 (0.161)	0.372 (0.376)	0.251 (0.320)
0.90	0.068 (0.298)	0.079 (0.259)	0.403 (0.591)	0.315 (0.510)
0.95	0.086 (0.538)	0.099 (0.471)	0.428 (1.172)	0.380 (1.094)
0.98	0.149 (1.477)	0.152 (1.315)	0.458 (2.893)	0.442 (2.839)
0.99	0.211 (2.985)	0.197 (2.625)	0.575 (6.076)	0.582 (6.234)

Table 4: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parenthesis) in terms of α for $\omega = 0.75$ fixed. We take 500 Montecarlo simulations.

n	Gumbel	Joe	Clayton	Frank
500	0.196 (1.590)	0.215 (1.097)	1.862 (9.903)	1.636 (9.030)
1000	0.137 (1.124)	0.187 (0.754)	1.732 (7.838)	1.447 (6.210)
1500	0.108 (0.885)	0.173 (0.576)	1.643 (5.754)	1.402 (4.827)
2000	0.103 (0.843)	0.171 (0.560)	1.651 (5.177)	1.388 (4.203)
2500	0.084 (0.684)	0.170 (0.472)	1.622 (4.610)	1.369 (3.918)
3000	0.074 (0.611)	0.163 (0.412)	1.564 (3.932)	1.363 (3.479)
5000	0.059 (0.482)	0.162 (0.360)	1.599 (3.392)	1.356 (2.741)

Table 5: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parenthesis) in terms of the size of the sample n for $\alpha = 0.9$, $\omega = 0.98$ fixed. We take 500 Montecarlo simulations.

contains $n = 1500$ observations. Each claim is composed of an indemnity payment (the loss, X) and an allocated loss adjustment expense (ALAE, Y). ALAE are insurance company expenses like the fees paid to lawyer and other experts to defend the claims. This data set is deeply studied in Frees and Valdez [16].

In Table 6 and 7, the $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X, Y)$ and $\widehat{\overline{\mathrm{CoVaR}}}_{\alpha,\omega}(X, Y)$ for Loss ALAE data are presented by considering different risk levels α, ω and different Archimedean copula models C_θ. Frees and Valdez [16], using the AIC criterion, proposed for Loss-ALAE data a Gumbel copula with parameter $\hat{\theta} = 1.453$. Then, in Table 6, we provide the $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X, Y)$ estimators for

(α, ω)	Clayton (0.51)	Frank (3.07)	Ali-Mikhail-Haq (0.79)	Gumbel (1.453)	Joe (1.64)
(0.75, 0.90)	(12.42, 10.96)	(12.43, 10.95)	(12.48, 11.01)	(**11.92**, **10.61**)	(11.84, 10.53)
(0.90, 0.95)	(13.12, 11.94)	(13.13, 11.99)	(13.13, 11.98)	(**12.95**, **11.50**)	(12.82, 11.27)
(0.95, 0.98)	(13.81, 12.82)	(13.82, 12.78)	(13.81, 12.94)	(**13.56**, **12.17**)	(13.12, 12.07)

Table 6: Coordinates of risk measure $\widehat{\underline{\text{CoVaR}}}_{\alpha,\omega}$ for Loss ALAE data, using different copula structures and risk levels (α, ω). $\hat{\theta}$ for each copula is in parenthesis.

(α, ω)	Clayton (0.78)	Frank (3.07)	Ali-Mikhail-Haq (0.96)	Gumbel (1.37)	Joe (1.39)
(0.75, 0.90)	(10.31, 9.37)	(10.31, 9.38)	(**10.31**, **9.36**)	(10.31, 9.34)	(10.31, 9.33)
(0.90, 0.95)	(11.48, 10.14)	(11.44, 10.13)	(**11.46**, **10.14**)	(11.43, 10.13)	(11.41, 10.13)
(0.95, 0.98)	(12.03, 10.72)	(11.99, 10.69)	(**12.03**, **10.72**)	(12.00, 10.69)	(12.00, 10.70)

Table 7: Coordinates of risk measure $\widehat{\overline{\text{CoVaR}}}_{\alpha,\omega}$ for Loss ALAE data, using different survival copula structures and risk levels (α, ω). $\hat{\theta}$ for each copula is in parenthesis.

Loss-ALAE data using the Gumbel model by Frees and Valdez [16] (bold column). Furthermore, $\widehat{\underline{\text{CoVaR}}}_{\alpha,\omega}(X, Y)$ from other Archimedean models are displayed in Table 6. Estimated parameters θ are obtained using `R` function `fitCopula`. Analogously for the survival structure of Loss-ALAE data the Ali-Mikhail-Haq copula with parameter $\hat{\theta} = 0.96$ is chosen. Hence, the $\widehat{\overline{\text{CoVaR}}}_{\alpha,\omega}(X, Y)$ using Ali-Mikhail-Haq copula (bold column) and some other Archimedean models are gathered in Table 7.

Loss-ALAE data in the log scale, the respective semiparametric estimated α-level sets ($\partial \underline{L}(\alpha)$ and $\partial \overline{L}(\alpha)$), the univariate empirical quantiles of Loss-ALAE data, and the estimators of multivariate CoVaRs measures are displayed in Figure 9. The univariate empirical quantiles of Loss-ALAE data are: $\widehat{\text{VaR}}_{0.75}(X) = 10.46$, $\widehat{\text{VaR}}_{0.9}(X) = 11.51$, $\widehat{\text{VaR}}_{0.95}(X) = 12.05$, $\widehat{\text{VaR}}_{0.75}(Y) = 9.44$, $\widehat{\text{VaR}}_{0.9}(Y) = 10.16$ and $\widehat{\text{VaR}}_{0.95}(Y) = 10.74$. It can be observed that the $\widehat{\underline{\text{CoVaR}}}_{\alpha,\omega}$ (star) is in the level set $\underline{L}(\alpha)$ due to the convexity. By contrast, $\widehat{\overline{\text{CoVaR}}}_{\alpha,\omega}$ can not be in the set $\overline{L}(\alpha)$ since this is a concave set. Furthermore, from Proposition 2.3, $\widehat{\overline{\text{CoVaR}}}^{1}_{\alpha,\boldsymbol{\omega}}(X, Y) \leq \widehat{\text{VaR}}_{\alpha}(X) \leq \widehat{\underline{\text{CoVaR}}}^{1}_{\alpha,\boldsymbol{\omega}}(X, Y)$, and $\widehat{\overline{\text{CoVaR}}}^{2}_{\alpha,\boldsymbol{\omega}}(X, Y) \leq \widehat{\text{VaR}}_{\alpha}(Y) \leq \widehat{\underline{\text{CoVaR}}}^{2}_{\alpha,\boldsymbol{\omega}}(X, Y)$.

Conclusion

In this paper, two multivariate extensions of the classic CoVaR are provided for continuous random vectors. These two risk measures are constructed by using the level set approach used in Embrechts and Puccetti [15], Cousin and Di Bernardino [6] and Cousin and Di Bernardino [7]. Since defined CoVaR are the minimizers of suitable expected losses (see (P7)), then they verify the *elicitability property*, which provides a natural methodology to perform backtesting. Moreover, since the two proposed measures are based on the corresponding quantile functions, they are more robust to extreme values than any other central tendency measures.

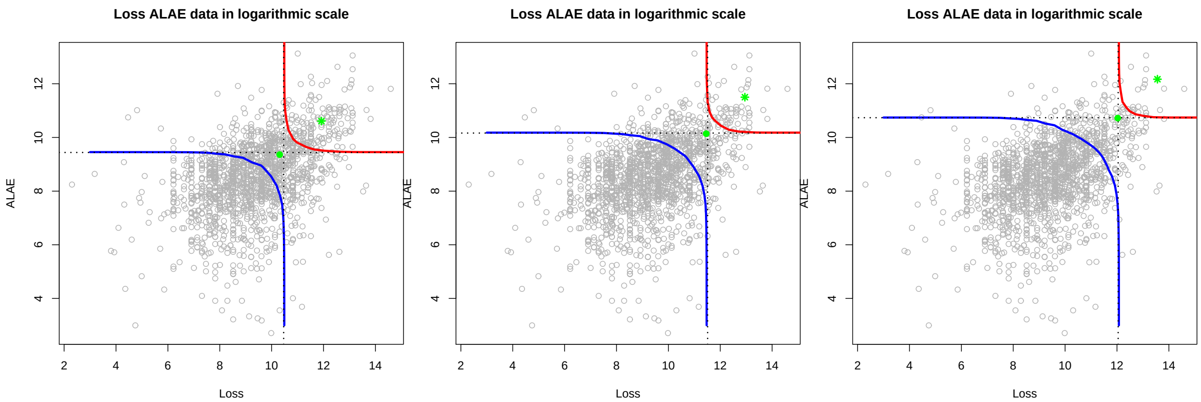

Figure 9: Loss ALAE data in log scale, boundary of estimated level sets ($\partial \underline{L}(\alpha)$, red line), boundary of estimated level sets ($\partial \overline{L}(\alpha)$, blue line), empirical quantile of Loss data (dotted black line), empirical quantile of ALAE data (dotted black line), $\widehat{\underline{\text{CoVaR}}}_{\alpha,\omega}$ (stars) and $\widehat{\overline{\text{CoVaR}}}_{\alpha,\omega}$ (solid circles) with ($\alpha = 0.75$, $\omega = 0.9$) (left panel); ($\alpha = 0.9$, $\omega = 0.95$) (center panel); ($\alpha = 0.95$, $\omega = 0.98$) (right panel).

The positive homogeneity and translation invariance properties are shown for the two proposed multivariate CoVaR. The relations between the univariate VaR and our CoVaR are also analysed as well as the relations between the multivariate VaR proposed by Cousin and Di Bernardino [6] and our multivariate CoVaR. Interestingly, both multivariate CoVaRs coincide with the univariate VaR when a comonotonic random vector is considered, and they verify the additivity property under π-comonotonic conditions. The behaviour of the multivariate CoVaR with respect to the risk level, the usual stochastic order of marginal distributions, and the dependence structure are studied. Unsurprisingly, the effect in the multivariate lower CoVaR (*resp.* upper CoVaR) with respect to a change in the risk level, a change in the dependence structure, or the usual stochastic order of marginal distributions, tends to be the same as for the multivariate lower VaR (*resp.* upper VaR) proposed in Cousin and Di Bernardino [6]. Important results and analytical expressions for our multivariate risk measures are obtained for random vectors with Archimedean copulas. In particular, certain subadditivity inequality is presented in the Archimedean case under regular variation conditions. Moreover, under Archimedean copula condition, estimators of the two proposed multivariate CoVaRs are provided in simulated data and insurance real data.

In a future perspective, quantile regression estimations in extreme theory for the two multivariate CoVaRs can be studied by adapting the works by Di Bernardino et al. [12] and by Daouia et al. [10]. Another approach could involve the evaluation of the proposed measures in certain multidimensional portfolios and the comparison between the results for these measures and the results for multivariate existent measures (see Cousin and Di Bernardino [6], Cousin and Di Bernardino [7] and Cai and Li [5]).

Acknowledgements The authors wish to thank the Editor and the two anonymous referees whose comments helped to improve a previous version of this paper. This work was partly supported by a grant from Junta de Andalucía (Spain) for research group (FQM- 328) and by a pre-doctoral contract (Palacios Rodríguez, F.) from the "V Plan Propio de Investigación" of the University of Seville.

References

[1] Adrian, T. and Brunnermeier, M. K. (2011). CoVaR. *NBER Working Papers 17454, National Bureau of Economic Research, Inc.*

[2] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. *Math. Finance*, 9(3):203–228.

[3] Barbe, P., Genest, C., Ghoudi, K., and Rémillard, B. (1996). On Kendall's process. *J. Multivariate Anal.*, 58(2):197–229.

[4] Bellini, F. and Bignozzi, V. (2013). Elicitable risk measures. *Available at SSRN 2334746.*

[5] Cai, J. and Li, H. (2005). Conditional tail expectations for multivariate phase-type distributions. *Journal of Applied Probability*, 42(3):810–825.

[6] Cousin, A. and Di Bernardino, E. (2013). On multivariate extensions of Value-at-Risk. *Journal of Multivariate Analysis*, 119:32 – 46.

[7] Cousin, A. and Di Bernardino, E. (2014). On multivariate extensions of Conditional-Tail-Expectation. *Insurance: Mathematics and Economics*, 55:272 – 282.

[8] Cuevas, A., González-Manteiga, W., and Rodríguez-Casal, A. (2006). Plug-in estimation of general level sets. *Australian & New Zealand Journal of Statistics*, 48(1):7–19.

[9] Daníelsson, J., Jorgensen, B., Samorodnitsky, G., Sarma, M., and De Vries, C. (2013). Fats tails, Var and subadditivity. *Journal of Econometrics*, 172:283–291.

[10] Daouia, A., Gardes, L., and Girard, S. (2013). On kernel smoothing for extremal quantile regression. *Bernouilli.*, 19(5B):2557.

[11] Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial Theory for Dependence Risks*. Wiley.

[12] Di Bernardino, E., Laloë, T., Maume-Deschamps, V., and Prieur, C. (2011). Plug-in estimation of level sets in a non-compact setting with applications in multivariable risk theory. *ESAIM: Probability and Statistics, http://dx.doi.org/10.1051/ps/2011161.*

[13] Embrechts, P. and Hofert, M. (2013). *Statistics and Quantitative Risk Management for Banking and Insurance*. ETH Zurich.

[14] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events for insurance and finance.* Springer. Berlin.

[15] Embrechts, P. and Puccetti, G. (2006). Bounds for functions of multivariate risks. *Journal of Multivariate Analysis*, 97(2):526–547.

[16] Frees, E. and Valdez, E. (1998). Understanding relationships using copulas. *North Amer. Actuar. J.*, 2:1–25.

[17] Genest, C., Ghoudi, K., and Rivest, L. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82(3):543–552.

[18] Genest, C. and Rivest, L.-P. (2001). On the multivariate probability integral transformation. *Statist. Probab. Lett.*, 53(4):391–399.

[19] Girardi, G. and Ergün, A. (2013). Systemic risk measurement: Multivariate GARCH estimation of CoVaR. *Journal of Banking and Finance.*, 37:3169.

[20] Gneiting, T. (2011). Making an evaluationg point forescasts. *Journal of American Statistical Association.*

[21] Goodhart, C. and Segoviano, M. (2009). Banking stability measures. *IMF Working Papers.*, pages 1 – 54.

[22] Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman & Hall, London.

[23] Jouini, E., Meddeb, M., and Touzi, N. (2004). Vector-valued coherent risk measures. *Finance Stoch.*, 8(4):531–552.

[24] Mainik, G. and Schaanning, E. (2014). On dependence consistency of CoVaR and some other systemic risk measures. *Statistics & Risk Modeling.*, 31(1):49–77.

[25] McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools.* Princeton Series in Finance. Princeton University Press, New Jersey.

[26] McNeil, A. J. and Nešlehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and l1-norm symmetric distributions. *The Annals of Statistics*, 37:3059–3097.

[27] Muñoz Pérez, J. and Sánchez-Gómez, A. (1990). A characterization of the distribution function: The dispersion function. *Statistics & Probability Letters*, 10(3):235 – 239.

[28] Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance Math. Econom.*, 21(3):219–223.

[29] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks.* Wiley Series in Probability and Statistics. John Wiley & Sons Inc.

[30] Nappo, G. and Spizzichino, F. (2009). Kendall distributions and level sets in bivariate exchangeable survival models. *Information Sciences*, 179:2878.

[31] Nelsen, R. (2006). *An Introduction to Copulas.* Springer Series in Statistics. Springer.

[32] Prékopa, A. (2012). Multivariate Value at Risk and related topics. *Annals of Operation Research*, 193:49.

[33] Puccetti, G. and Scarsini, M. (2010). Multivariate comonotonicity. *Journal of Multivariate Analysis*, 101(1):291 – 304.

[34] Resnick, S. I. (2007). *Heavy-Tail Phenomena. Probabilistic and Statistical Modeling.* Springer Series in Operations Research and Financial Engineering.

[35] Resnick, S. I. (2008). *Extreme Values, Regular Variation and Point Processes.* Springer. New York.

[36] Serfling, R. (1980). *Approximation Theorems of Mathematical Statistics.* John Wiley & Sons, New York.

[37] Shaked, M. and Shanthikumar, J. (2007). *Stochastic Orders.* Springer Series in Statistics. Physica-Verlag.

[38] Wang, A. and Oakes, D. (2008). Some properties of the Kendall distribution in bivariate Archimedean copula models under censoring. *Statistics & Probability Letters*, 78(16):2578 – 2583.

[39] Wei, G. and Hu, T. (2002). Supermodular dependence ordering on a class of multivariate copulas. *Statist. Probab. Lett.*, 57(4):375–385.

[40] Weng, C. and Zhang, Y. (2012). Characterization of multivariate heavy-tailed distribution families. *Journal of Multivariate Analysis*.

[41] Ziegel, J. (2014). Coherence and elicitability. *arXiv:1303.1690v3*.