Supporting Information for the estimation of extreme quantiles conditioning on multivariate critical layers

E. Di Bernardino* and F. Palacios-Rodríguez†‡

1 Proofs

Proof of Proposition 2.1 in the main document

We first prove item i). Let $F_i(\cdot|\alpha)$ as in Lemma 2.1 below. Since, by assumptions, $\phi \in RV_\rho(1)$, $\phi' \in RV_{\rho-1}(1)$ and F_i verifies the von Mises condition with index γ_i (see Definition 2.1 below),

*Département IMATH, Laboratoire Cédric EA4629, CNAM, Paris, France.

†Departamento de Estadística e Investigación Operativa, Facultad de Matemáticas, Universidad de Sevilla, Sevilla, España.

‡*Correspondence to: Ms. F. Palacios-Rodríguez, Departamento de Estadística e Investigación Operativa, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia sin número, Sevilla, 41012, España. E-mail: fpalacios2@us.es.*

then,

$$\lim_{x\uparrow x_{F_i}(\alpha)} \frac{(1-F_i(x|\alpha))F_i''(x|\alpha)}{(F_i')^2(x|\alpha)} = \lim_{z\uparrow 1}\frac{d-2}{d-1}\left[\left(1-\frac{\phi(z)}{\phi(\alpha)}\right)^{-(d-1)}-1\right]$$

$$+ \frac{\phi(\alpha)\left[(-\rho+1)-(\gamma_i+1)\right]}{(d-1)\phi'(z)(1-z)}\left[-\left(1-\frac{\phi(z)}{\phi(\alpha)}\right)^{2-d}+1\right]$$

$$- \left(-\frac{1}{\rho}\right)\frac{1}{d-1}\left[(-\rho+1)-(\gamma_i+1)\right].$$

Since $\phi(1)=0$, the first summand approaches 0 when z approaches 1. We denote $C = \left(-\frac{1}{\rho}\right)\frac{[(-\rho+1)-(\gamma_i+1)]}{d-1}$. For the second summand it is verified that:

$$\lim_{z\uparrow 1}\frac{\phi(\alpha)C}{\phi(z)}\left[-\left(1-\frac{\phi(z)}{\phi(\alpha)}\right)^{2-d}+1\right] = \frac{2\rho+2\gamma_i-d\rho-d\gamma_i}{\rho(d-1)}.$$

Hence,

$$\lim_{x\uparrow x_{F_i}(\alpha)} \frac{(1-F_i(x|\alpha))F_i''(x|\alpha)}{(F_i')^2(x|\alpha)} = -\left(\frac{\gamma_i}{\rho}+1\right).$$

The random variable T_i therefore verifies the von Mises condition with $\gamma^{T_i} = \frac{\gamma_i}{\rho}$. Similar to the proof of item *i*), the von Mises condition for T_i when $\rho=+\infty$ is satisfied with $\gamma^{T_i}=0$. Therefore item *ii*) is also proved. From Theorem 1.1.8 in de Haan and Ferreira (2006), others assertions of Proposition 2.1 in the main document are shown directly. □

Proof of Proposition 2.2 in the main document

For item *i*), since $k(n)/n \to 0$, as $n\to\infty$, and $\phi^{-1}(0)=1$, $k_U/n \to 0$ holds, as $n\to\infty$. Furthermore, we have the following asymptotic approximation

$$k_U(n) \sim n\left(\phi(\alpha)(d-1)\right)^{1/\rho}\left(\frac{k(n)}{n}\right)^{1/\rho}, \quad (1)$$

as $n \to +\infty$. From Equation (1), it holds that $k(n)/k_U(n) \to 0$, as $n \to \infty$, for $\rho \in (1, +\infty]$. Then, $k_U(n) \to +\infty$ as $n \to +\infty$.

Since $U_{X_i}(t) = F_i^{\leftarrow}(1 - 1/t)$ and using Lemma 2.1 below,

$$U_{T_i}\left(\frac{n}{k(n)}\right) = U_{X_i}\left(\frac{1}{1 - \phi^{-1}\left[\left(1 - (1 - k(n)/n)^{1/(d-1)}\right)\phi(\alpha)\right]}\right).$$

Therefore,

$$U_{T_i}\left(\frac{n}{k(n)}\right) = U_{X_i}\left(\frac{n}{k_U(n)}\right), \text{ where } k_U(n) = n\left\{1 - \phi^{-1}\left[\left(1 - \left(1 - \frac{k(n)}{n}\right)^{1/(d-1)}\right)\phi(\alpha)\right]\right\}.$$

Therefore item *ii)* of Proposition 2.2 in the main document is also proved. □

Proof of Theorem 3.1 in the main document

Firstly, we provide a normality result for the ratio $\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}}$. Since U_{X_i} satisfies Assumption 3.1 in the main document with $\gamma_i > 0$ and $\tau_i < 0$, from Theorem 3.2.5 in de Haan and Ferreira (2006) and Slutsky's Theorem (e.g., see Serfling (1980)), it is verified that

$$\sqrt{k_1}\left(\frac{\widehat{\gamma}_i}{\gamma_i} - 1\right) \xrightarrow{d} N\left(\mu/\gamma_i, 1\right), \tag{2}$$

with $\mu = \lambda/(1-\tau_i)$ and $\lim_{n\to\infty} \sqrt{k_1(n)} A_i(n/k_1(n)) = \lambda < +\infty$. Since the distribution function of the random vector $(X_1, \ldots, X_d)$ is given by a $d-$dimensional Archimedean copula C_ϕ with generator ϕ, then the distribution function of every bivariate subvector (X_i, X_j), $i \neq j$, is given by the same bivariate Archimedean copula. In addition, since under the assumptions of Theorem 3.1 in the main document, conditions of Corollary 2.1 below are satisfied, by using the Delta Method, it is verified that

$$\sqrt{k_2}\left(\frac{\rho}{\widehat{\rho}} - 1\right) \xrightarrow{d} N\left(0, \sigma^2/\rho^2\right), \tag{3}$$

with σ^2 provided by

$$\sigma^2 := \sigma_U^2 \left(\frac{\log(2)}{(2-\lambda_U)\log^2(2-\lambda_U)} \right)^2, \tag{4}$$

with $\lambda_U := \Lambda(1,1)$ the upper tail dependence coefficient and

$$\sigma_U^2 := \lambda_U + \left(\frac{\partial}{\partial x}\Lambda_U(1,1)\right)^2 + \left(\frac{\partial}{\partial y}\Lambda_U(1,1)\right)^2 + 2\lambda_U \left(\left(\frac{\partial}{\partial x}\Lambda_U(1,1) - 1\right)\left(\frac{\partial}{\partial y}\Lambda_U(1,1) - 1\right) - 1\right).$$

We now write

$$\frac{\widehat{\gamma}_i^T}{\gamma_i^T} = \frac{\widehat{\gamma}_i}{\gamma_i} \times \frac{\rho}{\widehat{\rho}} =: M_1 \times M_2$$

and we deal with the two factors separately.

- From (2), $M_1 = \frac{\Theta_1}{\sqrt{k_1}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_1}}\right) + 1$, with $\Theta_1 \sim N(\mu/\gamma_i, 1)$.

- From (3), $M_2 = \frac{\Theta_2}{\sqrt{k_2}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_2}}\right) + 1$, with $\Theta_2 \sim N\left(0, \sigma^2/\rho^2\right)$.

Hence,

$$\left(\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} - 1\right) = M_1 \times M_2 - 1 = \frac{\Theta_1}{\sqrt{k_1}} + \frac{\Theta_2}{\sqrt{k_2}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_1}}\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_2}}\right).$$

Let $r = \lim_{t\to+\infty} \frac{\sqrt{k_1(n)}}{\sqrt{k_2(n)}} \in [0,\infty]$ and $\gamma^{T_i} := \frac{\gamma_i}{\rho}$. Then, as $n \to \infty$, we get

$$\min(\sqrt{k_1}, \sqrt{k_2}) \left(\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} - 1\right) \xrightarrow{d} \begin{cases} \Theta_1 + r\Theta_2, & \text{if } r \le 1, \\ \frac{1}{r}\Theta_1 + \Theta_2, & \text{if } r > 1. \end{cases} \tag{5}$$

We now write

$$\frac{\widehat{x}_{p_n}^i}{x_{p_n}^i} = \frac{X_{n-\lfloor k_U \rfloor, n}^i}{U_{X_i}\left(\frac{n}{k_U}\right)} \times \left(\frac{k}{n\, p_n}\right)^{\widehat{\gamma}^{T_i} - \gamma^{T_i}} =: N_1 \times N_2.$$

From Theorem 2.1 below,

$$N_1 \xrightarrow{d} \frac{B}{\sqrt{k_U}} + 1 + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right), \quad \text{where} \quad B \sim N(0, \gamma_i^2). \tag{6}$$

By using the normality for the ratio $\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}}$ in Equation (5), we can get

$$\frac{\min(\sqrt{k_1}, \sqrt{k_2})}{\log(d_n)} \left(d_n^{\widehat{\gamma}^{T_i} - \gamma^{T_i}} - 1 \right) \xrightarrow{d} \begin{cases} \Theta_1 + r\Theta_2, & \text{if } r \leq 1, \\ \frac{1}{r}\Theta_1 + \Theta_2, & \text{if } r > 1, \end{cases}$$

where $d_n = \frac{k}{np_n}$. The interested reader is also referred to the proof of Theorem 4.3.8 in de Haan and Ferreira (2006). Consequently,

$$N_2 \xrightarrow{d} \begin{cases} \frac{\log(d_n)}{\sqrt{k_1}} \left(\Theta_1 + r\Theta_2\right) + 1 + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_1}}\right), & \text{if } r \leq 1, \\ \frac{\log(d_n)}{\sqrt{k_2}} \left(\frac{1}{r}\Theta_1 + \Theta_2\right) + 1 + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_2}}\right), & \text{if } r > 1. \end{cases} \tag{7}$$

Combining the asymptotic relations in (6) and (7), if $r \leq 1$, we have

$$\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 = \frac{B}{\sqrt{k_U}} + \frac{\log(d_n)}{\sqrt{k_1}} \left(\Theta_1 + r\Theta_2\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right) + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_1}}\right). \tag{8}$$

Similarly, if $r > 1$, then

$$\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 = \frac{B}{\sqrt{k_U}} + \frac{\log(d_n)}{\sqrt{k_2}} \left(\frac{1}{r}\Theta_1 + \Theta_2\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right) + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_2}}\right). \tag{9}$$

Hence, Theorem 3.1 in the main document comes down from Equations (8) and (9). □

2 Auxiliary results

In this section, some brief reminders and auxiliary results involved in the main document are described. These are only intended to outline some notation and references, and to help in the proofs developed in Section 1.

2.1 The von Mises condition and the distribution of T_i

Definition 2.1 and Lemma 2.1 introduced below are crucial in the proof of Propositions 2.1 and 2.2 in the main document. In particular, Lemma 2.1 can be obtained by adapting Lemma 3.4 in Brechmann (2014) in the case of $j = 1$.

Definition 2.1 (*the von Mises condition*)**.** *Let F be a distribution function and x^* its right endpoint. Let F' and F'' be the first and the second derivatives of F, respectively. Suppose $F''(x)$ exists and $F'(x)$ is positive for all x in some left neighborhood of x^*. The von Mises condition for F holds if*

$$\lim_{t \uparrow x^*} \frac{(1 - F(t))F''(t)}{(F'(t))^2} = -\gamma - 1.$$

Under the *von Mises condition* in Definition 2.1, the maximum domain of attraction (MDA) of the distribution function F can be determined by using the tail parameter γ (e.g., see Theorem 1.1.8 in de Haan and Ferreira (2006)).

Lemma 2.1. *Let $(X_1, \ldots, X_d)$ be a random vector which follows an Archimedean copula C_ϕ with generator ϕ. Let $F_i(x|\alpha) = \mathbb{P}[X_i \leq x | \boldsymbol{X} \in \partial L(\alpha)]$. Therefore, for $i = 1, \ldots, d$,*

$$F_i(x|\alpha) = \begin{cases} \left(1 - \frac{\phi(F_i(x))}{\phi(\alpha)}\right)^{d-1}, & \text{if } x > Q_i(\alpha); \\ 0, & \text{if } x \leq Q_i(\alpha), \end{cases}$$

where F_i is the marginal distribution of X_i and $Q_i(\alpha)$ is the associated quantile function at level $\alpha \in (0, 1)$.

Tail index and the von Mises condition for T_i In the following, certain tail indexes for T_i are derived. The ρ indexes for the classic bivariate Archimedean copulas are collected in Table 1 in Charpentier and Segers (2009). From this table and from Proposition 2.1 in the main document, Table 1 below is constructed. Table 1 (left panel) contains the tail index γ^{T_i} of T_i when X_i is in the Weibull domain (i.e., $\gamma_i < 0$), Gumbel domain (i.e., $\gamma_i = 0$) and Fréchet domain (i.e., $\gamma_i > 0$), for different values of ρ. In Table 1 (right panel), some specific models are considered.

[Table 1 about here.]

2.2 Estimation of the regularly varying index ρ

We now deal with the estimation of the regularly varying index ρ of the Archimedean generator ϕ. The corollary obtained below constitutes a major auxiliary result in the proof of Theorem 3.1 in the main document. In this paper, we use an estimator of ρ derived by the estimator of the *upper tail dependence coefficient* proposed by Schmidt and Stadtmüller (2006). This procedure is recalled in this section.

Let G be a $d-$dimensional distribution function with margin distributions G_i, $i = 1, \ldots, d$. If, for the subsets $I,\ J \in \{1, \ldots, d\}$, $I \cap J = \emptyset$, the following limit exists everywhere on $\overline{\mathbb{R}}^d_+ = [0, \infty]^d \setminus (\infty, \ldots, \infty)$

$$\Lambda_U^{I,J}(\mathbf{x}) := \lim_{t\to\infty} P\left[X_i > G_i^{-1}(1 - x_i/t),\ \forall\, i \in I \,|\, X_j > G_j^{-1}(1 - x_j/t),\ \forall\, j \in J\right],$$

then the function $\Lambda_U^{I,J} : \overline{\mathbb{R}}^d_+ \to \mathbb{R}$ is called an *upper tail copula* associated with G w.r. to I, J.

Let (X_i, X_j), $i \neq j$, be a bivariate random vector with distribution functions G_i and G_j. It is said to be *upper tail dependent* if $\Lambda_U(1,1)$ exists and

$$\lambda_U := \Lambda_U(1,1) = \lim_{t \to 1^-} P[X_i > G_i^{-1}(t) \,|\, X_j > G_j^{-1}(t)] > 0.$$

Conversely, if $\lambda_U = 0$, (X_i, X_j) is called *upper tail independent*. Further, λ_U is referred to as the *upper tail dependence coefficient*. If the distribution function of a random vector $(X_1, \ldots, X_d)$ is given by a $d-$dimensional Archimedean copula C_ϕ with generator ϕ, then the distribution function of every bivariate subvector (X_i, X_j) is given by the bivariate Archimedean copula with the same generator. As a consequence, to estimate ρ, we focus on the bivariate subvector (X_i, X_j). Furthermore, under our assumption, one can prove that $\lambda_U = 2 - 2^{1/\rho}$ (e.g., see Corollary 2.1. in Di Bernardino and Rullière (2014)). Therefore, we can consider the estimator $\widehat{\rho} := \frac{\log(2)}{\log(2-\widehat{\lambda}_U)}$. In this paper, we use the nonparametric rank-based estimator of λ_U proposed by Schmidt and Stadtmüller (2006). Assume that (X_i, X_j), $(X_i^{(1)}, X_j^{(1)})$, $\ldots$, $(X_i^{(n)}, X_j^{(n)})$, $i \neq j$, are *i.i.d.* bivariate random vectors with distribution function G having marginal distribution functions G_i and G_j. The estimator of λ_U in Schmidt and Stadtmüller (2006) is given by $\widehat{\lambda}_U = \widehat{\Lambda}_{U,n}(1,1)$, where

$$\widehat{\Lambda}_{U,n}(x,y) := \tfrac{1}{k_2} \textstyle\sum_{w=1}^n 1_{\{R_i^{(w)} > n - k_2\, x \text{ and } R_j^{(w)} > n - k_2\, y\}},$$

with $k_2 = k_2(n) \to \infty$, $k_2/n \to 0$, as $n \to \infty$ and $R_i^{(w)} = \sum_{h=1}^n 1_{\{X_i^{(h)} \leq X_i^{(w)}\}}$ (*resp.* $R_j^{(w)} = \sum_{h=1}^n 1_{\{X_j^{(h)} \leq X_j^{(w)}\}}$) is the rank of $X_i^{(w)}$ in $X_i^{(1)}, \ldots, X_i^{(n)}$ (*resp.* is the rank of $X_j^{(w)}$ in $X_j^{(1)}, \ldots, X_j^{(n)}$), for $w = 1, \ldots, n$.

Under a second-order condition for the bivariate *upper tail copula* $\Lambda_U(x,y)$ (see condition in

Equation (10)), we can obtain an asymptotic normality result for the estimator $\widehat{\rho}$ (see Corollary 2.1 below). The proof of Corollary 2.1 follows from Corollary 2 in Schmidt and Stadtmüller (2006) and the Delta Method technique.

Corollary 2.1 (Asymptotic normality of $\widehat{\rho}$)**.** *Let G be a bivariate distribution function of (X_i, X_j) with continuous marginal distribution functions G_i and G_j. Let C_ϕ be the copula of (X_i, X_j) with generator $\phi \in RV_\rho(1)$, with $\rho \in (1, +\infty)$. Let $k_2 = k_2(n) \to \infty$ and $k_2/n \to 0$ as $n \to \infty$. Assume that the bivariate upper tail copula $\Lambda_U(x, y)$ exists and has continuous partial derivatives. Furthermore, let $A_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be an auxiliary function such that $A_\rho(t) \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{\Lambda_U(x, y) - t\,\overline{C}(x/t,\, y/t)}{A_\rho(t)} = g(x,\, y) < \infty, \tag{10}$$

locally uniformly for $(x, y)^2 \in \overline{\mathbb{R}}_+^2$ for some nonconstant function g, where $\overline{C}$ represents the survival copula. Therefore, if $\sqrt{k_2} A_\rho(n/k_2) \to 0$ as $n \to \infty$, then

$$\sqrt{k}_2(\widehat{\rho} - \rho) \xrightarrow{d} N\left(0, \sigma^2\right),$$

where $N\left(0, \sigma^2\right)$ is a centred normal-distributed random variable with $\sigma^2 = \sigma_U^2 \left(\frac{\log(2)}{(2-\lambda_U)\log^2(2-\lambda_U)}\right)^2$ and $\sigma_U^2 = \lambda_U + \left(\frac{\partial}{\partial x}\Lambda_U(1,1)\right)^2 + \left(\frac{\partial}{\partial y}\Lambda_U(1,1)\right)^2 + 2\lambda_U\left(\left(\frac{\partial}{\partial x}\Lambda_U(1,1) - 1\right)\left(\frac{\partial}{\partial y}\Lambda_U(1,1) - 1\right) - 1\right)$.

Note that the asymptotic variance in Corollary 2.1, vanishes in the asymptotically independent case. Therefore, in the case $\Lambda_U = 0$, it is verified that $\widehat{\lambda}_U \xrightarrow{\mathbb{P}} 0$ (for more details see Theorem A.1. and Corollary A.1. in Di Bernardino et al. (2013)). Consequently, $\frac{\widehat{\rho}}{\rho} \xrightarrow{\mathbb{P}} 1$.

Second-order condition for the bivariate *upper tail copula* $\Lambda_U(x,y)$ In Table 2, the second-order condition for the bivariate *upper tail copula* $\Lambda_U(x,y)$ in Equation (10) is illustrated for some classic Archimedean copula models with $\Lambda_U(x,y) = x + y - (x^\theta + y^\theta)^{1/\theta}$. We consider the Gumbel copula, Joe copula, and Copulas (12), (14), (15) and (21) in Table 4.1 in Nelsen (2006). Observe that the property in Equation (10) is not verified for Copula (2) in Table 4.1 in Nelsen (2006).

[Table 2 about here.]

2.3 Intermediate Order Statistics

In the following, we adapt in our setting the well-known Central Limit Theorem for the intermediate order statistics. This result follows easily from Theorems 2.4.1 and 2.4.2 in de Haan and Ferreira (2006). Further details are given in Theorem 2.1 in Drees (1998). Theorem 2.1 below is crucial in the proof of our main result (see Theorem 3.1 in the main document).

Theorem 2.1 (Theorem 2.1 in Drees (1998))**.** *Let $(X_1, \dots, X_d)$ be a random vector with Archimedean copula C_ϕ with twice differentiable generator ϕ. Assume that $\phi \in RV_\rho(1)$, with $\rho \in [1, +\infty]$. Let $i \in \{1, \dots, d\}$. Assume that U_{X_i} satisfies Assumption 3.1 in the main document with auxiliary function $A_i(\cdot)$, $\gamma_i > 0$ and $\tau_i < 0$. Let $k = k(n) \to \infty$, $k/n \to 0$, $n \to \infty$ such that $\lim_{n\to\infty} \sqrt{k_U}\, A_i(n/k_U)$ exists and is finite with the sequence k_U defined by*

$k_U(n) := n\left\{1-\phi^{-1}\left[\left(1-\left(1-\frac{k(n)}{n}\right)^{1/(d-1)}\right)\phi(\alpha)\right]\right\}$. *Then, it holds that, as* $n\to\infty$,

$$\sqrt{k_U(n)}\left(\frac{X^i_{n-\lfloor k_U\rfloor,n}}{U_{X_i}\left(\frac{n}{k_U(n)}\right)}-1\right)\xrightarrow{d}\gamma_i\,N(0,1).$$

Proof of Theorem 2.1

From Proposition 2.2 in the main document, it is verified that $U_{T_i}\left(\frac{n}{k}\right)=U_{X_i}\left(\frac{n}{k_U}\right)$, and $k_U(n)\to\infty$, $k_U/n\to 0$ as $n\to\infty$. Since, by assumptions, U_{X_i} satisfies Assumption 3.1 in the main document with auxiliary function $A_i(\cdot)$, $\gamma_i>0$, $\tau_i<0$ and $\sqrt{k_U}A_i(n/k_U)\to\lambda'<\infty$, as $n\to\infty$, then from Theorem 2.4.2 in de Haan and Ferreira (2006), the result is attained. □

2.4 Adaptive version of the estimator $\widehat{x}^i_{p_n}$

The intermediate sequence $k_U(n)$ in Proposition 2.2 and Theorem 3.1 in the main document is an unknown sequence which depends on the generator of the considered Archimedean copula. In this section, a plug-in procedure based on the estimation of k_U is presented. This can be seen as an adaptive version of the results of Section 3 in the main document. For this purpose, the notion of self-nested diagonal of a copula and the associated nonparametric estimator proposed by Di Bernardino and Rullière (2013) are recalled in the following.

Recall that the diagonal section of a $d-$dimensional copula C is given by $\delta_1(u)=C(u,\dots,u)$, $u\in[0,1]$, and δ_{-1} is the inverse function of δ_1, such that $\delta_1\circ\delta_{-1}$ is the identity function. From Lemma 3.4 in Di Bernardino and Rullière (2013), one can write the family of self-nested diagonals of an Archimedean copula C_ϕ at each order $r\in\mathbb{R}$ as: $\delta_r(u)=\phi^{-1}(d^r\phi(u))$, for $u\in(0,1)$, $r\in\mathbb{R}$.

Di Bernardino and Rullière (2013) introduce the following estimation of a self-nested diagonal δ_r, by using an interpolation procedure (see also Lemma 3.6 in the aforementioned paper).

Definition 2.2 (Definition 4.2 in Di Bernardino and Rullière (2013))**.** *Let $\widehat{\delta}_1$ be an estimator of δ_1, and $\widehat{\delta}_{-1}$ be an estimator of the inverse function δ_{-1}. Estimators of δ_h and δ_{-h} can be obtained for any $h \in \mathbb{N}\backslash\{0\}$ by setting*

$$\begin{cases} \widehat{\delta}_h(u) & = & \widehat{\delta}_1 \circ \ldots \circ \widehat{\delta}_1(u), & (h \text{ times}) \\ \widehat{\delta}_{-h}(u) & = & \widehat{\delta}_{-1} \circ \ldots \circ \widehat{\delta}_{-1}(u), & (h \text{ times}) \\ \widehat{\delta}_0(u) & = & u. \end{cases}$$

At any order $r \in \mathbb{R}$, an estimator of $\widehat{\delta}_r$ of δ_r is

$$\widehat{\delta}_r(u) = z\left(\left(z^{-1} \circ \widehat{\delta}_h(u)\right)^{1-\eta}\left(z^{-1} \circ \widehat{\delta}_{h+1}(u)\right)^{\eta}\right), \quad \text{for } u \in [0,1], \tag{11}$$

with $\eta = r - \lfloor r \rfloor$ and $h = \lfloor r \rfloor$ where $\lfloor r \rfloor$ denotes the integer part of r, and where z is a strictly monotone function driving the interpolation, ideally the inverse of the generator of the copula.

Several different estimators for δ_1 can be found in the literature. In particular, one can propose $\widehat{\delta}_1(u) = F_{Y,n}(u)$, where $F_{Y,n}(u)$ is the empirical distribution function of $Y := \max(U_1, U_2, \ldots, U_d)$. Similarly, we consider $\widehat{\delta}_{-1}(u) = F^{-1}_{Y,n}(u)$, with $F^{-1}_{Y,n}(u)$ the empirical quantile function of Y.

Using the self-nested diagonal family δ_r, we write the sequence $k_U(n)$ as: $k_U(n) = n\,(1 - \delta_{r(n)}(\alpha))$, where $r(n) := \log\left(1 - \left(1 - \frac{k(n)}{n}\right)^{1/(d-1)}\right) / \log(d)$ is a negative real sequence. Therefore, using the nonparametric estimator $\widehat{\delta}_{r(n)}$ in Definition 2.2, we introduce the estimator

$$\widehat{k}_U(n) \;=\; n\,(1 - \widehat{\delta}_{r(n)}(\alpha)), \quad \text{for } \;\alpha \in (0,1). \tag{12}$$

The following consistency result for $\widehat{k}_U(n)$ can now be proved.

Lemma 2.2. *Let $k_U(n)$ be the intermediate sequence defined as in Theorem 2.1. Let $\widehat{\delta}_1(u) = F_{Y,n}(u)$, with $F_{Y,n}(u)$ the empirical distribution function of $Y := \max(U_1, U_2, \ldots, U_d)$ and $\widehat{\delta}_{-1}(u) = F^{-1}_{Y,n}(u)$, with $F^{-1}_{Y,n}(u)$ the empirical quantile function of Y. Let $\widehat{k}_U(n)$ be the associated estimator proposed in Equation (12) for a fixed $\alpha \in (0,1)$ and where z is a strictly monotone function driving the interpolation. Then,*

$$\frac{\widehat{k}_U(n)}{k_U(n)} \xrightarrow{\mathbb{P}} 1, \quad \text{as} \;\; n \to \infty.$$

Proof of Lemma 2.2

Firstly, we prove that $\frac{\widehat{\delta}_h(u)}{\delta_h(u)} \xrightarrow{\mathbb{P}} 1$, for $u \in (0,1)$ and for fixed $h \in \mathbb{Z}$, where δ_h is introduced at the beginning of this section and $\widehat{\delta}_h(u)$ is defined in Definition 2.2. Consider that $h \in \mathbb{Z}^+$. Since $\widehat{\delta}_1(u) := F_{Y,n}(u)$, where $F_{Y,n}(u)$ is the empirical distribution function of $Y := \max(U_1, U_2, \ldots, U_d)$, then from Glivenko Cantelli's Theorem, it is verified that

$$\sup_{u\in[0,1]} |\widehat{\delta}_1(u) - \delta_1(u)| = \sup_{u\in[0,1]} |F_{Y,n}(u) - F_Y(u)| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

By induction, we assume that $\sup_{u\in[0,1]} |\widehat{\delta}_{m-1}(u) - \delta_{m-1}(u)| \xrightarrow{\mathbb{P}} 0$. Since C is a Lipschitz function (see Definition 6.2.6 in Nelsen (2006)), from Theorem 1 in Kasy (2015) and from the uniformly convergence of $\widehat{\delta}_1(u)$, then $\sup_{u\in[0,1]} |\widehat{\delta}_m(u) - \delta_m(u)| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Let $h \in \mathbb{Z}^-$. We have $\widehat{\delta}_{-1}(u) := F^{-1}_{Y,n}(u)$, where $F^{-1}_{Y,n}(u)$ is the empirical quantile function of Y. From Theorem 3 in Mason (1982),

$$\sup_{u\in(0,1)} |\widehat{\delta}_{-1}(u) - \delta_{-1}(u)| = \sup_{u\in(0,1)} |F^{-1}_{Y,n}(u) - F^{-1}_Y(u)| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

By induction, we suppose that $\sup_{u\in(0,1)} |\widehat{\delta}_m(u) - \delta_m(u)| \xrightarrow{\mathbb{P}} 0$. Since C^{-1} is a uniformly continuous function in $[0,1]$, then from Theorem 1 in Kasy (2015) and from the uniformly convergence of $\widehat{\delta}_{-1}(u)$, we obtain $\sup_{u\in(0,1)} |\widehat{\delta}_{m-1}(u) - \delta_{m-1}(u)| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Therefore, $\frac{\widehat{\delta}_h(u)}{\delta_h(u)} \xrightarrow{\mathbb{P}} 1$, for $u \in (0,1)$ and for fixed $h \in \mathbb{Z}$. Furthermore, by using the Slutsky's Theorem (e.g., see Serfling (1980)), one can prove that $\frac{\widehat{\delta}_r(u)}{\delta_r(u)} \xrightarrow{\mathbb{P}} 1$, $\forall\, u \in (0,1)$ and $\forall\, r \in \mathbb{R}$ fixed. Therefore, since δ_r is also a continuous and bounded function in r, from Polya's Theorem (e.g., see Section A.1.1 in Embrechts et al. (1997)), then, for $u \in (0,1)$, $\sup_{r\in\mathbb{R}} |\widehat{\delta}_r(u) - \delta_r(u)| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. By using this uniform consistency we obtain the assertion of Lemma 2.2. □

Using Lemma 2.2, it can be proved that $X^i_{n-\lfloor \widehat{k}_U \rfloor, n}$ is asymptotically as efficient as $X^i_{n-\lfloor k_U \rfloor, n}$. To be more precise, an adaptive plug-in version of Theorem 2.1 can be obtained, i.e.,

$$\sqrt{\widehat{k}_U(n)} \left(\frac{X^i_{n-\lfloor \widehat{k}_U \rfloor, n}}{U_{X_i}\left(\frac{n}{k_U(n)}\right)} - 1 \right) \xrightarrow{d} \gamma_i \, N(0,1), \quad \text{as} \quad n \to \infty. \tag{13}$$

Further details are given in Hall and Welsh (1985), Drees and Kaufmann (1998), and Danielsson et al. (2001). Then, an adaptive version of Theorem 3.1 in the main document for $\widehat{x}^i_{p_n}$ can also be provided. The proof is a slightly modified version of the proof of Theorem 3.1, by using the result in Equation (13) instead of Theorem 2.1. Illustrations of this plug-in estimation of $\widehat{x}^i_{p_n}$, by using $\widehat{k}_U$ instead of k_U, can be found in Section 5 in the main document.

In particular, to estimate the adaptive sequence $\widehat{k}_U(n)$ in Section 5 of the main document, we consider $z(x) = \exp(-x)$. This choice is recommended in Di Bernardino and Rullière (2013) when there is positive dependence, since it is the best choice for any Gumbel copula, whatever the parameter of the copula (see Corollary 3.7 in Di Bernardino and Rullière (2013)). Another natural choice could be any estimator of the inverse of the generator of the copula. Finally, it should be borne in mind that this function z does not change values of any δ_k, for $k \in \mathbb{Z}$. Therefore, the global shape of δ_r, as a function of $r \in \mathbb{R}$, is not heavily impacted by the choice of z. For a in-depth analysis of the weak impact of the interpolation function z in the evaluation of δ_r, the reader is referred to Section 4.3.1 in Di Bernardino and Rullière (2013).

Illustrations of estimators $\widehat{\delta}_r$ and $\widehat{k}_U$ In Figure 1, illustrations of $\widehat{\delta}_r$ with $r \in \mathbb{R}$ are provided for two different interpolation functions. As in Di Bernardino and Rullière (2013), a 2−dimensional Gumbel copula is generated with $\theta = 3$ and sample size $n = 2000$ and $n = 7000$. We consider $z(x) = \exp(-x)$ (first row of Figure 1) and $z(x) = \exp(-x^{1/\theta})$, i.e., the inverse of the Gumbel generator copula (see second row of Figure 1) with $r = -3.5$, -2.4, -1.2, 0.6, 1.2, 2.4, 3.5. As pointed out before, it can be observed that the modification of the interpolation function z does not produce significant differences in the estimation of δ_r.

[Figure 1 about here.]

Finally, an illustration of Lemma 2.2 is provided in Figure 2 where the boxplots of the ratio $\widehat{k}_U(n)/k_U(n)$ are gathered for a Joe copula $\theta = 3$ with Fréchet margins $\beta = 3$ by considering different sample sizes, with $k(n) = \sqrt{n}$ (left panel) and $k(n) = n^{0.9}$ (right panel). In this case, we choose z the inverse of the generator of the considered Joe copula.

[Figure 2 about here.]

References

Brechmann, E. C. (2014). Hierarchical kendall copulas: properties and inference. *The Canadian Journal of Statistics*, 42(1):78—-108.

Charpentier, A. and Segers, J. (2009). Tails of multivariate Archimedean copulas. *Journal of Multivariate Analysis*, 100(7):1521—-1537.

Danielsson, J., de Haan, L., Peng, L., and de Vries, C. G. (2001). Using a bootstrap method to choose the sample fraction in tail index estimation. *Journal of Multivariate Analysis*, 76:226–248.

de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory. An Introduction.* Springer Series in Operations Research and Financial Engineering. Springer.

Di Bernardino, E., Maume-Deschamps, V., and Prieur, C. (2013). Estimating a bivariate tail: a copula based approach. *Journal of Multivariate Analysis*, 119:81—-100.

Di Bernardino, E. and Rullière, D. (2013). On certain transformations of Archimedean copulas: Application to the non-parametric estimation of their generators. *Dependence Modelling*, 1:1—-36.

Di Bernardino, E. and Rullière, D. (2014). On tail dependence coefficients of transformed multivariate Archimedean copulas. *Working paper, https://hal.archives-ouvertes.fr/hal-00992707v1/document.*

Drees, H. (1998). On smooth statistical tail functionals. *Scandinavian Journal of Statistics*, 25:187–210.

Drees, H. and Kaufmann, E. (1998). Selecting the optimal sample fraction in univariate extreme value estimation. *Stochastic Processes and their Applications*, 75:149–172.

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events for insurance and finance.* Springer. Berlin.

Hall, P. and Welsh, A. H. (1985). Adaptive estimates of parameters of regular variation. *The Annals of Statistics*, 13(1):331—-341.

Kasy, M. (2015). Uniformity and the delta method. *Harvard University Working Paper. http://scholar.harvard.edu/kasy/publications/uniformity-and-delta-method.*

Mason, D. M. (1982). Some Characterizations of Almost Sure Bounds for Weighted Multidimensional Empirical Distributions and a Glivenko-Cantelli Theorem for Sample Quantiles. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 59:505–513.

Nelsen, R. B. (2006). *An Introduction to Copulas.* Springer Series in Statistics. Springer.

Schmidt, R. and Stadtmüller, U. (2006). Non-parametric estimation of tail dependence. *Scandinavian Journal of Statistics. Theory and Applications*, 33(2):307—-335.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York.

List of Figures

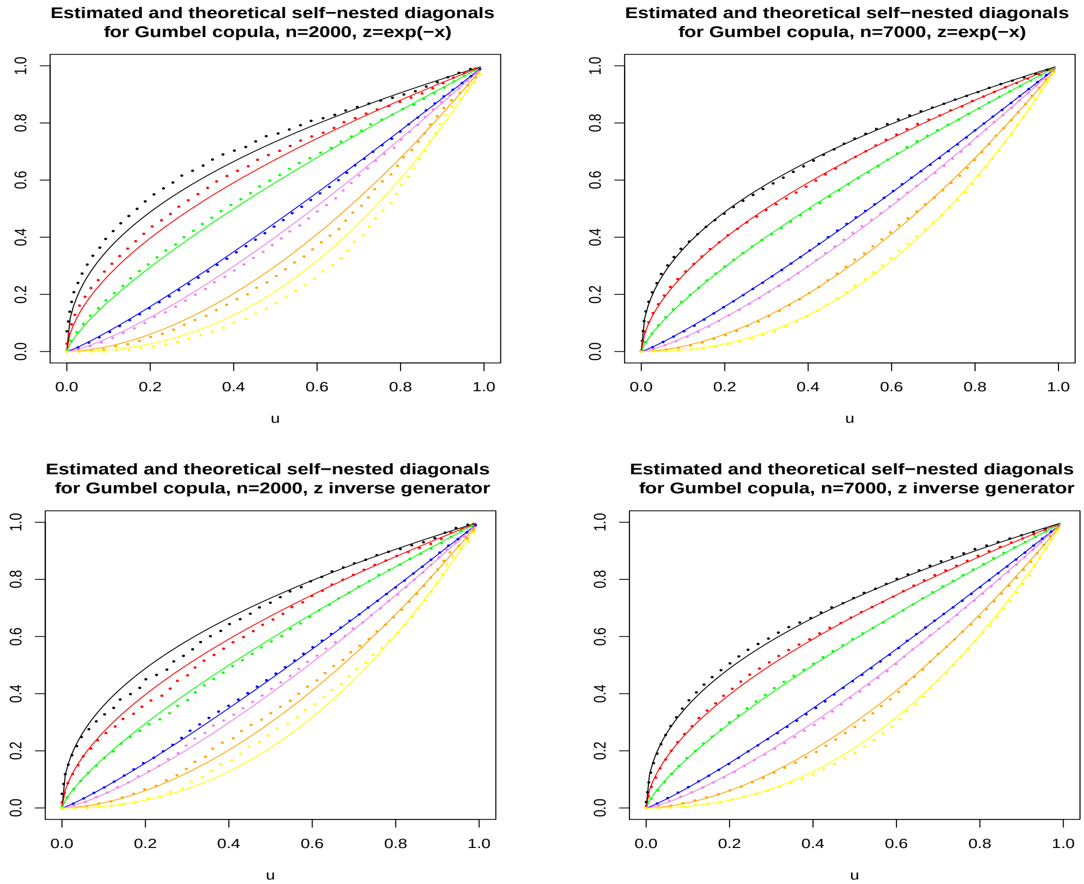

Figure 1: Gumbel copula with dependence parameter $\theta = 3$. Estimation of $\delta_r(x)$ by considering $z(x) = \exp(-x)$ (first row) and $z(x) = \exp(-x^{1/\theta})$ (second row) with $r = -3.5$, -2.4, -1.2, 0.6, 1.2, 2.4, 3.5, for $n = 2000$ (left panels) and $n = 7000$ (right panels).

Boxplots of $\widehat{k}_U(n)/k_U(n)$

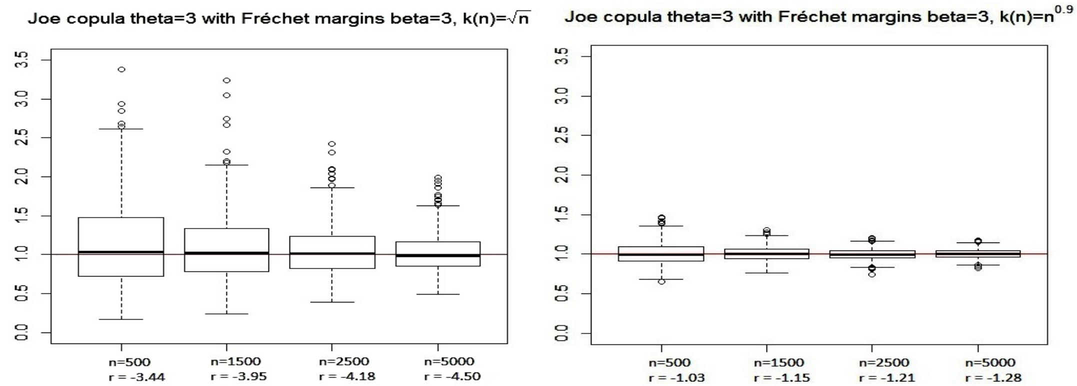

Figure 2: Joe copula with dependence parameter $\theta = 3$ and Fréchet margins with $\beta = 3$. Boxplots for the ratio $\widehat{k}_U(n)/k_U(n)$ for various values of n, $\alpha = 0.9$, $k(n) = \sqrt{n}$ (left panel) and $k(n) = n^{0.9}$ (right panel). 500 Monte Carlo simulations are taken.

List of Tables

ρ	$\gamma_i < 0$	$\gamma_i = 0$	$\gamma_i > 0$
$(1, +\infty)$	γ_i/ρ	0	γ_i/ρ
1	γ_i	0	γ_i
$+\infty$	0	0	0

Copula	$U(0,1)$	$Exp(\lambda)$	$Par(\delta, 1)$
Gumbel	$-1/\theta$	0	$1/\delta\theta$
Ali-Mikhail-Haq	-1	0	$1/\delta$
18	0	0	0

Table 1: Left panel: The tail index γ^{T_i} when $(X_1, \ldots, X_d)$ follows an Archimedean copula with $\phi \in RV_\rho(1)$ and F_i verifies the von Mises condition with index γ_i. Right panel: The tail index γ^{T_i} for some specific models.

Copula	$\phi(t)$	$A_\rho(t)$
Gumbel	$(-\log(t))^\theta$	t^{-1}
Joe	$-\log(1-(1-t)^\theta)$	$t^{-\theta}$
(12)	$(1/t-1)^\theta$	t^{-1}
(14)	$(t^{-1/\theta}-1)^\theta$	t^{-1}
(15)	$(1-t^{1/\theta})^\theta$	t^{-1}
(21)	$1-(1-(1-t)^\theta)^{1/\theta}$	$t^{-\theta}$

Table 2: Bivariate Archimedean copula models with $\rho = \theta$ and $\lambda_U = 2 - 2^{1/\theta}$.