

1 **SOME REGULARITY RESULTS FOR A DOUBLE**  
2 **TIME-DELAYED 2D-NAVIER-STOKES MODEL**

JULIA GARCÍA-LUENGO

Departamento de Matemática Aplicada a las TIC  
Universidad Politécnica de Madrid  
C/ Nikola Tesla s/n, 28031, Madrid, Spain

PEDRO MARÍN-RUBIO

Departamento de Ecuaciones Diferenciales y Análisis Numérico  
Universidad de Sevilla  
C/ Tarfia s/n, 41012, Sevilla, Spain

GABRIELA PLANAS

Departamento de Matemática  
Instituto de Matemática, Estatística e Computação Científica  
Universidade Estadual de Campinas  
Rua Sergio Buarque de Holanda 651, 13083-859 Campinas, SP, Brazil

Dedicated to Professor Peter E. Kloeden on occasion of his Seventieth Birthday

ABSTRACT. In this paper we analyze some regularity properties of a double time-delayed 2D-Navier-Stokes model, that includes not only a delay force but also a delay in the convective term. The interesting feature of the model -from the mathematical point of view- is that being in dimension two, it behaves similarly as a 3D-model without delay, and extra conditions in order to have uniqueness were required for well-posedness. This model was previously studied in several papers, being the existence of attractor in the  $L^2$ -framework obtained by the authors [Discrete Contin. Dyn. Syst. **34** (2014), 4085–4105]. Here regularization properties of the solutions and existence of (regular) attractors for several associated dynamical systems are established. Moreover, relationships among these objects are also provided.

3 **1. Introduction, statement of the problem, and previous results.** Consider  
4 a bounded domain  $\Omega \subset \mathbb{R}^2$ ,  $\tau \in \mathbb{R}$ , and the following non-autonomous functional  
5 Navier-Stokes model

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u(t - \rho(t)) \cdot \nabla)u + \nabla p = f(t) + g(t, u_t) & \text{in } \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u^\tau(x) & \text{in } \Omega, \\ u(x, \tau + s) = \phi(x, s) & \text{in } \Omega \times (-h, 0), \end{cases} \quad (1)$$

---

2010 *Mathematics Subject Classification.* Primary: 35Q30, 35Q35, 35B65, 35B41, 37L30.

*Key words and phrases.* Navier-Stokes equations; nonlinear delay terms; well-posed/ill-posed Navier-Stokes; delayed convective term; regular attractors; tempered universes for non-autonomous dynamical systems.

1 where  $\nu > 0$  is the kinematic viscosity,  $u = (u^1, u^2)$  is the velocity field of the fluid,  
 2  $p$  is the pressure,  $f$  is a non-delayed external force field,  $g$  is another external force  
 3 with some hereditary characteristics with memory length bounded by  $h > 0$ ,  $u_t$   
 4 denotes the delay function  $u_t(s) = u(t + s)$  where it has sense. The delay function  
 5  $\rho$  in the convective term is assumed to belong to  $C^1(\mathbb{R}; [0, h])$  with  $\rho'(t) \leq \rho^* < 1$   
 6 for all  $t \in \mathbb{R}$ , and  $u^\tau$  and  $\phi$  are the initial data in  $\tau$  and  $(\tau - h, \tau)$  respectively.

7  
 8 The study of Navier-Stokes models including delay terms –existence, uniqueness,  
 9 stationary solutions, exponential decay, existence of attractors, and other issues– is  
 10 initiated by Caraballo and Real [1, 2, 3], and after that, different questions have  
 11 been addressed (e.g., [19, 9, 13, 15, 16, 6, 7]).

12  
 13 In particular, the inclusion of a delay term in the convective part is firstly con-  
 14 sidered in [12] for a Burgers' equation. Then Planas and collaborators [17, 10, 11]  
 15 treat problem (1), the analysis of well-posedness (including uniqueness) and an  
 16 unbounded delay case too. The asymptotic behavior in the sense of attractors in  
 17  $L^2$ -norm is carried out in [4]. It is worth also to mention that in [20] the inclusion of  
 18 a delay is used as an approximation to a 3D Navier-Stokes model when the length  
 19 of the delay vanishes. Nevertheless, in the 2D case our interest in the problem is  
 20 just mathematical, due to the difficulties arising in controlling the norm of the de-  
 21 rivatives as cited in the abstract (see also [4]). Our goal in this paper is to improve  
 22 the results in [4] providing regularity for both solutions and attractors.

23  
 24 Let us first introduce some notation. As usual we will denote by  $H$  and  $V$  the  
 25 Hilbert spaces that are the closure of  $\mathcal{V}$  (infinitely differentiable functions in  $\Omega$  with  
 26 compact support and free divergence) in the  $L^2$  and  $H^1$  norm respectively, and  
 27 denote their inner products and norms by  $(\cdot, \cdot)$  and  $|\cdot|$  in  $H$ , and by  $((\cdot, \cdot))$  (product  
 28 of gradients, thanks to the Poincaré inequality) and  $\|\cdot\|$  in  $V$  (see [4] for more  
 29 details). We do the identification of the Hilbert space  $H$  with its dual, so we have  
 30 the chain of dense and compact inclusions  $V \subset H \equiv H' \subset V'$ . The duality between  
 31 elements in  $V'$  and  $V$  will be denoted by  $\langle \cdot, \cdot \rangle$ . For short, we introduce the notation  
 32  $L_X^p = L^p(-h, 0; X)$ , for several choices of  $p$  and  $X$ , and its norm will be denoted by  
 33  $\|\cdot\|_{L_X^p}$ . In the same sense, we denote  $C_X = C([-h, 0]; X)$ .

34  
 Recall that  $A : V \rightarrow V'$  given by  $\langle Av, w \rangle = ((v, w))$ , satisfies that  $Au = -P\Delta u$   
 for all  $u \in D(A)$  (the Stokes operator), where  $P$  is the Leray-Helmholtz projector  
 from  $(L^2(\Omega))^2$  onto  $H$ . The trilinear operator  $b$  associated to Navier-Stokes model  
 is given by

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

for every functions  $u, v, w : \Omega \rightarrow \mathbb{R}^2$  for which the right-hand side is well defined.  
 In particular,  $b$  is well defined on  $V \times V \times V$ , and therefore we can consider the  
 associated bilinear form

$$B : V \times V \rightarrow V'$$

35 given by  $\langle B(u, v), w \rangle := b(u, v, w)$ . It is well-known (e.g., cf. [18]) that  $b$  satisfies  
 36 the following inequalities (recall we are in dimension two) for a certain constant

1  $C > 0$  depending only on  $\Omega$ ,

$$|b(u, v, w)| \leq C|u|^{1/2}\|u\|^{1/2}\|v\|\|w\|^{1/2}\|w\|^{1/2} \quad \forall u, v, w \in V, \quad (2)$$

$$|b(u, v, w)| \leq C|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}\|w| \quad \forall u \in V, v \in D(A), w \in H. \quad (3)$$

2 The delay operator in the right-hand side is  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$ , and we  
 3 assume that it satisfies the following assumptions:

4 (H1) for all  $\xi \in C_H$ , the function  $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2$  is measurable,

5 (H2)  $g(t, 0) = 0$ , for all  $t \in \mathbb{R}$ ,

(H3) there exists  $L_g > 0$  such that for all  $t \in \mathbb{R}$ , and for all  $\xi, \eta \in C_H$ ,

$$|g(t, \xi) - g(t, \eta)| \leq L_g|\xi - \eta|_{C_H},$$

(H4) there exists  $C_g > 0$  such that for all  $\tau \leq t$  and for all  $u, v \in C([\tau - h, t]; H)$ ,

$$\int_{\tau}^t |g(r, u_r) - g(r, v_r)|^2 dr \leq C_g^2 \int_{\tau-h}^t |u(r) - v(r)|^2 dr.$$

6 Examples of several types of delay operators can be found in [1, Section 3], [3,  
 7 Sections 3.5 and 3.6] and [9, Section 3].

From (H1)–(H3), for  $T > \tau$  and  $u \in C([\tau - h, T]; H)$ , the function  $g_u : [\tau, T] \rightarrow (L^2(\Omega))^2$  given by  $g_u(t) = g(t, u_t)$  is measurable and belongs to  $L^\infty(\tau, T; (L^2(\Omega))^2)$ . By using (H4), the mapping

$$C([\tau - h, T]; H) \ni u \mapsto \mathcal{G}(u) := g_u \in L^2(\tau, T; (L^2(\Omega))^2)$$

8 has a unique extension to a mapping  $\tilde{\mathcal{G}}$  which is uniformly continuous from  $L^2(\tau - h, T; H)$  into  $L^2(\tau, T; (L^2(\Omega))^2)$ . We will still denote by  $g(t, u_t) = \tilde{\mathcal{G}}(u)(t)$  for each  
 9  $u \in L^2(\tau - h, T; H)$ , and therefore assumption (H4) will hold for all  $u, v \in L^2(\tau - h, T; H)$ .  
 10  
 11

12

13 Let us consider that  $u^\tau \in H$ ,  $\phi \in L_V^2$ , and  $f \in L_{loc}^2(\mathbb{R}; V')$ .

**Definition 1.** A weak solution to (1) is a function  $u \in L^\infty(\tau, T; H) \cap L^2(\tau - h, T; V)$  for all  $T > \tau$ , such that  $u(\tau) = u^\tau$ ,  $u_\tau = \phi$ , and satisfies

$$\frac{d}{dt}(u(t), v) + \nu \langle Au(t), v \rangle + b(u(t - \rho(t)), u(t), v) = \langle f(t), v \rangle + (g(t, u_t), v) \quad \forall v \in V,$$

14 where the equation must be understood in the sense of  $\mathcal{D}'(\tau, \infty)$ .

If  $u$  is a weak solution to (1), since there exists a constant  $\tilde{C} > 0$  such that for any  $v \in V$ ,

$$|b(u(t - \rho(t)), u(t), v)| \leq \tilde{C}\|u(t - \rho(t))\|\|u(t)\|^{1/2}|u(t)|^{1/2}\|v\|$$

15 (where we have used that  $b(u, v, w) = -b(u, w, v)$  for all  $u, v, w \in V$ , (2) and  
 16 the continuous embedding of  $V$  into  $H$ ), we conclude that  $B(u(\cdot - \rho(\cdot)), u(\cdot)) \in$   
 17  $L^{4/3}(\tau, T; V')$  and  $u' \in L^{4/3}(\tau, T; V')$  too. Thus,  $u \in C([\tau, \infty); V') \cap C_w([\tau, \infty); H)$ .  
 18 This continuity in time in  $V'$  and weakly in  $H$  does not seem enough to apply  
 19 an energy method to gain asymptotic compactness (in the study of the long time  
 20 behavior). Somehow this indicates the border between an ill-posed and well-posed  
 21 problems, as we recall now in a new phase-space.

22 When the initial (memory) data is more regular, namely  $\phi \in L_V^2 \cap L_H^\infty$ , we can  
 23 improve the above estimates again using (2).

24 Some existence results given in those settings (cf. [17, Theorem 2.1] and [4,  
 25 Theorems 1 and 2]) are summarized in the following result.

1 **Theorem 1.** Consider  $f \in L^2_{loc}(\mathbb{R}; V')$ , and  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  satisfying  
 2 assumptions (H1)–(H4). Then, the following statements hold:

- 3 (a) If  $u^\tau \in H$ ,  $\phi \in L^2_V$ , there exists at least one weak solution  $u(\cdot; \tau, u^\tau, \phi)$  to (1).  
 (b) If  $u^\tau \in H$  and  $\phi \in L^2_V \cap L^\infty_H$ , then there exists a unique weak solution to  
 (1),  $u(\cdot; \tau, u^\tau, \phi) \in C([\tau, \infty); H)$ , with  $u' \in L^2(\tau, T; V')$  for all  $T > \tau$ , and  
 satisfying the energy equality

$$\begin{aligned} & |u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr \\ & = |u(s)|^2 + 2 \int_s^t \langle f(r), u(r) \rangle dr + 2 \int_s^t (g(r, u_r), u(r)) dr \quad \forall \tau \leq s \leq t. \end{aligned} \quad (4)$$

**Remark 1.** As a by-product of the proof of the above result, it is not difficult to check that the following estimates hold for the weak solution obtained in statement (a) (unique under the assumptions of statement (b)) and also for suitable Galerkin approximations (see (6) below) for any  $T > \tau$ :

$$\begin{aligned} \|u\|_{L^\infty(\tau, T; H)}^2 & \leq (|u^\tau|^2 + C_g^2 \|\phi\|_{L^2_H}^2 + \nu^{-1} \|f\|_{L^2(\tau, T; V')}^2) \exp((1 + C_g^2)(T - \tau)) \\ & =: C_{L^\infty(H)}(\tau, T, u^\tau, \phi), \\ \|u\|_{L^2(\tau, T; V)}^2 & \leq \nu^{-1} (|u^\tau|^2 + C_g^2 \|\phi\|_{L^2_H}^2 + \nu^{-1} \|f\|_{L^2(\tau, T; V')}^2) \\ & \quad + \nu^{-1} (1 + C_g^2)(T - \tau) C_{L^\infty(H)}(\tau, T, u^\tau, \phi) =: C_{L^2(V)}(\tau, T, u^\tau, \phi). \end{aligned}$$

4 Our goal in this work is to improve some previous results obtained in [4, 17],  
 5 addressing to the existence of strong solutions and attractors in a higher norm.

6 The structure of the paper is the following: in Section 2 we establish the exist-  
 7 ence of strong solutions and the regularization effect in 2D. Estimates leading to  
 8 continuity of involved processes and absorbing properties are obtained in Section 3.  
 9 Section 4 is devoted to prove the asymptotic compactness via an energy method.  
 10 In order to do that, previous uniform estimates in several spaces are deduced. The  
 11 required computations for these results are more involved than in the non-delayed  
 12 case due mainly to the extra difficulty of the delay in the convective term. Finally,  
 13 in Section 5 all the previous results allow us to ensure the existence of several famil-  
 14 ies of pullback attractors in higher norms. Several relationships between them (and  
 15 also compared with those obtained in [4]) will be pointed out too.

16 **2. Regularization effect and strong solution.** One can also expect to introduce  
 17 a concept of strong solution for problem (1).

18 **Definition 2.** A strong solution to (1) is a weak solution that also satisfies  $u \in$   
 19  $L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A))$  for all  $T > \tau$ .

**Remark 2.** Observe that if  $\phi \in L^2_V \cap L^\infty_H$  and  $u$  is a strong solution for (1), from  
 (3) it yields that  $B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^2(\tau, T; H)$  for all  $T > \tau$ . Therefore, if  
 $f \in L^2_{loc}(\mathbb{R}; H)$ , then  $u' \in L^2(\tau, T; H)$  for all  $T > \tau$ ,  $u \in C([\tau, \infty); V)$ , and the  
 following energy equality holds,

$$\begin{aligned} & \|u(t)\|^2 + 2 \int_s^t [\nu |Au(r)|^2 + b(u(r - \rho(r)), u(r), Au(r))] dr \\ & = \|u(s)\|^2 + 2 \int_s^t (f(r) + g(r, u_r), Au(r)) dr \quad \forall \tau \leq s \leq t. \end{aligned} \quad (5)$$

20 The following regularizing effect holds for the problem.

- 1 **Theorem 2.** Consider  $u^\tau \in H$ ,  $\phi \in L^2_V \cap L^\infty_H$ ,  $f \in L^2_{loc}(\mathbb{R}; H)$ , and  $g : \mathbb{R} \times$   
 2  $C_H \rightarrow (L^2(\Omega))^2$  satisfying assumptions (H1)–(H4). Then, the weak solution  $u$  to  
 3 (1) regularizes to a strong solution in the sense that  $u \in L^\infty(\tau + \varepsilon, T; V) \cap L^2(\tau +$   
 4  $\varepsilon, T; D(A)) \cap C((\tau, \infty); V)$  for all  $T > \tau + \varepsilon > \tau$ .  
 5 Moreover, if  $u^\tau \in V$ , then  $u$  is indeed a strong solution to (1), so  $u \in C([\tau, \infty); V) \cap$   
 6  $L^2_{loc}(\tau, \infty; D(A))$ , and satisfies the energy equality (5).

*Proof.* Consider a special basis of  $H$  formed by normalized eigenfunctions of the Stokes operator,  $\{w_j\}_{j \geq 1}$ , with corresponding eigenvalues  $\{\lambda_j\}_{j \geq 1}$  being  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  with  $\lim_{j \rightarrow \infty} \lambda_j = \infty$ . Pose the approximate problems (for each  $k \geq 1$ ) of finding  $u^k \in V_k := \text{span}[w_1, \dots, w_k]$  with  $u^k(t) = \sum_{j=1}^k \gamma_{jk}(t) w_j$  such that

$$\begin{aligned} & \frac{d}{dt}(u^k(t), w_j) + \nu \langle Au^k(t), w_j \rangle + b(u^k(t - \rho(t)), u^k(t), w_j) \\ & = (f(t), w_j) + (g(t, u^k_t), w_j), \quad \text{a.e. } t > \tau, \forall 1 \leq j \leq k, \end{aligned} \quad (6)$$

- 7 fulfilled with the initial conditions

$$u^k(\tau) = P_k u^\tau \quad \text{and} \quad u^k(\tau + s) = P_k \phi(s) \quad \text{a.e. } s \in (-h, 0), \quad (7)$$

- 8 where  $P_k$  is the orthogonal projector from  $H$  onto  $V_k$ .

Multiplying each equation by  $\lambda_j \gamma_{jk}(t)$ , summing from  $j = 1$  to  $k$ , and taking into account (3), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^k(t)\|^2 + \nu |Au^k(t)|^2 & \leq C \|u^k(t - \rho(t))\|^{1/2} \|u^k(t - \rho(t))\|^{1/2} \|u^k(t)\|^{1/2} |Au^k(t)|^{3/2} \\ & + (f(t) + g(t, u^k_t), Au^k(t)) \quad \text{a.e. } t > \tau. \end{aligned}$$

Using the Cauchy-Schwartz and Young inequalities

$$\begin{aligned} \frac{d}{dt} \|u^k(t)\|^2 + \frac{\nu}{2} |Au^k(t)|^2 & \leq \tilde{C} \|u^k(t - \rho(t))\|^2 \|u^k(t - \rho(t))\|^2 \|u^k(t)\|^2 \\ & + \frac{2}{\nu} |f(t)|^2 + \frac{2}{\nu} |g(t, u^k_t)|^2 \quad \text{a.e. } t > \tau, \end{aligned} \quad (8)$$

where  $\tilde{C} = \frac{27C^4}{2\nu^3}$ . Integrating in time, using (H4) and Remark 1, we deduce

$$\begin{aligned} \|u^k(t)\|^2 + \frac{\nu}{2} \int_s^t |Au^k(r)|^2 dr & \leq \|u^k(s)\|^2 + \hat{C} \int_s^t \|u^k(r - \rho(r))\|^2 \|u^k(r)\|^2 dr \\ & + \frac{2}{\nu} \int_s^t |f(r)|^2 dr + \frac{2}{\nu} C_g^2 \int_{\tau-h}^t |u^k(r)|^2 dr \end{aligned} \quad (9)$$

- 9 for all  $\tau \leq s < t \leq T$ , where  $\hat{C} = \tilde{C} \max\{\|\phi\|_{L^\infty_H}^2, C_{L^\infty(H)}(\tau, T, u^\tau, \phi)\}$ .

In particular, from the above, integrating again with respect to  $s \in [\tau, \tau + \varepsilon]$  (with  $\tau + \varepsilon < T$ ), it yields

$$\begin{aligned} \|u^k(t)\|^2 & \leq \varepsilon^{-1} \int_\tau^{\tau+\varepsilon} \|u^k(s)\|^2 ds + \hat{C} \int_\tau^t \|u^k(r - \rho(r))\|^2 \|u^k(r)\|^2 dr \\ & + \frac{2}{\nu} \int_\tau^t |f(r)|^2 dr + \frac{2}{\nu} C_g^2 \int_{\tau-h}^t |u^k(r)|^2 dr \quad \forall \tau + \varepsilon \leq t \leq T. \end{aligned}$$

Now, applying the Gronwall lemma we obtain that

$$\begin{aligned} \|u^k(t)\|^2 \leq & \left( \varepsilon^{-1} \int_{\tau}^{\tau+\varepsilon} \|u^k(r)\|^2 dr + \frac{2}{\nu} \int_{\tau}^t |f(r)|^2 dr + \frac{2}{\nu} C_g^2 \int_{\tau-h}^t |u^k(r)|^2 dr \right) \\ & \times \exp \left( \widehat{C} \int_{\tau}^t \|u^k(r - \rho(r))\|^2 dr \right) \quad \forall t \in [\tau + \varepsilon, T]. \end{aligned}$$

- 1 From (7) and Remark 1 we conclude that  $\{u^k\}$  is bounded in  $L^\infty(\tau + \varepsilon, T; V)$ .  
 2 Turning back now to (9) we also obtain that  $\{u^k\}$  is bounded in  $L^2(\tau + \varepsilon, T; D(A))$ .  
 3 Passing through the limit in  $k$ , we conclude the first claim of the theorem.

4 Last claim is simpler. If  $u^\tau \in V$ , it only requires integration in (8) in  $[\tau, t]$  and  
 5 application of the Gronwall lemma. The details are omitted for brevity.  $\square$

6 An immediate consequence of the above is the following result.

7 **Corollary 1.** *Assume that  $f \in L^2_{loc}(\mathbb{R}; H)$  and  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  fulfills*  
 8 *(H1)–(H4). Then, for any bounded set  $B \subset H \times (L^2_V \cap L^\infty_H)$*

- 9 (i) *The set of weak solutions to (1)  $\{u(\cdot; \tau, u^\tau, \phi) : (u^\tau, \phi) \in B\}$  is bounded in*  
 10  *$L^\infty(\tau + \varepsilon, T; V) \cap L^2(\tau + \varepsilon, T; D(A))$  for any  $\varepsilon > 0$  and any  $T > \tau + \varepsilon$ .*  
 11 (ii) *Moreover, if  $B$  is bounded in  $V \times (L^2_V \cap L^\infty_H)$ , then  $\{u(\cdot; \tau, u^\tau, \phi) : (u^\tau, \phi) \in B\}$*   
 12 *is bounded in  $L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A))$  for any  $T > \tau$ .*

13 **3. Processes and their continuity and absorbing properties.** In this section  
 14 we start recalling the biparametric families of mappings  $S$  and  $U$  defined in [4]  
 15 through the solution to (1) given in Theorem 1, which in fact were proved to be  
 16 continuous processes. After that, and thanks to the improved regularity we have  
 17 several meaningful choices to restrict these mappings to others with higher norms.  
 18 Their continuity and absorbing properties will be established.

Recall that

$$S(t, \tau) : H \times (L^2_V \cap L^\infty_H) \rightarrow H \times (L^2_V \cap L^\infty_H) \text{ and } U(t, \tau) : L^2_V \cap C_H \rightarrow L^2_V \cap C_H$$

19 for any  $t \geq \tau$ , given by  $S(t, \tau)(u^\tau, \phi) = (u(t), u_t)$  for any  $(u^\tau, \phi) \in H \times (L^2_V \cap L^\infty_H)$   
 20 and  $U(t, \tau)\phi = u_t(\cdot; \tau, \phi(0), \phi)$  for any  $\phi \in L^2_V \cap C_H$  are well-defined mappings after  
 21 Theorem 1 (cf. [4]).

Actually, each of these mappings form a process (see definitions in [4, Section  
 3]), which for short we denote

$$(H \times (L^2_V \cap L^\infty_H), \{S(t, \tau)\}_{t \geq \tau}) \text{ and } (L^2_V \cap C_H, \{U(t, \tau)\}_{t \geq \tau})$$

22 respectively. They are continuous in their corresponding phase-spaces (cf. [4, Co-  
 23 rollary 2]) under the assumptions of Theorem 1.

Now, after Theorem 2 we may consider the corresponding restrictions (the nota-  
 tion for the operators will not be modified, since no confusion arises)

$$\begin{aligned} & (V \times (L^2_V \cap L^\infty_H), \{S(t, \tau)\}_{t \geq \tau}), (C_V, \{U(t, \tau)\}_{t \geq \tau}), (C^{\tilde{h}, V}_H, \{U(t, \tau)\}_{t \geq \tau}), \\ & (V \times (L^2_{D(A)} \cap L^\infty_V), \{S(t, \tau)\}_{t \geq \tau}), (L^2_{D(A)} \cap C_V, \{U(t, \tau)\}_{t \geq \tau}), \end{aligned}$$

24 which are well-defined processes, where  $C^{\tilde{h}, V}_H = \{\varphi \in C_H : \varphi|_{[-\tilde{h}, 0]} \in B([-\tilde{h}, 0]; V)\}$ ,  
 25 being  $B([-\tilde{h}, 0]; V)$  the space of bounded functions from  $[-\tilde{h}, 0]$  into  $V$  (with  $\tilde{h} \in$   
 26  $[0, h]$ ).

27 For the sake of clarity and brevity in the exposition, among all of the above  
 28 processes we restrict ourselves to the most interesting ones, namely  $(V \times (L^2_{D(A)} \cap$   
 29  $L^\infty_V), \{S(t, \tau)\}_{t \geq \tau})$  and  $(L^2_{D(A)} \cap C_V, \{U(t, \tau)\}_{t \geq \tau})$ .

1 In order to develop a more regular theory of attractors we start studying the  
2 continuity of these new processes. We have the following result.

**Proposition 1.** Consider  $f \in L^2_{loc}(\mathbb{R}; H)$  and  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  fulfilling (H1)–(H4),  $(u^\tau, \phi)$  and  $(v^\tau, \psi) \in V \times (L^2_V \cap L^\infty_H)$ , and denote  $u = u(\cdot; \tau, u^\tau, \phi)$  and  $v = v(\cdot; \tau, v^\tau, \psi)$  the corresponding solutions to (1) with (respective) initial data. Then,

$$\begin{aligned} \|u(s) - v(s)\|^2 &\leq \left( \|u^\tau - v^\tau\|^2 + \tilde{C} \int_{-h}^0 \|\phi(r) - \psi(r)\|^2 dr \right. \\ &\quad + \frac{C^4}{2\nu^2} \int_\tau^t |u(r - \rho(r)) - v(r - \rho(r))|^2 \|u(r)\|^2 |Au(r)|^2 dr \\ &\quad \left. \times \exp \left[ \int_\tau^t \left( \tilde{C} + \frac{27C^4}{2\nu^3} |v(r - \rho(r))|^2 \|v(r - \rho(r))\|^2 \right) dr \right] \right), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\nu}{2} \int_\tau^s |Au(r) - Av(r)|^2 dr &\leq \|u^\tau - v^\tau\|^2 + \frac{2C_g^2}{\nu} \int_{\tau-h}^s |u(r) - v(r)|^2 dr \\ &\quad + 2 \int_\tau^s \|u(r - \rho(r)) - v(r - \rho(r))\|^2 dr \\ &\quad + \frac{C^4}{2\nu^2} \int_\tau^s |u(r - \rho(r)) - v(r - \rho(r))|^2 \|u(r)\|^2 |Au(r)|^2 dr \\ &\quad + \frac{27C^4}{2\nu^3} \int_\tau^s |v(r - \rho(r))|^2 \|v(r - \rho(r))\|^2 \|u(r) - v(r)\|^2 dr \end{aligned} \quad (11)$$

3 for all  $\tau \leq s \leq t$ , where  $\tilde{C} = 2C_g^2/(\lambda_1\nu) + 2/(1 - \rho^*)$ .

*Proof.* Using the second energy equality (5) for  $w := u - v$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 + \nu |Aw(t)|^2 &= b(v(t - \rho(t)), v(t), Aw(t)) - b(u(t - \rho(t)), u(t), Aw(t)) \\ &\quad + (g(t, u_t) - g(t, v_t), Aw(t)) \text{ a.e. } t > \tau. \end{aligned}$$

Standard manipulations in the trilinear term  $b$  yields

$$\begin{aligned} &b(v(t - \rho(t)), v(t), Aw(t)) - b(u(t - \rho(t)), u(t), Aw(t)) \\ &= -b(w(t - \rho(t)), u(t), Aw(t)) - b(v(t - \rho(t)), w(t), Aw(t)) \text{ a.e. } t > \tau. \end{aligned}$$

Now from (3) and the Young inequality,

$$\begin{aligned} &|b(w(t - \rho(t)), u(t), Aw(t))| + |b(v(t - \rho(t)), w(t), Aw(t))| \\ &\leq C |w(t - \rho(t))|^{1/2} \|w(t - \rho(t))\|^{1/2} \|u(t)\|^{1/2} |Au(t)|^{1/2} |Aw(t)| \\ &\quad + C |v(t - \rho(t))|^{1/2} \|v(t - \rho(t))\|^{1/2} \|w(t)\|^{1/2} |Aw(t)|^{3/2} \\ &\leq \frac{C^2}{\nu} |w(t - \rho(t))| \|w(t - \rho(t))\| \|u(t)\| |Au(t)| + \frac{\nu}{4} |Aw(t)|^2 \\ &\quad + \frac{27C^4}{4\nu^3} |v(t - \rho(t))|^2 \|v(t - \rho(t))\|^2 \|w(t)\|^2 + \frac{\nu}{4} |Aw(t)|^2 \text{ a.e. } t > \tau. \end{aligned}$$

Combining the above with the Hölder inequality and integrating, we obtain

$$\begin{aligned} & \|w(s)\|^2 - \|w(\tau)\|^2 + \frac{\nu}{2} \int_{\tau}^s |Aw(r)|^2 dr \leq \frac{2C_g^2}{\nu} \int_{\tau-h}^s |w(r)|^2 dr \\ & + 2 \int_{\tau}^s \|w(r - \rho(r))\|^2 dr + \frac{C^4}{2\nu^2} \int_{\tau}^s |w(r - \rho(r))|^2 \|u(r)\|^2 |Au(r)|^2 dr \\ & + \frac{27C^4}{2\nu^3} \int_{\tau}^s |v(r - \rho(r))|^2 \|v(r - \rho(r))\|^2 \|w(r)\|^2 dr \quad \forall \tau \leq s. \end{aligned} \quad (12)$$

Splitting the first integral in the RHS, that on  $[\tau - h, s]$  in the initial datum part and the evolutive solution on  $[\tau, s]$ , and using the Poincaré inequality and an eventual change of variables for  $t - \rho(t)$ , in particular we conclude

$$\begin{aligned} \|w(s)\|^2 & \leq \|w(\tau)\|^2 + \tilde{C} \int_{-h}^0 \|\phi(r) - \psi(r)\|^2 dr + \frac{C^4}{2\nu^2} \int_{\tau}^s |w(r - \rho(r))|^2 \|u(r)\|^2 |Au(r)|^2 dr \\ & + \int_{\tau}^s \left( \tilde{C} + \frac{27C^4}{2\nu^3} |v(r - \rho(r))|^2 \|v(r - \rho(r))\|^2 \right) \|w(r)\|^2 dr \quad \forall s \geq \tau. \end{aligned}$$

1 The Gronwall lemma gives (10). Finally, (11) follows immediately from (12).  $\square$

2 From the continuity of  $(H \times (L_V^2 \cap L_H^\infty), \{S(t, \tau)\}_{t \geq \tau})$  and  $(L_V^2 \cap C_H, \{U(t, \tau)\}_{t \geq \tau})$   
3 (cf. [4, Corollary 2]) combined with Corollary 1 and Proposition 1, we deduce

4 **Corollary 2.** *Assume that  $f \in L_{loc}^2(\mathbb{R}; H)$  and  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  satisfies  
5 (H1)–(H4). Then  $(V \times (L_{D(A)}^2 \cap L_V^\infty), \{S(t, \tau)\}_{t \geq \tau})$  and  $(L_{D(A)}^2 \cap C_V, \{U(t, \tau)\}_{t \geq \tau})$   
6 are continuous processes.*

7 The first of the two key ingredients to ensure the existence of attractors is the  
8 absorbing property. We recall from [4] several concepts, definitions and extra con-  
9 dition on  $f$  such that the absorbing property holds in a natural universe associated  
10 to problem (1). These results will be improved in the sequel, in order to gain at-  
11 tractors in higher norms. Namely, we start recalling an additional assumption for  
12 extra energy estimates.

(H5) Assume that  $\nu\lambda_1 > C_g$ , and that there exists a value  $\eta \in (0, 2(\nu\lambda_1 - C_g))$  such that for every  $u \in L^2(\tau - h, t; H)$ ,

$$\int_{\tau}^t e^{\eta s} |g(s, u_s)|^2 ds \leq C_g^2 \int_{\tau-h}^t e^{\eta s} |u(s)|^2 ds \quad \forall t \geq \tau.$$

13 Now we recall the estimates leading to the introduction of one universe for the  
14 study of pullback attractors (cf. [4, Lemma 1]).

**Lemma 1.** *Consider given  $f \in L_{loc}^2(\mathbb{R}; V')$  and  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  satisfying conditions (H1)–(H5). Then, for any  $(u^\tau, \phi) \in H \times (L_V^2 \cap L_H^\infty)$ , the following inequalities hold for the solution  $u$  to (1) for all  $t \geq s \geq \tau$ :*

$$|u(t)|^2 \leq e^{-\eta(t-\tau)} (|u^\tau|^2 + C_g \|\phi\|_{L_H^2}^2) + \frac{e^{-\eta t}}{\beta} \int_{\tau}^t e^{\eta r} \|f(r)\|_*^2 dr, \quad (13)$$

$$\nu \int_s^t \|u(r)\|^2 dr \leq |u(s)|^2 + C_g \|u_s\|_{L_H^2}^2 + \frac{1}{\nu} \int_s^t \|f(r)\|_*^2 dr + 2C_g \int_s^t |u(r)|^2 dr, \quad (14)$$

15 where

$$\beta = 2\nu - (\eta + 2C_g)\lambda_1^{-1} > 0. \quad (15)$$



1 In the context of pullback attractors, a universe is a family of time-dependent  
 2 sections, i.e. family of subsets in the phase-space, that allows to establish good dy-  
 3 namical properties, acting as basis of attraction. In [4] it was introduced  $\mathcal{D}_\eta^{H, L_H^2}(H \times$   
 4  $(L_V^2 \cap L_H^\infty))$ , the class of all families of nonempty subsets  $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset$   
 5  $\mathcal{P}(H \times (L_V^2 \cap L_H^\infty))$  such that

$$\lim_{\tau \rightarrow -\infty} \left( e^{\eta\tau} \sup_{(\zeta, \varphi) \in D(\tau)} (|\zeta|^2 + \|\varphi\|_{L_H^2}^2) \right) = 0. \quad (16)$$

6 This notion of universe is naturally related to estimate (13), where the rate of growth  
 7 in  $-\infty$  is such that the initial data is killed by the dissipativity of the problem. Let  
 8 us also observe that  $\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))$  is inclusion-closed, which allows certain  
 9 advantages in the application of the theory.

10 We will denote by  $\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))$  the universe of fixed bounded sets in  
 11  $H \times (L_V^2 \cap L_H^\infty)$ .

12 Then, after the above comments and having in mind (13), we gain the first  
 13 absorbing family (cf. [4, Corollary 3]).

14 **Corollary 3.** *Assume that  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  fulfills conditions (H1)–(H5)*  
 15 *and that  $f \in L_{loc}^2(\mathbb{R}; V')$  satisfies*

$$\int_{-\infty}^0 e^{\eta r} \|f(r)\|_*^2 dr < \infty. \quad (17)$$

Then, the family  $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H \times (L_V^2 \cap L_H^\infty))$  defined by

$$D_0(t) = \overline{B}_H(0, R_H(t)) \times (\overline{B}_{L_V^2}(0, R_V(t)) \cap \overline{B}_{L_H^\infty}(0, R_H(t))),$$

where

$$R_H^2(t) = 1 + \beta^{-1} e^{-\eta(t-2h)} \int_{-\infty}^t e^{\eta r} \|f(r)\|_*^2 dr,$$

$$R_V^2(t) = \nu^{-1} \left[ (1 + 3C_g h) R_H^2(t) + \nu^{-1} \|f\|_{L^2(t-h, t; V')}^2 \right],$$

16 is pullback  $\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))$ –absorbing for the process  $S$  on  $H \times (L_V^2 \cap L_H^\infty)$   
 17 (and therefore pullback  $\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))$ –absorbing too), and  $\widehat{D}_0$  belongs to  
 18  $\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))$ .

19 As additional universe for the study of the problem in [4, Definition 5] it was  
 20 introduced  $\mathcal{D}_\eta^{C_H}(L_V^2 \cap C_H)$ , the class of all families of nonempty subsets  $\widehat{D} = \{D(t) :$   
 21  $t \in \mathbb{R}\} \subset \mathcal{P}(L_V^2 \cap C_H)$  such that

$$\lim_{\tau \rightarrow -\infty} \left( e^{\eta\tau} \sup_{\phi \in D(\tau)} |\phi|_{C_H}^2 \right) = 0. \quad (18)$$

22 **Remark 3.** After Corollary 3 it is immediate to realize that  $\widehat{D}_1 = \{D_1(t) : t \in$   
 23  $\mathbb{R}\} \subset \mathcal{P}(L_V^2 \cap C_H)$  given by  $D_1(t) = \overline{B}_{L_V^2}(0, R_V(t)) \cap \overline{B}_{C_H}(0, R_H(t))$  is pullback  
 24  $\mathcal{D}_\eta^{C_H}(L_V^2 \cap C_H)$ –absorbing for  $U$  on  $L_V^2 \cap C_H$ .

25 Since the pullback estimates necessary for the dissipativity only require the  
 26 tempered character in norms  $H \times L_H^2$  or  $C_H$ , we may consider more restricted  
 27 universes, with higher norms but the same tempered condition. So, analogously to  
 28 the previous definitions of  $\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))$  and  $\mathcal{D}_\eta^{C_H}(L_V^2 \cap C_H)$  we introduce  
 29 the following classes.

1 **Definition 3.** Denote by  $\mathcal{D}_\eta^{H, L^2_H}(V \times (L^2_{D(A)} \cap L^\infty_V))$  the class of all families of  
 2 nonempty subsets  $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times (L^2_{D(A)} \cap L^\infty_V))$  such that (16) holds.

3 Denote by  $\mathcal{D}_\eta^{C_H}(L^2_{D(A)} \cap C_V)$  the class of all families of nonempty subsets  $\widehat{D} =$   
 4  $\{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2_{D(A)} \cap C_V)$  such that (18) holds.

5 We also denote by  $\mathcal{D}_F(V \times (L^2_{D(A)} \cap L^\infty_V))$  and  $\mathcal{D}_F(L^2_{D(A)} \cap C_V)$  the universes of  
 6 fixed bounded sets in  $V \times (L^2_{D(A)} \cap L^\infty_V)$  and  $L^2_{D(A)} \cap C_V$  respectively.

7 **Remark 4.** After Corollary 3 and Remark 3, as an immediate consequence of  
 8 Theorem 2,  $\widehat{D}_0 \cap \mathcal{P}(V \times (L^2_{D(A)} \cap L^\infty_V))$  and  $\widehat{D}_1 \cap \mathcal{P}(L^2_{D(A)} \cap C_V)$  are pullback  
 9  $\mathcal{D}_\eta^{H, L^2_H}(V \times (L^2_{D(A)} \cap L^\infty_V))$ -absorbing and pullback  $\mathcal{D}_\eta^{C_H}(L^2_{D(A)} \cap C_V)$ -absorbing  
 10 for  $S$  and  $U$  on  $V \times (L^2_{D(A)} \cap L^\infty_V)$  and  $L^2_{D(A)} \cap C_V$  respectively.

11 **4. Asymptotic compactness.** The first goal of this section is to provide the  
 12 sufficient uniform estimates at any current time  $t$  such that the data is starting  
 13 pullback enough in time. The cumbersome choice of some intervals is due to the  
 14 necessity of controlling several delay terms appearing in the computations. This  
 15 will end up with the asymptotic compactness of the associated dynamical systems  
 16 introduced previously.

17 **Lemma 2.** Assume that  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  and  $f \in L^2_{loc}(\mathbb{R}; H)$  satisfy (H1)-  
 18 (H5) and (17) respectively. Then, for any  $t \in \mathbb{R}$  and  $\widehat{D} \in \mathcal{D}_\eta^{H, L^2_H}(H \times (L^2_V \cap L^\infty_H))$ ,  
 19 there exists  $\tau(\widehat{D}, t, h) < t - 5h - 2$  and functions  $\{\rho_i\}_{i=1}^4$  depending on  $t$  and  $h$ , such  
 20 that for any  $\tau \leq \tau(\widehat{D}, t, h)$  and any  $(u^\tau, \phi^\tau) \in D(\tau)$ ,

$$\left\{ \begin{array}{l} |u(r; \tau, u^\tau, \phi^\tau)|^2 \leq \rho_1(t) \quad \forall r \in [t - 5h - 2, t], \\ \|u(r; \tau, u^\tau, \phi^\tau)\|^2 \leq \rho_2(t) \quad \forall r \in [t - 3h - 1, t], \\ \frac{\nu}{2} \int_{r-1}^r |Au(s)|^2 ds \leq \rho_3(t) \quad \forall r \in [t - 2h, t], \\ \int_{r-1}^r |u'(s)|^2 ds \leq \rho_4(t) \quad \forall r \in [t - 2h, t], \end{array} \right. \quad (19)$$

where

$$\rho_1(t) = 1 + e^{-\eta(t-5h-2)} \beta^{-1} \int_{-\infty}^t e^{\eta s} \|f(s)\|_*^2 ds,$$

$$\begin{aligned} \rho_2(t) &= \left[ \frac{\rho_1(t)}{\nu} (1 + C_g(h + 2 + 2C_g(h + 1))) + \int_{t-3h-2}^t \left( \frac{\|f(s)\|_*^2}{\nu^2} + \frac{2}{\nu} |f(s)|^2 \right) ds \right] \\ &\quad \times \exp \left[ \frac{27C^4 \rho_1(t)}{2\nu^4(1 - \rho^*)} \left( \rho_1(t)(1 + C_g(3h + 2)) + \frac{1}{\nu} \int_{t-4h-2}^t \|f(s)\|_*^2 ds \right) + \frac{2C_g^2}{\lambda_1 \nu} \right], \end{aligned}$$

$$\rho_3(t) = \rho_2(t) + \frac{27C^4}{2\nu^3} \rho_1(t) \rho_2^2(t) + \frac{2}{\nu} \int_{t-2h-1}^t |f(s)|^2 ds + \frac{2C_g^2}{\nu} (h + 1) \rho_1(t),$$

$$\rho_4(t) = \nu \rho_2(t) + 4 \int_{t-2h-1}^t |f(s)|^2 ds + 4C_g^2(h + 1) \rho_1(t) + 2^{3/2} C^2 \rho_1^{1/2}(t) \rho_2(t) \left( \frac{\rho_3(t)}{\nu} \right)^{1/2}.$$

21 *Proof.* First estimate in (19) and the formula for  $\rho_1$  is a direct consequence of (13)  
 22 in the interval  $[t - 5h - 2, t]$  since the universe is tempered w.r.t. the exponential  
 23 with parameter  $\eta$ .

Using the Galerkin approximations introduced in Theorem 2, multiplying each equation by  $\lambda_j \gamma_{jk}(t)$ , summing from  $j = 1$  to  $k$ , and using the Cauchy-Schwartz and Young inequalities,

$$\begin{aligned} & \frac{d}{dt} \|u^k(t)\|^2 + 2\nu |Au^k(t)|^2 + 2b(u^k(t - \rho(t)), u^k(t), Au^k(t)) \\ & \leq \nu |Au^k(t)|^2 + \frac{2}{\nu} |f(t)|^2 + \frac{2}{\nu} |g(t, u_t^k)|^2 \text{ a.e. } t > \tau. \end{aligned}$$

From (3) and the Young inequality,

$$\begin{aligned} & |b(u^k(t - \rho(t)), u^k(t), Au^k(t))| \\ & \leq C |u^k(t - \rho(t))|^{1/2} \|u^k(t - \rho(t))\|^{1/2} \|u^k(t)\|^{1/2} |Au^k(t)|^{3/2} \\ & \leq \frac{27C^4}{4\nu^3} |u^k(t - \rho(t))|^2 \|u^k(t - \rho(t))\|^2 \|u^k(t)\|^2 + \frac{\nu}{4} |Au^k(t)|^2. \end{aligned}$$

Plugging this into the above inequality,

$$\begin{aligned} & \frac{d}{dt} \|u^k(t)\|^2 + \frac{\nu}{2} |Au^k(t)|^2 \tag{20} \\ & \leq \frac{27C^4}{2\nu^3} |u^k(t - \rho(t))|^2 \|u^k(t - \rho(t))\|^2 \|u^k(t)\|^2 + \frac{2}{\nu} |f(t)|^2 + \frac{2}{\nu} |g(t, u_t^k)|^2 \text{ a.e. } t > \tau. \end{aligned}$$

Integrating in  $[s, t]$  with  $s \geq \tau$ , and using (H4),

$$\begin{aligned} & \|u^k(t)\|^2 + \frac{\nu}{2} \int_s^t |Au^k(r)|^2 dr \\ & \leq \|u^k(s)\|^2 + \int_s^t \left( \frac{27C^4}{2\nu^3} |u^k(r - \rho(r))|^2 \|u^k(r - \rho(r))\|^2 + \frac{2C_g^2}{\lambda_1 \nu} \right) \|u^k(r)\|^2 dr \\ & \quad + \frac{2}{\nu} \int_s^t |f(r)|^2 dr + \frac{2C_g^2}{\nu} \int_{s-h}^s |u^k(r)|^2 dr. \end{aligned}$$

In particular, the Gronwall lemma yields

$$\begin{aligned} \|u^k(r)\|^2 & \leq \left( \|u^k(s)\|^2 + \frac{2}{\nu} \int_s^r |f(\theta)|^2 d\theta + \frac{2C_g^2}{\nu} \|u_s^k\|_{L^2_H}^2 \right) \\ & \quad \times \exp \left[ \int_s^r \left( \frac{27C^4}{2\nu^3} |u^k(\theta - \rho(\theta))|^2 \|u^k(\theta - \rho(\theta))\|^2 + \frac{2C_g^2}{\lambda_1 \nu} \right) d\theta \right] \tag{21} \end{aligned}$$

1 for any  $\tau \leq s \leq r$ .

Integrating w.r.t.  $s \in (r-1, r)$

$$\begin{aligned} \|u^k(r)\|^2 & \leq \left( \int_{r-1}^r \|u^k(s)\|^2 ds + \frac{2}{\nu} \int_{r-1}^r |f(\theta)|^2 d\theta + \frac{2C_g^2}{\nu} \int_{r-h-1}^r |u^k(\theta)|^2 d\theta \right) \\ & \quad \times \exp \left[ \int_{r-1}^r \left( \frac{27C^4}{2\nu^3} |u^k(\theta - \rho(\theta))|^2 \|u^k(\theta - \rho(\theta))\|^2 + \frac{2C_g^2}{\lambda_1 \nu} \right) d\theta \right] \end{aligned}$$

2 for any  $r \geq \tau + 1$ .

Using the first estimate in (19) by  $\rho_1$  proved above (which is valid also for the Galerkin approximations), and twice the estimate (14) for  $\|u^k\|_{L^2(r-1, r; V)}^2$  and  $\|u^k(\cdot - \rho(\cdot))\|_{L^2(r-1, r; V)}^2$  (this last one of order  $\|u^k\|_{L^2(r-h-1, r; V)}^2$  by a change of

variables), after some tedious computations (we omit the details for the sake of brevity) we conclude that

$$\begin{aligned} \|u^k(r)\|^2 &\leq \left[ \frac{\rho_1(t)}{\nu} (1 + C_g(h+2 + 2C_g(h+1))) + \int_{t-3h-2}^t \left( \frac{\|f(s)\|_*^2}{\nu^2} + \frac{2}{\nu} |f(s)|^2 \right) ds \right] \\ &\quad \times \exp \left[ \frac{27C^4 \rho_1(t)}{2\nu^4(1-\rho^*)} \left( \rho_1(t)(1+C_g(3h+2)) + \frac{1}{\nu} \int_{t-4h-2}^t \|f(s)\|_*^2 ds \right) + \frac{2C_g^2}{\lambda_1 \nu} \right] \end{aligned}$$

1 for all  $r \in [t-3h-1, t]$  and any  $k \in \mathbb{N}$ .

2 Since the convergence of the Galerkin approximations towards the solution  $u$   
3 holds weakly-star in  $L^\infty(t-3h-1, t; V)$  and  $u \in C([t-3h-1, t]; V)$  by Theorem  
4 2, taking the inferior limit when  $k$  goes to infinity above, we obtain the second  
5 estimate in (19).

Going back to (20), integrating in  $[r-1, r]$  we obtain

$$\begin{aligned} \frac{\nu}{2} \int_{r-1}^r |Au^k(s)|^2 ds &\leq \|u^k(r-1)\|^2 + \frac{2}{\nu} \int_{r-1}^r |f(s)|^2 ds + \frac{2}{\nu} \int_{r-1}^r |g(s, u_s^k)|^2 ds \\ &\quad + \frac{27C^4}{2\nu^3} \int_{r-1}^r |u^k(s-\rho(s))|^2 \|u^k(s-\rho(s))\|^2 \|u^k(s)\|^2 ds \\ &\leq \rho_2(t) + \frac{27C^4}{2\nu^3} \rho_1(t) \rho_2^2(t) + \frac{2}{\nu} \int_{t-2h-1}^t |f(s)|^2 ds + \frac{2C_g^2}{\nu} (h+1) \rho_1(t), \end{aligned}$$

6 for any  $r \in [t-2h, t]$ , where we have used (H4) and the two first estimates from  
7 (19) involving  $\rho_1$  and  $\rho_2$ .

8 Since the convergence of the Galerkin approximations towards the solution  $u$   
9 holds weakly in  $L^2(t-2h, t; D(A))$  by Theorem 2, taking the inferior limit when  $k$   
10 goes to infinity above, we obtain the third estimate in (19).

Finally, multiplying each equation in (6) by  $\gamma'_{jk}(t)$  and summing from  $j=1$  till  
 $k$ , after the Young inequality

$$\begin{aligned} |(u^k)'(s)|^2 + \frac{\nu}{2} \frac{d}{ds} \|u^k(s)\|^2 + b(u^k(s-\rho(s)), u^k(s), (u^k)'(s)) \\ = (f(s) + g(s, u_s^k), (u^k)'(s)) \leq 2|f(s)|^2 + 2|g(s, u_s^k)|^2 + \frac{1}{4} |(u^k)'(s)|^2 \text{ a.e. } s > \tau. \end{aligned}$$

From (3) and the Young inequality once more,

$$\begin{aligned} |b(u^k(s-\rho(s)), u^k(s), (u^k)'(s))| \\ \leq \frac{1}{4} |(u^k)'(s)|^2 + C^2 |u^k(s-\rho(s))| \|u^k(s-\rho(s))\| \|u^k(s)\| \|Au^k(s)\|. \end{aligned}$$

Plugging this into the above estimate gives

$$\begin{aligned} |(u^k)'(s)|^2 + \nu \frac{d}{ds} \|u^k(s)\|^2 \\ \leq 4|f(s)|^2 + 4|g(s, u_s^k)|^2 + 2C^2 |u^k(s-\rho(s))| \|u^k(s-\rho(s))\| \|u^k(s)\| \|Au^k(s)\| \text{ a.e. } s > \tau. \end{aligned}$$

In particular, integrating and combining (H4) with the previous estimates involving  $\{\rho_i\}_{i=1}^3$ ,

$$\begin{aligned} \int_{r-1}^r |(u^k)'(s)|^2 ds &\leq \nu \|u^k(r-1)\|^2 + 4 \int_{r-1}^r |f(s)|^2 ds + 4C_g^2 \int_{r-h-1}^r |u^k(s)|^2 ds \\ &\quad + 2C^2 \int_{r-1}^r |u^k(s-\rho(s))| \|u^k(s-\rho(s))\| \|u^k(s)\| |Au^k(s)| ds \\ &\leq \nu \rho_2(t) + 4 \int_{t-2h-1}^t |f(s)|^2 ds + 4C_g^2(h+1)\rho_1(t) \\ &\quad + 2C^2 \rho_1^{1/2}(t) \rho_2(t) \left( \frac{2}{\nu} \rho_3(t) \right)^{1/2} \end{aligned}$$

1 for any  $r \in [t-2h, t]$ .

2 Analogously to the previous arguments, following these computations one can  
3 check that the time derivatives  $(u^k)'$  of the Galerkin approximations given in Theorem  
4 2 converge towards  $u'$  weakly in  $L^2(t-2h, t; H)$ . Taking the inferior limit when  $k$   
5 goes to infinity above, we obtain the last estimate in (19).  $\square$

6 Now we may use the uniform estimates proved above to apply an energy method  
7 yielding the asymptotic compactness. The ideas are analogous to [6, Lemma 5.3]  
8 and [8, Lemma 4], but nevertheless the result here is not a verbatim copy of those.  
9 Actually, the presence of two delays make the arguments more involved. So we  
10 include the proof for the sake of clarity.

11 **Lemma 3.** *Assume that  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  and  $f \in L_{loc}^2(\mathbb{R}; H)$  satisfy (H1)–  
12 (H5) and (17) respectively. Then, for any  $t \in \mathbb{R}$ , any  $\widehat{D} \in \mathcal{D}_\eta^{H, L^2_H}(H \times (L^2_V \cap L^\infty_H))$   
13 and any sequences  $\{\tau_n\} \subset (-\infty, t]$  and  $\{(u^{\tau_n}, \phi^{\tau_n})\} \subset H \times (L^2_V \cap L^\infty_H)$  with  $\tau_n \rightarrow -\infty$   
14 and  $(u^{\tau_n}, \phi^{\tau_n}) \in D(\tau_n)$ , the sequence  $\{u(\cdot; \tau_n, u^{\tau_n}, \phi^{\tau_n})\}$  is relatively compact in  
15  $C([t-h, t]; V) \cap L^2(t-h, t; D(A))$ .*

16 *Proof.* Fix  $t \in \mathbb{R}$ , a family  $\widehat{D} \in \mathcal{D}_\eta^{H, L^2_H}(H \times (L^2_V \cap L^\infty_H))$ , and sequences  $\{\tau_n\} \subset$   
17  $(-\infty, t]$  and  $\{(u^{\tau_n}, \phi^{\tau_n})\} \subset H \times (L^2_V \cap L^\infty_H)$  as in the statement.

18 Denote for short  $u^n(\cdot)$  to  $u(\cdot; \tau_n, u^{\tau_n}, \phi^{\tau_n})$ . Firstly we will check the relative  
19 compactness of  $\{u^n\}$  in  $C([t-h, t]; V)$  and secondly in  $L^2(t-h, t; D(A))$ .

20 Indeed, thanks to Lemma 2, consider  $\tau(\widehat{D}, t, h) < t-5h-2$  such that the sequence  
21  $\{u^n : \tau_n \leq \tau(\widehat{D}, t, h)\}$  is bounded in  $L^\infty(t-3h-1, t; V) \cap L^2(t-2h-1, t; D(A))$   
22 with  $\{(u^n)'\}$  bounded in  $L^2(t-2h-1, t; H)$ . By using the Aubin-Lions compactness  
23 lemma there exists  $u \in L^\infty(t-3h-1, t; V) \cap L^2(t-2h-1, t; D(A))$  with  $u' \in$   
24  $L^2(t-2h-1, t; H)$  such that a sequence (labeled the same) satisfies

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-3h-1, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2h-1, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2h-1, t; H), \\ u^n \rightarrow u & \text{strongly in } L^2(t-2h-1, t; V), \\ u^n(s) \rightarrow u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-2h-1, t). \end{cases} \quad (22)$$

25 From these convergences we deduce that  $u \in C([t-2h-1, t]; V)$  is a strong solution  
26 to (1) in  $(t-h-1, t)$  with suitable initial data. Boundedness of  $\{u^n\}$  in  $L^\infty(t-2h-1, t; V)$   
27 jointly with equi-continuity on  $[t-2h-1, t]$  with values in  $H$ , leads by  
28 the Ascoli-Arzelà theorem (up to a subsequence, labeled the same) to

$$u^n \rightarrow u \text{ strongly in } C([t-2h-1, t]; H), \quad (23)$$

1 which also helps to identify the weak-limit in next property

$$u^n(s_n) \rightharpoonup u(s_*) \text{ weakly in } V \text{ for any } \{s_n\} \subset [t - 2h - 1, t] \text{ with } s_n \rightarrow s_*. \quad (24)$$

2 Now we can prove that

$$u^n \rightarrow u \text{ strongly in } C([t - h, t]; V). \quad (25)$$

3 Indeed, if (25) is false, there exists  $\varepsilon > 0$ ,  $t_* \in [t - h, t]$  and sequences (labeled the  
4 same)  $\{u^n\}$  and  $\{t_n\} \subset [t - h, t]$  with  $\lim_n t_n = t_*$  such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1. \quad (26)$$

5 However, from (24)

$$\|u(t_*)\| \leq \liminf_{n \rightarrow \infty} \|u^n(t_n)\|. \quad (27)$$

The second energy equality (5) for  $w = u^n$  or  $w = u$  reads

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\theta} \|w(\theta)\|^2 + \nu |Aw(\theta)|^2 + b(w(\theta - \rho(\theta)), w(\theta), Aw(\theta)) \\ & = (f(\theta) + g(\theta, w_\theta), Aw(\theta)) \text{ a.e. } \theta > t - h - 1. \end{aligned}$$

Combining the Young inequality for the RHS and for the trilinear term  $b$  (after (3))

$$\begin{aligned} & |b(w(\theta - \rho(\theta)), w(\theta), Aw(\theta))| \\ & \leq \frac{27C^4}{4\nu^3} |w(\theta - \rho(\theta))|^2 \|w(\theta - \rho(\theta))\|^2 \|w(\theta)\|^2 + \frac{\nu}{4} |Aw(\theta)|^2, \end{aligned}$$

we conclude

$$\begin{aligned} & \frac{d}{d\theta} \|w(\theta)\|^2 + \nu |Aw(\theta)|^2 \\ & \leq \frac{27C^4}{2\nu^3} |w(\theta - \rho(\theta))|^2 \|w(\theta - \rho(\theta))\|^2 \|w(\theta)\|^2 + \frac{4}{\nu} (|f(\theta)|^2 + |g(\theta, w_\theta)|^2) \end{aligned}$$

a.e.  $\theta > t - h - 1$ . Integrating, we have for any  $w = u^n$  or  $w = u$

$$\begin{aligned} & \|w(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Aw(r)|^2 dr \\ & \leq \|w(s_1)\|^2 + \frac{27C^4}{2\nu^3} \int_{s_1}^{s_2} |w(r - \rho(r))|^2 \|w(r - \rho(r))\|^2 \|w(r)\|^2 dr \\ & \quad + \frac{4}{\nu} \int_{s_1}^{s_2} (|f(r)|^2 + |g(r, w_r)|^2) dr \quad \forall t - h - 1 \leq s_1 \leq s_2 \leq t. \quad (28) \end{aligned}$$

Neglecting the integral term in the LHS, we may consider the functions

$$\begin{aligned} J_n(s) & = \|u^n(s)\|^2 - \frac{27C^4}{2\nu^3} \int_{t-h-1}^s |u^n(r - \rho(r))|^2 \|u^n(r - \rho(r))\|^2 \|u^n(r)\|^2 dr \\ & \quad - \frac{4}{\nu} \int_{t-h-1}^s (|f(r)|^2 + |g(r, u_r^n)|^2) dr \end{aligned}$$

and

$$\begin{aligned} J(s) & = \|u(s)\|^2 - \frac{27C^4}{2\nu^3} \int_{t-h-1}^s |u(r - \rho(r))|^2 \|u(r - \rho(r))\|^2 \|u(r)\|^2 dr \\ & \quad - \frac{4}{\nu} \int_{t-h-1}^s (|f(r)|^2 + |g(r, u_r)|^2) dr. \end{aligned}$$

These functions are continuous, and from the corresponding inequalities above, they are non-increasing. Moreover, from (22) and (H4),

$$J_n(s) \rightarrow J(s) \text{ a.e. } s \in (t-h-1, t).$$

It is now a standard matter to deduce (e.g., cf. [4, Lemma 2]) that

$$\limsup_{n \rightarrow \infty} J_n(t_n) \leq J(t_*),$$

whence, after (22) and (H4) again,

$$\limsup_{n \rightarrow \infty} \|u^n(t_n)\| \leq \|u(t_*)\|,$$

- 1 which combined with (27) and (24) gives that  $u^n(t_n) \rightarrow u(t_*)$  strongly in  $V$ , con-  
 2 tradicting (26). Therefore, (25) is proved.

Going back to the inequality (28), after the convergences proved of  $u^n$  towards  $u$  above, observing the integral term in the LHS, we deduce that  $u^n \rightarrow u$  in norm in  $L^2(t-h, t; D(A))$ , which jointly with (22) means that

$$u^n \rightarrow u \text{ strongly in } L^2(t-h, t; D(A)).$$

3

□

- 4 Observe that in particular the above gives immediately the following result.

5 **Corollary 4.** *Assume that  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  and  $f \in L^2_{loc}(\mathbb{R}; H)$  satisfy (H1)–  
 6 (H5) and (17) respectively. Then,  $(V \times (L^2_{D(A)} \cap L^\infty_V), \{S(t, \tau)\}_{t \geq \tau})$  and  $(L^2_{D(A)} \cap$   
 7  $C_V, \{U(t, \tau)\}_{t \geq \tau})$  are pullback asymptotically compact in  $\mathcal{D}_\eta^{H, L^2_H}(V \times (L^2_{D(A)} \cap L^\infty_V))$   
 8 and  $\mathcal{D}_\eta^{C_H}(L^2_{D(A)} \cap C_V)$  respectively.*

- 9 **5. Existence of attractors and their relationships.** Before establishing the  
 10 main result of the paper about the existence of attractors, it is useful to give an-  
 11 other one concerning the relation of tempered families in the two universes we are  
 12 considering. Namely, just allowing the solutions to (1) evolve for an elapsed time  
 13 bigger than the delay time  $h$ , the regularization effect makes, roughly speaking,  
 14 that a family tempered in one universe is mapped into a tempered family in a more  
 15 regular universe. This type of result is the analogous to [4, Lemma 3], and it is  
 16 helpful in order to establish comparison between attractors.

17 **Lemma 4.** *Under the assumptions of Corollary 3, for any  $\widehat{D} = \{D(\tau) : \tau \in \mathbb{R}\} \in$   
 18  $\mathcal{D}_\eta^{H, L^2_H}(H \times (L^2_V \cap L^\infty_H))$  and any  $r > h$ , the family  $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$   
 19 where  $D^{(r)}(\tau) = \{u_{\tau+r}(\cdot; \tau, u^\tau, \phi) : (u^\tau, \phi) \in D(\tau)\}$ , for any  $\tau \in \mathbb{R}$ , belongs to  
 20  $\mathcal{D}_\eta^{C_H}(L^2_{D(A)} \cap C_V)$ .*

*Proof.* The inclusion  $D^{(r)}(\tau) \subset L^2_{D(A)} \cap C_V$  follows from Theorem 2. The tempered character of any solution  $u_{\tau+r}$  in  $C_H$  for any  $\tau \in \mathbb{R}$  and  $(u^\tau, \phi) \in D(\tau)$  can be deduced from estimate (13) and assumption (17) for  $f$ . Indeed,

$$|u(\tau+s)|^2 \leq e^{-\eta s} (|u^\tau|^2 + C_g \|\phi\|_{L^2_H}^2) + \beta^{-1} e^{-\eta(\tau+s)} \int_\tau^{\tau+s} e^{\eta\theta} \|f(\theta)\|_*^2 d\theta \quad \forall s \geq 0.$$

Therefore,

$$\begin{aligned} |u_{\tau+r}|_{C_H}^2 &= \sup_{s \in [r-h, r]} |u(\tau+s)|^2 \\ &\leq e^{-\eta(r-h)} (|u^\tau|^2 + C_g \|\phi\|_{L_H^2}^2) + \beta^{-1} e^{-\eta(\tau+r-h)} \int_\tau^{\tau+r} e^{\eta\theta} \|f(\theta)\|_*^2 d\theta, \end{aligned}$$

1 whence  $\lim_{\tau \rightarrow -\infty} e^{\eta\tau} \sup_{u_{\tau+r} \in D(r)(\tau)} |u_{\tau+r}|_{C_H}^2 = 0$ .  $\square$

2 Now we may establish our main result.

3 **Theorem 3.** *Assume that  $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$  and  $f \in L_{loc}^2(\mathbb{R}; H)$  satisfy*  
 4 *(H1)–(H5) and (17) respectively. Then there exist the minimal pullback attractors*  
 5  *$\{\mathcal{A}_{\mathcal{D}_F(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s)\}_{s \in \mathbb{R}}$  and  $\{\mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s)\}_{s \in \mathbb{R}}$  for  $(V \times (L_{D(A)}^2 \cap$   
 6  $L_V^\infty), \{S(t, \tau)\}_{t \geq \tau})$  and  $\{\mathcal{A}_{\mathcal{D}_F(L_{D(A)}^2 \cap C_V)}(s)\}_{s \in \mathbb{R}}$  and  $\{\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)}(s)\}_{s \in \mathbb{R}}$  for  
 7  $(L_{D(A)}^2 \cap C_V, \{U(t, \tau)\}_{t \geq \tau})$ .*

The following relations hold for any  $s \in \mathbb{R}$

$$\mathcal{A}_{\mathcal{D}_F(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s) \subset \mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s) \subset D_0(s), \quad (29)$$

$$\mathcal{A}_{\mathcal{D}_F(L_{D(A)}^2 \cap C_V)}(s) \subset \mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)}(s) \subset D_1(s), \quad (30)$$

$$\tilde{j}(\mathcal{A}_{\mathcal{D}_F(L_{D(A)}^2 \cap C_V)}(s)) \subset \mathcal{A}_{\mathcal{D}_F(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s), \quad (31)$$

$$\tilde{j}(\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)}(s)) = \mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s), \quad (32)$$

8 where  $\tilde{j} : L_{D(A)}^2 \cap C_V \rightarrow V \times (L_{D(A)}^2 \cap C_V)$  is defined by  $\tilde{j}(\phi) = (\phi(0), \phi)$ .

9 Finally, if  $f$  satisfies

$$\sup_{s \leq 0} \int_{s-1}^s |f(\theta)|^2 d\theta < \infty, \quad (33)$$

10 then (31) becomes an equality,  $\mathcal{A}_{\mathcal{D}_F(V \times (L_{D(A)}^2 \cap L_V^\infty))} = \mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))}$  and

11  $\mathcal{A}_{\mathcal{D}_F(L_{D(A)}^2 \cap C_V)} = \mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)}$ .

12 *Proof.* The abstract result ensuring existence of pullback attractors for the tempered  
 13 universes (e.g., cf. [5, Theorem 3.11] or [4, Theorem 3]) can be applied for both  
 14  $(V \times (L_{D(A)}^2 \cap L_V^\infty), \{S(t, \tau)\}_{t \geq \tau})$  and  $(L_{D(A)}^2 \cap C_V, \{U(t, \tau)\}_{t \geq \tau})$  since they are  
 15 continuous processes (cf. Corollary 2), with absorbing families (cf. Remark 4) and  
 16 fulfilling the pullback asymptotic compact property (cf. Corollary 4) w.r.t. the  
 17 universes  $\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))$  and  $\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)$  respectively.

18 The existence of pullback attractors for the case of universes of fixed bounded  
 19 sets is simpler (e.g., cf. [14] or [4, Corollary 1]) since they are contained in the  
 20 tempered universes. This also implies relations (29) and (30).

21 Relations (31) and (32) via  $\tilde{j}$  follows by the characterization of minimal pullback  
 22 attractors and Lemma 4.

23 Last statement follows again from [4, Corollary 1] since absorbing families in  
 24 these phase-spaces are uniformly bounded in time (see expressions of  $\rho_2$  and  $\rho_3$  in  
 25 Lemma 2).  $\square$

26 Using comparison results for attractors (e.g., cf. [5, Theorem 3.15]) and relations  
 27 among tempered families of different universes after a time-shift (thanks to Lemma



4), it is possible to relate the attractors  $\mathcal{A}_{\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))}$  and  $\mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))}$  for  $(H \times (L_V^2 \cap L_H^\infty), \{S(t, \tau)\}_{t \geq \tau})$  and  $\mathcal{A}_{\mathcal{D}_F(L_V^2 \cap C_H)}$  and  $\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_V^2 \cap C_H)}$  for  $(L_V^2 \cap C_H, \{U(t, \tau)\}_{t \geq \tau})$  obtained in [4, Theorems 4 and 5] with the ones obtained here in Theorem 3. Arguments are analogous to the above proof so it is omitted.

**Theorem 4.** *Under the assumptions of Theorem 3, the attractors cited previously satisfy the following relations*

$$\mathcal{A}_{\mathcal{D}_F(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s) \subset \mathcal{A}_{\mathcal{D}_F(H \times (L_V^2 \cap L_H^\infty))}(s), \quad (34)$$

$$\mathcal{A}_{\mathcal{D}_F(L_{D(A)}^2 \cap C_V)}(s) \subset \mathcal{A}_{\mathcal{D}_F(L_V^2 \cap C_H)}(s), \quad (35)$$

$$\mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(H \times (L_V^2 \cap L_H^\infty))}(s) = \mathcal{A}_{\mathcal{D}_\eta^{H, L_H^2}(V \times (L_{D(A)}^2 \cap L_V^\infty))}(s),$$

$$\mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_V^2 \cap C_H)}(s) = \mathcal{A}_{\mathcal{D}_\eta^{C_H}(L_{D(A)}^2 \cap C_V)}(s)$$

for any  $s \in \mathbb{R}$ .

Moreover, if  $f$  satisfies (33), then (34) and (35) are in fact equalities.

**Acknowledgments.** The authors wish to congratulate Prof. Kloeden for his 70th birthday. Particularly, P.M.-R. thanks Prof. Kloeden for their relationship during all these years, both academic and friendship, and for his continuous encouragement from the beginning of his scientific career.

The authors also wish to thank the referees for their careful reading and comments on a previous version of the manuscript.

Partially supported by the projects MTM2015-63723-P (MINECO/FEDER, EU) and P12-FQM-1492 (Junta de Andalucía). G. Planas was partially supported by CNPq-Brazil, grants 306646/2015-3 and 402388/2016-0.

## REFERENCES

- [1] T. Caraballo and J. Real, [Navier-Stokes equations with delays](#), *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **457** (2001), 2441–2453.
- [2] T. Caraballo and J. Real, [Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays](#), *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **459** (2003), 3181–3194.
- [3] T. Caraballo and J. Real, [Attractors for 2D-Navier-Stokes models with delays](#), *J. Differential Equations*, **205** (2004), 271–297.
- [4] J. García-Luengo, P. Marín-Rubio and G. Planas, [Attractors for a double time-delayed 2D-Navier-Stokes model](#), *Discrete Contin. Dyn. Syst.*, **34** (2014), 4085–4105.
- [5] J. García-Luengo, P. Marín-Rubio and J. Real, [Pullback attractors in  \$V\$  for non-autonomous 2D-Navier-Stokes equations and their tempered behaviour](#), *J. Differential Equations*, **252** (2012), 4333–4356.
- [6] J. García-Luengo, P. Marín-Rubio and J. Real, [Pullback attractors for 2D Navier-Stokes equations with delays and their regularity](#), *Adv. Nonlinear Stud.*, **13** (2013), 331–357.
- [7] J. García-Luengo, P. Marín-Rubio and J. Real, [Regularity of pullback attractors and attraction in  \$H^1\$  in arbitrarily large finite intervals for 2D Navier-Stokes equations with infinite delay](#), *Discrete Contin. Dyn. Syst.*, **34** (2014), 181–201.
- [8] J. García-Luengo, P. Marín-Rubio and J. Real, [Some new regularity results of pullback attractors for 2D Navier-Stokes equations with delays](#), *Commun. Pure Appl. Anal.*, **14** (2015), 1603–1621.
- [9] M. J. Garrido-Atienza and P. Marín-Rubio, [Navier-Stokes equations with delays on unbounded domains](#), *Nonlinear Anal.*, **64** (2006), 1100–1118.
- [10] S. M. Guzzo and G. Planas, [On a class of three dimensional Navier-Stokes equations with bounded delay](#), *Discrete Contin. Dyn. Syst. Ser. B*, **16** (2011), 225–238.
- [11] S. M. Guzzo and G. Planas, [Existence of solutions for a class of Navier-Stokes equations with infinite delay](#), *Appl. Anal.* **94** (2015), 840–855.

- 1 [12] W. Liu, *Asymptotic behavior of solutions of time-delayed Burgers' equation*, *Discrete Contin.*  
2 *Dyn. Syst. Ser. B*, **2** (2002), 47–56.
- 3 [13] P. Marín-Rubio and J. Real, *Attractors for 2D-Navier-Stokes equations with delays on some*  
4 *unbounded domains*, *Nonlinear Anal.*, **67** (2007), 2784–2799.
- 5 [14] P. Marín-Rubio and J. Real, *On the relation between two different concepts of pullback*  
6 *attractors for non-autonomous dynamical systems*, *Nonlinear Anal.*, **71** (2009), 3956–3963.
- 7 [15] P. Marín-Rubio and J. Real, *Pullback attractors for 2D-Navier-Stokes equations with delays*  
8 *in continuous and sub-linear operators*, *Discrete Contin. Dyn. Syst.*, **26** (2010), 989–1006.
- 9 [16] P. Marín-Rubio, J. Real and J. Valero, *Pullback attractors for a two-dimensional Navier-*  
10 *Stokes model in an infinite delay case*, *Nonlinear Anal.*, **74** (2011), 2012–2030.
- 11 [17] G. Planas and E. Hernández, *Asymptotic behaviour of two-dimensional time-delayed Navier-*  
12 *Stokes equations*, *Discrete Contin. Dyn. Syst.*, **21** (2008), 1245–1258.
- 13 [18] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-  
14 Verlag, New York, 1997.
- 15 [19] T. Taniguchi, *The exponential behavior of Navier-Stokes equations with time delay external*  
16 *force*, *Discrete Contin. Dyn. Syst.*, **12** (2005), 997–1018.
- 17 [20] W. Varnhorn, *The Navier-Stokes equations with time delay*, *Applied Mathematical Sciences*,  
18 **2** (2008), 947–960.
- 19 *E-mail address*, Julia García-Luengo: [julia.gluengo@upm.es](mailto:julia.gluengo@upm.es)  
20 *E-mail address*, Pedro Marín-Rubio: [pmr@us.es](mailto:pmr@us.es)  
21 *E-mail address*, Gabriela Planas: [gplanas@ime.unicamp.br](mailto:gplanas@ime.unicamp.br)