Manuscript submitted to
AIMS' Journals
Volume **XX**, Number **0xx**, XXXXXX **20xx**

Website: http://AIMsciences.org

pp. –

SOME NEW REGULARITY RESULTS OF PULLBACK ATTRACTORS FOR 2D NAVIER-STOKES EQUATIONS WITH DELAYS

Julia García-Luengo, Pedro Marín-Rubio & José Real*

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
Apdo. de Correos 1160, 41080–Sevilla, Spain

(Communicated by XXXXX)

Abstract. In this paper we strengthen some results on the existence and properties of pullback attractors for a 2D Navier-Stokes model with finite delay formulated in [Caraballo and Real, J. Differential Equations **205** (2004), 271–297]. Actually, we prove that under suitable assumptions, pullback attractors not only of fixed bounded sets but also of a set of tempered universes do exist. Moreover, thanks to regularity results, the attraction from different phase spaces also happens in $C([-h,0];V)$. Finally, from comparison results of attractors, and under an additional hypothesis, we establish that all these families of attractors are in fact the same object.

1. Introduction and statement of the problem.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth enough boundary $\partial\Omega$, and consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following functional Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \quad \text{in } \ \Omega\times(\tau,\infty), \\ \text{div } u \ = 0 \quad \text{in } \ \Omega\times(\tau,\infty), \\ u = 0 \quad \text{on } \ \partial\Omega\times(\tau,\infty), \\ u(x,\tau) = u^\tau(x), \quad x\in\Omega, \\ u(x,\tau+s) = \phi(x,s), \quad x\in\Omega,\ s\in(-h,0), \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, f is a non-delayed external force field, g is another external force with some hereditary characteristics, and u^τ and $\phi(x, s-\tau)$ are the initial data in τ and $(\tau-h,\tau)$ respectively, where $h > 0$ is the time of memory effect. For each $t \geq \tau$, we denote by u_t the function defined a.e. on $(-h, 0)$ by the relation $u_t(s) = u(t+s)$, a.e. $s \in (-h, 0)$.

2010 Mathematics Subject Classification. Primary: 35B41, 35Q30, 37L30.

Key words and phrases. 2D Navier-Stokes equations; delay terms; pullback attractors.

*Deceased on January 27th, 2012. The first two coauthors dedicate this paper to the memory of their coauthor José Real.

The importance of physical models for fluid mechanic problems including delay terms is related, for instance, to real applications where devices to control properties of fluids (temperature, velocity, etc.) are inserted in domains and make a local influence on the behaviour of the system (e.g., cf. [13] for a wind-tunnel model).

The study of Navier-Stokes models including delay terms –existence, uniqueness, stationary solutions, exponential decay, and other asymptotic properties such as the existence of attractors– was initiated in the references [3, 4, 5], and after that, many different questions, as dealing with unbounded domains, and models (for instance in three dimensions for modified terms) have been addressed (e.g., cf. [10, 17, 21, 19, 14, 20, 11, 15, 16] among others).

In the recent paper [9], we have treated a relaxation on the assumptions for the delay operator involved, removing conditions related to the control of the L^2 norm of the delay terms (see assumptions (IV) and (V) below). Although this implies to restrict the phase space to continuous functions instead of square integrable in time, the delay functions driving the delayed time within this theory can be taken just measurable, without any additional assumption as continuity nor C^1 with bounded derivative, as usual in the literature.

Moreover, in [9] we were also able to establish attraction in a higher norm (namely, H^1 instead of L^2) making a sharp use of regularization of the equations in dimension two and by energy methods. Relationships among attractors in different metrics was successfully carried out there, too.

Our goal in this paper is to keep all usual conditions for the delay operator (including (IV) and (V)) and to compare both kind of attractors, for both possibilities of phase spaces (continuous in time, or just square integrable in time). Observe that in the autonomous framework this issue would be almost immediate since one inclusion is clear by continuous embedding, and the other is obtained after an elapsed time as long as the memory effect. However, in the non-autonomous case (that we are dealing with) this is not the case at all. Using the theory of attraction for universes (cf. [1, 2, 18]) we deal with different families and under different metrics. Namely, we consider universes of fixed (in time) bounded sets and also time-dependent families given by a tempered condition when time goes to $-\infty$.

Moreover, we also improve some results previously obtained in the literature (cf. [5]) since we can deal with the phase space $V \times L^2(-h, 0; V)$ and not only $H \times L^2(-h, 0; H)$. Finally, from comparison results of attractors and under an additional assumption, we establish that all these families of attractors are in fact the same object.

The structure of the paper is the following. We continue this section with the abstract setting of the problem, general definitions and some well-known results on existence of weak and strong solutions and regularity properties. In Section 2 we recall the basic theory of pullback attractors for non-autonomous dynamical systems within the framework of universes, and comparison results, when different metrics are involved, are also given. Section 3 is devoted to establish all possible attractors for different phase-spaces but taking into account the L^2 norm in space. Our main results, established in the higher norm H^1 (in space), are given in Section

4. In these two last sections, energy methods (introduced in this context by Rosa in [23]) are used to prove asymptotic compactness in the respective universes. As said before, relationships among all these objects are obtained.

To set our problem in the abstract framework, we consider the following usual function spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x)\, dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot,\cdot \rangle$ for the duality between V' and V. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and continuous, and, in fact, compact.

Now, we define the operator $A : V \to V'$ as

$$\langle Au, v\rangle = ((u,v)) \quad \forall\, u, v \in V.$$

Let us denote $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has that $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent (see [6] or [25]), and $D(A)$ is compactly and densely injected in V.

Let us define

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\, dx,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$.

Some useful properties concerning b that we will use in the next sections are the following (see [22] or [24]): $b(u,v,w) = -b(u,w,v)$ for all $u,\ v,\ w \in V$, which also implies that $b(u,v,v) = 0$ for all $u,\ v \in V$. Moreover, there exists a constant $C_1 > 0$, only dependent on Ω, such that (recall that we are in dimension two)

$$|b(u,v,w)| \le C_1 |u|^{1/2} |Au|^{1/2} \|v\| |w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H. \tag{2}$$

Now, we establish some suitable spaces in order to deal with the delay term, and some appropriate assumptions on the term in (1) containing the delay.

Let us denote $C_H = C([-h,0];H)$, with the norm $|\varphi|_{C_H} = \max_{s\in[-h,0]} |\varphi(s)|$, and $L^2_X = L^2(-h,0;X)$ for $X = H,\ V$. On the delay operator from (1), we consider that is well defined as $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable,
(II) $g(t,0) = 0$, for all $t \in \mathbb{R}$,
(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all ξ, $\eta \in C_H$,

$$|g(t,\xi) - g(t,\eta)| \le L_g |\xi - \eta|_{C_H},$$

(IV) there exists $C_g > 0$ such that for all $\tau \le t$, and for all u, $v \in C([\tau - h, t]; H)$,

$$\int_\tau^t |g(s,u_s) - g(s,v_s)|^2 \, ds \le C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 \, ds.$$

Examples of fixed, variable, and distributed delay operators can be found, for instance, in [3, Section 3], [5, Sections 3.5 and 3.6], and [10, Section 3], and we omit them here just for the sake of brevity.

Observe that $(I) - (III)$ imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$. Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C([\tau - h, T]; H) \to g_u \in L^2(\tau, T; (L^2(\Omega))^2)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. From now on, we will denote $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all u, $v \in L^2(\tau - h, T; H)$.

Assume that $u^\tau \in H$, $\phi \in L^2_H$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 1. A weak solution of (1) is a function u that belongs to $L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u^\tau$ and $u(t) = \phi(t - \tau)$ a.e. $t \in (\tau - h, \tau)$, and such that for all $v \in V$,

$$\frac{d}{dt}(u(t), v) + \nu \langle Au(t), v\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle + (g(t, u_t), v), \tag{3}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Remark 1. If u is a weak solution of (1), then from (3) we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and so $u \in C([\tau, \infty); H)$, whence the initial datum $u(\tau) = u^\tau$ has full sense. Moreover, in this case the following energy equality holds:

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr = |u(s)|^2 + 2\int_s^t \left[\langle f(r), u(r)\rangle + (g(r, u_r), u(r))\right] dr \;\; \forall\, \tau \le s \le t.$$

A notion of more regular solution is also suitable for problem (1).

Definition 2. A strong solution of (1) is a weak solution u of (1) such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

Remark 2. If $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ and u is a strong solution of (1), then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, \infty); V)$. In this case the following energy equality holds:

$$\begin{aligned} &\|u(t)\|^2 + 2\nu \int_s^t |Au(r)|^2 \, dr + 2\int_s^t b(u(r), u(r), Au(r)) \, dr \\ = \;& \|u(s)\|^2 + 2\int_s^t (f(r) + g(r, u_r), Au(r)) \, dr \quad \forall\, \tau \le s \le t. \end{aligned} \tag{4}$$

Concerning the existence and uniqueness of weak and strong solutions for (1), we have the following result which can be proved similarly as [3, Theorem 2.1] or [4, Theorem 2.5] (see also [10, Theorem 2.3] for a more general case).

Theorem 1. *Let us consider* $u^\tau \in H$, $\phi \in L^2_H$, $f \in L^2_{loc}(\mathbb{R}; V')$, *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfying (I)–(IV). Then, for each* $\tau \in \mathbb{R}$, *there exists a unique weak solution* $u = u(\cdot; \tau, u^\tau, \phi)$ *of (1).*

Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, *then*

(a) $u \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ *for all* $T > \tau + \varepsilon > \tau$.

(b) *If* $u^\tau \in V$, u *is in fact a strong solution of (1).*

Before establishing the original results about the regularity of pullback attractors, we recall the main existence results studied in [5, 17, 19]. Firstly, in order to do that, we remember briefly the abstract theory on pullback attractors in the next section.

2. **Abstract results on minimal pullback attractors.** Now, we present a summary of some results from [8] about the existence of minimal pullback attractors (see also [1, 2, 18]). In particular, we assume that the process U is closed (see Definition 3 below).

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \le t\}$.

A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \le r \le t$ and all $x \in X$.

Definition 3. Let U be a process on X.

(a) U is said to be continuous if for any pair $\tau \le t$, the mapping $U(t,\tau) : X \to X$ is continuous.

(b) U is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Remark 3. It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 4. We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) D_0(\tau)}^X \quad \forall\, t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x,y).$$

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 5. A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \le t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall\, \tau \le \tau_0(t,\widehat{D}).$$

With the above definitions, we may establish the main result of this section (cf. [8, Theorem 3.11]).

Theorem 2. *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}^X$, has the following properties:

(a) *for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$,*
(b) *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e., $\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,*
(c) *$\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $(t,\tau) \in \mathbb{R}^2_d$,*
(d) *if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.*

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\lim\limits_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 4. Under the assumptions of Theorem 2, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}$-attractor for the process U.

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 1. *Under the assumptions of Theorem 2, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.*

Remark 5. It can be proved (see [18]) that, under the assumptions of the preceding corollary, if for some $T \in \mathbb{R}$, the set $\cup_{t\le T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \le T$.

Now, and since it will be useful below, we establish an abstract result (cf. [8, Theorem 3.15]) that allows us to compare two attractors for a process under appropriate assumptions.

Theorem 3. *Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be two metric spaces such that $X_1 \subset X_2$ with continuous injection, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that we have a map U that acts as a process in both cases, i.e., $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$ is a process.*

For each $t \in \mathbb{R}$, let us denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \quad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence of the respective topology.

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Suppose moreover that the two following conditions are satisfied:

(i) *$\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$,*

(ii) *for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t^*_{\widehat{D}_1} \le t$ (both possibly depending on t and $\widehat{D}_2$), such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \le t^*_{\widehat{D}_1}$ there exists a $\tau_s \le s$ such that $U(s, \tau)D_2(\tau) \subset D_1(s)$ for all $\tau \le \tau_s$.*

Then, under all the conditions above, $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Remark 6. In the preceding theorem, if instead of assumption (ii) we consider the following condition:

(ii') for any $\widehat{D}_2 \in \mathcal{D}_2$ and any sequence $\tau_n \to -\infty$, there exist another family $\widehat{D}_1 \in \mathcal{D}_1$ and another sequence $\tau'_n \to -\infty$ with $\tau'_n \ge \tau_n$ for all n, such that U is pullback $\widehat{D}_1$-asymptotically compact, and

$$U(\tau'_n, \tau_n)D_2(\tau_n) \subset D_1(\tau'_n) \quad \forall\, n,$$

then, with a similar proof, one can obtain that the equality $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ also holds for all $t \in \mathbb{R}$.

Observe that a sufficient condition for (ii') is that there exists $T > 0$ such that for any $\widehat{D}_2 \in \mathcal{D}_2$, there exists a $\widehat{D}_1 \in \mathcal{D}_1$ satisfying that U is pullback $\widehat{D}_1$-asymptotically compact, and $U(\tau + T, \tau)D_2(\tau) \subset D_1(\tau + T)$ for all $\tau \in \mathbb{R}$.

3. **Previous results on processes and pullback attractors in H.** In this section we recall some known results (cf. [5, 17, 19]) on the existence of minimal pullback attractors in the H norm for suitable processes associated to problem (1).

In order to apply the theory of the above section, and following [5, 17, 19], we may consider the Banach space C_H, and the Hilbert space $M^2_H = H \times L^2_H$ with associated norm

$$\|(u^\tau, \phi)\|^2_{M^2_H} = |u^\tau|^2 + \int_{-h}^0 |\phi(s)|^2\, ds \quad \text{for } (u^\tau, \phi) \in M^2_H.$$

We can define two processes for problem (1).

Proposition 1. *Assume that $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfies (I)–(IV). Then, the bi-parametric families of mappings $U(t, \tau) : C_H \to C_H$ and $S(t, \tau) : M^2_H \to M^2_H$ given respectively by*

$$U(t, \tau)\phi = u_t(\cdot; \tau, \phi(0), \phi) \quad \textit{for } \phi \in C_H,\ \tau \le t, \tag{5}$$

and

$$S(t, \tau)(u^\tau, \phi) = (u(t; \tau, u^\tau, \phi), u_t(\cdot; \tau, u^\tau, \phi)) \quad \textit{for } (u^\tau, \phi) \in M^2_H,\ \tau \le t, \tag{6}$$

where u is the unique weak solution of (1), are well defined continuous processes on C_H and M_H^2 respectively.

Proof. The result follows from Theorem 1 above, and from [5, Theorem 9]. □

Now, in order to establish asymptotic estimates for the solutions of (1), we impose a fifth assumption on g and f.

Denote by λ_1 the first eigenvalue of the Stokes operator A.

(V) Assume that $\nu\lambda_1 > C_g$, and that there exists a value $\eta \in (0, 2(\nu\lambda_1 - C_g))$ such that for every $u \in L^2(\tau - h, t; H)$,

$$\int_\tau^t e^{\eta s}|g(s,u_s)|^2\,ds \quad \leq \quad C_g^2 \int_{\tau-h}^t e^{\eta s}|u(s)|^2\,ds \quad \text{for any } \tau \leq t, \quad \text{and}$$
$$\int_{-\infty}^0 e^{\eta s}\|f(s)\|_*^2\,ds \quad < \quad \infty.$$

Lemma 1. *Suppose that $f \in L^2_{loc}(\mathbb{R}; V')$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, for any $(u^\tau, \phi) \in M_H^2$, the following estimate holds for the solution u to (1) for all $t \geq \tau$:*

$$|u(t)|^2 \leq e^{-\eta(t-\tau)} \max\{1, C_g\}\|(u^\tau,\phi)\|^2_{M_H^2} + \beta^{-1}e^{-\eta t}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\,ds, \tag{7}$$

where

$$\beta = 2\nu - (\eta + 2C_g)\lambda_1^{-1}. \tag{8}$$

Proof. By the energy equality (see Remark 1), and Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2\\ \leq\ & \beta\|u(t)\|^2 + \beta^{-1}\|f(t)\|_*^2 + C_g|u(t)|^2 + C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\left(e^{\eta t}|u(t)|^2\right) + e^{\eta t}\left(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\right)\|u(t)\|^2\\ \leq\ & e^{\eta t}\beta^{-1}\|f(t)\|_*^2 + e^{\eta t}C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau,\end{aligned}$$

and therefore, integrating above and using property (V), we obtain

$$\begin{aligned}&e^{\eta t}|u(t)|^2 + \left(2\nu - \beta - (\eta + C_g)\lambda_1^{-1}\right)\int_\tau^t e^{\eta s}\|u(s)\|^2\,ds\\ \leq\ & e^{\eta\tau}|u^\tau|^2 + \beta^{-1}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\,ds + C_g\int_{\tau-h}^t e^{\eta s}|u(s)|^2\,ds\\ \leq\ & e^{\eta\tau}\max\{1, C_g\}\|(u^\tau,\phi)\|^2_{M_H^2} + \beta^{-1}\int_\tau^t e^{\eta s}\|f(s)\|_*^2\,ds + C_g\int_\tau^t e^{\eta s}|u(s)|^2\,ds,\end{aligned}$$

for all $t \geq \tau$, and from this last inequality and (8), in particular we deduce (7). □

After the above result, it turns out appropriate the introduction of the following tempered universes.

Definition 6. For any $\eta > 0$, we will denote by $\mathcal{D}_\eta(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{\varphi\in D(\tau)}|\varphi|^2_{C_H}\right) = 0.$$

Analogously, we will denote by $\mathcal{D}_\eta(M_H^2)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H^2)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{(w,\varphi)\in D(\tau)}\|(w,\varphi)\|^2_{M_H^2}\right) = 0.$$

Furthermore, accordingly to the notation introduced in the previous section, $\mathcal{D}_F(C_H)$ and $\mathcal{D}_F(M_H^2)$ will denote the universes of fixed bounded sets in C_H and M_H^2 respectively.

Remark 7. (i) The choices of the above universes are right and convenient to keep, in the sense that, on the one hand, M_H^2 is more general as phase space for the initial data of problem (1). On the other hand, the regularity of the solution to (1) (cf. Theorem 1) makes that, after an elapsed time h, every solution is continuous with values on H. Indeed, as it was observed in [5], in the case of the universes of fixed bounded sets, pullback attractors in both spaces do exist, and they are intrinsically related through the canonical embedding $j : C_H \to M_H^2$ defined by $j(\varphi) = (\varphi(0), \varphi)$ (see Theorem 4 below).

(ii) The universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_\eta(M_H^2)$, which are inclusion-closed, contain respectively the universes $\mathcal{D}_F(C_H)$ and $\mathcal{D}_F(M_H^2)$.

Now, we obtain pullback absorbing families for $U : \mathbb{R}^2_d \times C_H \to C_H$ and $S : \mathbb{R}^2_d \times M_H^2 \to M_H^2$.

Corollary 2. *Under the assumptions of Lemma 1, the family $\widehat{D}_{1,\eta} = \{D_{1,\eta}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ defined by $D_{1,\eta}(t) = \overline{B}_{C_H}(0, r_\eta(t))$, the closed ball in C_H of center zero and radius $r_\eta(t)$, where*

$$r_\eta^2(t) = 1 + \beta^{-1}e^{-\eta(t-h)}\int_{-\infty}^t e^{\eta s}\|f(s)\|_*^2\,ds,$$

with β given by (8), is pullback $\mathcal{D}_\eta(C_H)$-absorbing for the process U on C_H defined by (5) (and therefore pullback $\mathcal{D}_F(C_H)$-absorbing too), and $\widehat{D}_{1,\eta}$ belongs to $\mathcal{D}_\eta(C_H)$.

Besides, the family $\widehat{D}_{2,\eta} = \{D_{2,\eta}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H^2)$ defined by $D_{2,\eta}(t) = \overline{B}_{M_H^2}(0, R_\eta(t))$, the closed ball in M_H^2 of center zero and radius $R_\eta(t)$, with

$$R_\eta^2(t) = 1 + \beta^{-1}(1 + he^{\eta h})e^{-\eta t}\int_{-\infty}^t e^{\eta s}\|f(s)\|_*^2\,ds,$$

is pullback $\mathcal{D}_\eta(M_H^2)$-absorbing for the process S on M_H^2 given by (6) (and thus also pullback $\mathcal{D}_F(M_H^2)$-absorbing), and $\widehat{D}_{2,\eta}$ belongs to $\mathcal{D}_\eta(M_H^2)$.

Since it will be useful in order to compare the pullback attractors defined in the spaces C_H and M_H^2, we consider the bi-parametric family of mappings $\widetilde{U}(t,\tau) : M_H^2 \to L_H^2$ defined as

$$\widetilde{U}(t,\tau)(u^\tau, \phi) = u_t(\cdot;\tau, u^\tau, \phi) \quad \text{for } (u^\tau, \phi) \in M_H^2,\ \tau \le t.$$

Remark 8. Observe that $\widetilde{U}(t,\tau)$ maps M_H^2 into C_H if $t \ge \tau + h$, and therefore we can write

$$S(t,\tau)(u^\tau, \phi) = j(\widetilde{U}(t,\tau)(u^\tau, \phi)) \quad \text{for } (u^\tau, \phi) \in M_H^2,\ t \ge \tau + h,$$

where $S(\cdot,\cdot)$ is given by (6).

Moreover, it is clear that

$$U(t,\tau)\phi = \widetilde{U}(t,\tau)j(\phi) \quad \text{for } \phi \in C_H,\ t \ge \tau,$$

with $U(\cdot,\cdot)$ defined in (5).

Lemma 2. *Under the assumptions of Lemma 1, for any $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ and any $r \geq h$, the family $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$, where $D^{(r)}(\tau) = \widetilde{U}(\tau + r, \tau)D(\tau)$ for any $\tau \in \mathbb{R}$, belongs to $\mathcal{D}_\eta(C_H)$.*

Proof. From (7), we obtain

$$\begin{aligned}\sup_{\psi \in D^{(r)}(\tau)} \left(e^{\eta\tau}|\psi|_{C_H}^2\right) \quad &\leq \quad e^{-\eta(r-h)} \max\{1, C_g\} \sup_{(u^\tau,\phi)\in D(\tau)} \left(e^{\eta\tau}\|(u^\tau,\phi)\|_{M_H^2}^2\right) \\ &\quad + \beta^{-1} e^{-\eta(r-h)} \int_\tau^{\tau+r} e^{\eta s}\|f(s)\|_*^2 \, ds.\end{aligned}$$

From this inequality and assumption (V), we deduce the result. □

Now, we are able to establish the main result of this section.

Theorem 4. *Assume that $f \in L^2_{loc}(\mathbb{R}; V')$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, there exist the minimal pullback attractors $\{\mathcal{A}_{\mathcal{D}_F(C_H)}(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t)\}_{t\in\mathbb{R}}$, and $\{\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)\}_{t\in\mathbb{R}}$, in C_H and M_H^2 respectively, for the universes of fixed bounded sets and for those with tempered condition given in Definition 6.*

Besides, the following relations hold:

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t), \quad \text{and} \quad \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \ \forall\, t \in \mathbb{R}, \tag{9}$$

$$j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}, \quad \text{and} \tag{10}$$

$$j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}, \tag{11}$$

where the map j is the canonical injection of C_H into M_H^2 defined in Remark 7 (i).

Finally, if f also satisfies

$$\sup_{s\leq 0} \left(e^{-\eta s} \int_{-\infty}^{s} e^{\eta\theta}\|f(\theta)\|_*^2 \, d\theta\right) < \infty, \tag{12}$$

then, the inclusions in (9) and (10) are in fact equalities.

Proof. For the existence of the four minimal pullback attractors see [5, Theorem 17, Remark 18], [17, Theorem 20], and [19, Theorem 4].

The relations in (9) follow from Remark 7 (ii) and Corollary 1, and the inclusion in (10) can be proved analogously as in [19, Theorem 5].

Now, we analyze the equality (11). On the one hand, the inclusion $j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ can be obtained again in a similar way as in [19, Theorem 5]. On the other hand, from Remark 8 and Lemma 2, we have that for any $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ and any $\tau < t - h$,

$$\begin{aligned}&\operatorname{dist}_{M_H^2}(S(t,\tau)D(\tau), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))) \\ = \ &\operatorname{dist}_{M_H^2}(S(t,\tau+h)(S(\tau+h,\tau)D(\tau)), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))) \\ = \ &\operatorname{dist}_{M_H^2}(S(t,\tau+h)(j(\widetilde{U}(\tau+h,\tau)D(\tau))), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))) \\ = \ &\operatorname{dist}_{M_H^2}(j(U(t,\tau+h)D^{(h)}(\tau)), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))) \\ \leq \ &(1+h)^{1/2}\operatorname{dist}_{C_H}(U(t,\tau+h)D^{(h)}(\tau), \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)),\end{aligned}$$

where in the last inequality we have used that $j \in \mathcal{L}(C_H, M_H^2)$ with $\|j\|_{\mathcal{L}(C_H,M_H^2)} \leq (1+h)^{1/2}$. Therefore, the inclusion $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \subset j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))$ follows since

$\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ is the minimal closed set in M_H^2 that attracts any family $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ in the pullback sense.

Finally, if moreover f satisfies (12), the coincidences of the pullback attractors in (9) is a consequence of Remark 5, and the fact that (12) is equivalent to have that $\sup_{t\le T} r_\eta(t)$ and $\sup_{t\le T} R_\eta(t)$ are bounded for any $T \in \mathbb{R}$, with $r_\eta(t)$ and $R_\eta(t)$ defined in Corollary 2. Now, from these identities and (11), the equality in (10) follows. □

Remark 9. Under the assumptions of Theorem 4, as a consequence of Remarks 4 and 7 (ii), and Corollary 2, we have that $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$ and $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$ belong to the universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_\eta(M_H^2)$ respectively.

Actually, if in addition f satisfies (12), one can see that for each $T \in \mathbb{R}$, the sets $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t) : t \le T\}$ and $\{\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) : t \le T\}$ are bounded in C_H and M_H^2 respectively.

4. **Regularity of the pullback attractors and V attraction.** Now, we will improve in a certain way the main result of the previous section, Theorem 4, in the sense that we will establish the existence of minimal pullback attractors in the V norm, using some new phase spaces which will be defined below. Moreover, we will check that under suitable assumptions all these families of attractors are in fact the same (here Theorem 3 will play an essential role).

For any $\tilde{h} \in [0, h]$, let us denote

$$C_H^{\tilde{h},V} = \{\varphi \in C_H : \varphi|_{[-\tilde{h},0]} \in B([-\tilde{h}, 0]; V)\},$$

where $B([-\tilde{h}, 0]; V)$ is the space of bounded functions from $[-\tilde{h}, 0]$ into V. The space $C_H^{\tilde{h},V}$ is a Banach space with the norm

$$\|\varphi\|_{\tilde{h},V} = |\varphi|_{C_H} + \sup_{\theta\in[-\tilde{h},0]} \|\varphi(\theta)\|.$$

Observe that the space $C_V = C([-h, 0]; V)$ is a Banach subspace of $C_H^{h,V}$.

We also consider the Hilbert space $M_V^2 = V \times L_V^2$ with associated norm

$$\|(u^\tau, \phi)\|_{M_V^2}^2 = \|u^\tau\|^2 + \int_{-h}^0 \|\phi(s)\|^2 \, ds \quad \text{for } (u^\tau, \phi) \in M_V^2.$$

We have the following result.

Proposition 2. *Suppose that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfies (I)–(IV). Then, for any $\tilde{h} \in [0, h]$, the bi-parametric families of mappings $U(t, \tau)|_{C_H^{\tilde{h},V}}$ and $S(t, \tau)|_{M_V^2}$, with $\tau \le t$, are well defined continuous processes on $C_H^{\tilde{h},V}$ and M_V^2 respectively.*

Proof. The fact that, for any $\tilde{h} \in [0, h]$ and $\tau \le t$, $U(t, \tau)|_{C_H^{\tilde{h},V}}$ and $S(t, \tau)|_{M_V^2}$ are well defined processes follows from Theorem 1. The continuity can be proved similarly as [9, Proposition 5.2], using property (IV) instead of (III). □

We introduce the following universes in $\mathcal{P}(C_H^{\tilde{h},V})$ and in $\mathcal{P}(M_V^2)$.

Definition 7. For any $\eta > 0$ and $\tilde{h} \in [0, h]$, we will denote by $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(C_H)$ such that for any $t \in \mathbb{R}$ and for any $\varphi \in D(t)$, it holds that $\varphi|_{[-\tilde{h},0]} \in B([-\tilde{h}, 0]; V)$.

Analogously, we will denote by $\mathcal{D}_F^{\tilde{h},V}(C_H)$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H such that for any $\varphi \in D$, it holds that $\varphi|_{[-\tilde{h},0]} \in B([-\tilde{h}, 0]; V)$.

Finally, we will denote by $\mathcal{D}_F(C_H^{\tilde{h},V})$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_H^{\tilde{h},V}$.

Remark 10. The chain of inclusions for the universes in the above definition and the universes in $\mathcal{P}(C_H)$ introduced in Section 3, is the following:

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_F^{\tilde{h},V}(C_H) \subset \mathcal{D}_\eta^{\tilde{h},V}(C_H) \subset \mathcal{D}_\eta(C_H),$$

and

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_F^{\tilde{h},V}(C_H) \subset \mathcal{D}_F(C_H) \subset \mathcal{D}_\eta(C_H),$$

for all $\eta > 0$ and any $\tilde{h} \in [0, h]$.

It must also be pointed out that all the classes $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$, with $\tilde{h} \in [0, h]$, are inclusion-closed, which will be important (cf. Remark 4).

Finally, it is clear that if $0 \le \tilde{h}_1 < \tilde{h}_2 \le h$, then

$$\mathcal{D}_F(C_H^{\tilde{h}_2,V}) \subset \mathcal{D}_F(C_H^{\tilde{h}_1,V}), \quad \mathcal{D}_F^{\tilde{h}_2,V}(C_H) \subset \mathcal{D}_F^{\tilde{h}_1,V}(C_H), \quad \mathcal{D}_\eta^{\tilde{h}_2,V}(C_H) \subset \mathcal{D}_\eta^{\tilde{h}_1,V}(C_H).$$

Definition 8. For any $\eta > 0$, we will denote by $\mathcal{D}_\eta^V(M_H^2)$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(M_H^2)$ such that for any $t \in \mathbb{R}$ and for any $(w, \varphi) \in D(t)$, it holds that $(w, \varphi) \in M_V^2$.

Moreover, we will denote by $\mathcal{D}_F(M_V^2)$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of M_V^2.

Remark 11. In this case, the relations among the universes introduced above and those in $\mathcal{P}(M_H^2)$ defined in Section 3, are the following:

$$\mathcal{D}_F(M_V^2) \subset \mathcal{D}_\eta^V(M_H^2) \subset \mathcal{D}_\eta(M_H^2),$$

and

$$\mathcal{D}_F(M_V^2) \subset \mathcal{D}_F(M_H^2) \subset \mathcal{D}_\eta(M_H^2),$$

for any $\eta > 0$.

Observe also that $\mathcal{D}_\eta^V(M_H^2)$ is inclusion-closed.

Now, we establish the existence of pullback absorbing families for the processes $U : \mathbb{R}_d^2 \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ and $S : \mathbb{R}_d^2 \times M_V^2 \to M_V^2$.

Proposition 3. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, for any $\tilde{h} \in [0, h]$, the family $\widehat{D}_{1,\eta,\tilde{h}} = \{D_{1,\eta,\tilde{h}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$, with*

$$D_{1,\eta,\tilde{h}}(t) = D_{1,\eta}(t) \cap C_H^{\tilde{h},V},$$

where $D_{1,\eta}(t)$ is defined in Corollary 2, is a family of closed sets of $C_H^{\tilde{h},V}$, which is pullback $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$-absorbing for the process U on $C_H^{\tilde{h},V}$ given by (5), and $\widehat{D}_{1,\eta,\tilde{h}}$ belongs to $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$.

Besides, the family $\widehat{D}_{2,\eta,V} = \{D_{2,\eta,V}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_V^2)$*, where*

$$D_{2,\eta,V}(t) = D_{2,\eta}(t) \cap M_V^2,$$

with $D_{2,\eta}(t)$ *also given in Corollary 2, is a family of closed sets of* M_V^2*, that is pullback* $\mathcal{D}_\eta^V(M_H^2)$*-absorbing for the process* S *on* M_V^2 *defined by (6), and* $\widehat{D}_{2,\eta,V}$ *belongs to* $\mathcal{D}_\eta^V(M_H^2)$*.*

Proof. It is a direct consequence of Corollary 2. □

The following result can be obtained analogously as [9, Lemma 5.2] (see also [7, 8] for close results).

Lemma 3. *Under the assumptions of Proposition 3, for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$*, there exist* $\tau_1(\widehat{D}, t, h) < t - 3h - 2$ *and functions* $\{\rho_i\}_{i=1}^4$ *depending on* t *and* h*, such that for any* $\tau \leq \tau_1(\widehat{D}, t, h)$ *and any* $(u^\tau, \phi) \in D(\tau)$*, it holds*

$$\begin{cases} |u(r;\tau,u^\tau,\phi)|^2 \leq \rho_1(t) & \forall\, r \in [t-3h-2, t], \\ \|u(r;\tau,u^\tau,\phi)\|^2 \leq \rho_2(t) & \forall\, r \in [t-2h-1, t], \\ \nu \displaystyle\int_{r-1}^{r} |Au(\theta;\tau,u^\tau,\phi)|^2 \, d\theta \leq \rho_3(t) & \forall\, r \in [t-2h, t], \\ \displaystyle\int_{r-1}^{r} |u'(\theta;\tau,u^\tau,\phi)|^2 \, d\theta \leq \rho_4(t) & \forall\, r \in [t-2h, t], \end{cases} \tag{13}$$

where

$$\rho_1(t) = 1 + \beta^{-1} e^{-\eta(t-3h-2)} \int_{-\infty}^{t} e^{\eta s} \|f(s)\|_*^2 \, ds,$$

$$\rho_2(t) = \nu^{-1}\Big(\big(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2 + 4L_g^2\big)\rho_1(t) + \big(4 + 2\nu^{-1}\lambda_1^{-1}\big) \int_{t-2h-2}^{t} |f(\theta)|^2 \, d\theta \Big)$$
$$\times \exp\Big\{ 2\nu^{-1} C^{(\nu)} \rho_1(t) \Big[\big(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2\big)\rho_1(t) + 2\nu^{-1}\lambda_1^{-1} \int_{t-2h-2}^{t} |f(\theta)|^2 d\theta \Big] \Big\},$$

$$\rho_3(t) = \rho_2(t) + 4\nu^{-1} \int_{t-2h-1}^{t} |f(\theta)|^2 \, d\theta + 4L_g^2\nu^{-1}\rho_1(t) + 2C^{(\nu)}\rho_1(t)\rho_2^2(t),$$

$$\rho_4(t) = \nu\rho_2(t) + 4 \int_{t-2h-1}^{t} |f(\theta)|^2 \, d\theta + 4L_g^2\rho_1(t) + 2C_1^2\nu^{-1}\rho_2(t)\rho_3(t),$$

with β *given by (8), and*

$$C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}. \tag{14}$$

Remark 12. Under the assumptions of Lemma 3, $\lim_{t\to-\infty} e^{\eta t}\rho_1(t) = 0$.

Now, we apply an energy method that relies on the continuity of the solutions and some non-increasing functions (e.g., cf. [9, Lemma 5.3] for a similar proof) in order to obtain the pullback asymptotic compactness in $C_H^{\tilde{h},V}$ and M_V^2 for the universes $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$ and $\mathcal{D}_\eta^V(M_H^2)$ respectively.

Lemma 4. *Under the assumptions of Proposition 3, for any* $t \in \mathbb{R}$*, any* $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$*, and any sequences* $\{\tau_n\} \subset (-\infty, t]$ *and* $\{(u^{\tau_n}, \phi^n)\} \subset M_H^2$ *such that* $\tau_n \to -\infty$ *and* $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ *for all* n*, the sequence* $\{u(\cdot;\tau_n,u^{\tau_n},\phi^n)\}$ *is relatively compact in* $C([t-h,t];V)$*.*

Proof. Let us fix $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$, and sequences $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n. Denote for short $u^n(\cdot) = u(\cdot; \tau_n, u^{\tau_n}, \phi^n)$.

From Lemma 3, we have that there exists a $\tau_1(\widehat{D}, t, h) < t - 3h - 2$ such that the subsequence $\{u^n : \tau_n \le \tau_1(\widehat{D}, t, h)\} \subset \{u^n\}$ is bounded in $L^\infty(t-2h-1, t; V) \cap L^2(t-2h-1, t; D(A))$ with $\{(u^n)'\}$ also bounded in $L^2(t-2h-1, t; H)$. Therefore, using in particular the Aubin-Lions compactness lemma (e.g., cf. [12]), there exists a function $u \in L^\infty(t-2h-1, t; V) \cap L^2(t-2h-1, t; D(A))$ with $u' \in L^2(t-2h-1, t; H)$ such that, for a subsequence which we relabel the same, the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-2h-1, t; V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2h-1, t; D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2h-1, t; H), \\ u^n \to u & \text{strongly in } L^2(t-2h-1, t; V), \\ u^n(s) \to u(s) & \text{strongly in } V, \text{ a.e. } s \in (t-2h-1, t). \end{cases} \tag{15}$$

Observe that $u \in C([t-2h-1, t]; V)$ satisfies, thanks to (15), the equation (3) in the interval $(t-h-1, t)$.

Moreover, from (15) we can also deduce that $\{u^n\}$ is equi-continuous on $[t-2h-1, t]$ with values in H. Thus, since $\{u^n\}$ is bounded in $C([t-2h-1, t]; V)$ and the injection of V into H is compact, by the Ascoli-Arzelà theorem, we obtain that (once more, up to a subsequence)

$$u^n \to u \quad \text{strongly in } C([t-2h-1, t]; H). \tag{16}$$

Indeed, again from the boundedness of $\{u^n\}$ in $C([t-2h-1, t]; V)$, one has that for any sequence $\{s_n\} \subset [t-2h-1, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{17}$$

where we have used (16) to identify the weak limit.

Our goal now is to show that

$$u^n \to u \quad \text{strongly in } C([t-h, t]; V), \tag{18}$$

which in particular will imply the relative compactness.

If it were not true, there would exist $\varepsilon > 0$, a value $t_* \in [t-h, t]$, and subsequences (relabelled the same) $\{u^n\}$ and $\{t_n\} \subset [t-h, t]$, with $\lim_{n\to\infty} t_n = t_*$, such that

$$\|u^n(t_n) - u(t_*)\| \ge \varepsilon \quad \forall\, n \ge 1. \tag{19}$$

Recall that by (17) we already have that

$$\|u(t_*)\| \le \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{20}$$

On the other hand, applying the energy equality (4) to $w = u^n$ or $w = u$, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\theta}\|w(\theta)\|^2 + \nu|Aw(\theta)|^2 + b(w(\theta), w(\theta), Aw(\theta)) \\ = \; & (f(\theta) + g(\theta, w_\theta), Aw(\theta)) \\ \le \; & \frac{2}{\nu}|f(\theta)|^2 + \frac{2L_g^2}{\nu}|w_\theta|_{C_H}^2 + \frac{\nu}{4}|Aw(\theta)|^2, \quad \text{a.e. } \theta > t-h-1, \end{aligned}$$

where we have used Young's inequality and the assumptions (II) and (III) of g.

Observing that the trilinear term b can be estimated by (2) as follows

$$\begin{aligned} |b(w(\theta), w(\theta), Aw(\theta))| &\leq C_1 |w(\theta)|^{1/2} \|w(\theta)\| |Aw(\theta)|^{3/2} \\ &\leq C^{(\nu)} |w(\theta)|^2 \|w(\theta)\|^4 + \frac{\nu}{4} |Aw(\theta)|^2, \end{aligned}$$

with $C^{(\nu)}$ given in (14), we deduce from above that

$$\begin{aligned} &\frac{d}{d\theta} \|w(\theta)\|^2 + \nu |Aw(\theta)|^2 \\ \leq \ & 2C^{(\nu)} |w(\theta)|^2 \|w(\theta)\|^4 + \frac{4}{\nu} |f(\theta)|^2 + \frac{4L_g^2}{\nu} |w_\theta|_{C_H}^2, \quad \text{a.e. } \theta > t-h-1. \end{aligned}$$

Therefore, we have that for all $t-h-1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} \|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2 \, dr &\leq \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 \, dr \\ &\quad + \frac{4}{\nu} \int_{s_1}^{s_2} |f(r)|^2 \, dr + \frac{4L_g^2}{\nu} \int_{s_1}^{s_2} |u_r^n|_{C_H}^2 \, dr, \end{aligned}$$

and

$$\begin{aligned} \|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2 \, dr &\leq \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4 \, dr \\ &\quad + \frac{4}{\nu} \int_{s_1}^{s_2} |f(r)|^2 \, dr + \frac{4L_g^2}{\nu} \int_{s_1}^{s_2} |u_r|_{C_H}^2 \, dr. \end{aligned}$$

Now, consider the functions $J_n, J : [t-h-1, t] \to \mathbb{R}$ defined by

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u^n(r)|^2 \|u^n(r)\|^4 \, dr - \frac{4}{\nu} \int_{t-h-1}^{s} |f(r)|^2 \, dr \\ &\quad - \frac{4L_g^2}{\nu} \int_{t-h-1}^{s} |u_r^n|_{C_H}^2 \, dr, \\ J(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u(r)|^2 \|u(r)\|^4 \, dr - \frac{4}{\nu} \int_{t-h-1}^{s} |f(r)|^2 \, dr \\ &\quad - \frac{4L_g^2}{\nu} \int_{t-h-1}^{s} |u_r|_{C_H}^2 \, dr. \end{aligned}$$

From the regularity of u and all u^n, it is clear that these functions are continuous, and from the corresponding inequalities above, both J_n and J are non-increasing. Actually, by (15) and (16),

$$J_n(s) \to J(s) \quad \text{a.e. } s \in (t-h-1, t).$$

Hence, there exists a sequence $\{\tilde{t}_k\} \subset (t-h-1, t_*)$ satisfying that $\lim_{k\to\infty} \tilde{t}_k = t_*$, and

$$\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \forall\, k.$$

Fix an arbitrary value $\delta > 0$. Due to the continuity of J, there exists k_δ such that

$$|J(\tilde{t}_k) - J(t_*)| < \delta/2 \quad \forall\, k \geq k_\delta.$$

Now, let us take $n(k_\delta)$ such that for all $n \geq n(k_\delta)$ it holds

$$t_n \geq \tilde{t}_{k_\delta} \quad \text{and} \quad |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| < \delta/2.$$

Then, since all J_n are non-increasing, we deduce that for all $n \geq n(k_\delta)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\delta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta. \end{aligned}$$

Therefore, as $\delta > 0$ is arbitrary, we obtain that

$$\limsup_{n\to\infty} J_n(t_n) \leq J(t_*),$$

and consequently, again by (15) and (16),

$$\limsup_{n\to\infty} \|u^n(t_n)\| \leq \|u(t_*)\|,$$

which combined with (20) and (17) allows us to claim that $u^n(t_n) \to u(t_*)$ strongly in V, in contradiction with (19). Thus, (18) is proved as desired. □

As an immediate consequence of the previous lemma, we have the following result.

Corollary 3. *Under the assumptions of Lemma 4, it holds:*

(a) *For any $\tilde{h} \in [0, h]$, the process $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ is pullback $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$-asymptotically compact.*

(b) *The process $S : \mathbb{R}^2_d \times M^2_V \to M^2_V$ is pullback $\mathcal{D}^V_\eta(M^2_H)$-asymptotically compact.*

We establish now the following result about the existence of minimal pullback attractors for the process U on $C_H^{\tilde{h},V}$, which can be proved in a same way as [9, Theorem 5.1].

Theorem 5. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, for any $\tilde{h} \in [0, h]$, the process U on $C_H^{\tilde{h},V}$ possesses a minimal pullback $\mathcal{D}_\eta^{\tilde{h},V}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}$, a minimal pullback $\mathcal{D}_F^{\tilde{h},V}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)}$, and a minimal pullback $\mathcal{D}_F(C_H^{\tilde{h},V})$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}$. Besides, the following relations hold:*

$$\begin{aligned} \mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t) &\subset \mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)}(t) \\ &\subset \mathcal{A}_{\mathcal{D}_F(C_H)}(t) \\ &\subset \mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t) \\ &\subset C_V \quad \forall\, t \in \mathbb{R}, \end{aligned} \tag{21}$$

and for any family $\widehat{D} \in \mathcal{D}_\eta(C_H)$,

$$\lim_{\tau\to-\infty} \mathrm{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) = 0 \quad \forall\, t \in \mathbb{R}.$$

Finally, if moreover f satisfies

$$\sup_{s\leq 0} \left(e^{-\eta s} \int_{-\infty}^{s} e^{\eta\theta} |f(\theta)|^2 \, d\theta \right) < \infty, \tag{22}$$

then all attractors in (21) coincide, and this family is tempered in C_V, in the sense that

$$\lim_{t\to-\infty} \left(e^{\eta t} \sup_{v \in \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)} \|v\|^2_{C_V} \right) = 0,$$

where $\|v\|_{C_V} = \max_{s\in[-h,0]} \|v(s)\|$ for any $v \in C_V$.

Remark 13. Observe that, under the assumptions of Theorem 5, one has that $\mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)} \equiv \mathcal{A}_{\mathcal{D}_\eta^{h,V}(C_H)}$ for any $\tilde{h} \in [0,h]$, i.e., the pullback attractor $\mathcal{A}_{\mathcal{D}_\eta^{\tilde{h},V}(C_H)}$ is independent of $\tilde{h}$.

Actually, if f also satisfies (22), then $\mathcal{A}_{\mathcal{D}_F^{\tilde{h},V}(C_H)} \equiv \mathcal{A}_{\mathcal{D}_F^{h,V}(C_H)}$, and $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})} \equiv \mathcal{A}_{\mathcal{D}_F(C_H^{h,V})}$.

Remark 14. Under the assumptions of Theorem 5, since $\widehat{D}_{1,\eta,h}$ belongs to $\mathcal{D}_\eta^{h,V}(C_H)$, and the set $D_{1,\eta,h}(t)$ is closed in $C_H^{h,V}$ for all $t \in \mathbb{R}$, from Remarks 4 and 10, and the equality in (21), we deduce that $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$ belongs to $\mathcal{D}_\eta^{h,V}(C_H)$.

In fact, if in addition f satisfies (22), then, for each $T \in \mathbb{R}$, the set $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t) : t \le T\}$ is bounded in $C_H^{h,V}$.

We are also able to obtain the existence of minimal pullback attractors for the process S on M_V^2.

Theorem 6. *Suppose that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, and that f and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V). Then, there exist the minimal pullback $\mathcal{D}_F(M_V^2)$-attractor $\mathcal{A}_{\mathcal{D}_F(M_V^2)}$, and the minimal pullback $\mathcal{D}_\eta^V(M_H^2)$-attractor $\mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}$ for the process S on M_V^2, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) \subset \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t) \quad \forall\, t \in \mathbb{R}. \tag{23}$$

In particular, for any family $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_{M_V^2}(S(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) = 0 \quad \forall\, t \in \mathbb{R}. \tag{24}$$

Finally, if f also satisfies (22), then

$$\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) = \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t) \quad \forall\, t \in \mathbb{R},$$

and this family is tempered in M_V^2, i.e.,

$$\lim_{t \to -\infty} \left(e^{\eta t} \sup_{(w,\varphi) \in \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)} \|(w,\varphi)\|^2_{M_V^2} \right) = 0. \tag{25}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F(M_V^2)}$ and $\mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}$ is a direct consequence of Theorem 2, Corollary 1, Proposition 2, Proposition 3, and Corollary 3.

In (23), the inclusions follow from Corollary 1, Theorem 3, and Remark 11. The equality holds by applying Theorem 3 and Remark 6, using Theorem 1, Lemma 2, Remark 11, and Corollary 3.

The pullback attraction result (24) comes from Remark 8, Lemma 2, and the fact that by the regularity property (a) in Theorem 1, for any $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ and any $\tau < t - h - 1$,

$$\begin{aligned} & \operatorname{dist}_{M_V^2}(S(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\ = \; & \operatorname{dist}_{M_V^2}(S(t,\tau+h+1)(S(\tau+h+1,\tau)D(\tau)), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\ = \; & \operatorname{dist}_{M_V^2}(S(t,\tau+h+1)(j(\widetilde{U}(\tau+h+1,\tau)D(\tau))), \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)) \\ = \; & \operatorname{dist}_{M_V^2}(S(t,\tau+h+1)(j(D^{(h+1)}(\tau))), \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t)), \end{aligned}$$

since it is clear that the family $\{j(D^{(h+1)}(\tau)) : \tau \in \mathbb{R}\}$ belongs to $\mathcal{D}_\eta^V(M_H^2)$.

If moreover f satisfies (22), the equality $\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ follows from Remark 5, and the equality $\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t) = \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t)$ is again a consequence of

Theorem 3, by using the second estimate in (13), Remark 11, and Corollary 3, since (22) is equivalent to

$$\sup_{s\le 0}\int_{s-1}^{s}|f(\theta)|^2\,d\theta<\infty. \tag{26}$$

Lastly, the tempered property (25) comes from (22) (and therefore (26)) and the tempered character of $\rho_2(t)$ defined in Lemma 3. □

Remark 15. Under the assumptions of Theorem 6, reasoning analogously as in Remark 14, one has that $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$ belongs to $\mathcal{D}_\eta^V(M_H^2)$.

To conclude, we relate the minimal pullback attractors obtained in $C_H^{\tilde h,V}$ and M_V^2 through the canonical injection j.

Theorem 7. *Assume that $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, and that f and $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$ satisfy (I)–(V). Then, the following relations hold:*

$$j(\mathcal{A}_{\mathcal{D}_F(C_H^{h,V})}(t)) \subset \mathcal{A}_{\mathcal{D}_F(M_V^2)}(t)\quad \forall\, t\in\mathbb{R},\quad \textit{and} \tag{27}$$

$$j(\mathcal{A}_{\mathcal{D}_\eta^{\tilde h,V}(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_\eta^V(M_H^2)}(t)\quad \forall\,\tilde h\in[0,h],\ t\in\mathbb{R}. \tag{28}$$

Actually, if f also satisfies (22), then, for any $\tilde h\in[0,h]$,

$$j(\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde h,V})}(t))=j(\mathcal{A}_{\mathcal{D}_F^{\tilde h,V}(C_H)}(t))=\mathcal{A}_{\mathcal{D}_F(M_V^2)}(t)\quad\forall\, t\in\mathbb{R}. \tag{29}$$

Proof. In order to prove the inclusion in (27) we proceed similarly as in [19, Theorem 5], taking into account that the map j is continuous from $C_H^{h,V}$ into M_V^2, and that $j(\mathcal{D}_F(C_H^{h,V}))\subset\mathcal{D}_F(M_V^2)$.

The equality in (28) is a consequence of property (11) in Theorem 4, using the equalities (21) and (23).

Finally, the equalities in (29) follow from (28) and the known facts that, under the additional assumption (22), all attractors in (21) and (23) coincide. □

Acknowledgments. While finishing this paper among other projects, our coauthor Prof. José Real deceased. This work is dedicated to his memory, with our deepest and most sincere admiration, gratitude, and love. He was P.M.-R. and J.G.-L.'s PhD-advisor, clever mathematician with a sharp view on problems, generous and wonderful colleague, and better friend. He passed away too soon, being only 60 years old. We miss him deeply, but he will stay forever in our hearts.

This work has been partially supported by Ministerio de Ciencia e Innovación (Spain) under project MTM2011-22411. J.G.-L. is a fellow of Programa de FPU del Ministerio de Educación (Spain).

The authors thank one of the referees by his/her comments, which led to improvements in the presentation of this paper.

REFERENCES

[1] (MR2191992) T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.* **64** (2006), 484–498.

[2] (MR2196010) T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, *C. R. Math. Acad. Sci. Paris* **342** (2006), 263–268.

[3] (MR1862662) T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), 2441–2453.

[4] (MR2027360) T. Caraballo and J. Real, Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), 3181–3194.

[5] (MR2091818) T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations* **205** (2004), 271–297.

[6] (MR0138894) L. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, *Rend. Sem. Mat. Univ. Padova* **31** (1961), 308–340.

[7] (MR2810726) J. García-Luengo, P. Marín-Rubio, and J. Real, H^2-boundedness of the pullback attractors for non-autonomous 2D Navier-Stokes equations in bounded domains, *Nonlinear Anal.* **74** (2011), 4882–4887.

[8] (MR2881040) J. García-Luengo, P. Marín-Rubio, and J. Real, Pullback attractors in V for non-autonomous 2D-Navier-Stokes equations and their tempered behaviour, *J. Differential Equations* **252** (2012), 4333–4356.

[9] J. García-Luengo, P. Marín-Rubio, and J. Real, Pullback attractors for 2D Navier-Stokes equations with delays and their regularity, *Adv. Nonlinear Stud.* To appear.

[10] (MR2196812) M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.* **64** (2006), 1100–1118.

[11] (MR2799549) S. M. Guzzo and G. Planas, On a class of three dimensional Navier-Stokes equations with bounded delay, *Discrete Contin. Dyn. Syst. Ser. B* **16** (2011), 225–238.

[12] (MR0259693) J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, 1969.

[13] (MR0771394) A. Z. Manitius, Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulation, *IEEE Trans. Automat. Control* **29** (1984), 1058–1068.

[14] (MR2660876) P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, Three dimensional system of globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010), 655–673.

[15] (MR2882145) P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, On the convergence of solutions of globally modified Navier-Stokes equations with delays to solutions of Navier-Stokes equations with delays, *Adv. Nonlinear Stud.* **11** (2011), 917–927.

[16] (MR2825639) P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, Pullback attractors for globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst.* **31** (2011), 779–796.

[17] (MR2347436) P. Marín-Rubio and J. Real, Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains, *Nonlinear Anal.* **67** (2007), 2784–2799.

[18] (MR2536301) P. Marín-Rubio and J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Anal.* **71** (2009), 3956–3963.

[19] (MR2600726) P. Marín-Rubio and J. Real, Pullback attractors for 2D-Navier-Stokes equations with delays in continuous and sub-linear operators, *Discrete Contin. Dyn. Syst.* **26** (2010), 989–1006.

[20] (MR2764398) P. Marín-Rubio, J. Real, and J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case, *Nonlinear Anal.* **74** (2011), 2012–2030.

[21] (MR2399458) G. Planas and E. Hernández, Asymptotic behaviour of two-dimensional time-delayed Navier-Stokes equations, *Discrete Contin. Dyn. Syst.* **21** (2008), 1245–1258.

[22] (MR1881888) J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, 2001.

[23] (MR1491614) R. Rosa, The global attractor for the 2D Navier-Stokes flow on some unbounded domains, *Nonlinear Anal.* **32** (1998), 71–85.

[24] (MR0953967) R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1988.

[25] (MR0603444) R. Temam, *Navier-Stokes equations, Theory and Numerical Analysis*, 2nd. ed., North Holland, Amsterdam, 1979.

Received April 2011; revised July 2012.

E-mail address, J. García-Luengo: `luengo@us.es`
E-mail address, P. Marín-Rubio: `pmr@us.es`
E-mail address, J. Real: `jreal@us.es`