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Existence and regularity of pullback attractors for a 3D non-autonomous Navier–Stokes–Voigt model with finite delay

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Dedicated to the memory of Isabel Morillo Montaño 'Beli', with love

Abstract. In this manuscript previous results [*Nonlinearity* **25**(2012), 905–930] are extended to a non-autonomous 3D Navier–Stokes–Voigt model in which a forcing term contains memory effects. Under suitable assumptions on the function driving the delay time, the existence and uniqueness of weak solution are proved. Existence and relationships among pullback attractors in several phase-spaces are analyzed for two possible choices of the attracted universes, namely, the standard one of fixed bounded sets, and another one given by a tempered condition. Some regularity results for these attractors are also established. Compactness and attraction norms are strengthened. Since the model does not have a regularizing effect, obtaining asymptotic compactness for the associated process is a more involved task. Our proofs rely on a sharp use of the energy equality, an energy method, bootstrapping arguments and by using bi-space attractors results.

Keywords: 3D Navier–Stokes–Voigt equations, delay terms, pullback attractors, bi-space attractors.

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1 Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth enough (e.g. C^2) boundary $\partial\Omega$. We consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following non-autonomous functional Navier–Stokes–

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Voigt problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(u-\alpha^2\Delta u\right)-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f(t)+g(t,u_t) \quad \text{in } \Omega\times(\tau,\infty), \\ \operatorname{div} u=0 \quad \text{in } \Omega\times(\tau,\infty), \\ u=0 \quad \text{on } \partial\Omega\times(\tau,\infty), \\ u(x,\tau)=u^\tau(x), \quad x\in\Omega, \\ u(x,\tau+s)=\phi(x,s), \quad x\in\Omega,\ s\in(-h,0), \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $\alpha > 0$ is a characterizing parameter of the elasticity of the fluid, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, f is a non-delayed external force field, g is another external force containing some hereditary characteristics, and u^τ and $\phi(x, s-\tau)$ are the initial data in τ and $(\tau-h, \tau)$ respectively, where $h > 0$ is the time of memory effect. For each $t \geq \tau$, we denote by u_t the function defined a.e. on $(-h, 0)$ by the relation $u_t(s) = u(t+s)$, a.e. $s \in (-h, 0)$.

The Navier–Stokes–Voigt (NSV for short in the sequel) model of viscoelastic incompresible fluid, introduced by Oskolkov in [29], gives an approximate description of the Kelvin-Voigt fluid (see [22, 30]), and was proposed as a regularization of the 3D-Navier–Stokes equations for the purpose of direct numerical simulations in [2]. The extra regularizing term $-\alpha^2\Delta u_t$ changes the parabolic character of the equation, making it a well-posed (forward and backward) problem in 3D, but one does not observe any immediate smoothing of the solution, as expected in parabolic PDEs. Moreover, the generated semigroup is only asymptotically compact, similarly to damped hyperbolic systems. One of the studied topics about the problem is the inviscid question in some different senses. It is also worth observing that, when $\nu = 0$, the inviscid equation that one recovers is the simplified Bardina subgrid scale model of turbulence. The relationship between the original and inviscid models was also addressed in [2]. On other hand, some questions on the inviscid regularization were used for the study of a 2D surface quasi-geostrophic model in [21].

With respect to the non-delayed NSV model, the long-time behaviour of the solutions has been studied by different authors. Namely, in the autonomous case, the existence of a global compact attractor was proved by Kalantarov and Titi in [20]. Other related results have been also analyzed, as the Gévrey regularity of the global attractor (again for the autonomous model) when the force term is analytic of Gévrey type (see [19]), and the establishment of similar statistical properties (and invariant measures) as for the 3D-Navier–Stokes equations (cf. [23, 31]). Moreover, in the non-autonomous case, the existence of minimal pullback attractors in both V and $D(A)$ norms, and some regularity properties of these attractors, were obtained in [14]. We may also cite in this non-autonomous framework the paper [40], where the existence of uniform attractor for a NSV model is studied.

On the other hand, in many physical experiments, the inclusion of measurement devices to control properties of fluids (such as temperature, velocity, etc.) may incorporate additional external forces to the model including also delay effects (e.g. for a wind-tunnel model). In this sense, the study of 2D-Navier–Stokes models with delay terms – existence, uniqueness, stationary solutions, exponential decay, existence of attractors, et cetera – was initiated in the references [6–8] and, after that, many different questions, as dealing with unbounded domains, and models (for instance in three dimensions for modified terms) have been addressed (e.g., cf. [15, 17, 26, 28, 36] among others). In the past years, the asymptotic behaviour of the Navier–Stokes–Voigt equations with delays or with memory have been studied in [3, 12, 24, 34, 35, 38]. It is worth pointing out that, in [24], the authors establish the existence of pullback attractors in V norm for a three dimensional NSV model when the forcing term containing the delay is sublinear and only continuous. Since the uniqueness of solution is not guaranteed under these assumptions, they use the theory of multi-valued dynamical systems and similar arguments as in [28] for the proof of the asymptotic compactness of the process. In this work, we suppose more restrictive conditions on the delay operator that assure the uniqueness of solution, so we can apply the classical results of Dynamical Systems. However, in contrast with [24], we modify the phase-space enlarging the set of initial conditions. Moreover, for the associated single-valued process, we are able to obtain the existence of minimal pullback attractors, with richer compactness sections and not only in (roughly speaking) V norm, but also in $D(A)$ norm. Moreover, some regularity properties of these attractors are also successfully established. This analysis is carried out by applying similar techniques as in [14], but with the necessary modifications caused by the inclusion of a delay term.

As commented before, the difference between this model and the *standard* 2D-Navier–Stokes model is that there exists a regularizing effect in the 2D-Navier–Stokes model, while not here. For 2D-Navier–Stokes a continuous energy method can be applied thanks to the extra estimates that hold in higher norms (e.g., cf. [28]), which does not seem to hold for the NSV model. Some of the proofs in the previously cited references about NSV (e.g., cf. [20]) rely on splitting the problem in two, one with exponential decay, and the other with good asymptotic properties in the domain of a suitable fractional power of the Stokes operator. However, similarly as in [14], we will provide a simpler proof, which does not require the above mentioned technicalities, but a sharp use of the energy equality, and the energy method used by Rosa in [32]. Moreover, it is worth pointing out that our results in Section 3 do not use the regularity assumption on $\partial\Omega$ at all, and the force term may take values in V' instead of in L^2 as appears in [20].

The structure of the paper is the following. In Section 2 we recall some definitions of classical functional spaces to state our problem in an abstract form, basic properties and estimates of the involved operators. We also obtain a result on the existence, uniqueness and regularity of the weak solution for problem (1.1). We start Section 3 with a brief recall of the main definitions on the theory of minimal pullback attractors and bi-space attractors for non-autonomous dynamical systems within the framework of universes. Then, we prove the existence of pullback attractors in (roughly speaking) V norm and for two choices of the attracted universes, namely, the standard one of fixed bounded sets, and secondly, one given by a tempered growth condition. We also establish some relations among these families and improve compactness and attraction norm results. In Section 4, extra regularity for the obtained attractors will be deduced by using a bootstrapping argument that involves fractional powers of the Stokes operator. Finally, in Section 5, the problem of attraction in $D(A)$ norm is studied although it is more involved (namely it fits out from the standard theoretical results). Indeed under suitable assumptions, all attractors are proved to coincide.

2 Existence and uniqueness of solution

In this section we prove existence, uniqueness and regularity of the solutions to problem (1.1). These results will be obtained in a similar way as in [14], but with the necessary changes due to the inclusion of a delay term. We begin by stating the problem in an abstract setting, and to do so we recall several definitions of functional spaces, operators and some of their properties (for the details see [37]).

To start with, we consider the usual spaces in the variational theory of Navier–Stokes equations: H, the closure of $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}$ in $(L^2(\Omega))^3$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, and V, the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$, that is, the L^2-product of gradients, thanks to the Poincaré inequality.

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality $\langle V', V\rangle$. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and compact.

Let us define the linear continuous operator $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$ for all $u, v \in V$, and we denote $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has that $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^3}$ and $|\cdot|_{D(A)}$ are equivalent, and $D(A)$ is compactly and densely injected in V. We will also denote by $\{w_j\}_{j\geq 1} \subset D(A)$ a Hilbert basis of H formed by normalized eigenfunctions of the Stokes operator A, with corresponding eigenvalues $\{\lambda_j\}_{j\geq 1}$ being $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lim_{j\to\infty} \lambda_j = \infty$. Recall that the first eigenvalue of A satisfies

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2}. \tag{2.1}$$

For the fractional powers of A, we have the following inclusions with continuous injection (cf. [33, Chapter III, Lemmas 2.4.2 and 2.4.3]):

$$D(A^\beta) \subset (L^{6/(3-4\beta)}(\Omega))^3, \quad \forall\, 0 \leq \beta < 3/4, \tag{2.2}$$

$$D(A^{3/4}) \subset (L^p(\Omega))^3, \qquad \forall\, 1 \leq p < \infty, \tag{2.3}$$

and

$$D(A^\beta) \subset (L^\infty(\Omega))^3, \quad \forall\, 3/4 < \beta \leq 1. \tag{2.4}$$

Now, we define

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^3$ for which the right-hand side is well defined. In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$, i.e., there exists a constant $C_1 > 0$ such that

$$|b(u, v, w)| \leq C_1 \|u\| \|v\| \|w\|, \quad \forall u, v, w \in V. \tag{2.5}$$

Important properties concerning b are that

$$\begin{aligned} b(u, v, w) &= -b(u, w, v), \quad \forall u, v, w \in V, \\ b(u, v, v) &= 0, \quad \forall u, v \in V, \end{aligned} \tag{2.6}$$

and, using Agmon inequality (e.g. cf. [10]), we can assure that there exists a constant $C_2 > 0$ such that

$$|b(u, v, w)| \leq C_2 |Au|^{1/2} \|u\|^{1/2} \|v\| |w|, \quad \forall u \in D(A), v \in V, w \in H. \tag{2.7}$$

For any $u \in V$, we will use $B(u)$ to denote the element of V' given by $\langle B(u), w\rangle = b(u, u, w)$ for all $w \in V$. Thus, by (2.5),

$$\|B(u)\|_* \leq C_1 \|u\|^2, \quad \forall u \in V, \tag{2.8}$$

and in particular, by (2.7) and the identification of H' with H, if $u \in D(A)$, then $B(u) \in H$, with

$$|B(u)| \le C_2 |Au|^{1/2} \|u\|^{3/2}, \quad \forall u \in D(A). \tag{2.9}$$

In fact, from (2.4), one also deduces that if $u \in D(A^\beta)$ with $3/4 < \beta \le 1$, then $B(u) \in H$, and more exactly

$$|B(u)| \le C_{(\beta)} |A^\beta u| \|u\|, \quad \forall u \in D(A^\beta), \quad \forall\, 3/4 < \beta \le 1. \tag{2.10}$$

Analogously, if $0 \le \beta < 3/4$, from (2.2) one obtains that if $u \in D(A^\beta) \cap V$, $B(u) \in D(A^{\beta-3/4})$, and more exactly

$$|A^{\beta-3/4} B(u)| \le C_{(\beta)} |A^\beta u| \|u\|, \quad \forall u \in D(A^\beta) \cap V, \quad \forall\, 0 \le \beta < 3/4. \tag{2.11}$$

Finally, in the case $\beta = 3/4$, from (2.3) one can see that if $u \in D(A^{3/4})$, then $B(u) \in D(A^{-\delta})$ for all $\delta > 0$, and more exactly

$$|A^{-\delta} B(u)| \le C_{(3/4,\delta)} |A^{3/4} u| \|u\|, \quad \forall u \in D(A^{3/4}), \quad \forall \delta > 0.$$

Now, we establish some appropriate assumptions on the term in (1.1) containing the delay.

Let $(X, \|\cdot\|_X)$ be a Banach space. We will denote $C_X = C([-h, 0]; X)$, the space of continuous functions from $[-h, 0]$ into X, with the norm $\|\varphi\|_{C_X} = \max_{s \in [-h,0]} \|\varphi(s)\|_X$, and $L^2_X = L^2(-h, 0; X)$, where the norm will be denoted by $\|\cdot\|_{L^2_X}$. On the delay operator from (1.1), we consider that is well defined as $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^3$ is measurable,

(II) $g(t, 0) = 0$, for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all $\xi, \eta \in C_H$,

$$|g(t, \xi) - g(t, \eta)| \le L_g |\xi - \eta|_{C_H},$$

(IV) there exists $C_g > 0$ such that for all $\tau \le t$, and for all $u, v \in C([\tau - h, t]; H)$,

$$\int_\tau^t |g(s, u_s) - g(s, v_s)|^2 \, ds \le C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 \, ds.$$

Examples of fixed, variable and distributed delay operators can be found, for instance, in [6, Section 3], [8, Sections 3.5 and 3.6], and [17, Section 3], and we omit them here just for the sake of brevity.

Observe that (I)–(III) imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^3$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^3)$. Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C([\tau - h, T]; H) \to g_u \in L^2(\tau, T; (L^2(\Omega))^3)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^3)$. From now on, we will denote $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all $u, v \in L^2(\tau - h, T; H)$.

Since it will be used to deduce some estimates for the solutions of (1.1), we study the autonomous equation $u + \alpha^2 Au = \varphi$. From the Lax–Milgram lemma, we know that for each $\varphi \in V'$ there exists a unique $u_\varphi \in V$ such that $u_\varphi + \alpha^2 Au_\varphi = \varphi$. Therefore, the mapping

$$\mathcal{C} : u \in V \mapsto u + \alpha^2 Au \in V'$$

is linear and bijective, with $\mathcal{C}^{-1}\varphi = u_\varphi$. Moreover, by the definition of $D(A)$, we also have that $\mathcal{C}^{-1}(H) = D(A)$. Now, reasoning as in [14], we obtain that

$$\|u_\varphi\| \le \alpha^{-2}\|\varphi\|_*, \quad \forall \varphi \in V', \tag{2.12}$$

and

$$|Au_\varphi| \le 2\alpha^{-2}|\varphi|, \quad \forall \varphi \in H. \tag{2.13}$$

Let us consider that $u^\tau \in V$, $\phi \in L^2_H$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 2.1. A weak solution to (1.1) is a function u that belongs to $L^2(\tau - h, T; H) \cap L^2(\tau, T; V)$ for all $T > \tau$, such that $u(\tau) = u^\tau$, $u(t) = \phi(t - \tau)$ a.e. $t \in (\tau - h, \tau)$, and satisfies

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t) + g(t, u_t), \quad \text{in } \mathcal{D}'(\tau, \infty; V'). \tag{2.14}$$

Observe that if u is a weak solution to (1.1), then $u(t) + \alpha^2 Au(t) \in L^2(\tau, T; V')$ for all $T > \tau$, and by (2.8), $\frac{d}{dt}(u(t) + \alpha^2 Au(t)) \in L^1(\tau, T; V')$ for all $T > \tau$. Therefore, by using (2.12) and reasoning as in [14], we can deduce that $u \in C([\tau, \infty); V)$, whence the initial datum $u(\tau) = u^\tau$ has full sense, and $u' \in L^2(\tau, T; V)$ for all $T > \tau$.

Furthermore, the following energy equality holds:

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \nu\|u(t)\|^2 = \langle f(t), u(t)\rangle + (g(t, u_t), u(t)), \quad \text{a.e. } t > \tau. \tag{2.15}$$

Concerning the existence and uniqueness of weak solution to (1.1), we have the following result.

Theorem 2.2. *Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), be given. Then, for each $\tau \in \mathbb{R}$, $u^\tau \in V$ and $\phi \in L^2_H$, there exists a unique weak solution $u = u(\cdot; \tau, u^\tau, \phi)$ of (1.1).*

Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ and $u^\tau \in D(A)$, then u has the following regularity

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau, \tag{2.16}$$

and a.e. $t > \tau$ satisfies

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \alpha^2|Au(t)|^2) + \nu|Au(t)|^2 + (B(u(t)), Au(t)) = (f(t) + g(t, u_t), Au(t)). \tag{2.17}$$

Proof. Uniqueness. Consider two weak solutions $u^{(1)}$ and $u^{(2)}$ to problem (1.1), corresponding to the same initial data, and denote $\hat{u} = u^{(1)} - u^{(2)}$. Observe that by (2.5) and (2.6),

$$\begin{aligned}|b(u^{(1)}(s), u^{(1)}(s), \hat{u}(s)) - b(u^{(2)}(s), u^{(2)}(s), \hat{u}(s))| &= |b(\hat{u}(s), u^{(1)}(s), \hat{u}(s))| \\ &\le C_1\|u^{(1)}(s)\|\|\hat{u}(s)\|^2.\end{aligned}$$

Then, from the equation satisfied by $\hat{u}$ and the energy equality, it follows that

$$\begin{aligned}
&|\hat{u}(t)|^2 + \alpha^2\|\hat{u}(t)\|^2 + 2\nu\int_\tau^t \|\hat{u}(s)\|^2\,ds \\
&\quad = -2\int_\tau^t b(\hat{u}(s),u^{(1)}(s),\hat{u}(s))\,ds + 2\int_\tau^t (g(s,u_s^{(1)}) - g(s,u_s^{(2)}),\hat{u}(s))\,ds \\
&\quad \le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + 2\int_\tau^t |g(s,u_s^{(1)}) - g(s,u_s^{(2)})||\hat{u}(s)|\,ds
\end{aligned}$$

for all $t \ge \tau$. Now, by the Young inequality and the assumption (IV) on g, taking into account that $\hat{u}(s) = 0$ for $s \in (\tau - h, \tau)$, we obtain that

$$\begin{aligned}
&|\hat{u}(t)|^2 + \alpha^2\|\hat{u}(t)\|^2 + 2\nu\int_\tau^t \|\hat{u}(s)\|^2\,ds \\
&\quad \le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + \int_\tau^t |g(s,u_s^{(1)}) - g(s,u_s^{(2)})|^2\,ds + \int_\tau^t |\hat{u}(s)|^2\,ds \\
&\quad \le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + \lambda_1^{-1}(C_g^2+1)\int_\tau^t \|\hat{u}(s)\|^2\,ds
\end{aligned}$$

for all $t \ge \tau$, and in particular

$$\|\hat{u}(t)\|^2 \le \alpha^{-2}\big(2C_1 + \lambda_1^{-1}(C_g^2+1)\big)\int_\tau^t \big(\|u^{(1)}(s)\| + 1\big)\|\hat{u}(s)\|^2\,ds$$

for all $t \ge \tau$. Thus, from the Gronwall lemma, we conclude uniqueness.

Existence. We will follow a Galerkin scheme similarly as in [14, Theorem 4]. Let $\{w_j\}_{j\ge1} \subset D(A)$ be the Hilbert basis of H formed by normalized eigenfunctions of the Stokes operator A introduced before.

For each integer $m \ge 1$, we pose the approximate problems of finding $u^m \in V_m := \operatorname{span}[w_1,\dots,w_m]$ with $u^m(t) = \sum_{j=1}^m \gamma_{m,j}(t)w_j$, where the coefficients $\gamma_{m,j}$ are required to satisfy the system

$$\begin{aligned}
&\frac{d}{dt}(u^m(t) + \alpha^2 Au^m(t), w_j) + \nu((u^m(t),w_j)) + b(u^m(t),u^m(t),w_j) \\
&\qquad = \langle f(t), w_j\rangle + (g(t,u_t^m),w_j), \quad \text{a.e. } t > \tau, \quad 1 \le j \le m,
\end{aligned} \tag{2.18}$$

and the initial conditions

$$u^m(\tau) = P_m u^\tau \quad \text{and} \quad u^m(\tau+s) = P_m\phi(s) \quad \text{a.e. } s \in (-h,0),$$

where P_m is the orthogonal projector from H onto V_m. Observe that, by the choice of the basis $\{w_j\}_{j\ge1}$, the restriction $P_{m|_V}$ of P_m to V belongs to $\mathcal{L}(V)$, $\|P_{m|_V}\|_{\mathcal{L}(V)} \le 1$ for all $m \ge 1$, and $\lim_{m\to\infty}\|u^\tau - P_m u^\tau\| = 0$.

The above system of ordinary functional differential equations with finite delay fulfills the conditions for existence and uniqueness of local solution (see for example [18]).

Next, we will deduce a priori estimates that in particular assure that the solutions u^m do exist for all time $t \in [\tau - h, \infty)$.

Multiplying each equation in (2.18) by $\gamma_{m,j}(t)$ and summing from $j = 1$ to $j = m$, we obtain that a.e. $t > \tau$,

$$\begin{aligned}
\frac{d}{dt}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) + 2\nu\|u^m(t)\|^2 &= 2\langle f(t),u^m(t)\rangle + 2(g(t,u_t^m),u^m(t)) \\
&\le \nu\|u^m(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + |g(t,u_t^m)|^2 + |u^m(t)|^2,
\end{aligned}$$

where we have used (2.6) to remove the nonlinear term b, and the Young inequality.

By integrating in time, from the assumptions on the delay operator g, in particular we deduce that

$$\begin{aligned} &|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2 \\ &\quad \le |P_m u^\tau|^2 + \alpha^2\|P_m u^\tau\|^2 + \nu^{-1}\int_\tau^t \|f(s)\|_*^2\,ds + C_g^2\int_{\tau-h}^t |u^m(s)|^2\,ds + \int_\tau^t |u^m(s)|^2\,ds \\ &\quad \le |u^\tau|^2 + \alpha^2\|u^\tau\|^2 + C_g^2\|\phi\|_{L^2_H}^2 + \nu^{-1}\int_\tau^t \|f(s)\|_*^2\,ds + \lambda_1^{-1}(C_g^2+1)\int_\tau^t \|u^m(s)\|^2\,ds \end{aligned}$$

for all $t \ge \tau$, and any $m \ge 1$. Now, by the Gronwall lemma we conclude that the sequence $\{u^m\}_{m\ge1}$ is bounded in $C([\tau,T];V)$ for all $T > \tau$. Moreover, since $u^m_\tau = P_m\phi$ converges to ϕ in $L^2(-h,0;H)$, in particular, thanks to (IV), the sequence $\{g(\cdot,u^m_\cdot)\}_{m\ge1}$ is bounded in $L^2(\tau,T;(L^2(\Omega))^3)$ for all $T > \tau$.

Now from (2.8), (2.18) and by the choice of the basis, we obtain that $v^m = \mathcal{C}u^m$ satisfies

$$\|(v^m)'(t)\|_* \le \nu\|u^m(t)\| + C_1\|u^m(t)\|^2 + \|f(t)\|_* + \lambda_1^{-1/2}|g(t,u^m_t)|, \quad \text{a.e. } t > \tau,$$

which implies that the sequence $\{dv^m/dt\}_{m\ge1}$ is bounded in $L^2(\tau,T;V')$ for all $T > \tau$. Therefore, taking into account that $du^m/dt = \mathcal{C}^{-1}(dv^m/dt)$, we have that the sequence $\{du^m/dt\}_{m\ge1}$ is bounded in $L^2(\tau,T;V)$ for all $T > \tau$.

Thus, by the compactness of the injection of V into H and the Ascoli–Arzelà theorem, we deduce that there exist a subsequence $\{u^{m'}\}_{m'\ge1} \subset \{u^m\}_{m\ge1}$ and a function $u \in W^{1,2}(\tau,T;V)$ for all $T > \tau$, with $u_\tau = \phi$, such that

$$\begin{cases} u^{m'} \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(\tau,T;V), \\ u^{m'} \to u \text{ strongly in } C([\tau,T];H), \\ u^{m'} \to u \text{ a.e. in } \Omega\times(\tau,T), \\ g(\cdot,u^{m'}_\cdot) \to g(\cdot,u_\cdot) \text{ strongly in } L^2(\tau,T;(L^2(\Omega))^3), \\ \dfrac{du^{m'}}{dt} \rightharpoonup \dfrac{du}{dt} \text{ weakly in } L^2(\tau,T;V), \\ \dfrac{dv^{m'}}{dt} = \mathcal{C}\left(\dfrac{du^{m'}}{dt}\right) \rightharpoonup \mathcal{C}\left(\dfrac{du}{dt}\right) \text{ weakly in } L^2(\tau,T;V'), \end{cases} \tag{2.19}$$

for all $T > \tau$.

Now, using the same reasoning as in [14], we can obtain that $B(u^{m'}) \rightharpoonup B(u)$ weakly in $L^2(\tau,T;V')$, for all $T > \tau$. So, from all the convergences above, we can take limits in (2.18) and conclude that u satisfies (2.14).

Notice also that $u(\tau) = \lim_{m'\to\infty} u^{m'}(\tau) = \lim_{m'\to\infty} P_{m'}u^\tau = u^\tau$. Thus, u is the weak solution to (1.1).

Finally, the regularity property (2.16) and the identity (2.17) follow from the corresponding results proved in [14, Theorem 4] and the fact that, if $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, then the function $f(\cdot) + g(\cdot,u_\cdot)$ belongs to $L^2_{loc}(\tau,\infty;(L^2(\Omega))^3)$. □

Remark 2.3. Observe that in the above proof, using the uniqueness of solution to the problem, for any $T > \tau$ we have that the whole sequence of the Galerkin approximations $\{u^m\}$ converges to u in $C([\tau,T];H)$. Actually, all the convergences in (2.19), except the third one, hold for the whole sequence. Analogously, one also deduces that for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ weakly in V.

In addition, if $u^\tau \in D(A)$ and $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, then for any $T > \tau$ the sequence $\{u^m\}$ converges to u in $C([\tau, T]; V)$, and weakly-star in $L^\infty(\tau, T; D(A))$, for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ in $D(A)$, and the sequence $\{du^m/dt\}$ converges to du/dt weakly in $L^2(\tau, T; D(A))$.

Remark 2.4. (i) The solution depends continuously on the initial data in the strong topology of $V \times L^2_H$. Moreover, when $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, the solution depends continuously on the initial data in the strong topology of $D(A) \times L^2_V$. Indeed, this can be proved similarly to the proof of uniqueness of weak solution to (1.1), considering the difference of two solutions and using the Gronwall lemma.

(ii) The existence and uniqueness part of Theorem 2.2 do not need any regularity assumption on the boundary of the domain. In fact, this assumption is only required for the additional regularity results.

3 Existence of minimal pullback attractors in V norm

Before to start, let us recall some abstract definitions and results on pullback attractors and bi-space attractors theories. In fact, abstract existence results are omitted for the sake of brevity. For instance, they can be found in [4, 5, 13, 27] for pullback attractors (and references therein) and in [11] for bi-space pullback attractors (see also [1, 9, 39] for the autonomous bi-space attractors theory). They will be applied to a suitable dynamical system associated to (1.1), or to a restricted version involving more regularity or because of better properties.

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \le t\}$.

A process $\mathcal{U}$ on X is a mapping $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto \mathcal{U}(t, \tau)x \in X$ such that $\mathcal{U}(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $\mathcal{U}(t, r)(\mathcal{U}(r, \tau)x) = \mathcal{U}(t, \tau)x$ for any $\tau \le r \le t$ and all $x \in X$.

A process $\mathcal{U}$ is said to be continuous if for any pair $\tau \le t$, the mapping $\mathcal{U}(t, \tau) : X \to X$ is continuous. It is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $\mathcal{U}(t, \tau)x_n \to y \in X$, then $\mathcal{U}(t, \tau)x = y$. It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

The process $\mathcal{U}$ is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{\mathcal{U}(t, \tau_n)x_n\}$ is relatively compact in X.

A process $\mathcal{U}$ on X being pullback $\widehat{D}_0$-asymptotically compact possesses a family of non-empty compact subsets of X, namely the atomized structure for the asymptotic behavior, the omega-limit family $\Lambda_X(\widehat{D}_0) = \{\Lambda_X(\widehat{D}_0, t) : t \in \mathbb{R}\}$ with

$$\Lambda_X(\widehat{D}_0, t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} \mathcal{U}(t, \tau) D_0(\tau)}^X .$$

It pullback attracts in X norm to $\widehat{D}_0$ (cf. [13, Proposition 3.4]), i.e.

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(\mathcal{U}(t, \tau)D_0(\tau), \Lambda_X(\widehat{D}_0, t)) = 0, \qquad \forall t \in \mathbb{R},$$

where $\operatorname{dist}_X(\cdot, \cdot)$ denotes the Hausdorff semi-distance in X. In fact, it is the minimal family of closed sections in X that attracts $\widehat{D}_0$. Moreover, if $\mathcal{U}$ is also a closed process on X, then (cf. [13, Proposition 3.5]) it is invariant, i.e. $\mathcal{U}(t, \tau)\Lambda_X(\widehat{D}_0, \tau) = \Lambda_X(\widehat{D}_0, t)$ for all $\tau \le t$.

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.1. A process $\mathcal{U}$ on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for $\mathcal{U}$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that $\mathcal{U}(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \leq \tau_0(t, \widehat{D})$.

The suitable combination of the above two ingredients leads to

Definition 3.2. Given a metric space X, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a process $\mathcal{U}$ on X, a family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is called a pullback $\mathcal{D}$-attractor for $\mathcal{U}$ if (i) $\mathcal{A}_{\mathcal{D}}(t)$ is compact in X for any $t \in \mathbb{R}$, (ii) $\mathcal{A}_{\mathcal{D}}$ pullback $\mathcal{D}$-attracts in X and (iii) it is invariant (i.e. $\mathcal{U}(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for any $\tau \leq t$).

Besides, it is said the minimal pullback $\mathcal{D}$-attractor for $\mathcal{U}$ on X if given any family $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets that pullback $\mathcal{D}$-attracts under $\mathcal{U}$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Without minimality, pullback attractors are not unique in general (cf. [27]). Minimality involves uniqueness and a clear candidate, after the definition of omega-limit families. Namely, the following result is well-known.

Theorem 3.3 (cf. [13, Theorem 3.11])**.** *Consider a closed process $\mathcal{U} : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for $\mathcal{U}$, and assume also that $\mathcal{U}$ is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda_X(\widehat{D}, t)}^X$ is the minimal pullback $\mathcal{D}$-attractor for $\mathcal{U}$ in X.*

Remark 3.4. Under the assumptions of Theorem 3.3, the family $\mathcal{A}_{\mathcal{D}}$ satisfies $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda_X(\widehat{D}_0, t)$ for any $t \in \mathbb{R}$. Actually, if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}} = \Lambda_X(\widehat{D}_0)$. Moreover, if $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (ii)–(iii) in Definition 3.2. A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 3.5 (cf. [27, Corollaries 20 and 21])**.** *Under the assumptions of Theorem 3.3, if $\mathcal{D}$ contains $\mathcal{D}_F(X)$, then the minimal pullback attractor $\mathcal{A}_{\mathcal{D}_F(X)}$ also exists and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$. Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \leq T} D_0(t)$ is bounded, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.*

Comparison results with different universes are also possible if the process $\mathcal{U}$ is well-posed in several metric spaces with a connection between them. Namely, Theorem 3.15 in [13] allows us to gain additional regularity about attractors. For the sake of brevity, we omit such statement. Nevertheless, we will recall another one with previous definitions, which will be analogously useful for our results (this is inspired from another study, cf. [25, Section 5]).

Theory of bi-space attractors (cf. [1,9,39] for autonomous setting and the references therein) is close to the previous results but joining extra regularity of the solution operator involving two spaces. Since our context is non-autonomous, we borrow some of these results from [11], settled in this framework also for closed processes. Consider given two metric spaces (X_i, d_{X_i}), $i = 1,2$ (not necessarily related) and a process $\mathcal{U}$ on X_1. It is said (cf. [11, Definition 2.12]) that $\mathcal{U}$ is (X_1, X_2) closed if for any $\tau \le t$ and $\{x_n\} \subset X_1 \cap X_2$ with $\mathcal{U}(t,\tau)x_n \in X_1 \cap X_2$, if $x_n \to x$ in X_2 and $\mathcal{U}(t,\tau)x_n \to y \in X_2$, then $x \in X_1$ and $\mathcal{U}(t,\tau)x = y$.

Given a parameterized-in-time family $\widehat{D}_0 \subset \mathcal{P}(X_1)$, a process $\mathcal{U}$ on X_1 is said (X_1, X_2) pullback $\widehat{D}_0$-asymptotically compact (cf. [11, Definition 2.4]) if for any $t \in \mathbb{R}$, sequence $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X_1$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$, the sequence $\{\mathcal{U}(t,\tau_n)x_n\}$ is relatively compact in X_2. Analogously to Definition 3.1, a process $\mathcal{U}$ on X_1 is said to be (X_1, X_2) pullback $\mathcal{D}$-asymptotically compact if it is (X_1, X_2) pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

We may run parallel the construction of a family with the desired properties of minimal pullback attractor for a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$, provided that for any $\widehat{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $s_{\widehat{D},t} \le t$ such that $\mathcal{U}(t,s)D(s) \subset X_2$ for all $s \le s_{\widehat{D},t}$ (cf. [11, (2.2)]). In this case, data comes from X_1 and the arrival attracting space is X_2 with its corresponding metric [11, Definition 2.2].

Definition 3.6. Let be given a process $\mathcal{U}$ on X_1 and a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$. The family $\widehat{\mathcal{A}}_{\mathcal{D}} = \{\widehat{\mathcal{A}}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is called a (X_1, X_2) pullback $\mathcal{D}$-attractor if (i) $\widehat{\mathcal{A}}_{\mathcal{D}}(t) \subset X_1 \cap X_2$ is a nonempty compact set in X_2 for each $t \in \mathbb{R}$, (ii) it is pullback $\mathcal{D}$-attracting using the Hausdorff semidistance in X_2 and (iii) it is invariant. Besides, it is said minimal if for any other family $\widehat{C}$ of nonempty closed time-sections with values in X_2 and pullback $\mathcal{D}$-attracting in X_2, then $\widehat{\mathcal{A}}_{\mathcal{D}}(t) \subset \widehat{C}(t)$ for any $t \in \mathbb{R}$.

Similarly to Theorem 3.3, we may ensure the existence of the minimal (X_1, X_2) pullback $\mathcal{D}$-attractor under rather general conditions (cf. [11, Theorem 2.16]).

Theorem 3.7. *Let be given two metric spaces X_i, $i = 1,2$, a process $\mathcal{U}$ on X_1, and a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$. Suppose that there exists a family $\widehat{\mathcal{B}}_0$ in $\mathcal{P}(X_1)$ that is pullback $\mathcal{D}$-absorbing, such that for any $t \in \mathbb{R}$ there exists $s_{\widehat{\mathcal{B}}_0,t} \le t$ such that $\mathcal{U}(t,s)\widehat{\mathcal{B}}_0(s) \subset X_2$ for any $s \le s_{\widehat{\mathcal{B}}_0,t}$. If the process $\mathcal{U}$ is (X_1, X_2) closed and (X_1, X_2) pullback $\widehat{\mathcal{B}}_0$-asymptotically compact, then there exists $\widehat{\mathcal{A}}_{\mathcal{D}}$ the minimal (X_1, X_2) pullback $\mathcal{D}$-attractor for $\mathcal{U}$, and it is given by $\widehat{\mathcal{A}}_{\mathcal{D}}(t) = \overline{\cup_{\widehat{D} \in \mathcal{D}} \Lambda_{X_2}(\widehat{D}, t)}^{X_2} \subset \Lambda_{X_2}(\widehat{\mathcal{B}}_0, t)$.*

Remark 3.8. If $X_2 \subset X_1$ with continuous injection, the following consequences are immediate: (i) A process $\mathcal{U}$ on X_1 that is X_1 closed, it is also (X_1, X_2) closed. (ii) Given a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$ and a process $\mathcal{U}$ (X_1, X_2) pullback $\mathcal{D}$-asymptotically compact, then $\Lambda_{X_1}(\widehat{D}) = \Lambda_{X_2}(\widehat{D})$ for any $\widehat{D} \in \mathcal{D}$ thanks to the minimality properties of omega-limit families and that a compact set in X_2 is compact in X_1. (iii) A process $\mathcal{U}$ that has a (X_1, X_2) pullback $\mathcal{D}$-attractor $\widehat{\mathcal{A}}_{\mathcal{D}}$, it also has a (X_1, X_1) pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}}$ just using the embedding $X_2 \subset X_1$ (same arguments of minimality and compact sets than in (ii), even using different clousures). In this case we make an abuse of notation, identifying both families without any extra notation, gaining extra regularity in X_2 for the sections of the attractor.

In view of Theorem 2.2 and Remark 2.4 (i), we will apply the above abstract results in the phase-space $X = V \times L^2_H$, which is a Hilbert space with the norm $\|(u^\tau, \phi)\|^2_X = \|u^\tau\|^2 + \|\phi\|^2_{L^2_H}$ for a pair $(u^\tau, \phi) \in X$.

The first consequence after the Theorem 2.2 and Remark 2.4 (i) is the following

Corollary 3.9. *Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), be given. Then, the bi-parametric family of maps $S(t,\tau) : V \times L^2_H \to V \times L^2_H$, with $\tau \le t$, given by*

$$S(t,\tau)(u^\tau, \phi) = (u(t;\tau,u^\tau,\phi), u_t(\cdot;\tau,u^\tau,\phi)), \tag{3.1}$$

where $u = u(\cdot;\tau,u^\tau,\phi)$ *is the unique weak solution to* (1.1), *defines a continuous process on* $V \times L^2_H$.

We will need the following continuity result for the process S in a weak sense.

Proposition 3.10. *Let* $f \in L^2_{loc}(\mathbb{R};V')$, $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *satisfying (I)–(IV), and* $\tau < t$ *be given. Then, for any sequence such that*

$$(u^{\tau,n},\phi^n) \rightharpoonup (u^\tau,\phi) \quad \text{weakly in } V \times L^2_V$$

and

$$\frac{d\phi^n}{ds} \rightharpoonup \frac{d\phi}{ds} \quad \text{weakly in } L^2_V,$$

the following convergences hold for the sequence of solutions $u(\cdot;\tau,u^{\tau,n},\phi^n)$ *towards the solution* $u(\cdot;\tau,u^\tau,\phi)$:

$$\begin{aligned} &u(\cdot;\tau,u^{\tau,n},\phi^n) \overset{*}{\rightharpoonup} u(\cdot;\tau,u^\tau,\phi) \quad \text{weakly-star in } L^\infty(\tau,t;V),\\ &u(\cdot;\tau,u^{\tau,n},\phi^n) \to u(\cdot;\tau,u^\tau,\phi) \quad \text{strongly in } C([\tau-h,t];H),\\ &u(t;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(t;\tau,u^\tau,\phi) \quad \text{weakly in } V,\\ &u(\cdot;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(\cdot;\tau,u^\tau,\phi) \quad \text{weakly in } L^2(\tau-h,t;V). \end{aligned} \tag{3.2}$$

Proof. Taking into account that $\{\phi^n\}$ is bounded in $W^{1,2}(-h,0;V) \subset C([-h,0];V)$, and the compactness of the injection of V into H, by the Ascoli–Arzelà theorem we deduce that $\phi^n \to \phi$ strongly in C_H. Therefore, the *a priori* estimates obtained for the Galerkin approximations in Theorem 2.2 also hold for the sequence of solutions $\{u(\cdot;\tau,u^{\tau,n},\phi^n)\}$, and then all the convergences in (3.2) hold. Finally, the fact that the whole sequence satisfies the above convergences is a consequence of the uniqueness of solution for the problem (cf. Remark 2.3). □

Now, we introduce an additional assumption on g in order to obtain some asymptotic estimates for the solutions to (1.1).

(V) Assume that $\nu\lambda_1 > C_g$, and that there exists a value $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1} + \alpha^2)^{-1}$ such that for every $u \in L^2(\tau-h,t;H)$,

$$\int_\tau^t e^{\sigma s}|g(s,u_s)|^2\,ds \le C_g^2 \int_{\tau-h}^t e^{\sigma s}|u(s)|^2\,ds, \quad \forall t \ge \tau.$$

Lemma 3.11. *Consider given* $f \in L^2_{loc}(\mathbb{R};V')$ *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *satisfying conditions (I)–(V). Then, for any* $(u^\tau,\phi) \in V \times L^2_H$, *the following estimate holds for the solution* u *to* (1.1) *for all* $t \ge \tau$,

$$\|u(t)\|^2 \le \alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{\sigma(\tau-t)}\|(u^\tau,\phi)\|^2_{V\times L^2_H} + \alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}\|f(s)\|_*^2\,ds, \tag{3.3}$$

where

$$\varepsilon = 2\nu - \sigma(\lambda_1^{-1}+\alpha^2) - 2\lambda_1^{-1}C_g > 0. \tag{3.4}$$

Proof. By the energy equality (2.15) and the Young inequality, we have

$$\begin{aligned} &\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + 2\nu\|u(t)\|^2\\ &\quad \le \varepsilon\|u(t)\|^2 + \varepsilon^{-1}\|f(t)\|_*^2 + C_g|u(t)|^2 + C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus,

$$\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) + e^{\sigma t}\big(2\nu - \varepsilon - \sigma(\lambda_1^{-1} + \alpha^2) - \lambda_1^{-1}C_g\big)\|u(t)\|^2$$
$$\leq e^{\sigma t}\varepsilon^{-1}\|f(t)\|_*^2 + e^{\sigma t}C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau,$$

and therefore, integrating in time above and using property (V), we obtain

$$e^{\sigma t}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \big(2\nu - \varepsilon - \sigma(\lambda_1^{-1} + \alpha^2) - \lambda_1^{-1}C_g\big)\int_\tau^t e^{\sigma s}\|u(s)\|^2\,ds$$
$$\leq e^{\sigma\tau}(\lambda_1^{-1} + \alpha^2)\|u^\tau\|^2 + \varepsilon^{-1}\int_\tau^t e^{\sigma s}\|f(s)\|_*^2\,ds + C_g\int_{\tau-h}^t e^{\sigma s}|u(s)|^2\,ds$$
$$\leq e^{\sigma\tau}\left((\lambda_1^{-1} + \alpha^2)\|u^\tau\|^2 + C_g\int_{-h}^0 |\phi(s)|^2\,ds\right) + \varepsilon^{-1}\int_\tau^t e^{\sigma s}\|f(s)\|_*^2\,ds + \lambda_1^{-1}C_g\int_\tau^t e^{\sigma s}\|u(s)\|^2\,ds$$

for all $t \geq \tau$, and from this last inequality and (3.4), in particular we deduce (3.3). □

From now on, being $\sigma > 0$ given in (V), we will assume that $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies

$$\int_{-\infty}^0 e^{\sigma s}\|f(s)\|_*^2\,ds < \infty. \tag{3.5}$$

At the light of the previous result, we now define an appropriate concept of (tempered) universe for problem (1.1).

Definition 3.12. Denote by $\mathcal{D}_\sigma(V \times L^2_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{V\times L^2_H}\right) = 0.$$

According to the notation introduced in the previous section, we will denote by $\mathcal{D}_F(V \times L^2_H)$ the universe of fixed bounded sets in $V \times L^2_H$. Observe that trivially $\mathcal{D}_F(V \times L^2_H) \subset \mathcal{D}_\sigma(V \times L^2_H)$ and that $\mathcal{D}_\sigma(V \times L^2_H)$ is inclusion-closed.

Remark 3.13. Although from Lemma 3.11 it is easy to see that the family $\{\overline{B}_{V\times L^2_H}(0,\rho_\sigma(t)) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ is pullback $\mathcal{D}_\sigma(V \times L^2_H)$-absorbing for the process S, where

$$\rho_\sigma^2(t) = 1 + \alpha^{-2}\varepsilon^{-1}(1 + \lambda_1^{-1}he^{\sigma h})e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\|f(s)\|_*^2\,ds,$$

we will need, in order to apply Proposition 3.10, to obtain a different pullback $\mathcal{D}_\sigma(V \times L^2_H)$-absorbing family.

Lemma 3.14. *Assume that $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V), and $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (3.5). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\sigma(V \times L^2_H)$, there exist $\tau_1(\widehat{D}, t, h) < t - 2h$ and functions $\{\rho_i\}_{i=1}^2$ such that for any $\tau \leq \tau_1(\widehat{D}, t, h)$ and any $(u^\tau, \phi) \in D(\tau)$, it holds*

$$\|u(r;\tau,u^\tau,\phi)\|^2 \leq \rho_1^2(t), \quad \forall\, r \in [t-2h, t], \tag{3.6}$$

$$\int_{t-h}^t \|u'(\theta;\tau,u^\tau,\phi)\|^2\,d\theta \leq \rho_2^2(t), \tag{3.7}$$

where

$$\rho_1^2(t) = 1 + \alpha^{-2}\varepsilon^{-1}e^{-\sigma(t-2h)}\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|_*^2\,ds, \tag{3.8}$$

$$\rho_2^2(t) = 4\alpha^{-4}h\rho_1^2(t)\left(\nu^2 + C_1^2\rho_1^2(t) + 2\lambda_1^{-2}C_g^2\right) + 4\alpha^{-4}\int_{t-h}^{t}\|f(s)\|_*^2\,ds, \tag{3.9}$$

and ε is given by (3.4).

Proof. Let $\tau_1(\widehat{D},t,h) < t-2h$ be such that

$$\alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{-\sigma(t-2h)}e^{\sigma\tau}\|(u^\tau,\phi)\|^2_{V\times L^2_H} \le 1 \quad \forall\,\tau \le \tau_1(\widehat{D},t,h),\ (u^\tau,\phi)\in D(\tau).$$

Consider fixed $\tau \le \tau_1(\widehat{D},t,h)$ and $(u^\tau,\phi)\in D(\tau)$. The estimate (3.6) follows directly from (3.3), using the increasing character of the exponential.

Now, from (2.8), (2.14), (2.1) and the fact that A is an isometric isomorphism, we obtain that $v = Cu$ satisfies

$$\|v'(\theta)\|_* \le \nu\|u(\theta)\| + C_1\|u(\theta)\|^2 + \|f(\theta)\|_* + \lambda_1^{-1/2}|g(\theta,u_\theta)|, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\|v'(\theta)\|_*^2 \le 4\nu^2\|u(\theta)\|^2 + 4C_1^2\|u(\theta)\|^4 + 4\|f(\theta)\|_*^2 + 4\lambda_1^{-1}|g(\theta,u_\theta)|^2, \quad \text{a.e. } \theta > \tau.$$

Integrating in time and using properties (II) and (IV), we deduce

$$\int_{t-h}^{t}\|v'(\theta)\|_*^2\,d\theta \le 4\nu^2\int_{t-h}^{t}\|u(\theta)\|^2\,d\theta + 4C_1^2\int_{t-h}^{t}\|u(\theta)\|^4\,d\theta$$

$$+ 4\int_{t-h}^{t}\|f(\theta)\|_*^2\,d\theta + 4\lambda_1^{-2}C_g^2\int_{t-2h}^{t}\|u(\theta)\|^2\,d\theta,$$

whence, by (2.12) and (3.6), the estimate (3.7) follows. □

Remark 3.15. Observe that $\lim_{t\to-\infty} e^{\sigma t}\rho_1(t) = 0$.

Corollary 3.16. *Under the assumptions of Lemma 3.14, the family $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ defined by*

$$D_\sigma(t) = \left\{(w,\psi)\in V\times L^2_V : \exists\frac{d\psi}{ds}\in L^2_V,\ \|(w,\psi)\|_{V\times L^2_V} \le \widetilde{\rho}_\sigma(t),\ \left\|\frac{d\psi}{ds}\right\|_{L^2_V} \le \rho_2(t)\right\} \tag{3.10}$$

is pullback $\mathcal{D}_\sigma(V\times L^2_H)$-absorbing for the process S on $V\times L^2_H$ defined by (3.1), where $\widetilde{\rho}_\sigma(t)$ satisfies

$$\widetilde{\rho}_\sigma^2(t) = (1+h)\rho_1^2(t), \tag{3.11}$$

with $\rho_1(t)$ and $\rho_2(t)$ given by (3.8) and (3.9) respectively. Moreover, $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V\times L^2_H)$.

Now, we prove that the process S is $(V\times L^2_H, V\times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact. To this end, we will apply an energy method used by Rosa (cf. [32], see also [26] and [14]), which does not require any additional estimates on the solutions in higher norms in contrast with the *energy continuous method* (e.g., cf. [28]), or the method used in [20] with the fractional powers of the operator A. Our proof here relies on a sharp use of the differential equality that leads to the existence of an absorbing family, the use of weak limits in $V\times L^2_V$ in a diagonal argument, and the convergences established in Proposition 3.10.

Lemma 3.17. *Under the assumptions of Lemma 3.14, the process S defined by (3.1) is $(V \times L^2_H, V \times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact, where $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ is defined in Corollary 3.16.*

Proof. Let us consider $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D_\sigma(\tau_n)$ for all n. We must prove that the sequence

$$\{S(t,\tau_n)(u^{\tau_n}, \phi^n)\} = \{(u(t;\tau_n, u^{\tau_n}, \phi^n), u_t(\cdot;\tau_n, u^{\tau_n}, \phi^n))\}$$

is relatively compact in $V \times C_H$.

First, we check the asymptotic compactness in the first component of S. By Corollary 3.16, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}_\sigma}(k) \leq t - k$ such that $S(t-k,\tau)D_\sigma(\tau) \subset D_\sigma(t-k)$ for all $\tau \leq \tau_{\widehat{D}_\sigma}(k)$. From this and a diagonal argument, we can extract a subsequence $\{(u^{\tau_{n'}}, \phi^{n'})\} \subset \{(u^{\tau_n}, \phi^n)\}$ such that

$$S(t-k,\tau_{n'})(u^{\tau_{n'}}, \phi^{n'}) \rightharpoonup (w^k, \psi^k) \quad \text{weakly in } V \times L^2_V, \tag{3.12}$$

$$\frac{d}{ds} u_{t-k}(\cdot;\tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^k \quad \text{weakly in } L^2_V, \tag{3.13}$$

for all integer $k \geq 0$, where $(w^k, \psi^k) \in D_\sigma(t-k)$.

Now, applying Proposition 3.10 on each fixed interval $[t-k, t]$, we deduce that

$$\begin{aligned}
(w^0, \psi^0) &= (V \times L^2_V) - \text{weak} \lim_{n' \to \infty} S(t, \tau_{n'})(u^{\tau_{n'}}, \phi^{n'}) \\
&= (V \times L^2_V) - \text{weak} \lim_{n' \to \infty} S(t, t-k) S(t-k, \tau_{n'})(u^{\tau_{n'}}, \phi^{n'}) \\
&= S(t, t-k) \left[(V \times L^2_V) - \text{weak} \lim_{n' \to \infty} S(t-k, \tau_{n'})(u^{\tau_{n'}}, \phi^{n'}) \right] \\
&= S(t, t-k)(w^k, \psi^k).
\end{aligned}$$

From (3.12) with $k = 0$, we obtain in particular that $\|w^0\| \leq \liminf_{n' \to \infty} \|u(t;\tau_{n'}, u^{\tau_{n'}}, \phi^{n'})\|$. We will prove now that it also holds that

$$\limsup_{n' \to \infty} \|u(t;\tau_{n'}, u^{\tau_{n'}}, \phi^{n'})\| \leq \|w^0\|, \tag{3.14}$$

which combined with the weak converge of $u(t;\tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$ to w^0 in V, will imply the convergence in the strong topology of V.

Observe that, as we already used in Lemma 3.11, for any $\tau \in \mathbb{R}$ and $(u^\tau, \phi) \in V \times L^2_H$, the solution $u(\cdot;\tau, u^\tau, \phi)$, for short denoted $u(\cdot)$, satisfies the differential equality

$$\begin{aligned}
\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) &= \sigma e^{\sigma t}|u(t)|^2 + \alpha^2 \sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 \\
&\quad + 2e^{\sigma t}\langle f(t), u(t)\rangle + 2e^{\sigma t}(g(t, u_t), u(t)), \qquad \text{a.e. } t > \tau.
\end{aligned} \tag{3.15}$$

Since in particular $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$, notice that $[\cdot]$, with $[v]^2 = (2\nu - \alpha^2\sigma)\|v\|^2 - \sigma|v|^2$, defines an equivalent norm to $\|\cdot\|$ in V.

We integrate the above expression in the interval $[t-k, t]$ for the solutions $u(\cdot;\tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$

with $\tau_{n'} \leq t-k$, which yields

$$
\begin{aligned}
&|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2+\alpha^2\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2\\
&\quad=|u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))|^2+\alpha^2\|u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\|^2\\
&\quad=e^{-\sigma k}\left(|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2+\alpha^2\|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2\right)\\
&\qquad+2\int_{t-k}^{t}e^{\sigma(s-t)}\langle f(s),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rangle\,ds\\
&\qquad+2\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\
&\qquad-\int_{t-k}^{t}e^{\sigma(s-t)}[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]^2\,ds. \qquad (3.16)
\end{aligned}
$$

On other hand, by (3.12), (3.13) and Proposition 3.10, we deduce that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rightharpoonup u(\cdot;t-k,w^k,\psi^k)\quad\text{weakly in } L^2(t-k,t;V).$$

From this, as $e^{\sigma(\cdot-t)}f(\cdot)\in L^2(t-k,t;V')$, it yields

$$
\begin{aligned}
\lim_{n'\to\infty}&\int_{t-k}^{t}e^{\sigma(s-t)}\langle f(s),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rangle\,ds\\
&=\int_{t-k}^{t}e^{\sigma(s-t)}\langle f(s),u(s;t-k,w^k,\psi^k)\rangle\,ds.
\end{aligned}
$$

Since $\int_{t-k}^{t}e^{\sigma(s-t)}[v(s)]^2\,ds$ defines an equivalent norm in $L^2(t-k,t;V)$, we also deduce from above that

$$\int_{t-k}^{t}e^{\sigma(s-t)}[u(s;t-k,w^k,\psi^k)]^2\,ds\leq\liminf_{n'\to\infty}\int_{t-k}^{t}e^{\sigma(s-t)}[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]^2\,ds.$$

Finally, again by (3.12), (3.13) and Proposition 3.10, it holds that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\to u(\cdot;t-k,w^k,\psi^k)\quad\text{strongly in } L^2(t-k-h,t;H),$$

and therefore,

$$
\begin{aligned}
\lim_{n'\to\infty}&\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\
&=\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),u(s;t-k,w^k,\psi^k))\,ds \qquad (3.17)
\end{aligned}
$$

From (3.16)–(3.17), taking into account (3.12) with $k=0$, the compactness of the injection of V into H, and (3.10), we conclude that

$$
\begin{aligned}
&|w^0|^2+\alpha^2\limsup_{n'\to\infty}\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2\\
&\quad\leq e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)\widetilde{\rho}_\sigma^2(t-k)+2\int_{t-k}^{t}e^{\sigma(s-t)}\langle f(s),u(s;t-k,w^k,\psi^k)\rangle\,ds\\
&\qquad+2\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),u(s;t-k,w^k,\psi^k))\,ds\\
&\qquad-\int_{t-k}^{t}e^{\sigma(s-t)}[u(s;t-k,w^k,\psi^k)]^2\,ds.
\end{aligned}
$$

Now, taking into account that $w^0 = u(t; t-k, w^k, \psi^k)$, integrating again in (3.15), we obtain

$$|w^0|^2 + \alpha^2\|w^0\|^2 = e^{-\sigma k}(|w^k|^2 + \alpha^2\|w^k\|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s), u(s; t-k, w^k, \psi^k)\rangle\, ds$$
$$+ 2\int_{t-k}^{t} e^{\sigma(s-t)}(g(s, u_s(\cdot; t-k, w^k, \psi^k)), u(s; t-k, w^k, \psi^k))\, ds$$
$$- \int_{t-k}^{t} e^{\sigma(s-t)}[u(s; t-k, w^k, \psi^k)]^2\, ds.$$

Comparing the above two expressions, in particular we conclude that

$$|w^0|^2 + \alpha^2 \limsup_{n'\to\infty} \|u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})\|^2 \le e^{-\sigma k}(\lambda_1^{-1} + \alpha^2)\tilde{\rho}_\sigma^2(t-k) + |w^0|^2 + \alpha^2\|w^0\|^2.$$

But from Remark 3.15 and (3.11), we have that $\lim_{k\to\infty} e^{-\sigma k}\tilde{\rho}_\sigma^2(t-k) = 0$, so (3.14) holds, and we conclude that

$$u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \to w^0 \quad \text{strongly in } V.$$

Finally, we prove the asymptotic compactness in the second component of S. From (3.12) and (3.13) with $k = 0$, we have that

$$u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \rightharpoonup \psi^0 \qquad \text{weakly in } L_V^2,$$
$$\frac{d}{ds}u_t(\cdot; \tau_{n'}, u^{\tau'_n}, \phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^0 \quad \text{weakly in } L_V^2.$$

Thus, by applying the Ascoli–Arzelà theorem, we can deduce that there exists a subsequence (relabelled the same) such that $u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$ converges to ψ^0 in C_H. So, the proof is finished. □

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process S on $V \times L_H^2$ defined by (3.1).

Theorem 3.18. *Assume that $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V), and $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (3.5). Then, there exist the $(V \times L_H^2, V \times C_H)$ minimal pullback $\mathcal{D}_\sigma(V \times L_H^2)$ and $\mathcal{D}_F(V \times L_H^2)$-attractors $\{\mathcal{A}_{\mathcal{D}_\sigma(V\times L_H^2)}(t) : t \in \mathbb{R}\}$ and $\{\mathcal{A}_{\mathcal{D}_F(V\times L_H^2)}(t) : t \in \mathbb{R}\}$ respectively, both belonging to $\mathcal{D}_\sigma(V \times L_H^2)$, which means that they have compact sections in $V \times C_H$ and pullback attracts in this norm, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V\times L_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma(V\times L_H^2)}(t) = \Lambda_{V\times C_H}(\widehat{D}_\sigma, t), \quad \forall\, t \in \mathbb{R}. \tag{3.18}$$

Moreover, if f satisfies the stronger requirement

$$\sup_{r\le 0}\left(e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\|f(s)\|_*^2\, ds\right) < \infty, \tag{3.19}$$

then both attractors coincide, i.e.,

$$\mathcal{A}_{\mathcal{D}_F(V\times L_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\sigma(V\times L_H^2)}(t), \quad \forall\, t \in \mathbb{R}. \tag{3.20}$$

Proof. The process S is continuous on $V \times L_H^2$ by Corollary 3.9. By Remark 3.8, S is $(V \times L_H^2, V \times C_H)$ closed. There exists a pullback absorbing family $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V \times L_H^2)$ by Corollary 3.16, and the process S is $(V \times L_H^2, V \times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact by

Lemma 3.17. The existence of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ and $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ follows from Theorem 3.7 (actually Theorem 3.3 and Corollary 3.5 could also be applied, but using the bi-space attractors theory we strengthen compactness and attraction norm).

Moreover, the inclusion relation in (3.18) follows from Corollary 3.5.

The fact that $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ belongs to $\mathcal{D}_\sigma(V\times L^2_H)$ is due to Remark 3.4, since the pullback absorbing family $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V\times L^2_H)$ has closed sections and this universe is inclusion-closed.

Finally, the equality (3.20) is a consequence of Corollary 3.5, since $D_\sigma(t)\subset \overline{B}_{V\times L^2_V}(0,\widetilde{\rho}_\sigma(t))$ for all $t\in\mathbb{R}$, and the assumption (3.19) is equivalent to have that $\sup_{t\le T}\widetilde{\rho}_\sigma(t)$ is bounded for any $T\in\mathbb{R}$. □

Just splitted for the sake of clarity, with the same arguments as above, we obtain the following result, which relates the above attractors for the universes $\mathcal{D}_F(V\times L^2_H)\subset\mathcal{D}_\sigma(V\times L^2_H)$ with new ones for the universes $\mathcal{D}_F(V\times C_H)\subset\mathcal{D}_\sigma(V\times C_H)$.

Corollary 3.19. *Under the assumptions of Theorem 3.18 there exist the minimal pullback attractors $\mathcal{A}_{\mathcal{D}_F(V\times C_H)}$ and $\mathcal{A}_{\mathcal{D}_\sigma(V\times C_H)}$, both belonging to $\mathcal{D}_\sigma(V\times C_H)$, all time sections are compact subsets in $V\times C_H$, they attract in $V\times C_H$ norm, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V\times C_H)}(t)\subset\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t)\subset\mathcal{A}_{\mathcal{D}_\sigma(V\times C_H)}(t)=\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t),\quad \forall\, t\in\mathbb{R}.$$

Proof. Observe that S is well-defined on $V\times C_H$ by Theorem 2.2 and closed by Remark 2.4 (i). Observe that $\widehat{D}_\sigma \subset \mathcal{P}(V\times C_H)$. Then the existence of attractors and its inclusion in $\mathcal{D}_\sigma(V\times C_H)$ follows from Theorem 3.3 and Remark 3.4.

The equality relation of pullback $\mathcal{D}_\sigma(V\times C_H)$ and $\mathcal{D}_\sigma(V\times L^2_H)$-attractors follows from [13, Theorem 3.15]. Indeed, observe that after an elapsed time h, by (3.3), $S(\cdot+h,\cdot)$ maps elements from $\mathcal{D}_\sigma(V\times L^2_H)$ into $\mathcal{D}_\sigma(V\times C_H)$.

The rest of inclusions follows from Corollary 3.5 or by minimality arguments. □

Remark 3.20. The stronger attraction and compactness properties of these results also apply to several previous ones concerning asymptotic behavior of PDE with delays (e.g., cf. [16]).

Remark 3.21. Observe that by the invariance of the minimal pullback attractors under the process S, and the regularity of the solutions, it is clear that the second component of any time section of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ and $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ lives in C_V. In fact, denoting $R^2_\sigma(t)=2\rho^2_1(t)$, from (3.6) it holds that

$$\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t)\subset\overline{B}_{V\times C_V}(0,R_\sigma(t)),\quad\forall\, t\in\mathbb{R}.$$

4 Regularity of the pullback attractors

The main goal of this paragraph is to provide some extra regularity for the attractors obtained in the previous section. This will be obtained by a bootstrapping argument, and making the most out of a representation of the solutions to the problem splitting it in two parts, the linear part with an exponential decay, and the nonlinear part with good enough estimates. In order to achieve these results, we will use the fractional powers of the Stokes operator, introduced in Section 2.

Observe that for every $\tau\in\mathbb{R}$, $(u^\tau,\phi)\in V\times L^2_H$, $f\in L^2_{loc}(\mathbb{R};V')$, and $g:\mathbb{R}\times C_H\to(L^2(\Omega))^3$ satisfying (I)–(IV), by Theorem 2.2, there exists a unique weak solution u to problem (1.1). Moreover, let us point out that the following representation of the solution holds:

$$u(t;\tau,u^\tau,\phi)=y(t;\tau,u^\tau,\phi)+z(t;\tau,0,0),\quad\forall\, t\ge\tau,$$

where $y = y(\cdot;\tau,u^\tau,\phi)$ and $z = z(\cdot;\tau,0,0)$ are solutions of

$$\begin{cases} y \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \text{ for all } T > \tau, \\ \dfrac{d}{dt}(y(t) + \alpha^2 Ay(t)) + \nu Ay(t) = 0, \quad \text{in } \mathcal{D}'(\tau,\infty;V'), \\ y(\tau) = u^\tau, \\ y(t) = \phi(t-\tau) \quad \text{a.e. } t \in (\tau-h,\tau) \end{cases} \tag{4.1}$$

and

$$\begin{cases} z \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \text{ for all } T > \tau, \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = f(t) + g(t,u_t) - B(u(t)), \quad \text{in } \mathcal{D}'(\tau,\infty;V'), \\ z(\tau) = 0, \\ z(t) = 0 \quad \text{a.e.} t \in (\tau-h,\tau) \end{cases} \tag{4.2}$$

respectively.

The existence and uniqueness of weak solution to (4.1) and to (4.2) can be obtained reasoning as in the proof of Theorem 2.2.

For the problem (4.1) we have the following result.

Lemma 4.1. *For any $\tau \in \mathbb{R}$, $(u^\tau,\phi) \in V \times L^2_H$ and σ fulfilling that $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1} + \alpha^2)^{-1}$, the solution $y = y(\cdot;\tau,u^\tau,\phi)$ of (4.1) satisfies*

$$\|y(t)\|^2 \le \alpha^{-2}(\lambda_1^{-1} + \alpha^2)e^{\sigma(\tau-t)}\|(u^\tau,\phi)\|^2_{V\times L^2_H} \quad \textit{for all } t \ge \tau. \tag{4.3}$$

Proof. It is analogous to the proof of (3.3), and we omit it. □

For the study of the problem (4.2), we will make use of the following lemma.

Lemma 4.2. *Let me given $F \in L^2_{loc}(\mathbb{R};D(A^{-\beta}))$ with $0 \le \beta \le 1/2$, $\tau \in \mathbb{R}$ and σ fulfilling that $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1} + \alpha^2)^{-1}$. Then, the problem*

$$\begin{cases} z \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \textit{ for all } T > \tau, \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = F(t), \quad \textit{in } \mathcal{D}'(\tau,\infty;V'), \\ z(\tau) = 0, \\ z(t) = 0 \quad \text{a.e. } t \in (\tau-h,\tau) \end{cases}$$

has a unique solution z, which also satisfies $z \in C([\tau,\infty);D(A^{1-\beta}))$, and

$$|A^{1-\beta}z(t)|^2 \le \alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|A^{-\beta}F(s)|^2\,ds \quad \textit{for all } t \ge \tau,$$

where ε is given by (3.4).

Proof. It can be done analogously as in [14, Lemma 26] with $z = 0$ in $(\tau-h,\tau)$. □

Now we can prove the following regularity result for the pullback attractors in V norm.

Theorem 4.3. *Consider given $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying conditions (I)–(V). Assume that $f \in L^2_{loc}(\mathbb{R}; D(A^{-\beta}))$ for some $0 \leq \beta \leq 1/2$, and that*

$$\sup_{r \leq 0} \int_{r-1}^{r} \|f(s)\|_*^2 ds < \infty. \tag{4.4}$$

Then:

(1) *If f also satisfies*

$$\int_{-\infty}^{0} e^{\sigma s} |A^{-\beta} f(s)|^2 ds < \infty, \tag{4.5}$$

and

$$\begin{cases} \sup_{r \leq 0} \int_{r-1}^{r} |A^{-1/4-\beta} f(s)|^2 ds < \infty, & \text{if } 0 < \beta < 1/4, \\ \sup_{r \leq 0} \int_{r-1}^{r} |A^{-\delta} f(s)|^2 ds < \infty & \text{for some } 0 < \delta < 1/4, \text{ if } \beta = 0, \end{cases} \tag{4.6}$$

then, for any $t_1 < t_2$, the pullback attractor $\mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)} = \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}$ fulfills that

$$\bigcup_{t_1 \leq t \leq t_2} \mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}(t) = \bigcup_{t_1 \leq t \leq t_2} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \text{ is a bounded subset of } D(A^{1-\beta}) \times C_{D(A^{1-\beta})}. \tag{4.7}$$

(2) *If f also satisfies*

$$\sup_{r \leq 0} \int_{r-1}^{r} |A^{-\beta} f(s)|^2 ds < \infty, \tag{4.8}$$

then, for any $t_2 \in \mathbb{R}$, it holds that

$$\bigcup_{t \leq t_2} \mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}(t) = \bigcup_{t \leq t_2} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \text{ is a bounded subset of } D(A^{1-\beta}) \times C_{D(A^{1-\beta})}. \tag{4.9}$$

Proof. Let us fix $t \in \mathbb{R}$ and $(v, \psi) \in \mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}(t) = \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t)$. By Remark 3.21 and (4.4), we see that

$$\bigcup_{r \leq t} \mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}(r) \subset \overline{B}_{V \times C_V}(0, \widetilde{R}_\sigma(t)), \tag{4.10}$$

where $\widetilde{R}_\sigma^2(t) = 2 + 2\alpha^{-2}\varepsilon^{-1} e^{2\sigma h} \sup_{r \leq t}(e^{-\sigma r} \int_{-\infty}^{r} e^{\sigma s} \|f(s)\|_*^2 ds)$, with ε given by (3.4).

Let $\{\tau_n\}_{n \geq 1} \subset (-\infty, t-h]$ be a sequence with $\tau_n \to -\infty$ as $n \to \infty$. By the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}$, for each $n \geq 1$ there exists $(u^{\tau_n}, \phi^n) \in \mathcal{A}_{\mathcal{D}_\sigma(V \times L^2_H)}(\tau_n)$ such that $(v, \psi) = S(t, \tau_n)(u^{\tau_n}, \phi^n)$, and therefore,

$$(v, \psi) = Y(t, \tau_n)(u^{\tau_n}, \phi^n) + Z(t, \tau_n)(0, 0),$$

where

$$Y(t, \tau_n)(u^{\tau_n}, \phi^n) = (y(t; \tau_n, u^{\tau_n}, \phi^n), y_t(\cdot; \tau_n, u^{\tau_n}, \phi^n))$$

and

$$Z(t, \tau_n)(0, 0) = (z(t; \tau_n, 0, 0), z_t(\cdot; \tau_n, 0, 0))$$

are continuous processes on $V \times L^2_H$ associated to problems (4.1) and (4.2), respectively.

From (4.3) and (4.10) we deduce that $\|Y(t, \tau_n)(u^{\tau_n}, \phi^n)\|_{V \times C_V} \to 0$ as $n \to \infty$. Thus,

$$\lim_{n \to \infty} \|Z(t, \tau_n)(0, 0) - (v, \psi)\|_{V \times C_V} = 0. \tag{4.11}$$

Let us denote $(u^n(r), u^n_r(\cdot)) = S(r,\tau_n)(u^{\tau_n},\phi^n)$ for $r \geq \tau_n$ and $n \geq 1$. By (4.10) and the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$,

$$(u^n(r), u^n_r(\cdot)) \in \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r) \subset \overline{B}_{V\times C_V}(0,\widetilde{R}_\sigma(t)), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1. \tag{4.12}$$

Now we distinguish three cases.

Case 1. If $1/4 \leq \beta \leq 1/2$.

In this case, from (2.11), the continuous injection of V in $D(A^{3/4-\beta})$ and (4.12), we deduce that

$$\begin{aligned}|A^{-\beta}B(u^n(r))| &\leq C_{(3/4-\beta)}|A^{3/4-\beta}u^n(r)|\|u^n(r)\| \\ &\leq \widetilde{C}_{(3/4-\beta)}\|u^n(r)\|^2 \\ &\leq \widetilde{C}_{(3/4-\beta)}\widetilde{R}^2_\sigma(t), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1.\end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2, condition (V) on g, and the continuous injection of H in $D(A^{-\beta})$, we obtain that

$$\begin{aligned}|A^{1-\beta}z(\theta;\tau_n,0,0)|^2 &\leq 3\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t) \\ &\quad + \int_{\tau_n}^t e^{\sigma(s-t)}|A^{-\beta}g(s,u^n_s)|^2\,ds\Big) \\ &\leq 3\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t) \\ &\quad + C_\beta C_g^2\lambda_1^{-1}\Big(\int_{\tau_n-h}^{\tau_n} e^{\sigma(s-t)}\|\phi^n(s-\tau_n)\|^2\,ds + \int_{\tau_n}^t e^{\sigma(s-t)}\|u^n(s)\|^2\,ds\Big)\Big)\end{aligned}$$

for all $\theta \in [t-h,t]$, and then, from (4.12), we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}} \leq M^2_{\sigma,\beta}(t), \tag{4.13}$$

where

$$M^2_{\sigma,\beta}(t) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\left(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\right).$$

From (4.11), (4.13) and the weak lower semi-continuity of the norm, we deduce that (v,ψ) belongs to $\overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0, M_{\sigma,\beta}(t))$, and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t\leq t_2}\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) \subset \overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0,\widetilde{M}_{\sigma,\beta}(t_2)), \quad \text{for all } t_2 \in \mathbb{R}, \tag{4.14}$$

where

$$\begin{aligned}\widetilde{M}^2_{\sigma,\beta}(t_2) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(&\sup_{t\leq t_2}\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds \\ &+ \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t_2) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t_2)\Big).\end{aligned}$$

Case 2. If $0 < \beta < 1/4$.

In this case, if f satisfies (4.6), as $1/4 < 1/4 + \beta < 1/2$, from (4.14) we have that

$$\bigcup_{r \le t} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r) \subset \overline{B}_{D(A^{3/4-\beta})\times C_{D(A^{3/4-\beta})}}(0, \widetilde{M}_{\sigma,1/4+\beta}(t)).$$

Thus, by (2.11) and (4.12), we obtain that

$$\begin{aligned}|A^{-\beta}B(u^n(r))| &\le C_{(3/4-\beta)}|A^{3/4-\beta}u^n(r)|\|u^n(r)\| \\ &\le C_{(3/4-\beta)}\widetilde{M}_{\sigma,1/4+\beta}(t)\widetilde{R}_\sigma(t), \quad \forall \tau_n \le r \le t, \quad \forall n \ge 1.\end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2 we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}} \le R^2_{\sigma,\beta}(t), \tag{4.15}$$

where

$$R^2_{\sigma,\beta}(t) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds \\ + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t)\widetilde{R}^2_\sigma(t) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\Big).$$

Again, from (4.11), (4.15) and the weak lower semi-continuity of the norm, we deduce that (v,ψ) belongs to $\overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0, R_{\sigma,\beta}(t))$, and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t \le t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) \subset \overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0, \widetilde{R}_{\sigma,\beta}(t_2)), \quad \text{for all } t_2 \in \mathbb{R}, \tag{4.16}$$

where

$$\widetilde{R}^2_{\sigma,\beta}(t_2) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(\sup_{t\le t_2}\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds \\ + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t_2)\widetilde{R}^2_\sigma(t_2) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t_2)\Big).$$

Case 3. If $\beta = 0$.

In this case, if f satisfies (4.6), as $0 < \delta < 1/4$, from (4.16) we see that

$$\bigcup_{r \le t} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r) \subset \overline{B}_{D(A^{1-\delta})\times C_{D(A^{1-\delta})}}(0, \widetilde{R}_{\sigma,\delta}(t)).$$

So, by (2.10) and (4.12), we deduce that

$$\begin{aligned}|B(u^n(r))| &\le C_{(1-\delta)}|A^{1-\delta}u^n(r)|\|u^n(r)\| \\ &\le C_{(1-\delta)}\widetilde{R}_{\sigma,\delta}(t)\widetilde{R}_\sigma(t), \quad \forall \tau_n \le r \le t, \quad \forall n \ge 1.\end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2 we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A)\times C_{D(A)}} \le R^2_{\sigma,\delta,0}(t) \tag{4.17}$$

where

$$R^2_{\sigma,\delta,0}(t) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Big(\int_{-\infty}^t e^{\sigma(s-t)}|f(s)|^2\,ds + \sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t)\widetilde{R}^2_\sigma(t) + 2C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\Big).$$

Again, from (4.11), (4.17) and the weak lower semi-continuity of the norm, we deduce that

$$(v,\psi) \in \overline{B}_{D(A)\times C_{D(A)}}(0, R_{\sigma,\delta,0}(t)),$$

and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) \subset \overline{B}_{D(A)\times C_{D(A)}}(0, \widetilde{R}_{\sigma,\delta,0}(t_2)), \text{ for all } t_2 \in \mathbb{R},$$

where

$$\widetilde{R}^2_{\sigma,\delta,0}(t_2) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Bigg(\sup_{t\le t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|f(s)|^2\,ds + \sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t_2)\widetilde{R}^2_{\sigma}(t_2) + 2C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_{\sigma}(t_2)\Bigg). \quad \square$$

5 Attraction in $D(A)$ norm

By the previous results, when $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, the restriction to $D(A)\times L^2_V$ of the process S defined by (3.1) is a process on $D(A)\times L^2_V$. Now, we will prove that under suitable assumptions on f and g, we can obtain the existence of minimal pullback attractors for S on $D(A)\times L^2_V$ and even more.

Proposition 5.1. *Assume that $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, and $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), are given. Then, the restriction to $D(A)\times L^2_V$ of the bi-parametric family of maps $S(t,\tau)$, with $\tau \le t$, given by (3.1), is a continuous process on $D(A)\times L^2_V$.*

Proof. It is a consequence of Theorem 2.2 and Remark 2.4 (i). $\square$

As in the previous section, we will need the following continuity result for the process S in a weak sense.

Proposition 5.2. *Let $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), and $\tau < t$ be given. Then, for any sequence such that*

$$(u^{\tau,n},\phi^n) \rightharpoonup (u^\tau,\phi) \text{ weakly in } D(A)\times L^2_{D(A)}$$

and

$$\frac{d\phi^n}{ds} \rightharpoonup \frac{d\phi}{ds} \text{ weakly in } L^2_{D(A)},$$

the following convergences hold for the sequence of solutions $u(\cdot;\tau,u^{\tau,n},\phi^n)$ towards the solution $u(\cdot;\tau,u^\tau,\phi)$:

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \overset{*}{\rightharpoonup} u(\cdot;\tau,u^\tau,\phi) \text{ weakly-star in } L^\infty(\tau,t;D(A)),$$

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \to u(\cdot;\tau,u^\tau,\phi) \text{ strongly in } C([\tau-h,t];V),$$

$$u(t;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(t;\tau,u^\tau,\phi) \text{ weakly in } D(A),$$

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(\cdot;\tau,u^\tau,\phi) \text{ weakly in } L^2(\tau-h,t;D(A)).$$

Proof. It can be done analogously to that of Proposition 3.10. □

For the obtention of a pullback absorbing family for the process S restricted to $D(A)\times L^2_V$, we first have the following result.

Lemma 5.3. *Suppose that $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$ satisfies (4.4) and that $g:\mathbb{R}\times C_H\to(L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $\tau\in\mathbb{R}$, $(u^\tau,\phi)\in D(A)\times L^2_V$, and $0<\underline{\sigma}<\sigma/3$, the solution $u=u(\cdot;\tau,u^\tau,\phi)$ of (1.1) satisfies*

$$|Au(t)|^2\le\alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2,C_g\}e^{\sigma(\tau-t)}\|(u^\tau,\phi)\|^2_{D(A)\times L^2_V}+2\alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|f(s)|^2\,ds$$
$$+4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}\left(e^{-3\underline{\sigma}(t-\tau)}\|(u^\tau,\phi)\|^6_{V\times L^2_H}+M^3_{t,\underline{\sigma}}\right) \tag{5.1}$$

for all $t\ge\tau$, where $\varepsilon>0$ is given by (3.4),

$$C_\varepsilon=27C_2^4(2\varepsilon^3)^{-1}, \tag{5.2}$$

$$C_{\underline{\sigma}}=\alpha^{-2}\max\left\{\max\{\lambda_1^{-1}+\alpha^2,C_g\},\left(2\nu-\underline{\sigma}(\lambda_1^{-1}+\alpha^2)-2\lambda_1^{-1}C_g\right)^{-1}\right\}, \tag{5.3}$$

and

$$M_{t,\underline{\sigma}}=\sup_{r\le t}\int_{-\infty}^r e^{\underline{\sigma}(s-r)}\|f(s)\|^2_*ds. \tag{5.4}$$

Proof. From Lemma 3.11, we have that

$$\|u(s)\|^2\le C_{\underline{\sigma}}\left(e^{\underline{\sigma}(\tau-s)}\|(u^\tau,\phi)\|^2_{V\times L^2_H}+M_{t,\underline{\sigma}}\right),\quad\forall\tau\le s\le t. \tag{5.5}$$

On the other hand, by (2.17),

$$\frac{d}{dt}(e^{\sigma t}\|u(t)\|^2+\alpha^2e^{\sigma t}|Au(t)|^2)+2\nu e^{\sigma t}|Au(t)|^2+2e^{\sigma t}(B(u(t)),Au(t))$$
$$=\sigma e^{\sigma t}\|u(t)\|^2+\alpha^2\sigma e^{\sigma t}|Au(t)|^2+2e^{\sigma t}(f(t)+g(t,u_t),Au(t)),\quad\text{a.e. } t>\tau.$$

Thus, taking into account that $\|u(t)\|^2\le\lambda_1^{-1}|Au(t)|^2$,

$$2|\,(B(u(t)),Au(t))\,|\le 2C_2\|u(t)\|^{3/2}|Au(t)|^{3/2}$$
$$\le C_\varepsilon\|u(t)\|^6+\frac{\varepsilon}{2}|Au(t)|^2,$$
$$2|\,(f(t),Au(t))\,|\le\frac{\varepsilon}{2}|Au(t)|^2+\frac{2}{\varepsilon}|f(t)|^2,$$

and

$$2|(g(t,u_t),Au(t))|\le\frac{C_g}{\lambda_1}|Au(t)|^2+\frac{\lambda_1}{C_g}|g(t,u_t)|^2,$$

we deduce that

$$e^{\sigma t}(\|u(t)\|^2+\alpha^2|Au(t)|^2)+(2\nu-\varepsilon-\sigma(\lambda_1^{-1}+\alpha^2)-\lambda_1^{-1}C_g)\int_\tau^t e^{\sigma s}|Au(s)|^2\,ds$$
$$\le e^{\sigma\tau}(\lambda_1^{-1}+\alpha^2)|Au^\tau|^2+2\varepsilon^{-1}\int_\tau^t e^{\sigma s}|f(s)|^2\,ds+\lambda_1C_g\int_{\tau-h}^t e^{\sigma s}|u(s)|^2\,ds$$
$$+C_\varepsilon\int_\tau^t e^{\sigma s}\|u(s)\|^6\,ds$$
$$\le e^{\sigma\tau}\left((\lambda_1^{-1}+\alpha^2)|Au^\tau|^2+C_g\|\phi\|^2_{L^2_V}\right)+2\varepsilon^{-1}\int_\tau^t e^{\sigma s}|f(s)|^2\,ds+\lambda_1^{-1}C_g\int_\tau^t e^{\sigma s}|Au(s)|^2\,ds$$
$$+C_\varepsilon\int_\tau^t e^{\sigma s}\|u(s)\|^6\,ds$$

for all $t \geq \tau$.

From this inequality, since the choice of ε makes the term $\int_\tau^t e^{\sigma s}|Au(s)|^2\,ds$ disappear, using (5.5) we easily obtain (5.1). $\square$

Definition 5.4. For any $\sigma, \underline{\sigma} > 0$, consider the universe $\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$ formed by the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A)\times L^2_V)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{D(A)\times L^2_V}\right) = \lim_{\tau\to-\infty}\left(e^{\underline{\sigma}\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{V\times L^2_H}\right) = 0.$$

Accordingly to the notation introduced in Section 3, $\mathcal{D}_F(D(A)\times L^2_V)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $D(A)\times L^2_V$. Observe that the universe $\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$, which is inclusion-closed, contains the universe $\mathcal{D}_F(D(A)\times L^2_V)$.

Remark 5.5. Under the additional assumption

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds < \infty, \tag{5.6}$$

from Lemma 5.3 it is easy to see that, for $0 < \underline{\sigma} < \sigma/3$, the family $\{\overline{B}_{D(A)\times L^2_V}(0,\widetilde{R}_{\sigma,\underline{\sigma}}(t)) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A)\times L^2_V)$ is pullback $\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$-absorbing for the process S on $D(A)\times L^2_V$, where

$$\widetilde{R}^2_{\sigma,\underline{\sigma}}(t) = 1 + 2\alpha^{-2}\varepsilon^{-1}(1+\lambda_1^{-1}he^{\sigma h})e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}|f(s)|^2\,ds + (1+\lambda_1^{-1}h)4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}M^3_{t,\underline{\sigma}}.$$

However, in order to apply Proposition 5.2, we need to obtain a different pullback $\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$-absorbing family.

Lemma 5.6. *Assume that $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$ satisfies (4.4) and (5.6), and $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for $0 < \underline{\sigma} < \sigma/3$ and for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$, there exist $\tau_2(\widehat{D},t,h) < t-2h$ and functions $\{\rho_i\}_{i=3}^4$ such that for any $\tau \leq \tau_2(\widehat{D},t,h)$ and any $(u^\tau,\phi) \in D(\tau)$, it holds*

$$|Au(r;\tau,u^\tau,\phi)|^2 \leq \rho_3^2(t), \quad \forall r \in [t-2h,t], \tag{5.7}$$

$$\int_{t-h}^t |Au'(\theta;\tau,u^\tau,\phi)|^2\,d\theta \leq \rho_4^2(t), \tag{5.8}$$

where

$$\rho_3^2(t) = 1 + 2\alpha^{-2}\varepsilon^{-1}e^{-\sigma(t-2h)}\int_{-\infty}^t e^{\sigma s}|f(s)|^2\,ds + 4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}M^3_{t,\underline{\sigma}}, \tag{5.9}$$

$$\rho_4^2(t) = 16\alpha^{-4}h\rho_3^2(t)\left(\nu^2 + C_2^2\lambda_1^{-3/2}\rho_3^2(t) + 2\lambda_1^{-2}C_g^2\right) + 16\alpha^{-4}\int_{t-h}^t |f(s)|^2\,ds, \tag{5.10}$$

where $\varepsilon, C_\varepsilon, C_{\underline{\sigma}}$ and $M_{t,\underline{\sigma}}$ are given by (3.4), (5.2), (5.3) and (5.4), respectively.

Proof. Let $\tau_2(\widehat{D},t,h) < t-2h$ be such that

$$\begin{aligned}&\alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{-\sigma(t-2h)}e^{\sigma\tau}\|(u^\tau,\phi)\|^2_{D(A)\times L^2_V}\\&\quad + 4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}e^{-3\underline{\sigma}(t-2h)}e^{3\underline{\sigma}\tau}\|(u^\tau,\phi)\|^6_{V\times L^2_H} \leq 1 \quad \forall\tau \leq \tau_2(\widehat{D},t,h),\ (u^\tau,\phi) \in D(\tau).\end{aligned}$$

Consider fixed $\tau \le \tau_2(\widehat{D}, t, h)$ and $(u^\tau, \phi) \in D(\tau)$. The estimate (5.7) follows directly from (5.1), using the increasing character of the exponential.

Now, from (2.9), (2.14) and (2.1), we obtain that $v = \mathcal{C}u$ satisfies

$$\begin{aligned}|v'(\theta)| &\le \nu|Au(\theta)| + C_2|Au(\theta)|^{1/2}\|u(\theta)\|^{3/2} + |f(\theta)| + |g(\theta, u_\theta)| \\ &\le \nu|Au(\theta)| + C_2\lambda_1^{-3/4}|Au(\theta)|^2 + |f(\theta)| + |g(\theta, u_\theta)|, \quad \text{a.e. } \theta > \tau,\end{aligned}$$

and therefore,

$$|v'(\theta)|^2 \le 4\nu^2|Au(\theta)|^2 + 4C_2^2\lambda_1^{-3/2}|Au(\theta)|^4 + 4|f(\theta)|^2 + 4|g(\theta, u_\theta)|^2, \quad \text{a.e. } \theta > \tau.$$

Integrating in time above and using properties (II) and (IV) on g, we deduce

$$\begin{aligned}\int_{t-h}^t |v'(\theta)|^2\,d\theta \le 4\nu^2 \int_{t-h}^t |Au(\theta)|^2\,d\theta + 4C_2^2\lambda_1^{-3/2}\int_{t-h}^t |Au(\theta)|^4\,d\theta \\ + 4\int_{t-h}^t |f(\theta)|^2\,d\theta + 4\lambda_1^{-2}C_g^2\int_{t-2h}^t |Au(\theta)|^2\,d\theta,\end{aligned}$$

whence, by (2.13) and (5.7), the estimate (5.8) follows. □

Corollary 5.7. *Under the assumptions of Lemma 5.6, for* $0 < \underline{\sigma} < \sigma/3$, *the family* $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A) \times L_V^2)$ *defined by*

$$\begin{aligned}D_{\sigma,\underline{\sigma}}(t) = \Big\{(w,\psi) \in D(A) \times L^2_{D(A)} : \exists \frac{d\psi}{ds} \in L^2_{D(A)}, \\ \|(w,\psi)\|_{D(A)\times L^2_{D(A)}} \le R_{\sigma,\underline{\sigma}}(t), \left\|\frac{d\psi}{ds}\right\|_{L^2_{D(A)}} \le \rho_4(t)\Big\}\end{aligned} \tag{5.11}$$

is pullback $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$*-absorbing for the process* S *on* $D(A) \times L_V^2$ *defined by* (3.1), *(and therefore* $\mathcal{D}_F(D(A) \times L_V^2)$*-absorbing too), where* $R_{\sigma,\underline{\sigma}}(t)$ *satisfies*

$$R^2_{\sigma,\underline{\sigma}}(t) = (1+h)\rho_3^2(t), \tag{5.12}$$

with $\rho_3(t)$ *and* $\rho_4(t)$ *given by* (5.9) *and* (5.10) *respectively.*

Now, we prove that the process S is $(D(A) \times L_V^2, D(A) \times C_V)$ pullback $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$-asymptotically compact. We will apply, under the natural necessary changes, the same energy method used in the proof of Lemma 3.17.

Lemma 5.8. *Assume that* $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ *satisfies* (4.4) *and* (5.6), *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *fulfills conditions (I)–(V). Then, for any* $0 < \underline{\sigma} < \sigma/3$, *the restriction to* $D(A) \times L_V^2$ *of the process* S *defined by* (3.1) *is* $(D(A) \times L_V^2, D(A) \times C_V)$ *pullback* $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$*-asymptotically compact.*

Proof. Let us fix $0 < \underline{\sigma} < \sigma/3$. Let be given $\widehat{D} \in \mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$, $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n. We must prove that the sequence

$$\{S(t,\tau_n)(u^{\tau_n}, \phi^n)\} = \{(u(t;\tau_n, u^{\tau_n}, \phi^n), u_t(\cdot;\tau_n, u^{\tau_n}, \phi^n))\}$$

is relatively compact in $D(A) \times C_V$.

First, we check the asymptotic compactness in the first component of S. By Corollary 5.7, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}}(k) \leq t-k$ such that $S(t-k,\tau)D(\tau) \subset D_{\sigma,\underline{\sigma}}(t-k)$ for all $\tau \leq \tau_{\widehat{D}}(k)$. From this and a diagonal argument, we can extract a subsequence $\{(u^{\tau_{n'}},\phi^{n'})\} \subset \{(u^{\tau_n},\phi^n)\}$ such that

$$S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}) \rightharpoonup (w^k,\psi^k) \quad \text{weakly in } D(A)\times L^2_{D(A)}, \tag{5.13}$$

$$\frac{d}{ds}u_{t-k}(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^k \qquad \text{weakly in } L^2_{D(A)}, \tag{5.14}$$

for all integer $k \geq 0$, where $(w^k,\psi^k) \in D_{\sigma,\underline{\sigma}}(t-k)$.

Now, applying Proposition 5.2 on each fixed interval $[t-k,t]$, we deduce that

$$\begin{aligned}(w^0,\psi^0) &= (D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\\ &= (D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t,t-k)S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\\ &= S(t,t-k)\left[(D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\right]\\ &= S(t,t-k)(w^k,\psi^k).\end{aligned}$$

From (5.13) with $k=0$, we obtain in particular that $|Aw^0| \leq \liminf_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|$. We will prove now that it also holds that

$$\limsup_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})| \leq |Aw^0|, \tag{5.15}$$

which combined with the weak converge of $u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$ to w^0 in $D(A)$, will imply the convergence in the strong topology of $D(A)$.

Observe that, as we already used in Lemma 5.3, for any $\tau \in \mathbb{R}$ and $(u^\tau,\phi) \in D(A)\times L^2_V$, the solution $u(\cdot;\tau,u^\tau,\phi)$, for short denoted $u(\cdot)$, satisfies the differential equality

$$\begin{aligned}\frac{d}{dt}(e^{\sigma t}\|u(t)\|^2+\alpha^2 e^{\sigma t}|Au(t)|^2) &= \sigma e^{\sigma t}\|u(t)\|^2+\alpha^2\sigma e^{\sigma t}|Au(t)|^2-2\nu e^{\sigma t}|Au(t)|^2\\ &\quad -2e^{\sigma t}(B(u(t)),Au(t))+2e^{\sigma t}(f(t)+g(t,u_t),Au(t))\end{aligned} \tag{5.16}$$

a.e. $t > \tau$. Since in particular $0 < \sigma < 2\nu(\lambda_1^{-1}+\alpha^2)^{-1}$, notice that $[[\cdot]]$, with $[[v]]^2 = (2\nu-\alpha^2\sigma)|Av|^2-\sigma\|v\|^2$, defines an equivalent norm to $|\cdot|_{D(A)}$ in $D(A)$.

We integrate the above expression in the interval $[t-k,t]$ for the solutions $u(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$ with $\tau_{n'} \leq t-k$, which yields

$$\begin{aligned}&\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2+\alpha^2|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2\\ &\quad = \|u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\|^2+\alpha^2|Au(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))|^2\\ &\quad = e^{-\sigma k}\left(\|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2+\alpha^2|Au(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2\right)\\ &\qquad +2\int_{t-k}^t e^{\sigma(s-t)}(f(s),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad +2\int_{t-k}^t e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad -2\int_{t-k}^t e^{\sigma(s-t)}(B(u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad -\int_{t-k}^t e^{\sigma(s-t)}[[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]]^2\,ds.\end{aligned} \tag{5.17}$$

From (5.13), (5.14) and Proposition 5.2, in particular we have that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\to u(\cdot;t-k,w^k,\psi^k)\quad\text{strongly in } C([t-k,t];V),\tag{5.18}$$

and also

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rightharpoonup u(\cdot;t-k,w^k,\psi^k)\quad\text{weakly in } L^2(t-k,t;D(A)).\tag{5.19}$$

Then, it is not difficult to see that

$$\lim_{n'\to\infty}\int_{t-k}^{t}e^{\sigma(s-t)}(B(u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds$$
$$=\int_{t-k}^{t}e^{\sigma(s-t)}(B(u(s;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds.\tag{5.20}$$

On other hand, as $e^{\sigma(\cdot-t)}f(\cdot)\in L^2(t-k,t;(L^2(\Omega))^3)$, it yields

$$\lim_{n'\to\infty}\int_{t-k}^{t}e^{\sigma(s-t)}(f(s),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds$$
$$=\int_{t-k}^{t}e^{\sigma(s-t)}(f(s),Au(s;t-k,w^k,\psi^k))\,ds.$$

Moreover, from (5.18), in particular we also have that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\to u(\cdot;t-k,w^k,\psi^k)\quad\text{strongly in } L^2(t-k-h,t;H),$$

which jointly with (5.19), implies that

$$\lim_{n'\to\infty}\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds$$
$$=\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds$$

Finally, as $\int_{t-k}^{t}e^{\sigma(s-t)}[[v(s)]]^2\,ds$ defines an equivalent norm in $L^2(t-k,t;D(A))$, we also deduce from above that

$$\int_{t-k}^{t}e^{\sigma(s-t)}[[u(s;t-k,w^k,\psi^k)]]^2\,ds$$
$$\le\liminf_{n'\to\infty}\int_{t-k}^{t}e^{\sigma(s-t)}[[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]]^2\,ds.\tag{5.21}$$

From (5.17), (5.20)–(5.21), taking into account (5.13) with $k=0$, the compactness of the injection of $D(A)$ into V, and (5.11), we conclude that

$$\|w^0\|^2+\alpha^2\limsup_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2$$
$$\le e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k)+2\int_{t-k}^{t}e^{\sigma(s-t)}(f(s),Au(s;t-k,w^k,\psi^k))\,ds$$
$$+2\int_{t-k}^{t}e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds$$
$$-2\int_{t-k}^{t}e^{\sigma(s-t)}(B(u(s;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds$$
$$-\int_{t-k}^{t}e^{\sigma(s-t)}[[u(s;t-k,w^k,\psi^k)]]^2\,ds.$$

Now, taking into account that $w^0 = u(t; t-k, w^k, \psi^k)$, integrating again in (5.16), we obtain

$$\begin{aligned}\|w^0\|^2 + \alpha^2|Aw^0|^2 = e^{-\sigma k}(\|w^k\|^2 + \alpha^2|Aw^k|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), Au(s; t-k, w^k, \psi^k))\, ds \\ + 2\int_{t-k}^{t} e^{\sigma(s-t)}(g(s, u_s(\cdot; t-k, w^k, \psi^k)), Au(s; t-k, w^k, \psi^k))\, ds \\ - 2\int_{t-k}^{t} e^{\sigma(s-t)}(B(u(s; t-k, w^k, \psi^k)), Au(s; t-k, w^k, \psi^k))\, ds \\ - \int_{t-k}^{t} e^{\sigma(s-t)}[[u(s; t-k, w^k, \psi^k)]]^2\, ds.\end{aligned}$$

Comparing the above two expressions, we conclude that

$$\|w^0\|^2 + \alpha^2 \limsup_{n'\to\infty} |Au(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})|^2 \le e^{-\sigma k}(\lambda_1^{-1} + \alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k) + \|w^0\|^2 + \alpha^2|Aw^0|^2.$$

But from (5.9) and (5.12), we have that $\lim_{k\to\infty} e^{-\sigma k}R^2_{\sigma,\underline{\sigma}}(t-k) = 0$, so (5.15) holds, and we conclude that

$$u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \to w^0 \quad \text{strongly in } D(A).$$

Finally, we prove the asymptotic compactness in the second component of S. From (5.13) and (5.14) with $k = 0$, we have that

$$\begin{aligned} u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) &\rightharpoonup \psi^0 && \text{weakly in } L^2_{D(A)}, \\ \frac{d}{ds}u_t(\cdot; \tau_{n'}, u^{\tau'_n}, \phi^{n'}) &\rightharpoonup \frac{d}{ds}\psi^0 && \text{weakly in } L^2_{D(A)}. \end{aligned}$$

Thus, by applying the Ascoli–Arzelà theorem, we can deduce that there exists a subsequence (relabeled the same) such that $u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$ converges to ψ^0 in C_V. So, the proof is finished. □

Remark 5.9. Since $S : \mathbb{R}^2_d \times D(A) \times L^2_V \to D(A) \times L^2_V$ is a continuous process, by the regularity properties established in Theorem 2.2 and Remark 2.4 (i), $S : \mathbb{R}^2_d \times D(A) \times C_V \to D(A) \times C_V$ is a well-defined closed process. In particular, $\{\Lambda_{D(A)\times C_V}(\widehat{D}, t)\}_{t\in\mathbb{R}}$ is meaningful for any $\widehat{D} \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ by Lemma 5.8. Actually, by the embedding $C_V \subset L^2_V$, recalling Remark 3.8 (ii), it holds that $\Lambda_{D(A)\times C_V}(\widehat{D}, t) = \Lambda_{D(A)\times L^2_V}(\widehat{D}, t)$ for any $t \in \mathbb{R}$, which is therefore invariant for S.

In general, the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (5.11) does not belong to the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, and we do not know whether or not S is pullback $\widehat{D}_{\sigma,\underline{\sigma}}$-asymptotically compact. Thus, we cannot apply Theorem 3.3 nor Theorem 3.7 to the family $\widehat{D}_{\sigma,\underline{\sigma}}$. Nevertheless, collecting the proved results, we may construct *by hand* a minimal pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-attractor in a better norm than the natural one for the phase-space $D(A) \times L^2_V$, namely in the $D(A) \times C_V$ norm.

Theorem 5.10. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (4.4) and (5.6), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $0 < \underline{\sigma} < \sigma/3$, the family $\mathcal{A}_{\sigma,\underline{\sigma}} = \{\mathcal{A}_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$, given by*

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)} \Lambda_{D(A)\times C_V}(\widehat{D}, t)}^{D(A)\times C_V}, \quad \forall t \in \mathbb{R},$$

satisfies the following properties:

(a) $\lim_{\tau\to-\infty} dist_{D(A)\times C_V}(S(t,\tau)D(\tau), \mathcal{A}_{\sigma,\underline{\sigma}}(t)) = 0$ *for all* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\sigma(D(A)\times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ *(pullback attraction).*

(b) $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ *is compact in* $D(A)\times C_V$ *for all* $t \in \mathbb{R}$.

(c) It is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A)\times C_V)$ *(resp.* $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A)\times L^2_V)$*) is a family of closed subsets of* $D(A)\times C_V$ *(resp.* $D(A)\times L^2_V$*) such that* $\lim_{\tau\to-\infty} dist_{D(A)\times C_V}(S(t,\tau)D(\tau), C(t)) = 0$ *(resp.* $\lim_{\tau\to-\infty} dist_{D(A)\times L^2_V}(S(t,\tau)D(\tau), C(t)) = 0$*) for all* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\sigma(D(A)\times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$*, then* $\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset C(t)$ *for all* $t \in \mathbb{R}$.

(d) $S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau) = \mathcal{A}_{\sigma,\underline{\sigma}}(t)$ *for all* $\tau \le t$ *(invariance).*

In particular, $\mathcal{A}_{\sigma,\underline{\sigma}}$ *is the* $(D(A)\times L^2_V, D(A)\times C_V)$ *minimal pullback* $\mathcal{D}_\sigma(D(A)\times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$*-attractor for the process* $S : \mathbb{R}^2_d \times D(A)\times L^2_V \to D(A)\times L^2_V$.

Proof. It suffices to check (a)–(d).

Claim (a). The pullback $D(A)\times C_V$-attraction property is an easy consequence of Lemma 5.8 (see also Remark 5.9).

Claim (b). Consider a sequence $\{y^n\} \subset \mathcal{A}_{\sigma,\underline{\sigma}}(t)$. We will extract a converging subsequence $\{y^{n'}\} \subset \{y^n\}$ with $D(A)\times C_V - \lim_{n'} y^{n'} \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$.

By definition of $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ we may consider a sequence $\{x^n\}_n$ with $x^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, t)$, where $\widehat{D}_n \in \mathcal{D}_\sigma(D(A)\times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, with $\mathrm{dist}_{D(A)\times C_V}(x^n, y^n) < 1/n$. For each $n \in \mathbb{N}$, this means that there exist sequences $\{z^{m,n}\}_m$ and $\{\tau^n_m\}_m$ with $\lim_m \tau^n_m = -\infty$, $z^{m,n} \in D_n(\tau^n_m)$ and $x^n = D(A)\times C_V - \lim_m S(t,\tau^n_m)z^{m,n}$. We may consider $m(n)$ such that

$$\mathrm{dist}_{D(A)\times C_V}(x^n, S(t,\tau^n_{m(n)})z^{m(n),n}) < 1/n, \quad \forall n \ge 1.$$

It is obvious that we are done if we obtain a subsequence $\{x^{n'}\}$ converging in $D(A)\times C_V$ since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is closed in $D(A)\times C_V$ and then $\lim_{n'} y^{n'} = \lim_{n'} x^{n'} \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$.

Now we rearrange the arguments of Lemma 5.8. For each integer $k \ge 0$, by the absorbing property established in Corollary 5.7, there exists $\tau_{\widehat{D}_n}(k) \le t-k$ such that

$$S(t-k,\tau)D_n(\tau) \subset D_{\sigma,\underline{\sigma}}(t-k), \quad \forall \tau \le \tau_{\widehat{D}_n}(k).$$

From this and a diagonal argument we can extract subsequences (the notation $\tau^{n'}_{m(n')}$ and $z^{m(n'),n'}$ is shorten for simplicity) $\{\tau_{n'}\}$ and $\{z^{n'}\}$ with $\tau_{n'} \to -\infty$ and $z^{n'} \in D_{n'}(\tau_{n'})$ such that

$$S(t-k,\tau_{n'})z^{n'} \rightharpoonup (w^k, \psi^k) \quad \text{weakly in } D(A)\times L^2_{D(A)}, \text{ for all } k \ge 0,$$

where $(w^k, \psi^k) \in D_{\sigma,\underline{\sigma}}(t-k)$.

Now we can repeat verbatim the arguments from Lemma 5.8 to conclude that

$$D(A)\times C_V - \lim_{n'} S(t,\tau_{n'})z^{n'} = (w^0, \psi^0)$$

which is also the limit of $x^{n'}$ and $y^{n'}$, so $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is relatively compact and closed, therefore compact in $D(A)\times C_V$.

Claim (c). The minimality among the families of closed subsets in $D(A)\times C_V$ is obvious, since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is the closure of omega-limit sets in this topology. For the case of $D(A)\times L^2_V$, observe that the omega-limit sets in this topology are those obtained in the $D(A) \times C_V$ topology (see Remark 5.9 (ii)). Besides, from (b), we have that $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is compact in $D(A) \times C_V$, and therefore also compact (in particular closed) in $D(A) \times L^2_V$. So the minimality argument is analogous.

Claim (d). We prove it by double inclusion. Let us first check that $\mathcal{A}_{\sigma,\underline{\sigma}}$ is negatively invariant, that is,

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau), \quad \forall \tau \le t. \tag{5.22}$$

Consider $y \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$. Then $y = D(A) \times C_V - \lim_n y^n$ with $y^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, t)$, where $\widehat{D}_n \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$. As long as each $\Lambda_{D(A)\times C_V}(\widehat{D}_n, t)$ is invariant for the process S, there exists $x^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$ with $y^n = S(t,\tau)x^n$. Observe that $\mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$ is compact (proved previously in (b)) in $D(A) \times C_V$. Therefore there exists a subsequence $\{x^{n'}\} \subset \{x^n\}$ with $D(A) \times C_V - \lim_{n'} x^{n'} = x \in \mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$. In particular, by the $D(A) \times L^2_V$ continuity of $S(t,\tau)$ (in fact, it is also continuous in $D(A) \times C_V$) it holds that $y^{n'} = S(t,\tau)x^{n'}$ converges to $S(t,\tau)x$ in $D(A) \times L^2_V$. So, by the uniqueness of the limit, $y = S(t,\tau)x$ and (5.22) holds.

Let us check the converse inclusion

$$S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau) \subset \mathcal{A}_{\sigma,\underline{\sigma}}(t), \quad \forall \tau \le t.$$

Fix $\tau \le t$ and $x \in \mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$. Then $x = D(A) \times C_V - \lim_n x^n$ with $x^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$, where $\widehat{D}_n \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$. Using again the invariance property $\Lambda_{D(A)\times C_V}(\widehat{D}_n, t) = S(t,\tau)\Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$, denote $y^n := S(t,\tau)x^n$. As long as $S(t,\tau)$ is continuous in $D(A) \times C_V$,

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \supset \Lambda_{D(A)\times C_V}(\widehat{D}_n, t) \ni y^n = S(t,\tau)x^n \to S(t,\tau)x,$$

and since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is closed in $D(A) \times C_V$, we obtain that $S(t,\tau)x \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$, which concludes the positive invariance of $\mathcal{A}_{\sigma,\underline{\sigma}}$. □

Remark 5.11. Observe that [14, Theorem 35] can be improved analogously as we have proceeded here. The notation $X_{\sigma,\underline{\sigma}}$ coined in [14] -in a context without delay effects- for the analogous role of $\mathcal{A}_{\sigma,\underline{\sigma}}$ here, was used because at that moment we did not realize that this family already had compact sections and therefore it was the minimal pullback attractor (in several topologies).

Under the additional assumption

$$\sup_{r \le 0} \int_{r-1}^{r} |f(s)|^2 ds < \infty, \tag{5.23}$$

the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (5.11) does belong to $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, whence now we can apply Theorem 3.3, and actually we have the following result.

Theorem 5.12. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (5.23), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $0 < \underline{\sigma} < \sigma/3$,*

$$\mathcal{A}_{\sigma,\underline{\sigma}} = \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}.$$

Actually, $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ is the unique family of closed subsets in $D(A) \times L^2_V$ in the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ that is invariant for S and pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-attracting.

Proof. Consider a fixed value $\underline{\sigma} \in (0, \sigma/3)$.

Observe that under the above assumption on f, the family $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$ defined by (5.11)–(5.12) belongs to $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$.

Let us prove the equality $\mathcal{A}_{\sigma,\underline{\sigma}} = \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}$ by double inclusion.

By Theorem 5.10, $\mathcal{A}_{\sigma,\underline{\sigma}}$ is well defined, and indeed, $\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset D_{\sigma,\underline{\sigma}}(t)$ for any $t \in \mathbb{R}$. By (5.23), for any fixed $t \in \mathbb{R}$, the set $\bigcup_{s \le t} D_{\sigma,\underline{\sigma}}(s)$ is bounded in $D(A) \times L^2_{D(A)}$ since $\sup_{s \le t} R^2_{\sigma,\underline{\sigma}}(s) < \infty$. In particular, from the invariance of $\mathcal{A}_{\sigma,\underline{\sigma}}$, we conclude that

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t), \quad \forall t \in \mathbb{R}.$$

On the other hand, again by (5.23), from Theorem 4.3 we have that for any $\tau \in \mathbb{R}$, $\bigcup_{r \le \tau} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(r)$ is a bounded subset of $D(A) \times L^2_V$, and therefore,

$$\mathrm{dist}_{D(A) \times L^2_V}(\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t), \mathcal{A}_{\sigma,\underline{\sigma}}(t)) \le \mathrm{dist}_{D(A) \times L^2_V}(S(t,\tau) \bigcup_{r \le \tau} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(r), \mathcal{A}_{\sigma,\underline{\sigma}}(t)),$$

where the right-hand side goes to zero as $\tau \to -\infty$. So we conclude that

$$\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \subset \mathcal{A}_{\sigma,\underline{\sigma}}(t).$$

The final statement about uniqueness is a direct consequence of Remark 3.4. □

Remark 5.13. Observe that, in particular, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (5.23), by Corollary 3.5 the minimal pullback attractor $\mathcal{A}_{\mathcal{D}_F(D(A) \times L^2_V)}$ does exist, and it also coincides with the family $\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}$. They have compact sections in $D(A) \times C_V$ and pullback attract in this metric. Moreover, from Theorem 4.3 we have that

$$\bigcup_{t \le t_2} \mathcal{A}_{\sigma,\underline{\sigma}}(t) = \bigcup_{t \le t_2} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \text{ is a bounded subset of } D(A) \times C_{D(A)} \text{ for any } t_2 \in \mathbb{R}.$$

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