Regular graphs of girth 5 from elliptic semiplanes of type C

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A well-known technique to construct regular graphs with girth 5 is the amalgamation into the incidence graphs $\mathcal{C}_q$ and $\mathcal{L}_q$, elliptic semiplanes of type C and L respectively, where q is a prime power. The case q odd has extensively been studied by means of amalgamations into $\mathcal{L}_q$. In this paper we provide new families of small regular graphs of girth 5 constructed by amalgamation into $\mathcal{C}_q$ using finite fields of even order.

1. Introduction

A k-regular graph with girth g is called a (k, g)-graph. A classic problem in graph theory is the construction of (k, g)-*cages*, that is, (k, g)-graphs with the minimum number $n(k, g)$ of vertices. We are interested in the case $g = 5$, where there are only known cages for $k = 2, \ldots, 7$ [9,12–16].

For values of $k \geq 8$, upper bounds on $n(k, 5)$ are obtained by explicit construction of $(k, 5)$-graphs. To reach this aim, a useful technique consists in injecting (or *amalgamating*) edges from a pair of r_q-regular graphs with girth at least 5 into a large $(q, 6)$-graph.

With a prime power q and the incidence graph $\mathcal{L}_q$ of the elliptic semiplane of type L, the technique of amalgamation provides the bound

$$n\big(q + r_q, 5\big) \leq 2(q^2 - 1),$$

[1,2,8,10,11]. Moreover, specific deletion of vertices from the graph resulting after amalgamation leads to

$$n\big(k, 5\big) \leq 2(q - 1)(k + 1 - r_q) \quad \text{for } k \leq q + r_q.$$

In 2005, for an odd prime power q, the first general construction was established in [10] with

$$r_q = \lfloor \frac{1}{4}\sqrt{q - 1} \rfloor.$$

In [1,2] some better degrees r_q were obtained by distinct amalgamations, including the high value

$$r_q = \frac{1}{\sqrt{2}}\sqrt{q + 1} + 2,$$

only admissible when $q = 2(p^2 - 1) + 1$ is a prime power related to another prime power $p > 7$.

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Let us notice that no general result associated to even prime powers has been so far established. That is the goal of the paper. To do that, in Section 2, we describe the incidence graph $\mathcal{C}_q$ of the elliptic semiplane of type C together with the notation and technique of amalgamation. It provides the inequality

$$n\big(q + r_q, 5\big) \leq 2q^2,$$

for a prime power q, [1–3,6,8]. Reduction operations after amalgamation give

$$n\big(k, 5\big) \leq 2q(k - r_q) \ \text{ for } k \leq q + r_q.$$

The main results are shown in the third section, where we focus on even prime powers $q = 2^{2s+1}$ (Theorems 3.1, 3.2) and $q = 2^{2s}$ (Theorem 3.3), for $s \geq 3$. We amalgamate into $\mathcal{C}_q$ a pair of *suitable* r_q-regular graphs G_L and G_P, isomorphic to subgraphs of $\mathcal{C}_p$, with $p = 2^s$. This way we establish two families of bounds with degree of amalgamation

$$\begin{cases} r_q = \frac{7}{8\sqrt{2}}\sqrt{q} & \text{if } q = 2^{2s+1}; \\ r_q = \frac{1}{2}\sqrt{q} & \text{if } q = 2^{2s}. \end{cases}$$

By means of new amalgamations into G_L and G_P, in Section 4 we present refinements of these bounds which, in Section 5, are applied to small values of even prime powers q and compared with the ones in [1,2].

2. Notations and known results

For undefined terminology we refer the reader to [5]. Given a graph G we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively. If $A, B \subset V(G)$, let $[A, B] \subset E(G)$ hold for the set of edges joining A and B. If S is a set and $\phi : V(G) \to S$ is a bijection, we write $\phi(G)$ for the graph with vertex set S and edge set $\{\phi(v)\phi(v') : vv' \in E(G)\}$.

Given a prime power γ, let $\mathbb{F}_\gamma$ be the finite field of order γ. The following graph $\mathcal{C}_\gamma$, first introduced by Cronheim in [6], is the key structure throughout this paper.

Definition 2.1. The bipartite graph $\mathcal{C}_\gamma$ has vertex set $\mathbb{L}_\gamma \cup \mathbb{P}_\gamma$, where

$$\mathbb{L}_\gamma = \bigcup_{a\in\mathbb{F}_\gamma} L_a = \bigcup_{a\in\mathbb{F}_\gamma} \{\ell_{a,b} : b \in \mathbb{F}_\gamma\}, \quad \mathbb{P}_\gamma = \bigcup_{x\in\mathbb{F}_\gamma} P_x = \bigcup_{x\in\mathbb{F}_\gamma} \{p_{x,y} : y \in \mathbb{F}_\gamma\}$$

and edge set

$$E(\mathcal{C}_\gamma) = \bigcup_{a,x\in\mathbb{F}_\gamma} [L_a, P_x] = \bigcup_{a,x\in\mathbb{F}_\gamma} \{\ell_{a,b}p_{x,ax+b} : b \in \mathbb{F}_\gamma\}.$$

That is, the edge $\ell_{a,b}p_{x,y} \in E(\mathcal{C}_\gamma)$ iff $y = ax + b$.

It is well known [4,6–8] that $\mathcal{C}_\gamma$ is the incidence graph of the elliptic semiplane of type C, obtained from the projective plane by removing the points at the infinity together with the vertical lines. It is a γ-regular graph with order $2\gamma^2$, girth $g = 6$ and diameter 4. Two vertices are at distance 4 if and only if both of them belong to the same set L_a or P_x.

Our basic goal is to add as many edges as possible to the graph $\mathcal{C}_\gamma$ in order to construct a new graph with greater degree and girth 5. To do that, in [8] the author introduces the following technique.

Definition 2.2 (*[8]*)**.** Let γ be a prime power. Given two graphs G_L, G_P with vertex set $\mathbb{F}_\gamma$, the *amalgamation* $\mathcal{C}_\gamma(G_L, G_P)$ is the graph obtained from $\mathcal{C}_\gamma$ by adding to $E(\mathcal{C}_\gamma)$ the set of edges

$$\big\{\ell_{a,b}\ell_{a,b'} : a \in \mathbb{F}_\gamma, bb' \in E(G_L)\big\} \cup \big\{p_{x,y}p_{x,y'} : x \in \mathbb{F}_\gamma, yy' \in E(G_P)\big\}.$$

The pair of graphs G_L, G_P must satisfy a certain restriction to ensure that $\mathcal{C}_\gamma(G_L, G_P)$ has girth 5. Let us recall that given a graph G with vertex set $\mathbb{F}_\gamma$, the Cayley color of an edge $vv' \in E(G)$ is the pair $\pm(v - v')$ if γ is odd or the single element $v - v' = v + v'$ if γ is a power of two. The set of Cayley colors of the edges of G is denoted by $\omega(G)$.

Theorem 2.1 (*[8]*)**.** *If G_L, G_P are r-regular graphs with vertex set $\mathbb{F}_\gamma$, girth $g \geq 5$ and with no common Cayley color, then $\mathcal{C}_\gamma(G_L, G_P)$ is a $(\gamma + r)$-regular graph with girth $g \geq 5$. Then,*

$$n(\gamma + r, 5) \leq 2\gamma^2.$$

Moreover, deleting from $\mathcal{C}_\gamma(G_L, G_P)$ pairs of sets L_a, P_a, for $a \in \mathbb{F}_\gamma$, we have

$$n(k, 5) \leq 2\gamma(k - r) \quad \textit{for } k \leq \gamma + r.$$

We also need to introduce another result, which is a special case of Theorem 5 in [3]:

Theorem 2.2 *([3]). If G_L, G_P are $(r+1)$- and r-regular graphs respectively, with vertex set $\mathbb{F}_\gamma$, girth $g \geq 5$ and without any Cayley color in common, then $\mathcal{C}_\gamma(G_L, G_P)$ is a $(\gamma+r+1, \gamma+r)$-regular graph with girth $g \geq 5$. Deleting the set P_0, we have*

$$n(\gamma+r, 5) \leq 2\gamma^2 - \gamma = 2\gamma(\gamma - \tfrac{1}{2}).$$

Additionally, deleting pairs L_a, P_a, for $a \in \mathbb{F}_\gamma \setminus \{0\}$, we have

$$n(k, 5) \leq 2\gamma(k - (r + \tfrac{1}{2})) \quad \text{for } k \leq \gamma + r.$$

These results lead us to extend the definition of suitability given in [8].

Definition 2.3. A pair of regular graphs G_L, G_P with vertex set $\mathbb{F}_\gamma$, girth $g \geq 5$ and disjoint sets of Cayley colors (in particular, a pair satisfying the hypothesis of either Theorem 2.1 or Theorem 2.2) is said to be *suitable* for amalgamation into $\mathcal{C}_\gamma$ or, in this paper, simply suitable.

3. General results

In this section, for an integer $s \geq 2$, we first deal with the prime power $p = 2^s$, the finite field $\mathbb{F}_p$ and the graph $\mathcal{C}_p$.

Given a fixed *primitive element* η of $\mathbb{F}_p$ with minimal polynomial $g(X)$, the elements of the field $\mathbb{F}_p = \mathbb{F}_p[\eta] \approx \mathbb{F}[X]/(g(X))$ are given by the polynomial expressions $a_0 + a_1\eta + \cdots + a_{s-1}\eta^{s-1}$, with $a_0, \ldots, a_{s-1} \in \mathbb{F}_2$, and can be uniquely represented by the integers $a_0 + a_1\, 2 + \cdots + a_{s-1}\, 2^{s-1}$ in the range $0, \ldots, p-1$.

Remark 3.1. For the sake of simplicity, we use this integer representation and write "$a < p/2$" to refer to the subgroup $H = \{0, \ldots, \frac{p}{2} - 1\}$. Similarly, "$a \geq p/2$" refers to the coset $\frac{p}{2} + H = \{\frac{p}{2}, \ldots, p-1\}$. Notice that $a + x \in H$ if and only if $a, x \in H$ or $a, x \in \frac{p}{2} + H$.

We also consider the field $\mathbb{F}_q$ and the graph $\mathcal{C}_q$ for $q = p^2$ or $q = 2p^2$. Our target is to construct a pair of graphs G_L and G_P, suitable for amalgamation into $\mathcal{C}_q$. We develop this goal in two steps. First, we construct graphs $\widetilde{G}_L$ and $\widetilde{G}_P$ as certain subgraphs of $\mathcal{C}_p$. Secondly, we define isomorphic graphs G_L and G_P with vertex set $\mathbb{F}_q$.

The construction for odd powers of two $q = 2p^2 = 2^{2s+1}$ and for even ones $q = p^2 = 2^{2s}$ is different. Notice that $|V(\mathcal{C}_p)| = 2p^2$, so we begin with $q = 2p^2$.

3.1. Case $q = 2^{2s+1}$

Now we deal with an integer $s \geq 2$, and prime powers $p = 2^s$, $q = 2p^2 = 2^{2s+1}$. We have the relationship $|V(\mathcal{C}_p)| = 2p^2 = |\mathbb{F}_q|$.

As we only need computations in the *additive group* $\mathbb{F}_q^+$, we use the isomorphism $\mathbb{F}_q^+ \approx \mathbb{F}_2^+ \oplus \mathbb{F}_p^+ \oplus \mathbb{F}_p^+$ to identify the elements of $\mathbb{F}_q^+$ with triplets (d, α, β) with $d = 0, 1$ and $\alpha, \beta \in \mathbb{F}_p^+$. Notice that the Cayley color of an edge $(d_1, \alpha_1, \beta_1)(d_2, \alpha_2, \beta_2)$ joining two elements in $\mathbb{F}_q^+$ is the single element $(d_1, \alpha_1, \beta_1) - (d_2, \alpha_2, \beta_2) = (d_1, \alpha_1, \beta_1) + (d_2, \alpha_2, \beta_2) = (d_1 + d_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \in \mathbb{F}_q^+ \setminus \{0\}$.

To construct $\widetilde{G}_L$ and $\widetilde{G}_P$, regular subgraphs of $\mathcal{C}_p$, we decompose the set $E(\mathcal{C}_p)$ into p perfect matchings between the sets $\mathbb{L}_p$ and $\mathbb{P}_p$.

Definition 3.1. Let $p = 2^s$ be a prime power and $\mathcal{C}_p$ be the bipartite graph given in Definition 2.1. For $m \in \mathbb{F}_p$, define the subset $E(m) \subset E(\mathcal{C}_p)$ by

$$\begin{aligned} E(m) &= \bigcup_{\substack{a+x=m \\ a,x \in \mathbb{F}_p}} [L_a, P_x] = \bigcup_{\substack{a+x=m \\ a,x \in \mathbb{F}_p}} \left\{\ell_{a,b} p_{x,ax+b} : b \in \mathbb{F}_p\right\} \\ &= \bigcup_{a \in \mathbb{F}_p} [L_a, P_{m+a}] = \bigcup_{a \in \mathbb{F}_p} \left\{\ell_{a,b} p_{m+a,a(m+a)+b} : b \in \mathbb{F}_p\right\} \end{aligned}$$

Notice that $E(\mathcal{C}_p) = \cup_{m \in \mathbb{F}_p} E(m)$. Some subsets of $\mathbb{F}_p$ are immediately suggested:

Definition 3.2. For each $m, \delta \in \mathbb{F}_p^*$, define the sets

$$\begin{aligned} D(m) &= \{ax : a, x \in \mathbb{F}_p, a + x = m\} = \{a(m+a) : a \in \mathbb{F}_p\} \\ M_b(\delta) &= \{a + x : a, x \in \mathbb{F}_p, ax = \delta\} \\ M_g(\delta) &= \mathbb{F}_p \setminus M_b(\delta) \end{aligned}$$

The following properties, based on the fact that $\mathbb{F}_p$ is a characteristic-two field, are critical.

Lemma 3.1. *For $\delta \in \mathbb{F}_p^*$, the following assertions hold:*

(i) $|M_g(\delta)| = \frac{p}{2}$.
(ii) If $m \in M_g(\delta)$, then $m \neq 0$, and the sets $D(m)$, $\delta + D(m)$ are disjoint.

Proof. (i) If $a + a' = m$ and $aa' = \delta$, then a, a' is the unique (unordered) pair of roots of the equation $X^2 - mX + \delta = 0$, and $a = a'$ iff $m = a + a' = 0$. Then, given $\delta \neq 0$, the $p - 1$ elements of $\mathbb{F}_p^*$ are grouped into $p/2 - 1$ pairs a, a' with $aa' = \delta$ and different values of $m = a + a' \neq 0$, and one element a with $a^2 = \delta$, $m = a + a = 0$. That is, $|M_b(\delta)| = p/2$ and $|M_g(\delta)| = p/2$.

(ii) Since $a_0(m + a_0) + a_1(m + a_1) = (a_0 + a_1)(m + a_0 + a_1)$, the set $D(m)$ is a subgroup of the additive group $\mathbb{F}_p^+$. Then, $D(m)$ and $\delta + D(m)$ are disjoint (actually, complementary) sets, unless $\delta \in D(m)$, which is equivalent to $m \in M_b(\delta)$ ■

Attending the notation in Remark 3.1, for $\delta \in \mathbb{F}_p^*$, let us split the set $M_b(\delta)$ into disjoint sets $\underline{M}_b(\delta) = M_b(\delta) \cap H$ and $\overline{M}_b(\delta) = M_b(\delta) \cap \left(\frac{p}{2} + H\right)$. Analogously, denote $\underline{M}_g(\delta) = M_g(\delta) \cap H$ and $\overline{M}_g(\delta) = M_g(\delta) \cap \left(\frac{p}{2} + H\right)$. Clearly, $|\underline{M}_g(\delta)| + |\underline{M}_b(\delta)| = p/2$ and $|\overline{M}_g(\delta)| + |\overline{M}_b(\delta)| = p/2$.

We are going to select the largest possible set $\underline{M}_g(\delta)$ or $\overline{M}_g(\delta)$, for $\delta \in \mathbb{F}_p^*$.

Definition 3.3. With the notation above, denote

$$K = \max_{\delta \in \mathbb{F}_p^*}\Big(\max\big(\,|\underline{M}_g(\delta)|, |\overline{M}_g(\delta)|\,\big)\Big).$$

Notice that $p/4 \leq K \leq p/2$. Now, we are ready to prove the first main result of this section:

Theorem 3.1. *For an integer $s \geq 2$, denote $p = 2^s$ and $q = 2p^2 = 2^{2s+1}$. Determine the value K according to Definition 3.3. The following assertions hold:*

(i) If K is even, there exists a pair of suitable graphs G_L, G_P with vertex set $\mathbb{F}_q$ and degrees $r_L = r_P = \frac{3p}{4} + \frac{K}{2}$. Then, $n\left(q + \frac{3p}{4} + \frac{K}{2},\ 5\right) \leq 2q^2$.
(ii) If K is odd, there exists a pair of suitable graphs G_L, G_P with vertex set $\mathbb{F}_q$ and degrees $r_L = \frac{3p}{4} + \frac{K+1}{2}$, $r_P = r_L - 1$. Moreover, $n\left(q + \frac{3p}{4} + \frac{K-1}{2},\ 5\right) \leq 2q^2 - q$.

Proof. We construct a pair of graphs $\widetilde{G}_L$, $\widetilde{G}_P$ as regular subgraphs of $\mathcal{C}_p$ with girth $g \geq 5$, and bijections Φ_L, $\Phi_P : V(\mathcal{C}_p) \to \mathbb{F}_q$ such that $G_L = \Phi_L(\widetilde{G}_L)$ and $G_P = \Phi_P(\widetilde{G}_P)$. The key point is to ensure that these graphs share no Cayley color in $\mathbb{F}_q$.

The definition of the bijection Φ_L is rather simple:

$$\begin{cases} \Phi_L(\ell_{a,b}) &=& (0, a, b) \\ \Phi_L(p_{x,y}) &=& (1, x, y) \end{cases}$$

The rest of the construction depends on K and its associated value δ.

• $K = |\underline{M}_g(\delta)|$

As $|\underline{M}_b(\delta)| = \frac{p}{2} - K$, we split this set into disjoint subsets $\underline{M}_b^0(\delta)$ and $\underline{M}_b^1(\delta)$ only attending that $|\underline{M}_b^0(\delta)| = \lceil (\frac{p}{2} - K)/2 \rceil$ and $|\underline{M}_b^1(\delta)| = \lfloor (\frac{p}{2} - K)/2 \rfloor$.

Define subgraphs $\widetilde{G}_L$ and $\widetilde{G}_P$ with vertex set $V(\mathcal{C}_p)$ and edges:

$$\begin{aligned} E(\widetilde{G}_L) &= \bigcup_{m \in M_L} E(m) \quad \text{with } M_L = \{\tfrac{p}{2}, \ldots, p-1\} \cup \underline{M}_g(\delta) \cup \underline{M}_b^0(\delta) = \mathbb{F}_p \setminus \underline{M}_b^1(\delta) \\ E(\widetilde{G}_P) &= \bigcup_{m \in M_P} E(m) \quad \text{with } M_P = \{\tfrac{p}{2}, \ldots, p-1\} \cup \underline{M}_g(\delta) \cup \underline{M}_b^1(\delta) = \mathbb{F}_p \setminus \underline{M}_b^0(\delta). \end{aligned}$$

Consider the bijection Φ_P:

$$\begin{cases} \Phi_P(\ell_{a,b}) &=& (0,\ a,\ b) & \text{if } a < \frac{p}{2} \\ \Phi_P(\ell_{a,b}) &=& (1,\ a,\ b+\delta) & \text{if } a \geq \frac{p}{2} \\ \Phi_P(p_{x,y}) &=& (1,\ x,\ y+\delta) & \text{if } x < \frac{p}{2} \\ \Phi_P(p_{x,y}) &=& (0,\ x,\ y) & \text{if } x \geq \frac{p}{2} \end{cases}$$

and define $G_L = \Phi_L(\widetilde{G}_L)$ and $G_P = \Phi_P(\widetilde{G}_P)$.

Being isomorphic to subgraphs of $\mathcal{C}_p$, graphs G_L and G_P have $g \geq 6$. Their degrees are $|M_L|$ and $|M_P|$, both equal to $\frac{3p}{4} + \frac{K}{2}$ if K is even, or to $\frac{3p}{4} + \frac{K+1}{2}$ and $\frac{3p}{4} + \frac{K-1}{2}$ if K is odd. The last step to prove the suitability of the pair G_L, G_P is to see that both graphs have no Cayley color in common.

The computation of $E(G_L)$ and $\omega(G_L)$ is straightforward

$$E(G_L) = \bigcup_{m\in M_L} \bigcup_{a\in\mathbb{F}_p} \big\{(0,a,b)(1,\ m+a,a(m+a)+b) : b\in\mathbb{F}_p\big\},$$

$$\omega(G_L) = \bigcup_{m\geq\frac{p}{2}} \{(1,m,a(m+a)) : a\in\mathbb{F}_p\} \cup \bigcup_{m\in\underline{M}_g(\delta)\cup\underline{M}_b^0(\delta)} \{(1,m,a(m+a)) : a\in\mathbb{F}_p\}$$

To describe the set of edges of G_P we need to take into account Remark 3.1.

$$\begin{aligned} E(G_P) = & \bigcup_{m\geq\frac{p}{2}} \bigcup_{a<\frac{p}{2}} \big\{(0,a,b)(0,\ m+a,a(m+a)+b) : b\in\mathbb{F}_p\big\} \\ \cup & \bigcup_{m\geq\frac{p}{2}} \bigcup_{a\geq\frac{p}{2}} \big\{(1,a,b+\delta)(1,m+a,a(m+a)+b+\delta) : b\in\mathbb{F}_p\big\} \\ \cup & \bigcup_{m\in\underline{M}_g(\delta)\cup\underline{M}_b^1(\delta)} \bigcup_{a<\frac{p}{2}} \big\{(0,a,b)(1,m+a,a(m+a)+b+\delta) : b\in\mathbb{F}_p\big\} \\ \cup & \bigcup_{m\in\underline{M}_g(\delta)\cup\underline{M}_b^1(\delta)} \bigcup_{a\geq\frac{p}{2}} \big\{(1,a,b+\delta)(0,\ m+a,a(m+a)+b) : b\in\mathbb{F}_p\big\} \end{aligned}$$

and its set of Cayley colors

$$\omega(G_P) = \bigcup_{m\geq\frac{p}{2}} \{(0,m,a(m+a)) : a\in\mathbb{F}_p\} \cup \bigcup_{m\in\underline{M}_g(\delta)\cup\underline{M}_b^1(\delta)} \{(1,m,a(m+a)+\delta) : a\in\mathbb{F}_p\}$$

By Lemma 3.1, when $m\in\underline{M}_g(\delta)$, the sets $D(m)=\{a(m+a) : a\in\mathbb{F}_p\}$ and $D(m)+\delta=\{a(m+a)+\delta : a\in\mathbb{F}_p\}$ are disjoint. By definition, $\underline{M}_b^0(\delta)\cap\underline{M}_b^1(\delta)=\emptyset$. We conclude that $\omega(G_L)\cap\omega(G_P)=\emptyset$.

- $K=|\overline{M}_g(\delta)|$

Now we write $|\overline{M}_b(\delta)|=\overline{M}_b^0(\delta)\cup\overline{M}_b^1(\delta)$ with $|\overline{M}_b^0(\delta)|=\lceil(\frac{p}{2}-K)/2\rceil$ and $|\overline{M}_b^1(\delta)|=\lfloor(\frac{p}{2}-K)/2\rfloor$ and define subgraphs $\widetilde{G}_L$ and $\widetilde{G}_P$ with vertex set $V(\mathcal{C}_p)$ and edges:

$$\begin{aligned} E(\widetilde{G}_L) &= \bigcup_{m\in M_L} E(m) \quad \text{with } M_L=\{0,\dots,\tfrac{p}{2}-1\}\cup\overline{M}_g(\delta)\cup\overline{M}_b^0(\delta)=\mathbb{F}_p\setminus\overline{M}_b^1(\delta) \\ E(\widetilde{G}_P) &= \bigcup_{m\in M_P} E(m) \quad \text{with } M_P=\{0,\dots,\tfrac{p}{2}-1\}\cup\overline{M}_g(\delta)\cup\overline{M}_b^1(\delta)=\mathbb{F}_p\setminus\overline{M}_b^0(\delta). \end{aligned}$$

Consider the bijection Φ_P:

$$\begin{cases} \Phi_P(\ell_{a,b}) = (0,\ a,\ b) & \text{if } a<\frac{p}{2} \\ \Phi_P(\ell_{a,b}) = (1,\ a+\frac{p}{2},\ b+\delta) & \text{if } a\geq\frac{p}{2} \\ \Phi_P(p_{x,y}) = (0,\ x+\frac{p}{2},\ y) & \text{if } x<\frac{p}{2} \\ \Phi_P(p_{x,y}) = (1,\ x,\ y+\delta) & \text{if } x\geq\frac{p}{2} \end{cases} \tag{1}$$

and the graphs $G_L=\Phi_L(\widetilde{G}_L)$ and $G_P=\Phi_P(\widetilde{G}_P)$.

The girth and degrees of G_L and G_P are the same as in the previous case. The computation of the Cayley colors leads to:

$$\begin{aligned} \omega(G_L) &= \bigcup_{m<\frac{p}{2}} \{(1,m,a(m+a)) : a\in\mathbb{F}_p\} \cup \bigcup_{m\in\overline{M}_g(\delta)\cup\overline{M}_b^0(\delta)} \{(1,m,a(m+a)) : a\in\mathbb{F}_p\} \\ \omega(G_P) &= \bigcup_{m\geq\frac{p}{2}} \{(0,m+\tfrac{p}{2},a(m+a)) : a\in\mathbb{F}_p\} \cup \bigcup_{m\in\overline{M}_g(\delta)\cup\overline{M}_b^1(\delta)} \{(1,m,a(m+a)+\delta) : a\in\mathbb{F}_p\} \end{aligned}$$

Both sets are disjoint.

In both cases the graphs G_L and G_P satisfy the hypothesis of Theorem 2.1, when K is even, or Theorem 2.2, when K is odd. The proof is complete. ■

The result in Theorem 3.1 depends on K and it requires the computation of this value. By definition, $K\geq\frac{p}{4}$. Then, if $p=2^s\geq 8$, the conclusion of Theorem 3.1 holds when K is the even number $\frac{p}{4}$.

Theorem 3.2. *Given $s\geq 3$ and $q=2^{2s+1}$, the following inequality holds:*

$$n\big(q+\tfrac{7}{8}\sqrt{\tfrac{q}{2}},5\big)\leq 2q^2.$$

3.2. Case $q = 2^{2s}$

Now we consider an integer $s \geq 2$, the prime powers $p = 2^s$, $q = 2^{2s}$, the field $\mathbb{F}_p$, the additive group $\mathbb{F}_q^+$ and the isomorphism $\mathbb{F}_q^+ \approx \mathbb{F}_2^+ \oplus \mathbb{F}_{p/2}^+ \oplus \mathbb{F}_p^+$. As in Section 3.1, we are looking for suitable amalgamations into $\mathcal{C}_q$, but this time $|V(\mathcal{C}_p)| = 2p^2$ while $|\mathbb{F}_q| = p^2$.

Define $\widetilde{G}_L$ and $\widetilde{G}_P$ as the subgraphs of $\mathcal{C}_p$ induced by the disjoint sets of vertices $(\cup_{a<p/2} L_a) \cup (\cup_{x<p/2} P_x)$ and $(\cup_{a\geq p/2} L_a) \cup (\cup_{x\geq p/2} P_x)$, respectively. Clearly, $|V(\widetilde{G}_L)| = |V(\widetilde{G}_P)| = p^2 = q$. Also, $\widetilde{G}_L$ and $\widetilde{G}_P$ are $p/2$-regular graphs because

$$E(\widetilde{G}_L) = \bigcup_{a<p/2} \bigcup_{x<p/2} [L_a, P_x], \quad E(\widetilde{G}_P) = \bigcup_{a\geq p/2} \bigcup_{x\geq p/2} [L_a, P_x].$$

Writing $m = a + x$ and taking into account Remark 3.1, we have

$$E(\widetilde{G}_L) = \bigcup_{m<p/2} \bigcup_{a<p/2} [L_a, P_{m+a}], \quad E(\widetilde{G}_P) = \bigcup_{m<p/2} \bigcup_{a\geq p/2} [L_a, P_{m+a}]$$

With these constructions, we are ready to prove the following result:

Theorem 3.3. *For an integer $s \geq 2$ and prime powers $p = 2^s$, $q = 2^{2s}$, there is a pair of graphs G_L, G_P of degrees $r_L = r_P = \frac{p}{2}$ suitable for amalgamation into $\mathcal{C}_q$. Then, $n(q + \frac{1}{2}\sqrt{q}, 5) \leq 2q^2$.*

Proof. Consider the bijections $\Phi_L : V(\widetilde{G}_L) \to \mathbb{F}_q$ defined as

$$\begin{cases} \Phi_L(\ell_{a,b}) &= (0, a, b) \\ \Phi_L(p_{x,y}) &= (1, x, y), \end{cases}$$

and $\Phi_P : V(\widetilde{G}_P) \to \mathbb{F}_q$ such that

$$\begin{cases} \Phi_P(\ell_{a,b}) &= (0, a + \frac{p}{2}, b) \\ \Phi_P(p_{x,y}) &= (1, x + \frac{p}{2}, y). \end{cases}$$

Define the graphs $G_L = \Phi_L(\widetilde{G}_L)$ and $G_P = \Phi_P(\widetilde{G}_P)$. The computation of their Cayley colors is straightforward:

$$\begin{aligned}
E(G_L) &= \bigcup_{m<p/2} \bigcup_{a<p/2} \left\{(0,\ a,\ b)(1,\ m+a,\ a(m+a)+b) : b \in \mathbb{F}_p\right\} \\
\omega(G_L) &= \bigcup_{m<p/2} \{(1, m, a(m+a)) : a < \tfrac{p}{2}\} \\
E(G_P) &= \bigcup_{m<p/2} \bigcup_{a\geq p/2} \left\{(0,\ a+\tfrac{p}{2},\ b)(1,\ m+a+\tfrac{p}{2},\ a(m+a)+b) : b \in \mathbb{F}_p\right\} \\
\omega(G_P) &= \bigcup_{m<p/2} \{(1, m, a(m+a)) : a \geq \tfrac{p}{2}\},
\end{aligned}$$

If $a_0(m + a_0) = a_1(m + a_1)$, then $(a_0 + a_1)(m + a_0 + a_1) = 0$. Hence, $a_1 = a_0$ or $a_1 = m + a_0$. In both cases, $a_0, a_1 < p/2$ or $a_0, a_1 \geq p/2$ when $m < p/2$. Hence, $\omega(G_L) \cap \omega(G_P) = \emptyset$.

This proves that the $\frac{p}{2}$-regular graphs G_L and G_P are suitable for amalgamation into $\mathcal{C}_q$. By Theorem 2.1, the graph $\mathcal{C}_q(G_L, G_P)$ is $(q + \frac{p}{2})$-regular with girth at least five and order $2q^2$. ■

Before proceeding to obtain more accurate bounds, let us notice that no Cayley color $(0, 0, \beta)$ appears in the set $\omega(G_L) \cup \omega(G_P)$, where G_L and G_P are the graphs constructed in Theorem 3.1 and in Theorem 3.3.

4. Refined bounds

We are now interested in increasing the degree of the graphs G_L and G_P constructed in Section 3. The technique involves a recursive construction with *two* pairs of graphs, one to be amalgamated into $\widetilde{G}_L$ and the other into $\widetilde{G}_P$.

Proposition 4.1. *Let $s \geq 3$ and $p = 2^s$. If $q = 2p^2$, let K, r_L, r_P be defined as in Theorem 3.1; and if $q = p^2$, let r_L, r_P be defined as in Theorem 3.3. Assume that $H_L^0, H_L^1, H_P^0, H_P^1$ are graphs with vertex set $\mathbb{F}_p$ and girth $g \geq 5$. Additionally, consider*

(i) H_L^0, H_L^1 have common degree s_L.
(ii) H_P^0, H_P^1 have common degree s_P.
(iii) The sets of Cayley colors of $H_L^0, H_L^1, H_P^0, H_P^1$ are pairwise disjoint in $\mathbb{F}_p$.

Then, there exists a pair of suitable graphs G_L, G_P with vertex set $\mathbb{F}_q$ and degrees $r_L + s_L$, $r_P + s_P$ respectively.

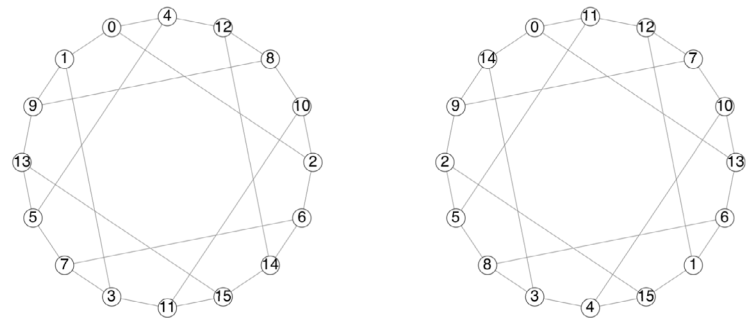

Fig. 1. One pair of graphs H_L^0 and H_L^1 with disjoint Cayley colors in $\mathbb{F}_{16}$.

Proof. Consider the bijections Φ_L, Φ_P and the graphs $\widetilde{G}_L$, $\widetilde{G}_P$ provided in Theorem 3.1 for $q = 2p^2$, or in Theorem 3.3 for $q = p^2$.

By (i) and (iii), the pair H_L^0, H_L^1 is suitable for amalgamation into $\mathcal{C}_p$, then also into its subgraph $\widetilde{G}_L$. Similarly, by (ii) and (iii), the pair H_P^0, H_P^1 can be suitably amalgamated into $\widetilde{G}_P$. In order not to introduce more notation, let $\widetilde{G}_L$ and $\widetilde{G}_P$ also denote the graphs resulting after these amalgamations. In the same way, let G_L and G_P hold for the graphs isomorphic to $\widetilde{G}_L$ and $\widetilde{G}_P$ after applying the bijections Φ_L and Φ_P. These new graphs G_L and G_P have girth 5 and degrees $r_L + s_L$ and $r_P + s_P$, respectively.

The colors in $\mathbb{F}_q$ of the new edges of G_L and G_P form the subsets $\{(0, 0, \beta) : \beta \in \omega(H_L^0) \cup \omega(H_L^1)\}$, $\{(0, 0, \beta) : \beta \in \omega(H_P^0) \cup \omega(H_P^1)\}$ of $\mathbb{F}_q$, respectively. By (iii) both sets are disjoint. Also, as we have mentioned above, these colors are different from the original ones of G_L and G_P. Hence, the new pair G_L, G_P is suitable for amalgamation into $\mathcal{C}_q$. ■

Proposition 4.1 allows a slight improvement of Theorems 3.1 and 3.3. There is an easy way to construct the four graphs required in this proposition.

Lemma 4.1. *Let $s \geq 3$ and $p = 2^s$. Assume H_L, H_P are graphs with vertex set $\mathbb{F}_{p/2}$, girth $g \geq 5$, degrees s_L, s_P, and sharing no Cayley color. Then, there exist four graphs H_L^0, H_L^1, H_P^0, H_P^1 verifying the hypothesis of Proposition 4.1.*

Proof. Let us identify $\mathbb{F}_p^+ \approx \mathbb{F}_{p/2}^+ \oplus \mathbb{F}_2^+$. Define graphs H_L^0, H_L^1 with vertex set $\mathbb{F}_p$ and edge sets

$$\begin{aligned} E(H_L^0) &= \{(u, 0)(v, 0) : uv \in E(H_L)\} \cup \{(u, 1)(v, 1) : uv \in E(H_L)\} \\ E(H_L^1) &= \{(u, 0)(v, 1) : uv \in E(H_L)\} \cup \{(u, 1)(v, 0) : uv \in E(H_L)\} \end{aligned}$$

Similarly, define graphs H_P^0, H_P^1. It is straightforward to see that H_L^0, H_L^1, H_P^0, H_P^1 satisfy the hypothesis of Proposition 4.1. ■

5. Small cases

We apply the refined bounds described in Section 4 to some particular cases. Moreover, we also describe an iterative process that makes feasible to provide better bounds.

In [1] the authors construct two pairs of suitable graphs with vertex sets $\mathbb{F}_{2^5}$ and $\mathbb{F}_{2^6}$ with degrees 5 and 6 respectively. Consequently, bounds $n(32 + 5, 5) \leq 2 \cdot 32^2$ and $n(64 + 6, 5) \leq 2 \cdot 64^2$ were established. In [2], (see Table 2 and inequality (2) in Remark 2.1), thanks to suitable amalgamations into the elliptic semiplanes $\mathcal{L}_{127}$ and $\mathcal{L}_{257}$, the bounds $n(127 + 10 - 1, 5) \leq 2(127 + 1 - 1)(127 - 1) = 32004$ and $n(257 + 12 - 3, 5) \leq 2(257 + 1 - 3)(257 - 1) = 130560$ were established. In this paper we get the less accurate bounds $n(2^7 + 8, 5) = n(136, 5) \leq 2 \cdot (2^7)^2 - 2^7 = 32640$ and $n(2^8 + 10, 5) = n(266, 5) \leq 2 \cdot (2^8)^2 - 2^8 = 130816$. Let us continue with some more values.

- $q = 2^9 = 512$
  We have $q = 2p^2$ with $p = 2^4$. Let us represent the elements of $\mathbb{F}_{16}$ by the integers $0, \ldots, 15$ and consider that $\mathbb{F}_{16} \simeq \mathbb{F}_2[X]/(X^4 + X + 1)$. Computations lead to the value $K = 6$ and, by Theorem 3.1, there is a suitable pair of regular graphs with degrees $r_L = r_P = 15$.
  To increase the degree of these graphs consider the 3-regular graphs H_L^0, H_L^1 displayed in Fig. 1 and the 2-regular graphs H_P^0, H_P^1 in Fig. 2. They have girth 6 and 8 respectively. Since $\omega(H_L^0) = \{1, 2, 4, 8\}$, $\omega(H_L^1) = \{7, 11, 13, 14\}$, $\omega(H_P^0) = \{3, 6, 12\}$ and $\omega(H_P^1) = \{5, 9, 10\}$ any two of these graphs have disjoint sets of colors in $\mathbb{F}_p$.

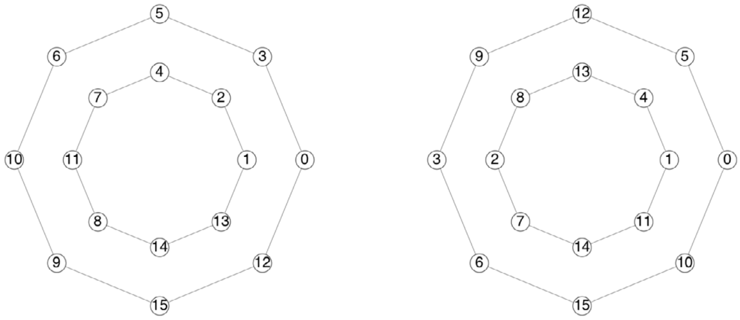

Fig. 2. Another pair of graphs H_P^0 and H_P^1 with disjoint Cayley colors in $\mathbb{F}_{16}$.

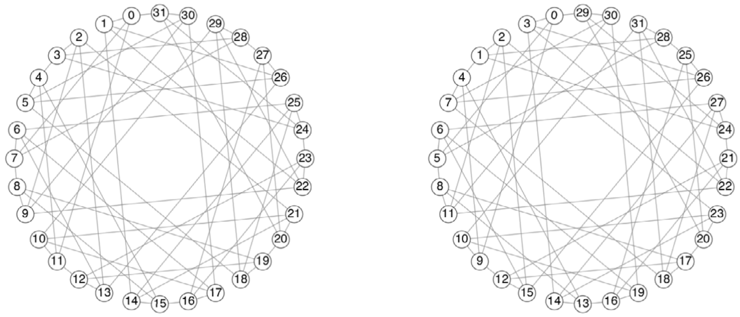

Fig. 3. $(4, 6)$-graphs H_L^0 and H_L^1 with disjoint Cayley colors in $\mathbb{F}_{32}$.

Using Proposition 4.1 we construct graphs G_L, G_P of degrees 18, 17 respectively, suitable for amalgamation into $\mathcal{C}_{2^9}$. According to Theorem 2.2, we have the bound $n(2^9 + 17) \le 2 \cdot (2^9)^2 - 2^9 = 2 \cdot 2^9(2^9 - \frac{1}{2})$ and, more generally, $n(k, 5) \le 2 \cdot 2^9(k - 17 - \frac{1}{2})$ for $k \le 2^9 + 17$. Actually, this is a record bound when $2^9 + 5 \le k \le 2^9 + 17$.

- $q = 2^{10} = 1024$

  Now $q = p^2$ with $p = 2^5 = 32$. By Theorem 3.3, there is a suitable pair of regular graphs with degrees $r_L = r_P = 16$. Consider the $(4, 6)$-regular graphs H_L^0, H_L^1 in Fig. 3 and the $(3, 6)$-regular graphs H_P^0, H_P^1 in Fig. 4. The sets of Cayley colors in $\mathbb{F}_{32}$ are $\omega(H_L^0) = \{1, 7, 11, 15, 23, 27, 31\}$, $\omega(H_L^1) = \{3, 5, 9, 13, 21, 25, 29\}$, $\omega(H_P^0) = \{17, 18, 20, 24\}$, $\omega(H_P^1) = \{2, 4, 6, 8, 10, 12, 14, 16, 22, 26, 28, 30\}$. Any two of these sets are disjoint.

  By using Proposition 4.1 we construct the new suitable pair of graphs G_L, G_P with degrees 20, 19 respectively. Theorem 2.2 states $n(2^{10} + 19) \le 2 \cdot 2^{10}(2^{10} - \frac{1}{2})$ and the new record bound $n(k, 5) \le 2 \cdot 2^{10}(k - 19 - \frac{1}{2})$ for $2^{10} + 16 \le k \le 2^{10} + 19$.

- $q = 2^{11} = 2048$

  Again, $p = 2^5$. Computations in $\mathbb{F}_{32} \simeq \mathbb{F}_2[X]/(X^5 + X^2 + 1)$ lead to the value $K = 11$. By Theorem 3.1 there are two suitable graphs G_L, G_P of degrees $3p/4 + \lceil K/2 \rceil = 30$ and $3p/4 + \lfloor K/2 \rfloor = 29$.

  By using again the graphs H_L^0, H_L^1, H_P^0, H_P^1 provided in Figs. 3 and 4 together with Proposition 4.1 we construct the new suitable 33-regular graphs G_L, G_P. Theorem 2.1 establishes the new record bound $n(k, 5) \le 2 \cdot 2^{11}(k - 33)$ for $2^{11} + 3 \le k \le 2^{11} + 33$.

- $q = 2^{12} = 4096$

  Now $p = 2^6 = 64$ and $q = p^2$. In [1] the authors give a pair of suitable graphs with vertex set $\mathbb{F}_{2^5}$ and degree 5. By Lemma 4.1, there are four graphs with vertex set $\mathbb{F}_{2^6}$ verifying the hypothesis of Proposition 4.1. Then, by Theorem 3.3, we obtain a suitable pair of graphs G_L and G_P, both with degree $2^6/2 + 5 = 37$, and the associated bound $n(2^{12} + 37, 5) \le 2(2^{12})^2$.

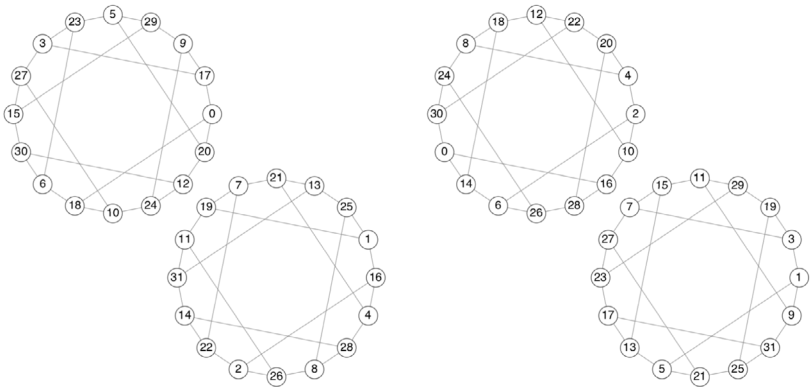

Fig. 4. $(3, 6)$-graphs H_P^0 and H_P^1 with disjoint Cayley colors in $\mathbb{F}_{32}$.

We summarize the last results:

Theorem 5.1. *The following upper bound on the minimum order of a* $(k, 5)$*-graph holds*

q	k	$rec(k, 5)$
2^9	$2^9 + 5 \leq k \leq 2^9 + 17$	$2 \cdot 2^9(k - 17 - \frac{1}{2})$
2^{10}	$2^{10} + 16 \leq k \leq 2^{10} + 19$	$2 \cdot 2^{10}(k - 19 - \frac{1}{2})$
2^{11}	$2^{11} + 3 \leq k \leq 2^{11} + 33$	$2 \cdot 2^{11}(k - 33)$
2^{12}	$2^{12} + 29 \leq k \leq 2^{12} + 37$	$2 \cdot 2^{12}(k - 37)$

Alternating Theorems 3.2 and 3.3, the process used for $q = 2^{12}$ could indefinitely be applied to obtain refined bounds associated to greater powers of two.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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