A decomposition result for the pressure of a fluid in a thin domain and extensions to elasticity problems

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Abstract. In order to study the asymptotic behavior of a fluid in a domain of small thickness ε, it is common to use that the norm of the pressure p_ε in L^q, $q>1$, is smaller than $C\|\nabla p_\varepsilon\|_{W^{-1,q}}/\varepsilon$. Our purpose in the present paper is to improve this estimate by showing that in fact p_ε can be decomposed as the sum of two terms: the first one is of order $1/\varepsilon$ with respect to ∇p_ε but it belongs to the Sobolev space $W^{1,q}$ and not only to L^q; the second one only belongs to L^q but it is of order one with respect to ∇p_ε. This result also allows us to improve the classical estimate for Korn's constant in an elastic thin domain providing a decomposition of the deformation which contains terms with a stronger regularity. The advantage of these expansions is that they simplify the study of the asymptotic behavior of continuum mechanics problems in thin domains since they give an additional compactness. As examples we provide two applications in fluid mechanics and linear elasticity.

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1 Introduction

In order to obtain an estimate for the pressure of a fluid in a given domain, it is usual to apply the following result (see e.g. [17], [23], [24]): for every smooth connected bounded domain $\Omega\subset\mathbb{R}^N$, there exists $C>0$, such that for every $p\in L^2(\Omega)$ with null mean value on Ω, one has

$$\|p\|_{L^2(\Omega)}\leq C\|\nabla p\|_{H^{-1}(\Omega)^N}. \tag{1.1}$$

However, this constant C depends on the geometry of Ω and then, if we deal with a sequence of partial differential problems in varying domains, it is necessary to study the variation of this constant with respect to the domain. An important example is the case of problems posed in thin domains. To fix ideas let us consider two smooth connected bounded open sets $\omega\subset\mathbb{R}^k$

and $\vartheta \subset \mathbb{R}^{N-k}$ with $N \geq 2$, $1 \leq k \leq N-1$. Then, for a small parameter $\varepsilon > 0$, we define the sequence of thin domains $\Omega_\varepsilon \subset \mathbb{R}^N$ by

$$\Omega_\varepsilon = \omega \times (\varepsilon\vartheta). \tag{1.2}$$

In this case it is known that the corresponding constant in (1.1), which we now denote as C_ε, can be estimated by

$$C_\varepsilon \leq \frac{C}{\varepsilon}, \tag{1.3}$$

with C independent of ε. As a classical application, we consider the Stokes problem in Ω_ε

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

with $f \in L^2(\omega)^N$. Multiplying the first equation by u_ε and using that the Poincaré constant in Ω_ε is of order ε we easily deduce

$$\frac{1}{\varepsilon^2} \fint_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \ \text{ bounded},$$

where $\fint_{\Omega_\varepsilon} v dx$ denotes the mean value $\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} v\, dx$. Then (1.3) shows that choosing p_ε with null integral, we also have

$$\fint_{\Omega_\varepsilon} |p_\varepsilon|^2 dx \ \text{ bounded}.$$

This proves that, at least for a subsequence, there exists $p \in L^2(\omega)$ such that (we use the notation $x' = (x_1, \ldots, x_k)$, $x'' = (x_{k+1}, \ldots, x_N)$)

$$\fint_{\varepsilon\vartheta} p_\varepsilon\, dx'' \rightharpoonup p \ \text{ in } L^2(\omega). \tag{1.4}$$

It is well known that in fact it is not necessary to extract any subsequence and that the function p is the solution of the Reynolds problem

$$\begin{cases} -\operatorname{div} M\big(\nabla p - f'\big) = 0 \text{ in } \omega, \\ M\big(\nabla p - f'\big)\cdot \nu = 0 \text{ on } \partial\omega, \end{cases} \tag{1.5}$$

with $f' = (f_1, ..., f_k)$, ν the outward unitary normal vector to ω and $M \in \mathbb{R}^{k^2}$ a definite positive symmetric matrix which depends on ϑ. By (1.4) we expect p to only belong to $L^2(\omega)$ but since it is the solution of (1.5) we get surprisingly that p is smoother: it is in $H^1(\omega)$. This phenomenon of regularity gain also appears in several other related examples.

Our aim in this paper is to show that in fact the estimate

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}, \quad \forall\, p_\varepsilon \in L^2(\Omega_\varepsilon), \ \int_{\Omega_\varepsilon} p_\varepsilon\, dx = 0,$$

for Ω_ε as above can be improved. Namely, we show that if $p_\varepsilon \in L^2(\Omega_\varepsilon)$ has null mean value, then it can be decomposed as

$$p_\varepsilon(x) = \frac{1}{\varepsilon} p^0_\varepsilon(x') + p^1_\varepsilon(x) \ \text{ a.e. } x \in \Omega_\varepsilon, \tag{1.6}$$

with $p_\varepsilon^0 \in H^1(\omega)$ and $p_\varepsilon^1 \in L^2(\Omega_\varepsilon)$ satisfying

$$\|p_\varepsilon^0\|_{H^1(\Omega_\varepsilon)} + \|p_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} \leq C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}, \tag{1.7}$$

with C independent of ε and p_ε. Applying this result to the previous example, we get that p_ε^1 can be neglected because

$$⨍_{\Omega_\varepsilon} |p_\varepsilon^1|^2 dx \leq C\varepsilon^2,$$

and that p is the limit in the weak topology of $H^1(\omega)$ of the sequence $p_\varepsilon^0/\varepsilon$. So, the fact that p is in $H^1(\omega)$ is just a consequence of the decomposition theorem, without the need for p to be the solution of any partial differential problem.

Our results refer to more general domains Ω_ε than the one described by (1.2) (see the beginning of Section 3 for the precise assumptions) which include thin domains with rough boundaries. Besides, more generally than the Hilbert framework $p_\varepsilon \in L^2(\Omega_\varepsilon)$ we just consider $p_\varepsilon \in L^q(\Omega_\varepsilon)$ with $1 < q < \infty$. A similar result was obtained in [11] in a simple framework.

A well known important application of (1.1) is that it provides an easy proof of Korn inequality in elasticity. In this way, we use the previous decomposition result (1.6), (1.7) to get a decomposition for a sequence of elastic deformations $u_\varepsilon \in W^{1,q}(\Omega_\varepsilon)^N$. For Ω_ε defined by (1.2) (the result holds for more general domains) the decomposition reads as

$$u_\varepsilon(x) = \phi_\varepsilon(x) + \begin{pmatrix} -D_{x'}u_\varepsilon^0(x')^t \dfrac{x''}{\varepsilon} \\ \dfrac{1}{\varepsilon}u_\varepsilon^0(x') + Z_\varepsilon(x')\dfrac{x''}{\varepsilon} \end{pmatrix} + v_\varepsilon(x), \quad \text{a.e. } x \in \Omega_\varepsilon, \tag{1.8}$$

where ϕ_ε is a rigid displacement, i.e. $\phi_\varepsilon(x) = a_\varepsilon + A_\varepsilon x$ with $a_\varepsilon \in \mathbb{R}^N$, $A_\varepsilon \in \mathbb{R}^{N^2}$ skew-symmetric, v_ε belongs to $W^{1,q}(\Omega_\varepsilon)^N$, Z_ε belongs to $W^{1,q}(\omega)^{(N-k)^2}$ and is skew-symmetric and the sequence u_ε^0 belongs to the space of smoother functions $W^{2,q}(\omega)^{N-k}$. Moreover, we have

$$\|u_\varepsilon^0\|_{W^{2,q}(\Omega_\varepsilon)^{N-k}} + \|Z_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon)^{(N-k)^2}} + \|v_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon)^N} \leq C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},$$

where as usual, $e(u_\varepsilon)$ denotes the symmetric part of the derivative matrix of u_ε, and where the constant C does not depend on u_ε or ε.

In the case $N = 3$, $k = 1, 2$ some similar decompositions to (1.8) have been proved in [6] and [18] but in these results all the terms are in the space $W^{1,q}(\Omega_\varepsilon)$. Here the main novelty is the stronger regularity $u_\varepsilon^0 \in W^{2,q}(\omega)^{N-k}$. This is related to the fact that the classical models for elastic beams and plates correspond to fourth order equations whose solutions are in $W^{2,q}(\omega)$.

The main interest of the results stated above is that they allow us to simplify the study of the asymptotic behavior of continuum mechanics problems posed in thin domains which can have a rough boundary. The presence of terms with a stronger regularity in the decompositions (1.6) and (1.8) provides an additional compactness which allow us to easily pass to the limit in the product of some terms which other would only converge in a weak topology. As an application we consider in Sections 5 and 6 of the paper the following two examples:

In the first example we study the asymptotic behavior of a viscous fluid in the rough domain

$$\Omega_\varepsilon = \left\{(x', x_3) \in \omega \times \mathbb{R} : \ \varepsilon\Psi_b\Big(\frac{x'}{\varepsilon}\Big) < x_3 < \varepsilon\Psi_t\Big(\frac{x'}{\varepsilon}\Big)\right\},$$

where ω is a smooth bounded open set in $\mathbb{R}^2$ and Ψ_b, Ψ_t are periodic Lispchtiz functions.

In the second example we consider an anisotropic non-homogeneous elastic beam whose geometry is more general than (1.2) with $k = 1$. For this example our results extend the ones obtained in [20] (see also [9], [12], [16], [25]).

2 Some notations and definitions

- We denote by $\{e_1, ..., e_N\}$ the usual basis in $\mathbb{R}^N$.
- For $k, N \in \mathbb{N}$, $N \geq 2$, $k < N$, we decompose the elements $x \in \mathbb{R}^N$ as $x = (x', x'')$ with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{N-k}$. By x' and x'' we also denote generic points in $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ respectively. Confusions are avoided by the context.
- We define $\pi' : \mathbb{R}^N \longrightarrow \mathbb{R}^k$, $\pi'' : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-k}$ as the projections of $\mathbb{R}^N$ into $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ respectively, i.e.
$$\pi'(x) = x', \quad \pi''(x) = x'', \quad \forall\, x = (x', x'') \in \mathbb{R}^N.$$
- By $B_d(\eta, r)$ and $\overline{B}_d(\eta, r)$ we denote the open and closed ball of $\mathbb{R}^d$ of center $\eta \in \mathbb{R}^d$ and radius $r > 0$.
- Given $h > 0$, $\theta \in (0, \pi/2)$ and a unitary vector $\xi' \in \mathbb{R}^k$, we denote by $\mathcal{C}(\xi', h, \theta)$ the open cone of $\mathbb{R}^k$ with vertex at the origin, angle 2θ, height h, and axis in the direction of ξ', i.e.
$$\mathcal{C}(\xi', h, \theta) = \left\{ z' \in \mathbb{R}^k \,:\, |z'| \cos\theta < z' \cdot \xi' < h \right\}.$$
- For $m \in \mathbb{N}$, we denote $\mathbb{R}^{m^2}_s$ and $\mathbb{R}^{m^2}_a$ the space of symmetric and skew-symmetric (or antisymmetric) $m \times m$ matrices. We also use the indexes s and a for spaces of functions with values in the space of symmetric or skew-symmetric matrices respectively. For example $L^p(\Omega)^{m^2}_s$ and $L^p(\Omega)^{m^2}_a$ denote the spaces $L^p(\Omega; \mathbb{R}^{m^2}_s)$ and $L^p(\Omega; \mathbb{R}^{m^2}_a)$ respectively.
- Given a mesurable set $A \subset \mathbb{R}^d$, we denote its d-dimensional measure by $|A|_d$. When the value of d is clear by the context, we simply write $|A|$. We denote $⨍_A v\, dx$ the mean value $\frac{1}{|A|} \int_A v dx$.
- For $u \in W^{1,q}(\Theta)^m$ with $\Theta \subset \mathbb{R}^m$ open, $m \geq 1$, we denote by Du the derivative of u, and by $e(u)$ and $a(u)$ the symmetric and skew-symmetric parts of Du, namely
$$e(u) = \frac{1}{2}\left(Du + Du^T\right), \qquad a(u) = \frac{1}{2}\left(Du - Du^T\right).$$
- $C > 0$ denotes a generic constant which can change from line to line.
- O_ε denotes a generic sequence depending on the positive parameter ε which tends to zero when ε goes to zero. This sequence can change from line to line.

3 Decomposition results for the pressure and the elastic deformation in thin domains

The goal of the present section is to state the main results of the paper which, as announced in the introduction, are concerned with a decomposition result for the pressure of a fluid and for the elastic deformation of a solid body in a domain Ω_ε of thickness of (small) order ε. The proof of these results will be carried out in Section 4.

We start by stating the precise assumptions that we impose to the sequence of thin domains.

For $\varepsilon > 0$ we denote by Ω_ε an open set of $\mathbb{R}^N$ and we define ω_ε and $\widehat{B}(x', \varepsilon)$ by

$$\omega_\varepsilon = \pi'(\Omega_\varepsilon), \tag{3.1}$$

$$\widehat{B}(x',\varepsilon)=\left\{z\in\Omega_\varepsilon\,:\,z'\in B_k(x',\varepsilon)\right\}\ \Big(=\Omega_\varepsilon\cap\big(B_k(x',\varepsilon)\times\mathbb{R}^{N-k}\big)\Big),\quad\forall\,x'\in\overline{\omega_\varepsilon}.\tag{3.2}$$

We assume that

i) There exist $r,R>0$ such that for every $\varepsilon>0$, we have

$$\omega_\varepsilon\times B_{N-k}(0,r\varepsilon)\subset\Omega_\varepsilon\subset\omega_\varepsilon\times B_{N-k}(0,R\varepsilon).\tag{3.3}$$

ii) There exist $h>0$, $\theta\in(0,\pi/2)$ such that for every $\varepsilon>0$ and every $x'\in\overline{\omega_\varepsilon}$ there exists $\xi'_{x'}\in\mathbb{R}^k$, with $|\xi'_{x'}|=1$, which satisfies

$$z'+\mathcal{C}(\xi'_{x'},\varepsilon h,\theta)\subset\omega_\varepsilon,\quad\forall z'\in B_k(x',\varepsilon)\cap\overline{\omega_\varepsilon}.\tag{3.4}$$

iii) There exist $\Lambda>0$ and $q>1$ such that for every $\varepsilon>0$ and every $x'\in\overline{\omega_\varepsilon}$, we have

$$\left\|p_\varepsilon-⨍_{\widehat{B}(x',\varepsilon)}p_\varepsilon\,dz\right\|_{L^q(\widehat{B}(x',\varepsilon))}\le\Lambda\|\nabla p_\varepsilon\|_{W^{-1,q}(\widehat{B}(x',\varepsilon))^N},\quad\forall\,p_\varepsilon\in L^q(\widehat{B}(x',\varepsilon)).\tag{3.5}$$

Remark 3.1 *Taking into account that for a smooth, connected open set the norm in L^q of a function can be estimated by the norm of its gradient in $W^{-1,q}$ (see e.g. [17], [23], [24]) an elementary example of Ω_ε satisfying (3.3)-(3.5) is given by*

$$\Omega_\varepsilon=\omega\times(\varepsilon\vartheta),\quad\forall\varepsilon>0,\tag{3.6}$$

with $\omega\subset\mathbb{R}^k$, $\vartheta\subset\mathbb{R}^{N-k}$ connected Lipschitz open sets such that $0\in\vartheta$.

Assumptions (3.3), (3.4) and (3.5) given above hold true for domains which are much more general than (3.6). For example we can take ω and ϑ in (3.6) varying with ε. Specially, we are interested in the case of rough domains, some examples of which are given in Sections 5 and 6.

Our first result is the following theorem which provides a special decomposition for any function whose gradient is in $W^{-1,q}(\Omega_\varepsilon)^N$.

Theorem 3.2 *Let Ω_ε be a sequence of open sets in $\mathbb{R}^N$ (not necessarily bounded) satisfying (3.3), (3.4) and (3.5). Then there exists $C>0$ such that for every $\varepsilon>0$ and every $p_\varepsilon\in L^q(\Omega_\varepsilon)$ there exist $p^0_\varepsilon\in H^1(\omega_\varepsilon)$ and $p^1_\varepsilon\in L^q(\Omega_\varepsilon)$ which satisfy*

$$p_\varepsilon(x)=\frac{1}{\varepsilon}p^0_\varepsilon(x')+p^1_\varepsilon(x)\quad a.e.\ x\in\Omega_\varepsilon,\tag{3.7}$$

$$\varepsilon^{\frac{N-k}{q}}\|\nabla p^0_\varepsilon\|_{L^q(\omega_\varepsilon)^k}+\|p^1_\varepsilon\|_{L^q(\Omega_\varepsilon)}\le C\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}.\tag{3.8}$$

Moreover, p^0_ε and p^1_ε can be chosen linearly dependent on p_ε.

Remark 3.3 *Since p^0_ε does not depend on x'' and (3.3) is satisfied, we can also write (3.8) as*

$$\|\nabla p^0_\varepsilon\|_{L^q(\Omega_\varepsilon)^N}+\|p^1_\varepsilon\|_{L^q(\Omega_\varepsilon)}\le C\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}.\tag{3.9}$$

The function p^0_ε which we obtain in the proof of Theorem 3.2 is not only in $H^1(\omega_\varepsilon)$ but in $C^\infty(\omega_\varepsilon)$. However, we can only estimate the norm of p^0_ε in $H^1(\omega_\varepsilon)$.

Inequality (3.8) only provides an estimate for ∇p^0_ε and not for p^0_ε. As usual (for ω_ε connected and bounded) we can also estimate p^0_ε in $L^q(\Omega_\varepsilon)$ if we take p^0_ε with null mean value, but for this purpose we need to assume that a Poincaré inequality holds in ω_ε. The corresponding result is given by

Corollary 3.4 *Let Ω_ε be a sequence of connected bounded open sets in $\mathbb{R}^N$ satisfying (3.3), (3.4) and (3.5). Let us also assume that there exists $C>0$ such that*

$$\left\| w_\varepsilon - \fint_{\omega_\varepsilon} w_\varepsilon \, dz' \right\|_{L^q(\omega_\varepsilon)} \leq C \|\nabla w_\varepsilon\|_{L^q(\omega_\varepsilon)^k}, \quad \forall w_\varepsilon \in W^{1,q}(\omega_\varepsilon), \quad \forall \varepsilon > 0. \tag{3.10}$$

Then, there exists another constant $C>0$ such that for every $\varepsilon>0$ and every $p_\varepsilon \in L^q(\Omega_\varepsilon)$ there exist $p_\varepsilon^0 \in H^1(\omega_\varepsilon)$ and $p_\varepsilon^1 \in L^q(\Omega_\varepsilon)$ which satisfy

$$p_\varepsilon(x) = \fint_{\Omega_\varepsilon} p_\varepsilon \, dz + \frac{1}{\varepsilon} p_\varepsilon^0(x') + p_\varepsilon^1(x) \qquad a.e.\ x \in \Omega_\varepsilon, \tag{3.11}$$

$$\varepsilon^{\frac{N-k}{q}} \|p_\varepsilon^0\|_{W^{1,q}(\omega_\varepsilon)} + \|p_\varepsilon^1\|_{L^q(\Omega_\varepsilon)} \leq C \|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}. \tag{3.12}$$

Remark 3.5 *Corollary 3.4 proves in particular the existence of $C>0$ such that*

$$\left\| p_\varepsilon - \fint_{\Omega_\varepsilon} p_\varepsilon \, dx \right\|_{L^q(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon} \|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}, \quad \forall p_\varepsilon \in L^q(\Omega_\varepsilon), \ \forall \varepsilon > 0. \tag{3.13}$$

In fluid mechanics, this is the classical estimate for the pressure when one deals with a thin domain of thickness ε. However, Corollary 3.4 provides a more precise information. The term $1/\varepsilon p_\varepsilon^0$ which appears in (3.11) is estimated not only in $L^q(\Omega_\varepsilon)$ but in $W^{1,q}(\Omega_\varepsilon)$. Moreover, it only depends on x'. To this term we need to add p_ε^1 which is just in $L^q(\Omega_\varepsilon)$ but it is of order one.

It is well known that the estimate of $\|p_\varepsilon\|_{L^q(\Omega_\varepsilon)}$ in terms of $\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}$ can be used to prove Korn's inequality. In this sense, we can use Theorem 3.2 to prove

Theorem 3.6 *Let Ω_ε be a sequence of connected open bounded sets in $\mathbb{R}^N$ satisfying (3.3), (3.4) and (3.5). Let us also assume that there exists $C>0$ such that*

$$\left\| w_\varepsilon - \fint_{\omega_\varepsilon} w_\varepsilon \, dz' \right\|_{L^q(\omega_\varepsilon)} \leq C \|\nabla w_\varepsilon\|_{W^{-1,q}(\omega_\varepsilon)^k}, \quad \forall w_\varepsilon \in L^q(\omega_\varepsilon), \quad \forall \varepsilon > 0. \tag{3.14}$$

Then there exists another constant $C>0$ such that for every $\varepsilon>0$ and every $u_\varepsilon \in W^{1,q}(\Omega_\varepsilon)^N$ there exist $a_\varepsilon \in \mathbb{R}^N$, $A_\varepsilon \in \mathbb{R}^{N^2}_a$, $u_\varepsilon^0 \in W^{2,q}(\omega_\varepsilon)^{N-k}$, $v_\varepsilon \in W^{1,q}(\Omega_\varepsilon)^N$, and $Z_\varepsilon \in W^{1,q}(\omega_\varepsilon)_a^{(N-k)^2}$ which satisfy

$$u_\varepsilon(x) = a_\varepsilon + A_\varepsilon x + \begin{pmatrix} -D_{x'} u_\varepsilon^0(x')^t \dfrac{x''}{\varepsilon} \\ \dfrac{1}{\varepsilon} u_\varepsilon^0(x') + Z_\varepsilon(x') \dfrac{x''}{\varepsilon} \end{pmatrix} + v_\varepsilon(x), \tag{3.15}$$

$$\varepsilon^{\frac{N-k}{q}} \|u_\varepsilon^0\|_{W^{2,q}(\omega_\varepsilon)^{N-k}} + \varepsilon^{\frac{N-k}{q}} \|Z_\varepsilon\|_{W^{1,q}(\omega_\varepsilon)_a^{(N-k)^2}} + \|v_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon)^N} \leq C \|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}. \tag{3.16}$$

Moreover, a_ε, A_ε, u_ε^0, v_ε, and Z_ε can be chosen linearly dependent on u_ε.

Remark 3.7 *Theorem 3.6 in particular proves the existence of $C>0$ such that*

$$\|u_\varepsilon - a_\varepsilon - A_\varepsilon x\|_{W^{1,q}(\Omega_\varepsilon)^N} \leq \frac{C}{\varepsilon} \|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}, \quad \forall\, u_\varepsilon \in W^{1,q}(\Omega_\varepsilon)^N, \ \forall\, \varepsilon > 0.$$

In linear elasticity problems, this is the estimate for the displacement when we deals with thin domains of thickness ε. However, Theorem 3.6 provides a more precise information. It shows *that the terms with derivatives of order $1/\varepsilon$ have a very particular structure, and that they correspond to a Bernouilli-Navier displacement and a torsion term given by*

$$\begin{pmatrix} -D_{x'}u^0_\varepsilon(x')^t\dfrac{x''}{\varepsilon} \\ \\ \dfrac{1}{\varepsilon}u^0_\varepsilon(x') \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \\ Z_\varepsilon(x')\dfrac{x''}{\varepsilon} \end{pmatrix},$$

respectively. Related results for $N = 3$ and $k = 1, 2$ have been obtained in [6] and [18]. The main difference is that our function u^0_ε belongs to the Sobolev space $W^{2,q}(\omega_\varepsilon)^{N-k}$, while the corresponding function in these references only belongs to $W^{1,q}(\omega_\varepsilon)^{N-k}$.

4 Proof of the decomposition results stated in Section 3

Let us now prove in this section the results stated in the previous one. We start with the proof of Theorem 3.2.

Proof of Theorem 3.2. The proof is divided into 5 steps. The first one is devoted to construct a suitable partition of the unity of $\overline{\omega_\varepsilon}$. This is necessary to deal with assumption (3.4) where the vertex and the axis of the cones vary on $\partial\omega_\varepsilon$. In the second step we define a certain mollifier sequence associated to the partition of the unity defined in Step 1. In the third step we use this mollifier sequence and the partition of the unity to define the functions p^0_ε and p^1_ε in the thesis of Theorem 3.2. In the fourth and fifth steps we prove estimate (3.8).

Before of starting the proof, we remark that in assumption (3.4) h can be chosen as small as we want. Thus, it is not restrictive to assume in the following that

$$h < \frac{1}{2}. \tag{4.1}$$

Step 1. Let us prove that there exists $m \in \mathbb{N}$ which only depends on k such that for every $\varepsilon > 0$ there exists a set of indices I_ε at most countable and points $x'_{\varepsilon,i} \in \overline{\omega_\varepsilon}$, $i \in I_\varepsilon$, satisfying that

$$\overline{\omega_\varepsilon} \subset \bigcup_{i\in I_\varepsilon} \overline{B}_k(x'_{\varepsilon,i}, \varepsilon/4), \tag{4.2}$$

and

$$\operatorname{card}\big(\,\{i \in I_\varepsilon\ :\ B_k(x'_{\varepsilon,i}, \varepsilon/2) \cap B_k(x'_{\varepsilon,l}, \varepsilon/2) \neq \emptyset\}\,\big) \le m, \quad \forall l \in I_\varepsilon, \tag{4.3}$$

i.e. each ball $B_k(x'_{\varepsilon,i}, \varepsilon/2)$ intersects at most m balls. Moreover, there exist $\phi^i_\varepsilon \in C^\infty(\overline{\omega_\varepsilon})$, $i \in I_\varepsilon$, such that

$$\operatorname{supp}(\phi^i_\varepsilon) \subset B_k(x'_{\varepsilon,i}, \varepsilon/2) \cap \overline{\omega_\varepsilon}, \qquad 0 \le \phi^i_\varepsilon \le 1,\ |\partial^\alpha_{x'}\phi^i_\varepsilon| \le \frac{C_\alpha}{\varepsilon^{|\alpha|}} \text{ in } \overline{\omega_\varepsilon},\ \forall\, \alpha \in \mathbb{N}^k,\ \ \forall\, i \in I_\varepsilon, \tag{4.4}$$

$$\sum_{i\in I_\varepsilon} \phi^i_\varepsilon = 1 \text{ in } \overline{\omega_\varepsilon}, \tag{4.5}$$

where C_α is a positive constant independent of i and ε, but depending on α.

For the proof, we apply Vitali's covering theorem to the closed balls $\overline{B}_k(x', \varepsilon/20)$, with $x' \in \overline{\omega_\varepsilon}$. It gives the existence of a set of indices I_ε at most countable and points $x'_{\varepsilon,i} \in \overline{\omega_\varepsilon}$, $i \in I_\varepsilon$, satisfying (4.2), and such that the balls $\overline{B}_k(x'_{\varepsilon,i}, \varepsilon/20)$, $i \in I_\varepsilon$, are disjoint. In particular, this proves the existence of m depending just on the dimension k such that (4.3) holds.

Now, for $\psi \in C_c^\infty(\mathbb{R}^k)$, $\psi \geq 0$, $\mathrm{supp}(\psi) \subset B_k(0,1/2)$, and $\psi > 0$ in $\overline{B}_k(0,1/4)$, we define

$$\psi_\varepsilon^l(x') = \psi\left(\frac{x' - x'_{\varepsilon,l}}{\varepsilon}\right), \quad \phi_\varepsilon^l(x') = \frac{\psi_\varepsilon^l(x')}{\displaystyle\sum_{i\in I_\varepsilon} \psi_\varepsilon^i(x')}, \qquad \forall x' \in \overline{\omega_\varepsilon}, \quad \forall l \in I_\varepsilon. \tag{4.6}$$

Then, it is easy to check that (4.4)-(4.5) hold true.

For the following, we observe that (4.5) also implies

$$\sum_{i\in I_\varepsilon} \partial_j \phi_\varepsilon^i = 0 \ \text{ in } \overline{\omega_\varepsilon}, \quad \forall j \in \{1, \cdots, k\}. \tag{4.7}$$

Step 2. Associated to the points $x'_{\varepsilon,i}$, $i \in I_\varepsilon$, of *Step 1*, we denote

$$\mathcal{C}_\varepsilon^i = \mathcal{C}(\xi'_{x'_{\varepsilon,i}}, \varepsilon h, \theta) \times B_{N-k}(0, r\varepsilon), \quad \forall i \in I_\varepsilon \quad \varepsilon > 0, \tag{4.8}$$

with $\xi_{x'_{\varepsilon,i}}$, h and θ given by (3.4).

Let us prove that for every $\varepsilon > 0$ and every $i \in I_\varepsilon$, there exists ρ_ε^i in $C_c^\infty(\mathbb{R}^N)$ satisfying

$$\mathrm{supp}(\rho_\varepsilon^i) \subset -\mathcal{C}_\varepsilon^i, \ \int_{\mathbb{R}^N} \rho_\varepsilon^i \, dx = 1, \ \int_{\mathbb{R}^N} x\, \rho_\varepsilon^i \, dx = 0, \ \|\rho_\varepsilon^i\|_{L^\infty(\mathbb{R}^N)} + \varepsilon \|\nabla \rho_\varepsilon^i\|_{L^\infty(\mathbb{R}^N)^N} \leq \frac{C}{\varepsilon^N}, \tag{4.9}$$

where C does not depend on ε or i.

In order to prove this property, let us first show that there exists $\rho \in C_c^\infty(B_N(e_1, \sin\theta))$ such that

$$\int_{\mathbb{R}^N} \rho \, dy = 1, \quad \int_{\mathbb{R}^N} y\, \rho \, dy = 0. \tag{4.10}$$

Once this is proved, taking $P_\varepsilon^i : \mathbb{R}^N \to \mathbb{R}^N$ a rotation such that $P_\varepsilon^i(\xi_{x'_{\varepsilon,i}}) = -e_1$, we get that

$$\rho_\varepsilon^i(x) = \frac{2^N}{(h\varepsilon)^N} \rho\left(P_\varepsilon^i\left(\frac{2x}{h\varepsilon}\right)\right),$$

satisfies (4.9). To show the existence of a such $\rho \in C_c^\infty(B_N(e_1, \sin\theta))$ satisfying (4.10), we define $L : C_c^\infty(B_N(e_1, \sin\theta)) \to \mathbb{R}^{1+N}$ by

$$L\psi = \left(\int_{\mathbb{R}^N} \psi \, dy, \int_{\mathbb{R}^N} y\, \psi \, dy\right), \quad \forall \psi \in C_c^\infty(B_N(e_1, \sin\theta)).$$

Our aim is to show that L is surjective. For this purpose, we take $(a_0, a) \in Range(L)^\perp$, then

$$\int_{\mathbb{R}^N} (a_0 + a \cdot y)\, \psi \, dy = 0, \quad \forall \psi \in C_c^\infty(B_N(e_1, \sin\theta)),$$

which shows that $a_0 + a \cdot y$ vanishes in $B_N(e_1, \sin\theta)$ and then that $(a_0, a) = (0, 0)$. This proves that the orthogonal of Range(L) is the null space. Since Range(L) is closed because it is of finite dimension we get L surjective.

Step 3. For every $\varepsilon > 0$ and $i \in I_\varepsilon$, we denote

$$B_\varepsilon^i = \omega_\varepsilon \cap B_k(x'_{\varepsilon,i}, \varepsilon) \subset \mathbb{R}^k, \quad E_\varepsilon^i = \omega_\varepsilon \cap B_k\big(x'_{\varepsilon,i}, \frac{\varepsilon}{2}\big) \subset \mathbb{R}^k$$

$$\hat{B}_\varepsilon^i = \big\{x \in \Omega_\varepsilon \, : \, x' \in B_k(x'_{\varepsilon,i}, \varepsilon)\big\} \subset \mathbb{R}^N, \quad \hat{E}_\varepsilon^i = \Big\{x \in \Omega_\varepsilon \, : \, x' \in B_k\big(x'_{\varepsilon,i}, \frac{\varepsilon}{2}\big)\Big\} \subset \mathbb{R}^N,$$

and we take $\phi_\varepsilon^i \in C^\infty(\overline{\omega_\varepsilon})$ and $\rho_\varepsilon^i \in C_c^\infty(\mathbb{R}^N)$ as the functions given in *Steps 1* and *2* respectively. Then, for p_ε in $L^q(\Omega_\varepsilon)$, with $q>1$, we define $p_\varepsilon^0 \in C^\infty(\omega_\varepsilon)$ and $p_\varepsilon^1 \in L^q(\Omega_\varepsilon)$ by

$$p_\varepsilon^0(x') = \varepsilon \sum_{i\in I_\varepsilon} \phi_\varepsilon^i(x')(\rho_\varepsilon^i * p_\varepsilon)(x',0), \quad \forall\, x' \in \overline{\omega_\varepsilon}, \tag{4.11}$$

$$p_\varepsilon^1(x) = p_\varepsilon(x) - \frac{1}{\varepsilon} p_\varepsilon^0(x'), \quad \forall\, x = (x',x'') \in \Omega_\varepsilon, \tag{4.12}$$

where as usual

$$(\rho_\varepsilon^i * p_\varepsilon)(x',0) = \int_{\Omega_\varepsilon} p_\varepsilon(y)\rho_\varepsilon^i(x'-y',-y'')\,dy, \quad \forall\, x' \in \overline{\omega_\varepsilon},\ \forall\, i \in I_\varepsilon. \tag{4.13}$$

Observe that p_ε^0 and p_ε^1 depend linearly on p_ε. Our aim is to prove that these functions satisfy the results stated in Theorem 3.2. Equality (3.7) is evident. Thus we just need to prove estimate (3.8).

We define η_ε as the unique solution of

$$-\mathrm{div}\,(|D\eta_\varepsilon|^{q'-2}D\eta_\varepsilon) = \nabla p_\varepsilon \ \text{ in } \Omega_\varepsilon, \quad \eta_\varepsilon \in W_0^{1,q'}(\Omega_\varepsilon)^N. \tag{4.14}$$

The existence of η_ε follows from (3.3), which implies Poincaré's inequality

$$\|v\|_{L^{q'}(\Omega_\varepsilon)^N} \le C\varepsilon \|Dv\|_{L^{q'}(\Omega_\varepsilon)^{N^2}}, \quad \forall\, v \in W_0^{1,q'}(\Omega_\varepsilon)^N.$$

The introduction of η_ε is necessary to estimate the norm of ∇p_ε in the negative Sobolev space $W^{-1,q}(\Omega_\varepsilon)$. Observe that η_ε satisfies:

$$\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N} = \|D\eta_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)^{N^2}}^{q'-1}, \tag{4.15}$$

$$\|\nabla p_\varepsilon\|_{W^{-1,q}(O)^N} \le \|D\eta_\varepsilon\|_{L^{q'}(O)^{N^2}}^{q'-1}, \quad \forall O \subset \Omega_\varepsilon \text{ open}. \tag{4.16}$$

Thus, to show (3.8) (see also (3.9)) it is equivalent to prove the existence of $C>0$ such that

$$\|p_\varepsilon^1\|_{L^q(\Omega_\varepsilon)} \le C\|D\eta_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)^{N^2}}^{q'-1}, \tag{4.17}$$

$$\|\nabla p_\varepsilon^0\|_{L^q(\omega_\varepsilon)^N} \le C\|D\eta_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)^{N^2}}^{q'-1}. \tag{4.18}$$

These estimates are proved in the remaining two steps.

Step 4. Proof of (4.17): For every $\varepsilon>0$ and $i\in I_\varepsilon$, we set

$$m_\varepsilon^i = \fint_{\hat{B}_\varepsilon^i} p_\varepsilon\,dz.$$

Thanks to (3.4), (4.5), (4.9), (4.11) and (4.12), we can write

$$p_\varepsilon^1(x) = \sum_{i\in I_\varepsilon}\phi_\varepsilon^i(x')(p_\varepsilon(x)-m_\varepsilon^i) + \sum_{i\in I_\varepsilon}\phi_\varepsilon^i(x')(\rho_\varepsilon^i*(m_\varepsilon^i - p_\varepsilon))(x',0), \quad \text{a.e. } x\in\Omega_\varepsilon. \tag{4.19}$$

By (4.4) and (4.5), the first term on the right hand side is a convex combination of the terms $p_\varepsilon - m_\varepsilon^i$, with $i\in I_\varepsilon$. Using also (3.5), (4.16), (4.4) and (4.3), we have

$$\begin{aligned}\int_{\Omega_\varepsilon}\left|\sum_{i\in I_\varepsilon}\phi_\varepsilon^i(x')(p_\varepsilon(x)-m_\varepsilon^i)\right|^q dx &\le \sum_{i\in I_\varepsilon}\int_{\hat{E}_\varepsilon^i}|p_\varepsilon(x)-m_\varepsilon^i|^q dx \\ \le C\sum_{i\in I_\varepsilon}\|\nabla p_\varepsilon\|_{W^{-1,q}(\hat{B}_\varepsilon^i)^N}^q &\le C\sum_{i\in I_\varepsilon}\|D\eta_\varepsilon\|_{L^{q'}(\hat{B}_\varepsilon^i)^{N^2}}^{q'} \le C\|D\eta_\varepsilon\|_{L^{q'}(\Omega_\varepsilon)^{N^2}}^{q'}.\end{aligned} \tag{4.20}$$

For the second term on the right hand side of (4.19), a similar reasoning using convexity, (4.4), (3.3), (3.5), (4.9), (4.16) and (4.3), provides

$$\begin{aligned}&\int_{\Omega_\varepsilon}\Big|\sum_{i\in I_\varepsilon}\phi^i_\varepsilon(x')(\rho^i_\varepsilon*(m^i_\varepsilon-p_\varepsilon))(x',0)\Big|^q dx\le\sum_{i\in I_\varepsilon}\int_{\hat E^i_\varepsilon}\big|(\rho^i_\varepsilon*(m^i_\varepsilon-p_\varepsilon))(x',0)\big|^q dx\\&\le\sum_{i\in I_\varepsilon}|\hat E^i_\varepsilon|\,\|(\rho^i_\varepsilon*(m^i_\varepsilon-p_\varepsilon))(\cdot,0)\|^q_{L^\infty(E^i_\varepsilon)}\le C\varepsilon^N\sum_{i\in I_\varepsilon}\|\rho^i_\varepsilon\|^q_{L^{q'}(-\mathcal{C}^i_\varepsilon)}\|m^i_\varepsilon-p_\varepsilon\|^q_{L^q(\hat B^i_\varepsilon)}\\&\le C\sum_{i\in I_\varepsilon}\|\nabla p_\varepsilon\|^q_{W^{-1,q}(\hat B^i_\varepsilon)^N}\le C\|D\eta_\varepsilon\|^{q'}_{L^{q'}(\Omega_\varepsilon)^{N^2}}.\end{aligned}\tag{4.21}$$

From (4.19), (4.20) and (4.21) we get (4.17).

Step 5 . Proof of (4.18): By (4.5), (4.7), (4.9) and (4.11) we have

$$\begin{aligned}\nabla_{x'}p^0_\varepsilon(x')&=\varepsilon\sum_{i\in I_\varepsilon}\nabla_{x'}\phi^i_\varepsilon(x')(\rho^i_\varepsilon*(p_\varepsilon-m^i_\varepsilon))(x',0)+\varepsilon\sum_{i\in I_\varepsilon}\nabla_{x'}\phi^i_\varepsilon(x')\left(m^i_\varepsilon-p_\varepsilon(x)\right)\\&+\varepsilon\sum_{i\in I_\varepsilon}\phi^i_\varepsilon(x')\left(\nabla_{x'}\rho^i_\varepsilon*p_\varepsilon\right)(x',0),\quad\text{a.e. } x\in\Omega_\varepsilon.\end{aligned}\tag{4.22}$$

To estimate the first term on the right hand side we reason similarly to (4.21) using Hölder's inequality combined with (4.3), instead of convexity. We get

$$\begin{aligned}&\int_{\Omega_\varepsilon}\Big|\sum_{i\in I_\varepsilon}\nabla_{x'}\phi^i_\varepsilon(x')(\rho^i_\varepsilon*(p_\varepsilon-m^i_\varepsilon))(x',0)\Big|^q dx\\&\le\int_{\Omega_\varepsilon}\Big(\sum_{i\in I_\varepsilon}|\nabla_{x'}\phi^i_\varepsilon(x')|^{q'}\Big)^{\frac{q}{q'}}\sum_{i\in I_\varepsilon}\chi_{\hat E^i_\varepsilon}(x)|(\rho^i_\varepsilon*(p_\varepsilon-m^i_\varepsilon))(x',0)|^q dx\\&\le\frac{C}{\varepsilon^q}\sum_{i\in I_\varepsilon}\int_{\hat E^i_\varepsilon}|(\rho^i_\varepsilon*(p_\varepsilon-m^i_\varepsilon))(x',0)|^q dx\le\frac{C}{\varepsilon^q}\|D\eta_\varepsilon\|^{q'}_{L^{q'}(\Omega_\varepsilon)^{N^2}}.\end{aligned}\tag{4.23}$$

For the second term on the right hand side of (4.22), we use again Hölder's inequality combined with (4.4), (4.3), (3.5) and (4.16), which provides

$$\int_{\Omega_\varepsilon}\Big|\sum_{i\in I_\varepsilon}\nabla_{x'}\phi^i_\varepsilon(x')(p_\varepsilon(x)-m^i_\varepsilon)\Big|^q dx\le\frac{C}{\varepsilon^q}\sum_{i\in I_\varepsilon}\int_{\hat E^i_\varepsilon}|p_\varepsilon(x)-m^i_\varepsilon|^q dx\le\frac{C}{\varepsilon^q}\|D\eta_\varepsilon\|^{q'}_{L^{q'}(\Omega_\varepsilon)^{N^2}}.\tag{4.24}$$

For the last term in (4.22), we define for every $x'\in B_k(x'_{\varepsilon,i},\varepsilon/2)$, the function h^i_ε by $h^i_\varepsilon(y)=\rho^i_\varepsilon(x'-y',-y'')$. By (4.9), (3.4) and (4.1), it has support contained in $(x',0)+\mathcal{C}^i_\varepsilon\subset\hat B^i_\varepsilon$ and satisfies

$$\|h^i_\varepsilon\|_{W^{1,q'}_0(\hat B^i_\varepsilon)}\le\|\nabla\rho^i_\varepsilon\|_{L^\infty(\mathbb{R}^N)^N}|\hat B^i_\varepsilon|^{1/q'}\le\frac{C}{\varepsilon^{\frac{N}{q}+1}}.$$

Using this function, for every $j\in\{1,\cdots,k\}$, we have

$$\begin{aligned}&\left|(\partial_{x_j}\rho^i_\varepsilon*p_\varepsilon)(x',0)\right|=\left|\int_{\Omega_\varepsilon}\partial_{x_j}\rho^i_\varepsilon(x'-y',-y'')p_\varepsilon(y)dy\right|=\left|\int_{(x',0)+\mathcal{C}^i_\varepsilon}\partial_{x_j}\rho^i_\varepsilon(x'-y',-y'')p_\varepsilon(y)dy\right|\\&=\left|-\int_{(x',0)+\mathcal{C}^i_\varepsilon}\partial_{y_j}h^i_\varepsilon p_\varepsilon\,dy\right|=\left|\langle\partial_{y_j}p_\varepsilon,h^i_\varepsilon\rangle_{W^{-1,q}(\hat B^i_\varepsilon),W^{1,q'}_0(\hat B^i_\varepsilon)}\right|\le\frac{C}{\varepsilon^{\frac{N}{q}+1}}\|\nabla p_\varepsilon\|_{W^{-1,q}(\hat B^i_\varepsilon)^N}.\end{aligned}$$

From this estimate, using convexity, (4.3) and (4.16), we deduce

$$\int_{\Omega_\varepsilon}\left|\sum_{i\in I_\varepsilon}\phi^i_\varepsilon(x')(\nabla_{x'}\rho^i_\varepsilon * p_\varepsilon)(x',0)\right|^q dx \le \frac{C}{\varepsilon^{N+q}}\sum_{i\in I_\varepsilon}|\hat{E}^i_\varepsilon|\|D\eta_\varepsilon\|^{q'}_{L^{q'}(\hat{B}^i_\varepsilon)^{N^2}} \le \frac{C}{\varepsilon^q}\|D\eta_\varepsilon\|^{q'}_{L^{q'}(\Omega_\varepsilon)^{N^2}}.$$

This inequality combined with (4.22), (4.23) and (4.24) proves (4.18).

□

Proof of Corollary 3.4. For every $\varepsilon > 0$ and p_ε in $L^q(\Omega_\varepsilon)$, we consider the functions $p^0_\varepsilon \in C^\infty(\omega_\varepsilon)$ and $p^1_\varepsilon \in L^q(\Omega_\varepsilon)$ given by Theorem 3.2. We define $\tilde{p}^0_\varepsilon \in C^\infty(\omega_\varepsilon)$, $\tilde{p}^1_\varepsilon \in L^q(\Omega_\varepsilon)$ by

$$\tilde{p}^0_\varepsilon(x') = p^0_\varepsilon(x') - \fint_{\Omega_\varepsilon} p^0_\varepsilon dz, \quad \text{a.e. } x' \in \omega_\varepsilon; \qquad \tilde{p}^1_\varepsilon(x) = p^1_\varepsilon(x) - \fint_{\Omega_\varepsilon} p^1_\varepsilon dz, \quad \text{a.e. } x \in \Omega_\varepsilon.$$

Thanks to (3.8), we easily deduce that

$$\varepsilon^{\frac{N-k}{q}}\|\nabla\tilde{p}^0_\varepsilon\|_{L^q(\omega_\varepsilon)^k} + \|\tilde{p}^1_\varepsilon\|_{L^q(\Omega_\varepsilon)} \le C\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega_\varepsilon)^N}.$$

On the other hand, thanks to (3.3), Holder's inequality, (3.10) and (3.8), we get

$$\begin{aligned}
\varepsilon^{N-k}\|\tilde{p}^0_\varepsilon\|^q_{L^q(\omega_\varepsilon)} &\le C\int_{B_{N-k}(0;r\varepsilon)}\int_{\omega_\varepsilon}\left|p^0_\varepsilon - \fint_{\Omega_\varepsilon} p^0_\varepsilon dz\right|^q dx'dx'' \le C\int_{\Omega_\varepsilon}\left|p^0_\varepsilon - \fint_{\Omega_\varepsilon} p^0_\varepsilon dz\right|^q dx \\
&\le C\int_{\Omega_\varepsilon}\left|p^0_\varepsilon - \fint_{\omega_\varepsilon} p^0_\varepsilon d\zeta'\right|^q dx + C\int_{\Omega_\varepsilon}\left|\fint_{\Omega_\varepsilon} p^0_\varepsilon dz - \fint_{\omega_\varepsilon} p^0_\varepsilon d\zeta'\right|^q dx \\
&\le C\int_{\Omega_\varepsilon}\left|p^0_\varepsilon - \fint_{\omega_\varepsilon} p^0_\varepsilon d\zeta'\right|^q dx + C|\Omega_\varepsilon|\left|\fint_{\Omega_\varepsilon}\left(p^0_\varepsilon - \fint_{\omega_\varepsilon} p^0_\varepsilon d\zeta'\right)dz\right|^q \\
&\le C\int_{\Omega_\varepsilon}\left|p^0_\varepsilon - \fint_{\omega_\varepsilon} p^0_\varepsilon d\zeta'\right|^q dz \le C\varepsilon^{N-k}\|\nabla p^0_\varepsilon\|^q_{L^q(\omega_\varepsilon)^k} \le C\|\nabla p_\varepsilon\|^q_{W^{-1,q}(\Omega_\varepsilon)^N},
\end{aligned}$$

Using then that by (3.7), we have

$$p_\varepsilon = \fint_{\Omega_\varepsilon} p_\varepsilon dx + \frac{1}{\varepsilon}\tilde{p}^0_\varepsilon + \tilde{p}^1_\varepsilon \qquad \text{in } \Omega_\varepsilon,$$

it is enough to rename $\tilde{p}^0_\varepsilon$ as p^0_ε and $\tilde{p}^1_\varepsilon$ as p^1_ε to finish the proof. □

Let us now start with the proof of Theorem 3.6. We need some preliminary lemmas

Lemma 4.1 *Let $O \subset \mathbb{R}^N$ be a bounded open set and $q \in (1,\infty)$ such that there exists $C > 0$ satisfying*

$$\left\|u - \fint_O u\,dx\right\|_{L^q(O)} \le C\|\nabla u\|_{W^{-1,q}(O)^N}, \qquad \forall\, u \in L^q(O). \tag{4.25}$$

Then, we have

$$\left\|u - \fint_O u\,dx\right\|_{L^q(O)} \le C\,\mathrm{diam}(O)\|\nabla u\|_{L^q(O)^N}, \qquad \forall\, u \in W^{1,q}(O). \tag{4.26}$$

Proof. It is enough to use that for every $u \in W^{1,q}(O)$, we have

$$\|\nabla u\|_{W^{-1,q}(O)^N} = \sup_{\substack{\varphi\in W^{1,q'}_0(O)^N\\ \varphi\neq 0}} \frac{\displaystyle\int_O \nabla u\cdot\varphi\,dx}{\|\varphi\|_{W^{1,q'}_0(O)^N}} \le \sup_{\substack{\varphi\in W^{1,q'}_0(O)^N\\ \varphi\neq 0}} \frac{\|\nabla u\|_{L^q(O)^N}\|\varphi\|_{L^{q'}(O)^N}}{\|\varphi\|_{W^{1,q'}_0(O)^N}},$$

where thanks to Poincaré's inequality, we have

$$\|\varphi\|_{L^{q'}(O)^N}\le \mathrm{diam}(O)\|\varphi\|_{W_0^{1,q'}(O)^N},$$

and then

$$\|\nabla u\|_{W^{-1,q}(O)^N}\le \mathrm{diam}(O)\|\nabla u\|_{L^q(O)^N}.$$

Using this inequality in (4.25) proves (4.26). □

Lemma 4.2 *There exists $C>0$ depending only on N such that for every open set $O\subset\mathbb{R}^N$ and every $u\in L^q(O)^N$, with $e(u)\in L^q(O)^{N^2}$, we have*

$$\|D^2u\|_{W^{-1,q}(O)^{N^3}}\le C\|e(u)\|_{L^q(O)^{N^2}}. \tag{4.27}$$

Proof. It is a classical result which just follows from

$$\partial^2_{ij}u_r=\partial_i e(u)_{jr}+\partial_j e(u)_{ir}-\partial_r e(u)_{ij},\quad \forall i,j,r\in\{1,\dots,N\},\ \forall u\in W^{1,q}(O)^N.$$

□

A first decomposition result for a sequence u_ε in the conditions of Theorem 3.6 is given by the following lemma.

Lemma 4.3 *Let Ω_ε be a family of bounded domains in $\mathbb{R}^N$ which satisfy (3.3), (3.4), (3.5) and (3.14). Then, there exist $C>0$ such that for every $u_\varepsilon\in W^{1,q}(\Omega_\varepsilon)^N$, there exist $\hat u''_\varepsilon\in W^{2,q}(\omega_\varepsilon)^{N-k}$, $w_\varepsilon\in W^{1,q}(\Omega_\varepsilon)^N$, $B_\varepsilon\in\mathbb{R}^{k^2}_a$ and $S_\varepsilon\in W^{1,q}(\omega_\varepsilon)_a^{(N-k)^2}$ such that*

$$u_\varepsilon(x)=\begin{pmatrix} B_\varepsilon x'-D\hat u''_\varepsilon(x')^t\dfrac{x''}{\varepsilon}\\[2mm] \dfrac{1}{\varepsilon}\hat u''_\varepsilon(x')+S_\varepsilon(x')\dfrac{x''}{\varepsilon}\end{pmatrix}+w_\varepsilon(x),\quad a.e.\ x\in\Omega_\varepsilon, \tag{4.28}$$

$$\varepsilon^{\frac{N-k}{q}}\|D^2\hat u''_\varepsilon\|_{L^q(\omega_\varepsilon)^{(N-k)k^2}}+\varepsilon^{\frac{N-k}{q}}\|DS_\varepsilon\|_{L^q(\omega_\varepsilon)^{(N-k)^2k}}+\|Dw_\varepsilon\|_{L^q(\Omega_\varepsilon)^{N^2}}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}. \tag{4.29}$$

Proof. We divide the proof in seven steps.

Step 1. For u_ε in $W^{1,q}(\Omega_\varepsilon)^N$, we define $\hat u_\varepsilon\in C^\infty(\omega_\varepsilon)^N$ by (in the statement of Lemma 4.3 we just use the last $N-k$ component of this function)

$$\hat u_\varepsilon(x')=\varepsilon\sum_{i\in I_\varepsilon}\phi^i_\varepsilon(x')(\rho^i_\varepsilon*u_\varepsilon)(x',0),\quad \forall x'\in\omega_\varepsilon, \tag{4.30}$$

where ϕ^i_ε and ρ^i_ε, $i\in I_\varepsilon$, are given in *Steps 1* and *2* of the proof of Theorem 3.2, respectively. Let us prove the following estimates

$$\left\|\partial_i u_\varepsilon-\frac{1}{\varepsilon}\partial_i\hat u_\varepsilon\right\|_{L^q(\Omega_\varepsilon)^N}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i\in\{1,\dots,k\}, \tag{4.31}$$

$$\|\partial^2_{ij}\hat u_\varepsilon\|_{L^q(\Omega_\varepsilon)^N}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i,j\in\{1,\dots,k\}. \tag{4.32}$$

We start deriving $\hat u_\varepsilon$ with respect to x_i, $i\in\{1,\dots,k\}$. Taking into account $\mathrm{supp}(\rho^i_\varepsilon)\subset -\mathcal{C}^i_\varepsilon$, (4.8) and (3.4), we have

$$\partial_i\hat u_\varepsilon(x')=\varepsilon\sum_{l\in I_\varepsilon}\phi^l_\varepsilon(x')(\rho^l_\varepsilon*\partial_i u_\varepsilon)(x',0)+\varepsilon\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(x')(\rho^l_\varepsilon*u_\varepsilon)(x',0), \tag{4.33}$$

for every $x' \in \omega_\varepsilon$ and every $i \in \{1, \dots, k\}$. We observe that the first term in this decomposition agrees with the expression of p_ε^0 in (4.11) applied to $\partial_i u_\varepsilon$ (which is now a vectorial function) instead of p_ε. Then, thanks to (3.9) and (4.27), we have

$$\left\| \partial_i u_\varepsilon - \sum_{l \in I_\varepsilon} \phi_\varepsilon^l (\rho_\varepsilon^l * \partial_i u_\varepsilon)(\cdot, 0) \right\|_{L^q(\Omega_\varepsilon)^N} \leq C \| D \partial_i u_\varepsilon \|_{W^{-1,q}(\Omega_\varepsilon)^N} \leq C \| e(u_\varepsilon) \|_{L^q(\Omega_\varepsilon)^{N^2}}, \tag{4.34}$$

$$\left\| D_{x'} \Big(\sum_{l \in I_\varepsilon} \phi_\varepsilon^l (\rho_\varepsilon^l * \partial_i u_\varepsilon) \Big)(\cdot, 0) \right\|_{L^q(\Omega_\varepsilon)^N} \leq \frac{C}{\varepsilon} \| D \partial_i u_\varepsilon \|_{W^{-1,q}(\Omega_\varepsilon)^{Nk}} \leq \frac{C}{\varepsilon} \| e(u_\varepsilon) \|_{L^q(\Omega_\varepsilon)^{N^2}}, \tag{4.35}$$

for every $i \in \{1, \dots, k\}$. Thanks to (4.34), (4.35) and decomposition (4.33) of $\partial_i \hat{u}_\varepsilon$, in order to prove (4.31) and (4.32), it is enough to show

$$\left\| \sum_{l \in I_\varepsilon} \partial_i \phi_\varepsilon^l (\rho_\varepsilon^l * u_\varepsilon)(\cdot, 0) \right\|_{L^q(\Omega_\varepsilon)^N} \leq C \| e(u_\varepsilon) \|_{L^q(\Omega_\varepsilon)^{N^2}}, \quad \forall i \in \{1, \dots, k\}, \tag{4.36}$$

$$\left\| D_{x'} \Big(\sum_{l \in I_\varepsilon} \partial_i \phi_\varepsilon^l (\rho_\varepsilon^l * u_\varepsilon) \Big)(\cdot, 0) \right\|_{L^q(\Omega_\varepsilon)^{Nk}} \leq \frac{C}{\varepsilon} \| e(u_\varepsilon) \|_{L^q(\Omega_\varepsilon)^{N^2}}, \quad \forall i \in \{1, \dots, k\}. \tag{4.37}$$

The proof of these estimates is carried out in the following two steps.

Step 2. Proof of (4.36): We define φ_ε^l, $l \in I_\varepsilon$, $\varepsilon > 0$, as

$$\varphi_\varepsilon^l(x) = u_\varepsilon(x) - \fint_{\hat{B}_\varepsilon^l} u_\varepsilon dz - \fint_{\hat{B}_\varepsilon^l} D u_\varepsilon dz \left(x - \fint_{\hat{B}_\varepsilon^l} z dz \right), \quad \text{a.e. } x \in \Omega_\varepsilon. \tag{4.38}$$

An estimate for this function in $L^q(\hat{B}_\varepsilon^l)$ can be obtained as follows: Using that the integral of φ_ε^l vanishes in $\hat{B}_\varepsilon^l$, (3.5) and (4.26), we get

$$\int_{\hat{B}_\varepsilon^l} \left| \varphi_\varepsilon^l \right|^q dx \leq C \varepsilon^q \int_{\hat{B}_\varepsilon^l} \left| D \varphi_\varepsilon^l \right|^q dx = C \varepsilon^q \int_{\hat{B}_\varepsilon^l} \left| D u_\varepsilon - \fint_{\hat{B}_\varepsilon^l} D u_\varepsilon dz \right|^q dx,$$

where we observe that (3.5) and (4.27) also imply

$$\int_{\hat{B}_\varepsilon^l} \left| D u_\varepsilon - \fint_{\hat{B}_\varepsilon^l} D u_\varepsilon dz \right|^q dx \leq C \| e(u_\varepsilon) \|_{L^q(\hat{B}_\varepsilon^l)^{N^2}}^q, \quad \forall \varepsilon > 0, \ \forall l \in I_\varepsilon. \tag{4.39}$$

Thus,

$$\int_{\hat{B}_\varepsilon^l} \left| \varphi_\varepsilon^l \right|^q dx \leq C \varepsilon^q \| e(u_\varepsilon) \|_{L^q(\hat{B}_\varepsilon^l)^{N^2}}^q, \quad \forall \varepsilon > 0, \ \forall l \in I_\varepsilon. \tag{4.40}$$

Taking into account the second and third statements in (4.9), we have

$$\begin{aligned} (\rho_\varepsilon^l * u_\varepsilon)(x', 0) &= (\rho_\varepsilon^l * \varphi_\varepsilon^l)(x', 0) + \fint_{\hat{B}_\varepsilon^l} u_\varepsilon dz + \fint_{\hat{B}_\varepsilon^l} D_{x'} u_\varepsilon dz \, x' - \fint_{\hat{B}_\varepsilon^l} D u_\varepsilon dz \fint_{\hat{B}_\varepsilon^l} z dz \\ &= (\rho_\varepsilon^l * \varphi_\varepsilon^l)(x', 0) + \fint_{\hat{B}_\varepsilon^l} u_\varepsilon dz + \fint_{\hat{B}_\varepsilon^l} D u_\varepsilon dz \left(x - \fint_{\hat{B}_\varepsilon^l} z \, dz \right) - \fint_{\hat{B}_\varepsilon^l} D_{x''} u_\varepsilon dz \, x''. \end{aligned}$$

Then, using (4.7), we can write a.e. in Ω_ε

$$
\begin{aligned}
&\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(\rho^l_\varepsilon * u_\varepsilon)(x',0)=\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(\rho^l_\varepsilon * \varphi^l_\varepsilon)(x',0)\\
&+\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\fint_{\hat B^l_\varepsilon}u_\varepsilon dz+\fint_{\hat B^l_\varepsilon}Du_\varepsilon dz\Big(x-\fint_{\hat B^l_\varepsilon}z\,dz\Big)-u_\varepsilon\Big)\\
&-\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\fint_{\hat B^l_\varepsilon}D_{x''}u_\varepsilon dz-D_{x''}u_\varepsilon\Big)x''\\
&=\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(\rho^l_\varepsilon * \varphi^l_\varepsilon)(x',0)-\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(x')\varphi^l_\varepsilon(x)\\
&+\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(D_{x''}u_\varepsilon(x)-\fint_{\hat B^l_\varepsilon}D_{x''}u_\varepsilon dz\Big)x''.
\end{aligned}
\tag{4.41}
$$

In the first term of the right-hand side, using (4.3), (4.4), Holder's inequality, (4.40) and reasoning as in (4.21) we deduce

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\Big|\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(\rho^l_\varepsilon * \varphi^l_\varepsilon)(x',0)\Big|^q dx\le\frac{C}{\varepsilon^q}\sum_{l\in I_\varepsilon}\int_{\hat E^l_\varepsilon}\big|(\rho^l_\varepsilon * \varphi^l_\varepsilon)(x',0)\big|^q dx\\
&\le\frac{C}{\varepsilon^q}\sum_{l\in I_\varepsilon}|\hat E^l_\varepsilon|\,\|(\rho^l_\varepsilon * \varphi^l_\varepsilon)(\cdot,0)\|^q_{L^\infty(B^l_\varepsilon)}\le C\|e(u_\varepsilon)\|^q_{L^q(\Omega_\varepsilon)^{N^2}}.
\end{aligned}
\tag{4.42}
$$

A similar reasoning also provides the following estimate for the second term in the right-hand side of (4.41)

$$
\int_{\Omega_\varepsilon}\Big|\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\varphi^l_\varepsilon\Big|^q dx\le\frac{C}{\varepsilon^q}\sum_{l\in I_\varepsilon}\int_{\hat E^l_\varepsilon}|\varphi^l_\varepsilon|^q dx\le C\|e(u_\varepsilon)\|^q_{L^q(\Omega_\varepsilon)^{N^2}}.
\tag{4.43}
$$

Finally, in the third term term in (4.41) we use (4.4), (4.39) and $|x''|<C\varepsilon$, for every $(x',x'')\in\Omega_\varepsilon$, to get

$$
\int_{\Omega_\varepsilon}\Big|\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(D_{x''}u_\varepsilon(x)-\fint_{\hat B^i_\varepsilon}D_{x''}u_\varepsilon dz\Big)x''\Big|^q dx\le C\|e(u_\varepsilon)\|^q_{L^q(\Omega_\varepsilon)^{N^2}}.
\tag{4.44}
$$

Taking into account (4.42), (4.43) and (4.44) in (4.41), we deduce (4.36).

Step 3. Proof of (4.37): For every $i,j\in\{1,\cdots,k\}$, Using (4.9) and (4.7), we have

$$
\begin{aligned}
&\partial_j\Big(\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon(\rho^l_\varepsilon * u_\varepsilon)(x',0)\Big)=\sum_{l\in I_\varepsilon}\partial_{ij}\phi^l_\varepsilon(\rho^l_\varepsilon * u_\varepsilon)(x',0)\\
&+\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\rho^l_\varepsilon * \big(\partial_j u_\varepsilon-\fint_{\hat B^l_\varepsilon}\partial_j u_\varepsilon dz\big)\Big)(x',0)+\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\fint_{\hat B^l_\varepsilon}\partial_j u_\varepsilon dz-\partial_j u_\varepsilon\Big).
\end{aligned}
\tag{4.45}
$$

The first term in the right-hand side is similar to the term we estimated in (4.36). Taking into account that now $|\partial_{ij}\phi^l_\varepsilon|\le C/\varepsilon^2$, we get

$$
\Big\|\sum_{l\in I_\varepsilon}\partial_{ij}\phi^l_\varepsilon(\rho^l_\varepsilon * u_\varepsilon)(\cdot,0)\Big)\Big\|_{L^q(\Omega_\varepsilon)^N}\le\frac{C}{\varepsilon}\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad\forall i,j\in\{1,\dots,k\}.
\tag{4.46}
$$

In the second term in the right hand side of (4.45) we reason as in (4.42) by using (4.39) instead of (4.40). This gives

$$\left\|\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\rho^l_\varepsilon*\big(\partial_j u_\varepsilon-\fint_{\hat B^l_\varepsilon}\partial_j u_\varepsilon dz\big)\Big)(\cdot,0)\right\|_{L^q(\Omega_\varepsilon)^N}\le\frac{C}{\varepsilon}\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\ \forall i,j\in\{1,\dots,k\}.\quad(4.47)$$

For the third term in the right hand side of (4.45) we just use (4.3), (4.4) and (4.39) to get

$$\left\|\sum_{l\in I_\varepsilon}\partial_i\phi^l_\varepsilon\Big(\partial_j u_\varepsilon-\fint_{\hat B^l_\varepsilon}\partial_j u_\varepsilon dz\Big)\right\|_{L^q(\Omega_\varepsilon)^N}\le\frac{C}{\varepsilon}\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad\forall i,j\in\{1,\dots,k\}.\qquad(4.48)$$

Equations (4.45), (4.46), (4.47) and (4.48) give (4.37). This finishes the proof of (4.31) and (4.32).

Step 4. Let us prove the existence of $C>0$ independent of ε such that the following partial Korn's inequality holds

$$\left\|\partial_j u_{\varepsilon,i}-\fint_{\Omega_\varepsilon}a(u_\varepsilon)_{ij}dz\right\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad\forall i,j\in\{1,\dots,k\},\quad\forall\varepsilon>0.\qquad(4.49)$$

Indeed, taking into account (4.31), it is enough to show that a similar inequality holds for u_ε replaced by $\hat u_\varepsilon$. Since this last function does not depend on the variable x'', the result is a simple consequence of (3.14), (4.27) and (3.3), which prove

$$\begin{aligned}&\left\|\partial_j\hat u_{\varepsilon,i}-\fint_{\Omega_\varepsilon}a(\hat u_\varepsilon)_{ij}dz\right\|_{L^q(\Omega_\varepsilon)}\le\left\|\partial_j\hat u_{\varepsilon,i}-\fint_{\omega_\varepsilon}\partial_j\hat u_{\varepsilon,i}dz\right\|_{L^q(\Omega_\varepsilon)}+\left\|\fint_{\omega_\varepsilon}e(\hat u_\varepsilon)_{ij}dz\right\|_{L^q(\Omega_\varepsilon)}\\&+\left\|\fint_{\omega_\varepsilon}a(\hat u_\varepsilon)_{ij}dz-\fint_{\Omega_\varepsilon}a(\hat u_\varepsilon)_{ij}dx\right\|_{L^q(\Omega_\varepsilon)}\\&\le C\varepsilon^{N-k}\left\|e(\hat u'_\varepsilon)\right\|_{L^q(\omega_\varepsilon)^{k^2}}+\left\|\fint_{\Omega_\varepsilon}\left|a(\hat u_\varepsilon)_{ij}-\fint_{\omega_\varepsilon}a(\hat u_\varepsilon)_{ij}dx\right|dz\right\|_{L^q(\Omega_\varepsilon)}\\&\le C\varepsilon^{N-k}\|e(\hat u'_\varepsilon)\|_{L^q(\omega_\varepsilon)^{k^2}}\le C\|e(\hat u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}.\end{aligned}\qquad(4.50)$$

Step 5. We define $w_{\varepsilon,i}\in W^{1,q}(\Omega_\varepsilon)$, $i\in\{1,\dots,k\}$, $\varepsilon>0$, by

$$w_{\varepsilon,i}(x)=u_{\varepsilon,i}(x)-\sum_{r=1}^k\fint_{\Omega_\varepsilon}a(u_\varepsilon)_{ir}dz\,x_r+\frac{1}{\varepsilon}\sum_{r=k+1}^N\partial_i\hat u_{\varepsilon,r}(x')\,x_r,\ \text{ a.e. } x\in\Omega_\varepsilon.\qquad(4.51)$$

Let us prove that there exists $C>0$ independent of ε such that

$$\|\nabla w_{\varepsilon,i}\|_{L^q(\Omega_\varepsilon)^N}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad\forall i\in\{1,\dots,k\},\,\forall\varepsilon>0.\qquad(4.52)$$

To estimate $\partial_j w_{\varepsilon,i}$, with $j\in\{1,\dots,k\}$, we use (4.49), (4.32) and $|x_r|\le C\varepsilon$, for $x\in\Omega_\varepsilon$, $r\in\{k+1,\dots,N\}$. We get

$$\|\partial_j w_{\varepsilon,i}\|_{L^q(\Omega_\varepsilon)}=\left\|\partial_j u_{\varepsilon,i}-\fint_{\Omega_\varepsilon}a(u_\varepsilon)_{ij}dz+\frac{1}{\varepsilon}\sum_{r=k+1}^N\partial^2_{ij}\hat u_{\varepsilon,r}\,x_r\right\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}.\qquad(4.53)$$

The estimate of $\partial_j w_{\varepsilon,i}$, $j\in\{k+1,\dots,N\}$, $i\in\{1,\dots,k\}$, just follows from (4.31), which gives

$$\|\partial_j w_{\varepsilon,i}\|_{L^q(\Omega_\varepsilon)}=\Big\|2e(u_\varepsilon)_{ij}+\frac{1}{\varepsilon}\partial_i\hat u_{\varepsilon,j}-\partial_i u_{\varepsilon,j}\Big\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}.$$

This finishes the proof of (4.52).

Step 6. For $\varepsilon>0$, we introduce $S_\varepsilon\in C^\infty(\omega_\varepsilon)_a^{(N-k)^2}$ as

$$S_\varepsilon(x')_{ij}=\varepsilon\sum_{l\in I_\varepsilon}\phi^l_\varepsilon(x')(\rho^l_\varepsilon*\partial_j u_{\varepsilon,i})(x',0),\quad \forall x'\in\omega_\varepsilon\quad \forall i,j\in\{k+1,\dots,N\}\text{ with } j<i, \tag{4.54}$$

and $w_{\varepsilon,i}\in W^{1,q}(\Omega_\varepsilon)$, $i\in\{k+1,\dots,N\}$, by

$$w_{\varepsilon,i}(x)=u_{\varepsilon,i}(x)-\frac{1}{\varepsilon}\hat u_{\varepsilon,i}(x')-\frac{1}{\varepsilon}\sum_{r=k+1}^N S_\varepsilon(x')_{ir}\,x_r,\quad \text{a.e. } x\in\Omega_\varepsilon,\ \forall i\in\{k+1,\dots,N\}. \tag{4.55}$$

The function $(S_\varepsilon)_{ij}$ agrees with the expression of p^0_ε in (4.11) applied to $\partial_j u_{\varepsilon,i}$ instead of p_ε for $j<i$. Thanks to (3.9), Lemma 4.2, the definition of $e(u_\varepsilon)$ and S_ε skew-symmetric is then simple to show

$$\Big\|\partial_j u_{\varepsilon,i}-\frac{1}{\varepsilon}(S_\varepsilon)_{ij}\Big\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i,j\in\{k+1,\dots,N\}, \tag{4.56}$$

$$\|\nabla(S_\varepsilon)_{ij}\|_{L^q(\Omega_\varepsilon)^N}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i,j\in\{k+1,\dots,N\}. \tag{4.57}$$

Let us prove that $w_{\varepsilon,i}$ satisfies

$$\|\nabla w_{\varepsilon,i}\|_{L^q(\Omega_\varepsilon)^N}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i\in\{k+1,\dots,N\}. \tag{4.58}$$

By (4.56) we have

$$\big\|\partial_j w_{\varepsilon,i}\big\|_{L^q(\Omega_\varepsilon)}=\Big\|\partial_j u_{\varepsilon,i}-\frac{1}{\varepsilon}(S_\varepsilon)_{ij}\Big\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall i,j\in\{k+1,\dots,N\}. \tag{4.59}$$

For the partial derivatives of $w_{\varepsilon,i}$, with respect to the k first variables we use

$$\partial_j w_{\varepsilon,i}(x)=\partial_j u_{\varepsilon,i}(x)-\frac{1}{\varepsilon}\partial_j\hat u_{\varepsilon,i}(x')-\frac{1}{\varepsilon}\sum_{r=k+1}^N\partial_j S_\varepsilon(x')_{ir}\,x_r,\quad \text{a.e. } x\in\Omega_\varepsilon,\ \forall j\in\{1,\dots,k\}. \tag{4.60}$$

Then, by (4.31), (4.57) and $|x''|\le C\varepsilon$, for $(x',x'')\in\Omega_\varepsilon$, we get

$$\|\partial_j w_{\varepsilon,i}\|_{L^q(\Omega_\varepsilon)}\le C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}},\quad \forall j\in\{1,\dots,k\},\ \forall i\in\{k+1,\dots,N\},\ \forall\varepsilon>0. \tag{4.61}$$

Step 7. From (4.51) and (4.55), we have

$$u_{\varepsilon,i}(x)=\begin{cases}\displaystyle\sum_{r=1}^k\fint_{\Omega_\varepsilon}a(u_\varepsilon)_{ir}dz\,x_r-\sum_{r=k+1}^N\partial_i\hat u_{\varepsilon,r}(x')\,\frac{x_r}{\varepsilon}+w_{\varepsilon,i}(x), & \text{if } i\in\{1,\dots,k\},\\[2ex] \displaystyle\frac{1}{\varepsilon}\hat u_{\varepsilon,i}(x')+\sum_{r=k+1}^N(S_\varepsilon)_{ir}(x')\,\frac{x_r}{\varepsilon}+w_{\varepsilon,i}(x), & \text{if } i\in\{k+1,\dots,N\},\end{cases} \tag{4.62}$$

for a.e. $x \in \Omega_\varepsilon$, where (4.32), (4.52), (4.57) and (4.58) prove

$$\varepsilon^{\frac{N-k}{q}}\|D^2\hat{u}''_\varepsilon\|_{L^q(\omega_\varepsilon)^{(N-k)k^2}} + \varepsilon^{\frac{N-k}{q}}\|DS_\varepsilon\|_{L^q(\omega_\varepsilon)^{(N-k)^2k}} + \|Dw_\varepsilon\|_{L^q(\Omega_\varepsilon)^{N^2}} \leq C\|e(u_\varepsilon)\|_{L^q(\Omega_\varepsilon)^{N^2}}, \quad (4.63)$$

for every $\varepsilon > 0$. Taking then

$$B_\varepsilon := \left(\fint_{\Omega_\varepsilon} a(u_\varepsilon)_{ir}dz\right)_{i,r\in\{1,\dots,k\}}, \qquad \forall\varepsilon > 0,$$

we get (4.28) and (4.29). □

Proof of Theorem 3.6. From Lemma 4.3 and assumption (3.14) it is enough to take

$$a'_\varepsilon := \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} w'_\varepsilon\,dy, \quad a''_\varepsilon := \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon}\hat{u}''_\varepsilon dy' - \frac{1}{|\omega_\varepsilon|^2}\int_{\omega_\varepsilon} D\hat{u}''_\varepsilon dy'\int_{\omega_\varepsilon} y'dy' + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} w''_\varepsilon\,dy$$

$$A_\varepsilon := \begin{pmatrix} B_\varepsilon & -\dfrac{1}{\varepsilon|\omega_\varepsilon|}\displaystyle\int_{\omega_\varepsilon}(D\hat{u}''_\varepsilon)^t dx' \\ \dfrac{1}{\varepsilon|\omega_\varepsilon|}\displaystyle\int_{\omega_\varepsilon} D\hat{u}''_\varepsilon dy' & \dfrac{1}{\varepsilon|\omega_\varepsilon|}\displaystyle\int_{\omega_\varepsilon} S^\varepsilon dy' \end{pmatrix}$$

$$u^0_\varepsilon(x') := \hat{u}''_\varepsilon(x') - \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon}\hat{u}''_\varepsilon dy' - \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} D\hat{u}''_\varepsilon dy'\Big(x' - \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} y'dy'\Big),$$

$$Z^\varepsilon(x') := S_\varepsilon(x') - \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} S_\varepsilon dy', \quad v_\varepsilon(x) := w_\varepsilon(x) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} w_\varepsilon\,dy,$$

a.e. $x \in \Omega_\varepsilon$, to get the result. □

5 Application to the behavior of a fluid in thin domains with oscillating boundaries

In this section we assume that $N = 3$ and we denote by $Y' = (-1/2, 1/2)^2$ the unitary cube of $\mathbb{R}^2$.

For a connected bounded open set $\omega \subset \mathbb{R}^2$ with boundary which is locally a Lipschitz continuous graph, and for $\Psi_b, \Psi_t \in W^{1,\infty}(\mathbb{R}^2)$, Y'-periodic, with $\Psi_b < \Psi_t$ (where b refers to bottom and t to top) we define

$$\Omega_\varepsilon = \left\{(x', x_3) \in \omega\times\mathbb{R} \,:\, \varepsilon\Psi_b\left(\frac{x'}{\varepsilon}\right) < x_3 < \varepsilon\Psi_t\left(\frac{x'}{\varepsilon}\right)\right\}, \qquad \forall\,\varepsilon > 0. \quad (5.1)$$

Figure 1 corresponds to the case

$$\Psi_b(y) = \cos\big(2\pi(y_1 + y_2)\big) + 0.5\cos(2\pi y_2), \quad \Psi_t(y) = 2 + \Psi_b(y),$$

and $\varepsilon = 0.2$. Our aim in this section is to study the asymptotic behavior of the solutions $(u_\varepsilon, p_\varepsilon)$ of the Navier- Stokes problem in Ω_ε

$$\begin{cases} -\mu\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \qquad \displaystyle\int_{\Omega_\varepsilon} p_\varepsilon\,dx = 0, \end{cases} \quad (5.2)$$

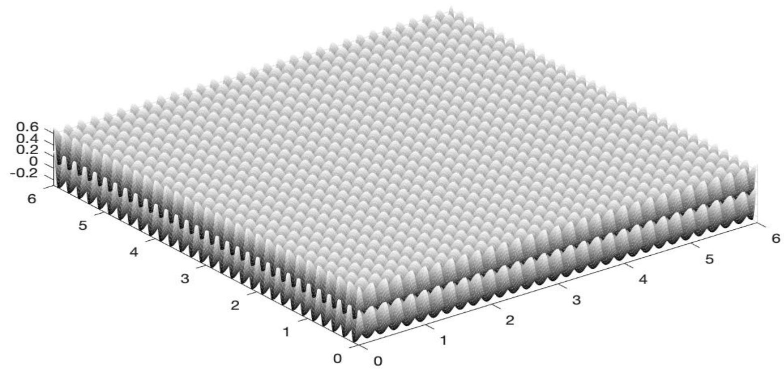

Figure 1: The domain Ω_ε

where the viscosity μ is strictly positive and the external force $f = f(x')$ is assumed to belong to $L^2(\omega)^3$. Some related problems have been considered for example in [4], [5] and [7].

We introduce the following notation:

The canonical basis in $\mathbb{R}^3$ is denoted by $\{e^1, e^2, e^3\}$.
The sets $\tilde{\Lambda}, \tilde{Y}, \tilde{\Gamma} \subset \mathbb{R}^3$ are defined by

$$\tilde{\Lambda} = \left\{y \in \mathbb{R}^3 \,:\, \Psi_b(y') < y_3 < \Psi_t(y')\right\}, \quad \tilde{Y} = \Big\{y \in \tilde{\Lambda} :\; y' \in Y'\Big\}, \quad \tilde{\Gamma} = \big\{y \in \partial\tilde{\Lambda} :\; u' \in Y'\big\}.$$

The spaces $L^2_\sharp(\tilde{Y})$, $H^1_\sharp(\tilde{Y})$, $H^1_{0,\sharp}(\tilde{Y})$ are defined by:

$$L^2_\sharp(\tilde{Y}) = \Big\{\tilde{w} \in L^2_{loc}(\tilde{\Lambda}) :\; \int_{\tilde{Y}} |\tilde{w}|^2 dy < +\infty, \quad \tilde{w}(y' + k', y_3) = \tilde{w}(y), \;\; \forall k' \in \mathbb{Z}^2, \text{ a.e. } y \in \tilde{\Lambda}\Big\},$$

$$H^1_\sharp(\tilde{Y}) = \Big\{\tilde{w} \in H^1_{loc}(\tilde{\Lambda}) :\; \tilde{w} \in L^2_\sharp(\tilde{Y}),\; \nabla\tilde{w} \in L^2_\sharp(\tilde{Y})^N\Big\},$$

$$H^1_{0,\sharp}(\tilde{Y}) = \Big\{\tilde{w} \in H^1_\sharp(\tilde{Y}) :\; \tilde{w} = 0 \;\text{ on } \tilde{\Gamma}\Big\}.$$

Our main result is given by the following theorem

Theorem 5.1 *Let $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ be a solution of (5.2) for $f \in L^2(\omega)^3$. For $i = 1, 2$, there exists a unique solution $(\tilde{w}^i, \tilde{\pi}^i)$ of the so called cell problem*

$$\begin{cases} -\mu\Delta\tilde{w}^i + \nabla\tilde{\pi}^i = e^i \quad \text{in } \tilde{\Lambda}, \\ \operatorname{div}\tilde{w}^i = 0 \quad \text{in } \tilde{\Lambda}, \\ \tilde{w}^i \in H^1_{0,\sharp}(\tilde{Y})^3,\; \tilde{\pi}^i \in L^2_\sharp(\tilde{Y})/\mathbb{R}. \end{cases} \tag{5.3}$$

Let $A \in \mathbb{R}^{2\times 2}_s$ be the matrix defined by

$$A_{ij} = \mu \int_{\tilde{Y}} D\tilde{w}^i : D\tilde{w}^j dy, \quad i, j \in \{1, 2\}. \tag{5.4}$$

Then, A is a positive definite matrix. Defining p as the solution of the Reynolds problem

$$\begin{cases} -\mathrm{div}_{x'}\Big(A(\nabla_{x'}p - f')\Big) = 0 \quad \text{in } \omega, \\ A(\nabla_{x'}p - f')\cdot\nu = 0 \quad \text{on } \partial\omega, \qquad \displaystyle\int_\omega p\,dx' = 0, \end{cases} \tag{5.5}$$

and $\tilde{u} \in L^2(\omega; H^1_{0,\sharp}(\tilde{Y}))^3$ by

$$\tilde{u}(x', y) = \sum_{i=1}^{2} \big(f_i(x') - \partial_i p(x')\big)\tilde{w}^i(y), \tag{5.6}$$

we have the following approximation result for $(u_\varepsilon, p_\varepsilon)$:

$$\lim_{\varepsilon\to 0} \fint_{\Omega_\varepsilon} \left(\Big|\frac{1}{\varepsilon^2}u_\varepsilon - \hat{u}_\varepsilon\big(x', \frac{x}{\varepsilon}\big)\Big|^2 + \Big|\frac{1}{\varepsilon}Du_\varepsilon - D_y\hat{u}_\varepsilon\big(x', \frac{x}{\varepsilon}\big)\Big|^2 + |p_\varepsilon - p|^2 \right) dx = 0, \tag{5.7}$$

where

$$\hat{u}_\varepsilon(x', y) := \sum_{k'\in\mathbb{Z}^2} \left(\fint_{\varepsilon k' + \varepsilon Y'} \tilde{u}(z', y)\,dz' \right) \chi_{\varepsilon k' + \varepsilon Y'}(x'), \qquad a.e.\ (x', y) \in \mathbb{R}^2 \times \tilde{Y}.$$

Remark 5.2 *In the approximation of u_ε given by (5.7), we have used in place of the function $\tilde{u}$, the regularization with respect to x' given by $\hat{u}_\varepsilon$. This is necessary because in general the functions $x \mapsto \tilde{u}(x', x/\varepsilon)$ and $x \mapsto D_y\tilde{u}(x', x/\varepsilon)$ are products of functions in $L^2(\Omega_\varepsilon)$, see (5.6), and therefore they belong only to $L^1(\Omega_\varepsilon)$. But we can replace $\hat{u}_\varepsilon$ by $\tilde{u}$ if we assume more regularity in the data. For example, this is the case if we assume that the functions Ψ_b, Ψ_t in the definition (5.1) of Ω_ε belong to $W^{2,\infty}_\sharp(Y')$, which implies that the solution $(\tilde{w}^i, \tilde{\pi}^i)$ of (5.3) belongs to $W^{1,\infty}(\tilde{Y})^3 \times L^\infty(\tilde{Y})$.*

Theorem 5.1 gives a strong approximation in $L^2(\Omega_\varepsilon)$ of u_ε which contains the quick variable $y = x/\varepsilon$. However, for related problems, it is usual in the literature to deal with an approximation which only depends on the macroscopic variable x'. This can be done by using the function

$$u(x') = \fint_{\tilde{Y}} \tilde{u}(x', y)\,dy. \tag{5.8}$$

The next corollary shows that this function u provides a "weak" approximation of u_ε and satisfies a Darcy law.

Corollary 5.3 *Let $(u_\varepsilon, p_\varepsilon) \in H^1_0(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ be a solution of (5.2) for $f \in L^2(\omega)^3$, and let A be defined by (5.4) and p by (5.5). Then u defined by (5.8) satisfies*

$$u'(x') = A(f' - \nabla_{x'}p), \quad u_3(x') = 0, \qquad a.e.\ x' \in \omega, \tag{5.9}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \fint_{\Omega_\varepsilon} u_\varepsilon \cdot \varphi\,dx = \fint_\omega u\cdot\varphi\,dx', \quad \forall\,\varphi \in L^2(\omega)^3. \tag{5.10}$$

The proof of Theorem 5.1 is based on the decomposition result obtained in Section 3 for the pressure and the unfolding method (see e.g. [3], [8], [13], [14]), which is closely related to the two scale convergence method ([1], [19], [21]) . We also refer to the Bloch-wave homogenization method as a related approach to deal with this type of problems (see e.g. [2], [15]).

We recall that the idea of the unfolding method is to use a convenient dilatation of the periodic cell. In the present case we introduce the following notation

Definition 5.4 *For $k' \in \mathbb{Z}^2$ and $\varepsilon > 0$, we define $C_\varepsilon^{k'} \subset \mathbb{R}^2$ as the square of center $\varepsilon k'$ and sides of length ε parallel to the coordinate axis, i.e.*

$$C_\varepsilon^{k'} = \varepsilon k' + \varepsilon Y', \tag{5.11}$$

and $\widehat{C}_\varepsilon^{k'}$ by

$$\widehat{C}_\varepsilon^{k'} = \left\{ (x', x_3) \in C_\varepsilon^{k'} \times \mathbb{R} : \quad \varepsilon \Psi_b \left(\frac{x'}{\varepsilon} \right) < x_3 < \varepsilon \Psi_t \left(\frac{x'}{\varepsilon} \right) \right\}. \tag{5.12}$$

We also introduce the function $\kappa' : \mathbb{R}^2 \to \mathbb{Z}^2$ defined by $x' \in C_1^{\kappa'(x')}$ for a.e. $x' \in \mathbb{R}^2$.

The idea to prove Theorem 5.1 will be to study the asymptotic behavior of the sequences of functions

$$\tilde{u}_\varepsilon(x', y) = u_\varepsilon \left(\varepsilon \kappa' \left(\frac{x'}{\varepsilon} \right) + \varepsilon y \right), \qquad \text{a.e. } (x', y) \in \mathbb{R}^2 \times \tilde{Y}, \tag{5.13}$$

$$\tilde{p}_\varepsilon^0(x', y') = p_\varepsilon^0 \left(\varepsilon \kappa' \left(\frac{x'}{\varepsilon} \right) + \varepsilon y' \right), \qquad \text{a.e. } (x', y') \in \mathbb{R}^2 \times Y', \tag{5.14}$$

and

$$\tilde{p}_\varepsilon^1(x', y) = p_\varepsilon^1 \left(\varepsilon \kappa' \left(\frac{x'}{\varepsilon} \right) + \varepsilon y \right), \qquad \text{a.e. } (x', y) \in \mathbb{R}^2 \times \tilde{Y}, \tag{5.15}$$

where p_ε^0, p_ε^1 are the functions defined by Theorem 3.2, from the pressure p_ε. The functions u_ε, p_ε^0 and p_ε^1 are assumed to be defined in the whole set

$$\mathcal{Z}_\varepsilon = \left\{ (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \varepsilon \Psi_b \left(\frac{x'}{\varepsilon} \right) < x_3 < \varepsilon \Psi_t \left(\frac{x'}{\varepsilon} \right) \right\}, \tag{5.16}$$

by extending them by zero outside Ω_ε.

Remark 5.5 *The introduction of the functions $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon^0$ and $\tilde{p}_\varepsilon^1$ is the main idea in the unfolding method. We observe that for $x' \in C_\varepsilon^{k'}$, with $k' \in \mathbb{Z}^2$, we have*

$$\tilde{u}_\varepsilon(x', y) = u_\varepsilon (\varepsilon k' + \varepsilon y), \quad \tilde{p}_\varepsilon^0(x', y') = p_\varepsilon^0 (\varepsilon k' + \varepsilon y'), \quad \tilde{p}_\varepsilon^1(x', y) = p_\varepsilon^1 (\varepsilon k' + \varepsilon y) \quad a.e. \ y \in \tilde{Y}.$$

Thus, in $C_\varepsilon^{k'} \times \tilde{Y}$, these functions only depend on y and are obtained by the change of variables

$$\frac{x - \varepsilon k'}{\varepsilon} = y, \tag{5.17}$$

which transforms the small set $\widehat{C}_\varepsilon^{k'}$ into $\tilde{Y}$. The variable y represents the microscopic variable and x' the macroscopic one.

The following compactness lemmas give a first result about the asymptotic behavior of the sequences $p_\varepsilon^0 \in H^1(\mathbb{R}^2)$, $p_\varepsilon^1 \in L^2(\mathcal{Z}_\varepsilon)$ and $u_\varepsilon \in H_0^1(\mathcal{Z}_\varepsilon)^3$, which are not necessarily the solution of any partial differential problem but satisfy some suitable a priori estimates.

The first lemma is just the classical compactness result for the unfolding method, and so, it is given without proof.

Lemma 5.6 *Let p_ε^0 be any bounded sequence in $H^1(\mathbb{R}^2)$. Define $\tilde{p}_\varepsilon^0 : \mathbb{R}^2 \times Y' \to \mathbb{R}$ by (5.14). Then there exist $p \in H^1(\mathbb{R}^2)$ and $\tilde{p}^0 \in L^2(\mathbb{R}^2; H_\sharp^1(Y'))$ such that for a subsequence of ε, still denoted by ε, we have*

$$p_\varepsilon^0 \rightharpoonup p \quad in\ H^1(\mathbb{R}^2), \tag{5.18}$$

$$\frac{1}{\varepsilon} \nabla_{y'} \tilde{p}_\varepsilon^0 \rightharpoonup \nabla_{x'} p + \nabla_{y'} \tilde{p}^0 \quad in\ L^2(\mathbb{R}^2; L^2(Y'))^2. \tag{5.19}$$

Lemma 5.7 *Let p^1_ε be any sequence in $L^2(\Omega_\varepsilon)$ such that there exists $C>0$, with*

$$\fint_{\Omega_\varepsilon} |p^1_\varepsilon|^2 dx \le C, \quad \forall\, \varepsilon > 0. \tag{5.20}$$

Extending p^1_ε by zero to the set $\mathcal{Z}_\varepsilon$ given by (5.16), define $\tilde{p}^1_\varepsilon : \mathbb{R}^2 \times \tilde{Y} \to \mathbb{R}$ by (5.15). Then, for a subsequence of ε, still denoted by ε, there exists a function $\tilde{p}^1 \in L^2(\mathbb{R}^2; L^2(\tilde{Y}))$, such that

$$\tilde{p}^1_\varepsilon \rightharpoonup \tilde{p}^1 \quad \text{in } L^2(\mathbb{R}^2; L^2(\tilde{Y})). \tag{5.21}$$

Proof. Using the change of variables (5.17) in each set $\widehat{C}^{k'}_\varepsilon$, $k' \in \mathbb{Z}^2$, we get

$$\fint_{\Omega_\varepsilon} |p^1_\varepsilon|^2 dx = \frac{1}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\widehat{C}^{k'}_\varepsilon} |p^1_\varepsilon|^2 dx = \frac{\varepsilon^3}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\tilde{Y}} \left|p^1_\varepsilon(\varepsilon k' + \varepsilon y)\right|^2 dy$$
$$= \frac{\varepsilon}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{C^{k'}_\varepsilon} \int_{\tilde{Y}} \left|\tilde{p}^1_\varepsilon\right|^2 dy dx' = \frac{\varepsilon}{|\Omega_\varepsilon|} \int_{\mathbb{R}^2 \times \tilde{Y}} \left|\tilde{p}^1_\varepsilon\right|^2 dx' dy, \quad \forall \varepsilon > 0,$$

which thanks to (5.20) and

$$\frac{|\Omega_\varepsilon|}{\varepsilon} = \int_\omega \left(\Psi_t\left(\frac{x'}{\varepsilon}\right) - \Psi_b\left(\frac{x'}{\varepsilon}\right) \right) dx' \to |\omega| \int_{Y'} (\Psi_t - \Psi_b)\, dy' = |\omega||\tilde{Y}|, \tag{5.22}$$

shows that $\tilde{p}^1_\varepsilon$ is bounded in $L^2(\mathbb{R}^2, L^2(\tilde{Y}))$ and then the existence of a subsequence of ε and $\tilde{p}^1 \in L^2(\mathbb{R}^2; L^2(\tilde{Y}))$ satisfying (5.21).

□

Lemma 5.8 *Let u_ε be any sequence in $H^1_0(\Omega_\varepsilon)^3$, with* $\operatorname{div} u_\varepsilon = 0$ *in Ω_ε, such that there exists $C > 0$ satisfying*

$$\fint_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \le C, \quad \forall\, \varepsilon > 0. \tag{5.23}$$

Extending u_ε by zero to the set $\mathcal{Z}_\varepsilon$ given by (5.16), define $\tilde{u}_\varepsilon : \mathbb{R}^2 \times \tilde{Y} \to \mathbb{R}^3$ by (5.13). Then, there exists a function $\tilde{u} \in L^2(\mathbb{R}^2; H^1_{0,\sharp}(\tilde{Y}))^3$, such that

$$\tilde{u} = 0 \quad \text{a.e. in } (\mathbb{R}^2 \setminus \omega) \times \tilde{Y}, \tag{5.24}$$

$$\operatorname{div}_y \tilde{u} = 0 \quad \text{a.e. in } \mathbb{R}^2 \times \tilde{Y}, \tag{5.25}$$

$$\operatorname{div}_{x'} \int_{\tilde{Y}} \tilde{u}' dy = 0 \quad \text{a.e. in } \mathbb{R}^2, \tag{5.26}$$

and such that, for a subsequence of ε still denoted by ε, we have

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup \tilde{u} \quad \text{in } L^2(\mathbb{R}^2; H^1(\tilde{Y}))^3. \tag{5.27}$$

Proof. Using the change of variables (5.17) in each set $\widehat{C}^{k'}_\varepsilon$, $k' \in \mathbb{Z}^2$, we have

$$\fint_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx = \frac{1}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\widehat{C}^{k'}_\varepsilon} |Du_\varepsilon|^2 dx$$
$$= \frac{\varepsilon^3}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\tilde{Y}} \left|Du_\varepsilon(\varepsilon k' + y)\right|^2 dy = \frac{\varepsilon}{|\Omega_\varepsilon|} \int_{\omega \times \tilde{Y}} \left| D_y \left(\frac{\tilde{u}_\varepsilon}{\varepsilon}\right)(x', y) \right|^2 dx' dy.$$

From (5.22) and (5.23), this proves that $\tilde{u}_\varepsilon/\varepsilon$ is bounded in $L^2(\mathbb{R}^2; H^1(\tilde{Y}))^3$. Since it vanishes on $\mathbb{R}^2 \times \tilde{\Gamma}$, we deduce the existence of $\tilde{u} \in L^2(\mathbb{R}^2; H^1(\tilde{Y}))^3$, which vanishes on $\mathbb{R}^2 \times \tilde{\Gamma}$ such that (5.27) holds, up to a subsequence. Moreover, by construction,

$$\tilde{u}_\varepsilon = 0 \text{ a.e. in } \left\{(x', y) \in \mathbb{R}^2 \times \tilde{Y} : \ \mathrm{dist}(x', \omega) > \sqrt{2}\varepsilon\right\},$$

and so $\tilde{u}$ satisfies (5.24). Using also

$$\mathrm{div}_y \tilde{u}_\varepsilon(x, y) = \varepsilon\, \mathrm{div}_x u_\varepsilon \left(\varepsilon\kappa' \left(\frac{x'}{\varepsilon}\right) + \varepsilon y\right) = 0,$$

we deduce (5.25).

Now for $\phi \in C^\infty(\mathbb{R}^2)$, we use that $\tilde{u}_\varepsilon/\varepsilon$ is bounded in $L^2(\mathbb{R}^2; H^1(\tilde{Y}))^3$ to deduce

$$\begin{aligned} 0 &= \frac{1}{\varepsilon} \fint_{\Omega_\varepsilon} u'_\varepsilon \cdot \nabla_{x'}\phi\, dx = \frac{1}{\varepsilon|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\widehat{C}_\varepsilon^{k'}} u'_\varepsilon \cdot \nabla_{x'}\phi\, dx \\ &= \frac{\varepsilon^2}{|\Omega_\varepsilon|} \sum_{k' \in \mathbb{Z}^2} \int_{\tilde{Y}} u'_\varepsilon(\varepsilon k' + \varepsilon y) \cdot \nabla_{x'}\phi(\varepsilon k' + \varepsilon y)\, dy = \frac{\varepsilon}{|\Omega_\varepsilon|} \int_{\mathbb{R}^2} \int_{\tilde{Y}} \frac{1}{\varepsilon} \tilde{u}'_\varepsilon(x', y) \cdot \nabla_{x'}\phi(x')\, dy dx' + O_\varepsilon, \end{aligned}$$

which passing to the limit thanks to (5.22) and (5.27) proves

$$\int_\omega \fint_{\tilde{Y}} \tilde{u}' \cdot \nabla_{x'}\phi\, dy dx' = 0, \quad \forall\, \phi \in C^\infty(\mathbb{R}^2).$$

This combined with (5.24) shows (5.26).

It remains to prove that $\tilde{u}$ is periodic in y'. This follows by passing to the limit in the equality

$$\frac{1}{\varepsilon}\tilde{u}_\varepsilon \left(x' + \varepsilon e^1, -\frac{1}{2}, y_2, y_3\right) = \frac{1}{\varepsilon}\tilde{u}_\varepsilon \left(x', \frac{1}{2}, y_2, y_3\right),$$

which is a consequence of definition (5.13). This shows

$$\tilde{u}\left(x', -\frac{1}{2}, y_2, y_3\right) = \tilde{u}\left(x', \frac{1}{2}, y_2, y_3\right),$$

and then the periodicity of $\tilde{u}$ with respect to y_1. Similarly one can show the periodicity with respect to y_2. □

Proof of Theorem 5.1. The proof is divided in two steps.

Step 1. It is well known that (5.2) has at least a solution $(u_\varepsilon, p_\varepsilon)$ in $H^1_0(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$. Let us obtain some a priori estimates for u_ε and ∇p_ε, which allow us to apply Lemmas 5.6, 5.7 and 5.8.

Using u_ε as test function in (5.2), and taking into account that $\mathrm{div}\, u_\varepsilon$ vanishes in Ω_ε, we get

$$\mu \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon\, dx. \tag{5.28}$$

Now, we observe that since the height of Ω_ε is of order ε, we have

$$\int_{\Omega_\varepsilon} |v|^2 dx \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\partial_3 v|^2 dx, \quad \forall\, v \in H^1_0(\Omega_\varepsilon)^3,\ \forall\, \varepsilon > 0, \tag{5.29}$$

which, combined with (5.28) and $f = f(x')$ implies

$$\int_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \le C\varepsilon^2 \int_{\Omega_\varepsilon} |f|^2 dx \le C\varepsilon^3, \quad \forall \varepsilon > 0. \tag{5.30}$$

Let us now obtain an estimate for the gradient of the pressure in $H^{-1}(\Omega_\varepsilon)^3$. For this purpose, we use (5.2), which gives

$$\langle \nabla p_\varepsilon, v\rangle = \int_{\Omega_\varepsilon} f\cdot v\, dx - \mu \int_{\Omega_\varepsilon} Du_\varepsilon : Dv\, dx - \int_{\Omega_\varepsilon} (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot v\, dx, \quad \forall\, v\in H^1_0(\Omega_\varepsilon)^3. \tag{5.31}$$

Thanks to (5.29) and (5.30), we have

$$\left|\int_{\Omega_\varepsilon} f\cdot v\, dx - \mu\int_{\Omega_\varepsilon} Du_\varepsilon : Dv\, dx\right| \le C\varepsilon^{\frac{3}{2}}\|v\|_{H^1_0(\Omega_\varepsilon)^3}. \tag{5.32}$$

In order to estimate the third term in (5.31) we use that, extending by zero the functions in $H^1_0(\Omega_\varepsilon)$, we have $H^1_0(\Omega_\varepsilon)\subset H^1_0(\omega\times(-1,1))$, for $\varepsilon>0$ small enough. Therefore, Sobolev's inequality applied to the fixed open set $\omega\times(-1,1)$ shows

$$\|v\|_{L^6(\Omega_\varepsilon)^3} \le C\|v\|_{H^1_0(\Omega_\varepsilon)^3}, \quad \forall\, v\in H^1_0(\Omega_\varepsilon)^3, \tag{5.33}$$

which combined with (5.28) proves the inequality

$$\left|\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot v\, dx\right| \le \|u_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|v\|_{L^6(\Omega_\varepsilon)^3}\|u_\varepsilon\|_{H^1_0(\Omega_\varepsilon)^3}|\Omega_\varepsilon|^{\frac{1}{6}} \le C\varepsilon^{\frac{19}{6}}\|v\|_{H^1_0(\Omega_\varepsilon)^3}.$$

Using this estimate and (5.32) in (5.31), we then have

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{3}{2}}, \quad \forall\varepsilon>0. \tag{5.34}$$

From (5.30) we can apply Lemma 5.8 with u_ε replaced by $u_\varepsilon/\varepsilon$ which proves the existence of a subsequence of ε, still denoted by ε, and a function $\tilde u\in L^2(\mathbb{R}^2; H^1_{0,\sharp}(\tilde Y))^3$ which satisfies (5.24), (5.25), (5.26) such that the sequence $\tilde u_\varepsilon$ defined by (5.13) satisfies

$$\frac{\tilde u_\varepsilon}{\varepsilon^2} \rightharpoonup \tilde u \ \text{ in } L^2(\mathbb{R}^2; H^1(\tilde Y))^3. \tag{5.35}$$

On the other hand, using (5.34) and ω Lipschitz and connected, we can apply Corollary 3.4 with $q=2$, $N=3$ and $k=2$ to deduce the existence of $p^0_\varepsilon\in H^1(\omega)$ and $p^1_\varepsilon\in L^2(\Omega_\varepsilon)$ such that

$$p_\varepsilon = p^0_\varepsilon + \varepsilon p^1_\varepsilon \ \text{ in } \Omega_\varepsilon, \tag{5.36}$$

$$\varepsilon^{\frac{3}{2}}\|p^0_\varepsilon\|_{H^1(\omega)} + \varepsilon\|p^1_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{3}{2}}. \tag{5.37}$$

Extending p^0_ε to a function in $H^1(\mathbb{R}^2)$, thanks to ω Lipschitz, this allows us to apply Lemmas 5.6 and 5.7 to p^0_ε and p^1_ε to deduce the existence of a subsequence of ε, still denoted by ε and functions $p\in H^1(\mathbb{R}^2)$, $\tilde p^0\in L^2(\mathbb{R}^2; H^1_\sharp(Y'))$, $\tilde p^1\in L^2(\mathbb{R}^2; L^2_\sharp(\tilde Y))$ such that $\tilde p^0_\varepsilon$ and $\tilde p^1_\varepsilon$ defined by (5.14) and (5.15) satisfy (5.18), (5.19) and (5.21). Moreover, passing to the limit when ε tends to zero in

$$0 = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} p_\varepsilon\, dx = \int_\omega \Big(\Psi_t\Big(\frac{x'}{\varepsilon}\Big) - \Psi_b\Big(\frac{x'}{\varepsilon}\Big)\Big)p^0_\varepsilon dx' + \int_{\Omega_\varepsilon} p^1_\varepsilon dx,$$

we deduce

$$0 = \int_{Y'} (\Psi_t - \Psi_b) dy' \int_\omega p \, dx',$$

and so, that p has null mean value in ω.

Step 2. Let us now show that p, $\tilde{u}$ are given by (5.5), (5.6) and that (5.7) holds.

For $\tilde{v} \in C^1_c(\omega; H^1_{0,\sharp}(\tilde{Y}))^3$, we consider $v_\varepsilon(x) = \varepsilon^{-1}\tilde{v}(x', x/\varepsilon)$ as test function in (5.2). Using (5.36), we get

$$\begin{aligned}
&\frac{\mu}{\varepsilon}\int_{\Omega_\varepsilon} Du_\varepsilon : D_x\tilde{v}(x', \frac{x}{\varepsilon})\, dx + \frac{\mu}{\varepsilon^2}\int_{\Omega_\varepsilon} Du_\varepsilon : D_y\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx \\
&+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} (u_\varepsilon\cdot\nabla)u_\varepsilon\, \tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \nabla p^0_\varepsilon \cdot \tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx - \int_{\Omega_\varepsilon} p^1_\varepsilon \mathrm{div}_x\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx \qquad (5.38)\\
&-\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} p^1_\varepsilon\, \mathrm{div}_y\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f\cdot\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx.
\end{aligned}$$

Let us pass to the limit in the different terms in (5.38).
Taking into account (5.30), (5.33) and (5.37), the first, third and fifth terms on the left-hand side tend to zero.

For the remaining terms in (5.38), we use the change of variables (5.17), the definitions (5.13), (5.14), (5.15) of $\tilde{u}_\varepsilon$, $\tilde{p}^0_\varepsilon$ and $\tilde{p}^1_\varepsilon$ respectively and the convergences (5.35), (5.19) and (5.21), to get

$$\frac{\mu}{\varepsilon^2}\int_{\Omega_\varepsilon} Du_\varepsilon : D_y\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx = \frac{\mu}{\varepsilon^2}\int_{\omega\times\tilde{Y}} D_y\tilde{u}_\varepsilon : D_y\tilde{v}\, dx'dy + O_\varepsilon = \mu\int_{\omega\times\tilde{Y}} D_y\tilde{u} : D_y\tilde{v}\, dx'dy + O_\varepsilon,$$

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \nabla p^0_\varepsilon\cdot\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx = \frac{1}{\varepsilon}\int_{\omega\times\tilde{Y}} \nabla_{y'}\tilde{p}^0_\varepsilon\cdot\tilde{v}\, dx'dy + O_\varepsilon = \int_{\omega\times\tilde{Y}} (\nabla_{x'}p + \nabla_{y'}\tilde{p}^0)\cdot\tilde{v}\, dx'dy + O_\varepsilon,$$

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} p^1_\varepsilon\, \mathrm{div}_y\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx = \int_{\omega\times\tilde{Y}} \tilde{p}^1_\varepsilon\, \mathrm{div}_y\tilde{v}\, dx'dy + O_\varepsilon = \int_{\omega\times\tilde{Y}} \tilde{p}^1 \mathrm{div}_y\tilde{v}\, dx'dy + O_\varepsilon,$$

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f\cdot\tilde{v}\left(x', \frac{x}{\varepsilon}\right) dx = \int_{\omega\times\tilde{Y}} f\cdot\tilde{v}\, dx'dy + O_\varepsilon.$$

Taking $\tilde{q} := \tilde{p}^0 + \tilde{p}^1 \in L^2(\omega; L^2_\sharp(\tilde{Y}))$, we then deduce from (5.38) that $\tilde{u}$, p and $\tilde{q}$ satisfy

$$\mu\int_{\omega\times\tilde{Y}} D_y\tilde{u} : D_y\tilde{v}\, dx'dy + \int_{\omega\times\tilde{Y}} \nabla_{x'}p\cdot\tilde{v}\, dx'dy - \int_{\omega\times\tilde{Y}} \tilde{q}\, \mathrm{div}_y\tilde{v}\, dx'dy = \int_{\omega\times\tilde{Y}} f\cdot\tilde{v}\, dx'dy, \quad (5.39)$$

for every $\tilde{v} \in C^1_c(\omega; H^1_{0,\sharp}(\tilde{Y}))^3$. By density, this equality holds true for every $\tilde{v} \in L^2(\omega; H^1_{0,\sharp}(\tilde{Y}))^3$.

Since $\tilde{u} \in L^2(\omega; H^1_{0,\sharp}(\tilde{Y}))^3$ and $\mathrm{div}_y\tilde{u} = 0$ in $\omega\times\tilde{Y}$, we then deduce that $\tilde{u}$, p and $\tilde{q}$ satisfy

$$\left\{\begin{array}{l}
-\mu\Delta_y\tilde{u} + \nabla_y\tilde{q} = f - \nabla_{x'}p \;\text{ in } \tilde{\Lambda},\\
\mathrm{div}_y\tilde{u} = 0 \;\text{ in } \tilde{\Lambda},\\
(\tilde{u}, \tilde{q}) \in H^1_{0,\sharp}(\tilde{Y})^3 \times L^2_\sharp(\tilde{Y})/\mathbb{R},
\end{array}\right. \qquad \text{a.e. in } \omega. \qquad (5.40)$$

From this equation we can obtain $\tilde{u}$ and $\tilde{q}$ from f and p. Namely: we define $(\tilde{w}^i, \tilde{\pi}^i)$, $i = 1, 2, 3$, as the unique solution of problem (5.3), where we observe that

$$\tilde{w}^3 = 0 \;\text{ in } H^1_{0,\sharp}(\tilde{Y})^3, \quad \tilde{\pi}^3 = y_3 \;\text{ in } L^2_\sharp(\tilde{Y})/\mathbb{R}. \qquad (5.41)$$

Then, reasoning by linearity and uniqueness we deduce (5.6) and

$$\tilde{q}(x',y)=\sum_{i=1}^{2}\big(f_i(x')-\partial_i p(x')\big)\tilde{\pi}^i(y)+f_3(x')y_3 \ \text{ in } L^2(\omega;L^2_\sharp(\tilde{Y})/\mathbb{R}). \tag{5.42}$$

Let us now prove (5.5), which in particular shows the existence and uniqueness of p and then of $\tilde{u}$, $\tilde{q}$. For this aim, we recall that $\tilde{u}$ also satisfies (5.24) and (5.26) and so that

$$\int_{\tilde{Y}}\tilde{u}'dy\cdot\nu=0 \ \text{ on } \partial\omega,$$

and therefore, using (5.6) we get

$$\begin{cases}\displaystyle\sum_{j=1}^{2}\partial_{x_j}\left[\sum_{i=1}^{2}(f_i-\partial_{x_i}p)\int_{\tilde{Y}}\tilde{w}^i_j dy\right]=\operatorname{div}_{x'}\int_{\tilde{Y}}\tilde{u}'(x,y)\,dy=0 \ \text{ in } \omega,\\ \displaystyle\sum_{i=1}^{2}(f_i-\partial_{x_i}p)\int_{\tilde{Y}}\tilde{w}^i dy\cdot\nu=\int_{\tilde{Y}}\tilde{u}'(x,y)\,dy\cdot\nu=0 \ \text{ on } \partial\omega.\end{cases} \tag{5.43}$$

On the other hand, we observe that taking $\tilde{w}^i$ as test function in the equation for $\tilde{w}^j$, we have

$$\int_{\tilde{Y}}\tilde{w}^i_j(y)\,dy=\mu\int_{\tilde{Y}}D\tilde{w}^i:D\tilde{w}^j\,dy,\quad 1\le i,j\le 3. \tag{5.44}$$

Thus, (5.43) proves (5.5).

In order to finish the proof of the theorem, it remains to show the corrector result (5.7). As usual this can be done by showing the convergence of the energies. Namely, using u_ε as test function in (5.2), taking $\tilde{u}$ as test function in (5.40), and using the change of variables (5.17) and (5.35) we easily have

$$\begin{aligned}&\frac{\mu}{\varepsilon^4}\int_{\mathbb{R}^2}\int_{\tilde{Y}}|D_y\tilde{u}_\varepsilon|^2dy\,dx'=\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon}|Du_\varepsilon|^2dx=\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon}f\cdot u_\varepsilon\,dx\\&=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\tilde{Y}}f\cdot\tilde{u}_\varepsilon\,dy\,dx'+O_\varepsilon=\mu\int_{\mathbb{R}^2}\int_{\tilde{Y}}|D_y\tilde{u}|^2dy\,dx'+O_\varepsilon.\end{aligned}$$

This shows that the convergence in (5.35) holds in fact in the strong topology of $L^2(\mathbb{R}^2;H^1(\tilde{Y}))^3$.

On the other hand, the change of variables (5.17) and the definition (5.13) of $\tilde{u}_\varepsilon$ give

$$\begin{aligned}&\fint_{\Omega_\varepsilon}\left(\left|\frac{1}{\varepsilon^2}u_\varepsilon-\hat{u}_\varepsilon\big(x',\frac{x}{\varepsilon}\big)\right|^2+\left|\frac{1}{\varepsilon}D_xu_\varepsilon-D_y\hat{u}_\varepsilon\big(x',\frac{x}{\varepsilon}\big)\right|^2\right)dx\\&\le C\int_{\mathbb{R}^2}\int_{\tilde{Y}}\left(\left|\frac{1}{\varepsilon^2}\tilde{u}_\varepsilon-\tilde{u}\right|^2+\left|\frac{1}{\varepsilon^2}D_y\tilde{u}_\varepsilon-D_y\tilde{u}\right|^2\right)dydx'.\end{aligned}$$

Therefore, the two first terms in (5.7) tend to zero. In order to prove that the last term also tends to zero we just use that the weak convergence in $H^1(\omega)$ of p^0_ε to p implies the strong convergence in $L^2(\omega)$, which combined with (5.36) and (5.37) proves the result. □

Proof of Corollary 5.3. By (5.6) we have

$$u(x')=\sum_{i=1}^{2}\big(f_i(x')-\partial_i p(x')\big)\fint_{\tilde{Y}}\tilde{w}^i dy.$$

From (5.44), (5.41), and definition (5.4) of A, we then deduce (5.9).

It remains to prove (5.10). Taking into account that thanks to (5.29) and (5.30), the sequence

$$\frac{1}{\varepsilon^3}\int_{\varepsilon\Psi_b(\frac{x'}{\varepsilon})}^{\varepsilon\Psi_t(\frac{x'}{\varepsilon})} u_\varepsilon(x',t)dt$$

is bounded in $L^2(\omega)^3$, we can assume that φ in (5.10) is smooth. In this case, (5.7) and the change of variables (5.17) give

$$\frac{1}{\varepsilon^2}\fint_{\Omega_\varepsilon} u_\varepsilon\cdot\varphi\,dx = \fint_{\Omega_\varepsilon}\hat{u}_\varepsilon\left(x',\frac{x}{\varepsilon}\right)\cdot\varphi\,dx + O_\varepsilon = \fint_\omega\fint_{\tilde{Y}}\tilde{u}\cdot\varphi\,dy\,dx' + O_\varepsilon = \fint_\omega u\cdot\varphi\,dx' + O_\varepsilon.$$

□

6 Application to the behavior of thin elastic beams with rough boundaries

As an application of Theorem 3.6 we study in this section the asymptotic behavior of an elastic beam with a varying profile. Namely, for a connected bounded open set with boundary which is locally a Lipschitz continuous graph, $\vartheta\subset\mathbb{R}^{N-1}$, $N\geq 2$, with $0\in\vartheta$ and a sequence of functions $\Psi_\varepsilon\in W^{1,\infty}(\mathbb{R}^{1+N})^{N-1}$, $\Psi_\varepsilon=\Psi_\varepsilon(x_1,y)$, such that

$$\lim_{\varepsilon\to 0}\left(\|\Psi_\varepsilon\|_{L^\infty(\mathbb{R}^{1+N})^{N-1}} + \big\|D_y\Psi_\varepsilon\big\|_{L^\infty(\mathbb{R}^{1+N})^{(N-1)N}} + \varepsilon\big\|\partial_{x_1}\Psi_\varepsilon\big\|_{L^\infty(\mathbb{R}^{1+N})^{N-1}}\right) = 0, \tag{6.1}$$

(see Remark 6.1 for an example of such a sequence of functions) we define the "thin beam" Ω_ε, with $\varepsilon>0$, by

$$\Omega_\varepsilon = \left\{(x_1,x'')\in(0,1)\times\mathbb{R}^{N-1}:\ \frac{x''}{\varepsilon}+\Psi_\varepsilon\left(x_1,\frac{x}{\varepsilon}\right)\in\vartheta\right\}, \tag{6.2}$$

and we denote

$$\Omega=(0,1)\times\vartheta. \tag{6.3}$$

The two bases of Ω_ε and Ω are denoted by

$$\Gamma_\varepsilon=\big\{(x_1,x'')\in\overline{\Omega_\varepsilon}:\ x_1\in\{0,1\}\big\},\qquad \Gamma=\{0,1\}\times\overline{\vartheta}. \tag{6.4}$$

We also consider a tensor valued function $A\in L^\infty(\mathbb{R}^N;\mathcal{L}(\mathbb{R}^{N^2}_s))$ such that there exists $\alpha>0$, with

$$A(x)\xi:\xi\geq\alpha|\xi|^2,\quad \forall\xi\in\mathbb{R}^{N^2}_s,\ \text{a.e. } x\in\mathbb{R}^N, \tag{6.5}$$

and two functions $f\in L^2(\mathbb{R}^N)^N$, $G\in L^2(\mathbb{R}^N)^{N^2}_s$.

Defining $A_\varepsilon\in L^\infty(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N^2}_s))$, $f_\varepsilon\in L^2(\Omega_\varepsilon)^N$ and $G_\varepsilon\in L^2(\Omega_\varepsilon)^{N^2}_s$ by

$$A_\varepsilon(x)=A\left(x_1,\frac{x''}{\varepsilon}\right),\quad f_\varepsilon(x)=\left(f_1\left(x_1,\frac{x''}{\varepsilon}\right),\varepsilon f''\left(x_1,\frac{x''}{\varepsilon}\right)\right),\quad G_\varepsilon(x)=G\left(x_1,\frac{x''}{\varepsilon}\right),$$

a.e. $x=(x_1,x'')\in\Omega_\varepsilon$, we are interested in the asymptotic behavior of the solutions u_ε of the elasticity problem

$$\begin{cases} -\mathrm{div}\,(A_\varepsilon e(u_\varepsilon)) = f_\varepsilon - \mathrm{div}\,G_\varepsilon \ \text{ in } \Omega_\varepsilon\\ u_\varepsilon=0 \text{ on } \Gamma_\varepsilon,\quad (A_\varepsilon e(u_\varepsilon)-G_\varepsilon)\nu=0 \ \text{ on } \partial\Omega_\varepsilon\setminus\Gamma_\varepsilon,\end{cases} \tag{6.6}$$

where ν denotes the unitary outward normal vector on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$.

An answer to the above question is given in Theorem 6.2 below. This result has been proved in [20] in the case where Ψ_ε is the null function. To take $\Psi_\varepsilon \not\equiv 0$ allows us to deal with slightly rough domains. In the case where $G = 0$ and where A_ε is less general (usually isotropic), problem (6.6) has been previously solved for example in [12], [16], [25]. We also refer to [9] for a related result to [20].

Remark 6.1 *As an example of sequence Ψ_ε we can take*

$$\Psi_\varepsilon(x_1, y) = r_\varepsilon \Psi\left(x_1, \frac{y}{\tau_\varepsilon}\right),$$

where $\Psi = \Psi(x_1, z)$ is a Lipschitz continuous function in $\mathbb{R}^{1+N}$, which is Y-periodic in z, with $Y = (-1/2, 1/2)^N$, and r_ε, τ_ε are two positive parameters such that

$$\lim_{\varepsilon\to 0} \tau_\varepsilon = \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\tau_\varepsilon} = 0.$$

In Figure 2 we represent Ω_ε in the case $N = 3$, $\vartheta = B((0,0),1) \subset \mathbb{R}^2$

$$\Psi(x_1, y) = \Big(\big(\sin\big((2\pi(y_1 + y_2)\big) + \sin(2\pi y_1)\big)x_1, \sin\big(2\pi(y_1 - y_2 + 2y_1)\big)(1 - x_1)\Big),$$

and $\varepsilon = 0.1$, $r_\varepsilon = 0.2$, $\tau_\varepsilon = 0.25$.

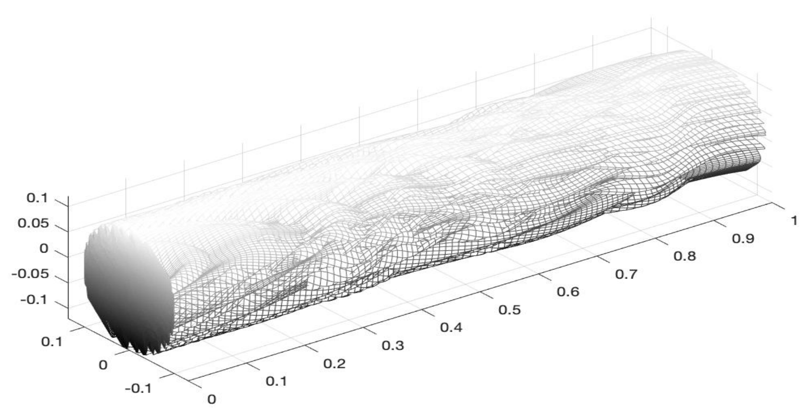

Figure 2: The elastic beam

In order to describe the asymptotic behavior of the solutions of (6.6) we need to introduce the following spaces:

We define the spaces W and $\mathcal{E}$ by

$$W = \big\{w \in L^2(\Omega)^N : \quad D_{y''} w \in L^2(\Omega)^{N\times(N-1)}\big\}, \tag{6.7}$$

$$\mathcal{E} = H_0^2(0,1)^{N-1} \times H_0^1(0,1) \times H_0^1(0,1)_a^{(N-1)^2} \times W. \tag{6.8}$$

For $(u^0, v_1, Z, w) \in \mathcal{E}$ we denote

$$E(u^0, v_1, Z, w) = \begin{pmatrix} \dfrac{dv_1}{dy_1} - \dfrac{d^2u^0}{dy_1^2} \cdot y'' & \dfrac{1}{2}\left(\nabla_{y''} w_1 + \dfrac{dZ}{dy_1} y''\right)^t \\ \dfrac{1}{2}\left(\nabla_{y''} w_1 + \dfrac{dZ}{dy_1} y''\right) & e_{y''}(w'') \end{pmatrix}. \tag{6.9}$$

The main result of the present section is given by the following theorem

Theorem 6.2 *For every $A \in L^\infty(\mathbb{R}^N; \mathcal{L}(\mathbb{R}^{N^2}_s))$, which satisfies (6.5) and every $f \in L^2(\mathbb{R}^N)^N$, $G \in L^2(\mathbb{R}^N)^{N^2}_s$, the solution u_ε of (6.6) satisfies*

$$\lim_{\varepsilon \to 0} \fint_{\Omega_\varepsilon} \left(\left| u_{\varepsilon,1} - v_1 + \frac{du^0}{dx_1} \cdot \frac{x''}{\varepsilon} \right|^2 + \left| \varepsilon u''_\varepsilon - u^0 \right|^2 \right) dx = 0, \tag{6.10}$$

$$\lim_{\varepsilon \to 0} \fint_{\Omega_\varepsilon} \left| e(u_\varepsilon) - E(u^0, v_1, Z, w) \left(x_1, \frac{x''}{\varepsilon} \right) \right|^2 dx = 0, \tag{6.11}$$

where (u^0, v_1, Z, w) is the solution of

$$\begin{cases} (u^0, v_1, Z, w) \in \mathcal{E} \\ \displaystyle\int_\Omega AE(u^0, v_1, Z, w) : E(\bar{u}^0, \bar{v}_1, \bar{Z}, \bar{w})\, dy \\ \displaystyle = \int_\Omega \left(f_1 \left(\bar{v}_1 - \frac{d\bar{u}^0}{dy_1} \cdot y'' \right) + f'' \cdot \bar{u}^0 + G : E(\bar{u}^0, \bar{v}_1, \bar{Z}, \bar{w}) \right) dy, \quad \forall\, (\bar{u}^0, \bar{v}_1, \bar{Z}, \bar{w}) \in \mathcal{E}. \end{cases} \tag{6.12}$$

Remark 6.3 *The existence of solution for problem (6.12) easily follows from Lax-Milgram Theorem. The functions u^0, v_1 and Z are unique while w is defined up to an additive function in $L^2(0,1)^N$. Observe that $E(u^0, v_1, Z, w)$ does not depend on such additive function.*

Consider the particular case of an elastic homogeneous isotropic material, i.e.

$$AM = \lambda\, \mathrm{tr}(M) I + 2\mu M, \quad \forall\, M \in \mathbb{R}^{N^2}_s,$$

with $\lambda, \mu > 0$ the Lamé coefficients, $G \equiv 0$ and f depending only on x_1, take the coordinate system such that

$$\int_\vartheta x_i\, dx' = 0, \qquad \int_\vartheta x_i x_j\, dx' = 0, \quad 2 \le i < j \le N.$$

Defining S_j with $j \ge 2$ and the Young's modulus E by

$$S_j = \fint_\vartheta x_j^2 dx', \quad E = \frac{2\mu(\lambda N + 2\mu)}{\lambda(N-1) + 2\mu},$$

one can check that v_1 and u^0 are the solutions of

$$-E\, \frac{d^2 v_1}{dx_1^2} = f_1 \quad en\ (0,1), \quad v_1(0) = v_1(1) = 0,$$

$$ES_j \frac{d^4 u_j^0}{dx_1^4} = f_j \quad en\ (0,1), \quad u_j^0(0) = u_j^0(1) = \frac{du_j^0}{dx_1}(0) = \frac{du_j^0}{dx_1}(1) = 0, \quad 2 \le j \le N,$$

while $Z = 0$, $w_1 = 0$ *and*

$$w_j = \frac{\lambda}{\lambda(N-1)+2\mu}\left(-\frac{dv_1}{dx_1}x_j + \frac{1}{2}\frac{d^2u_j^0}{dx_1^2}\left(x_j^2 - \sum_{\substack{k\geq 2\\ k\neq j}} x_k^2\right) + \sum_{\substack{k\geq 2\\ k\neq j}} \frac{d^2u_k^0}{dx_1^2}x_jx_k\right), \quad 2\leq j\leq N.$$

In particular, the equations for u_j^0 *(which are the terms of higher order in the approximation of* u_ε *given by (6.10)) are the classical equations for an elastic beam.*

The proof of Theorem 6.2 is based on the following Lemma. It complements Theorem 3.6 in the case of the thin beam Ω_ε, taking into account that the deformation u_ε vanishes in the extremities of the beam.

Lemma 6.4 *Consider a connected Lipschitz bounded open set* $\vartheta \subset \mathbb{R}^{N-1}$, $N \geq 2$, *with* $0 \in \vartheta$, *and a sequence of functions* $\Psi_\varepsilon \in W^{1,\infty}(\mathbb{R}^{N+1})^{N-1}$ *satisfying (6.1). Then, for* $r > 0$ *such that* $B(0, 2r) \subset \vartheta$ *and* $\varepsilon > 0$ *small enough to have* $(0, 1) \times B(0, \varepsilon r) \subset \Omega_\varepsilon$, *there exists a constant* $C > 0$ *such that every sequence* $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ *satisfying*

$$(u_\varepsilon)_{|x_1=0} = 0, \quad \forall\, \varepsilon > 0, \tag{6.13}$$

can be decomposed as

$$\begin{cases} u_{\varepsilon,1}(x) = -\dfrac{du_\varepsilon^0}{dx_1}(x_1)\cdot\dfrac{x''}{\varepsilon} + v_{\varepsilon,1}(x_1) + w_{\varepsilon,1}(x) \\[2ex] u_\varepsilon''(x) = \dfrac{1}{\varepsilon}u_\varepsilon^0(x_1) + Z_\varepsilon(x_1)\dfrac{x''}{\varepsilon} + v_\varepsilon''(x_1) + w_\varepsilon''(x), \end{cases} \tag{6.14}$$

with $u_\varepsilon^0 \in H^2(0,1)^{N-1}$, $Z_\varepsilon \in H^1(0,1)_a^{(N-1)^2}$, $v_\varepsilon \in H^1(0,1)^N$, $w_\varepsilon \in H^1(\Omega_\varepsilon)^N$

$$u_\varepsilon^0(0) = 0, \qquad v_\varepsilon(0) = 0, \qquad \left|\frac{du_\varepsilon^0}{dx_1}(0)\right| + \left|Z_\varepsilon(0)\right| \leq C\sqrt{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)_s^{N^2}}, \tag{6.15}$$

$$w_{\varepsilon,1}(0,x'') - \frac{du_\varepsilon^0}{dx_1}(0)\cdot\frac{x''}{\varepsilon} = 0, \quad Z_\varepsilon(0)\frac{x''}{\varepsilon} + w_\varepsilon''(0,x'') = 0, \qquad a.e.\ x'' \in \vartheta, \tag{6.16}$$

$$\begin{gathered}\varepsilon^{\frac{N-1}{2}}\big(\|u_\varepsilon^0\|_{H^2(0,1)^{N-1}} + \|Z_\varepsilon\|_{H^1(0,1)_a^{(N-1)^2}} + \|v_\varepsilon\|_{H^1(0,1)^N}\big) + \frac{1}{\varepsilon}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^N} + \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{N^2}} \\ \leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)_s^{N^2}}.\end{gathered} \tag{6.17}$$

Moreover, if u_ε *also satisfies*

$$(u_\varepsilon)_{|x_1=1} = 0, \quad \forall\, \varepsilon > 0, \tag{6.18}$$

then we can take u_ε^0, Z_ε, v_ε *and* w_ε *satisfying*

$$u_\varepsilon^0(1) = 0, \qquad v_\varepsilon(1) = 0, \qquad \left|\frac{du_\varepsilon^0}{dx_1}(1)\right| + \left|Z_\varepsilon(1)\right| \leq C\sqrt{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)_s^{N^2}}, \tag{6.19}$$

$$w_{\varepsilon,1}(1,x'') - \frac{du_\varepsilon^0}{dx_1}(1)\cdot\frac{x''}{\varepsilon} = 0, \quad Z_\varepsilon(1)\frac{x''}{\varepsilon} + w_\varepsilon''(1,x'') = 0, \qquad a.e.\ x'' \in \vartheta. \tag{6.20}$$

Proof. We first note that for every $z \in \mathbb{R}^N$, equation

$$x_1 = z_1, \quad \frac{x''}{\varepsilon} + \Psi_\varepsilon\Big(x_1, \frac{x}{\varepsilon}\Big) = z'' \tag{6.21}$$

is equivalent to

$$x_1 = z_1, \quad \frac{x''}{\varepsilon} + \Psi_\varepsilon\Big(z_1, \frac{z_1}{\varepsilon}, \frac{x''}{\varepsilon}\Big) = z'',$$

which taking into account that $\big\|D_{y''}\Psi_\varepsilon\|_{L^\infty(\mathbb{R}^{N+1})^{(N-1)^2}}$ tends to zero we can apply Banach fixed point theorem to get a unique solution. Therefore, the function $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\Phi_\varepsilon(x) = \bigg(x_1, \frac{x''}{\varepsilon} + \Psi_\varepsilon\Big(x_1, \frac{x}{\varepsilon}\Big)\bigg) \tag{6.22}$$

is bijective. On the other hand, we observe that assumptions (3.3) and (3.4) on Ω_ε are trivially satisfied with $\vartheta_\varepsilon = (0,1)$ while (3.5) can be easily proved using for every fixed $\tilde{x}_1 \in (0,1)$ the change of variables

$$y_1 = \frac{x_1 - \tilde{x}_1}{\varepsilon}, \quad y'' = \frac{x''}{\varepsilon} + \Psi_\varepsilon\Big(x_1, \frac{x}{\varepsilon}\Big),$$

which transforms

$$\hat{B}(\tilde{x}_1, \varepsilon) = \{x \in \Omega_\varepsilon : \; x_1 \in (\tilde{x}_1 - \varepsilon, \tilde{x}_1 + \varepsilon)\}$$

into the set

$$Q_{\varepsilon,\tilde{x}_1} := \big\{y \in (-1,1) \times \mathbb{R}^{N-1} : (\tilde{x}_1 + \varepsilon y_1, y'') \in \Omega\big\} = \Big(-\min\big\{1, \frac{\tilde{x}_1}{\varepsilon}\big\}, \min\big\{1, \frac{1-\tilde{x}_1}{\varepsilon}\big\}\Big) \times \vartheta,$$

and then using that $\big\|D_{y''}\Psi_\varepsilon\|_{L^\infty(\mathbb{R}^{N+1})^{(N-1)^2}}$ tends to zero combined with the existence of $C > 0$, independent of ε and $\tilde{x}_1$, such that

$$\bigg\|q - \fint_{Q_{\varepsilon,\tilde{x}_1}} q\,dy\bigg\|_{L^2(Q_{\varepsilon,\tilde{x}_1})} \leq C\|\nabla q\|_{H^{-1}(Q_{\varepsilon,\tilde{x}_1})^N}, \quad \forall\, q \in L^2(Q_{\varepsilon,\tilde{x}_1}),$$

where the last assertion follows from the smoothness assumptions on ϑ.

Thus, Ω_ε satisfies the conditions of Section 3. So, if $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ vanishes at $x_1 = 0$, we can apply Theorem 3.6 to get the decomposition

$$u_{\varepsilon,1}(x) = b_{\varepsilon,1} - \lambda''_\varepsilon \cdot x'' - \frac{d\hat{u}^0_\varepsilon}{dy_1}(x_1) \cdot \frac{x''}{\varepsilon} + \hat{v}_{\varepsilon,1}(x), \tag{6.23}$$

$$u''_\varepsilon(y) = b''_\varepsilon + \lambda''_\varepsilon x_1 + \Lambda_\varepsilon x'' + \frac{1}{\varepsilon}\hat{u}^0_\varepsilon(x_1) + \hat{Z}_\varepsilon(x_1)\frac{x''}{\varepsilon} + \hat{v}''_\varepsilon(x), \tag{6.24}$$

with $b_\varepsilon \in \mathbb{R}^N$, $\lambda''_\varepsilon \in \mathbb{R}^{N-1}$, $\Lambda_\varepsilon \in \mathbb{R}^{(N-1)^2}_a$, $\hat{u}^0_\varepsilon \in H^2(0,1)^{N-1}$, $\hat{Z}_\varepsilon \in H^1(0,1)^{(N-1)^2}_a$ and $v_\varepsilon \in H^1(\Omega_\varepsilon)^N$, satisfying

$$\varepsilon^{\frac{N-1}{2}}\Big(\|\hat{u}^0_\varepsilon\|_{H^2(0,1)^{N-1}} + \|\hat{Z}_\varepsilon\|_{H^1(0,1)^{(N-1)^2}_a}\Big) + \|\hat{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)^N} \leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N^2}} \tag{6.25}$$

and

$$0 = b_{\varepsilon,1} - \lambda''_\varepsilon \cdot x'' - \frac{d\hat{u}^0_\varepsilon}{dx_1}(0) \cdot \frac{x''}{\varepsilon} + \hat{v}_{\varepsilon,1}(0, x''),$$

$$0 = b''_\varepsilon + \Lambda_\varepsilon x'' + \frac{1}{\varepsilon}\hat{u}^0_\varepsilon(0) + \hat{Z}_\varepsilon(0)\frac{x''}{\varepsilon} + \hat{v}''_\varepsilon(0, x''),$$

which combined with (6.25) proves

$$\varepsilon^{\frac{N-1}{2}}|b_{\varepsilon,1}|+\varepsilon^{\frac{N+1}{2}}\big(|b''_\varepsilon|+|\lambda''_\varepsilon|\big)+\varepsilon^{\frac{N+3}{2}}|\Lambda_\varepsilon|\leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N^2}}.$$

Defining then

$$u^0_\varepsilon(x_1)=\hat u^0_\varepsilon(x_1)+\varepsilon b''_\varepsilon+\varepsilon\fint_{B_{N-1}(0,r\varepsilon)}\hat v''_\varepsilon(0,y'')dy''+\varepsilon\lambda''_\varepsilon x_1,$$

$$Z_\varepsilon(x_1)=\hat Z_\varepsilon(x_1)+\varepsilon\Lambda_\varepsilon,$$

$$v_{\varepsilon,1}(x_1)=b_{\varepsilon,1}+\fint_{B_{N-1}(0,r\varepsilon)}\hat v_{\varepsilon,1}(x_1,y'')dy'',\quad v''_\varepsilon(x_1)=\fint_{B_{N-1}(0,r\varepsilon)}\big(\hat v''_\varepsilon(x_1,y'')-\hat v''_\varepsilon(0,y'')\big)dy'',$$

$$w_\varepsilon(x)=\hat v_\varepsilon(x)-\fint_{B_{N-1}(0,r\varepsilon)}\hat v_\varepsilon(x_1,y'')dy'',$$

and taking into account that Poincaré-Wirtinger's inequality provides

$$\begin{aligned}&\int_{\{x'':(x_1,x'')\in\Omega_\varepsilon\}}\Big|\hat v_\varepsilon(x_1,x'')-\fint_{B_{N-1}(0,r\varepsilon)}\hat v_\varepsilon(x_1,y'')dy''\Big|^2dx''\\&\leq C\varepsilon^2\int_{\{x'':(x_1,x'')\in\Omega_\varepsilon\}}|D_{x''}v_\varepsilon(x_1,x'')|^2\,dx'',\quad \text{a.e. } x_1\in(0,1),\end{aligned}$$

we deduce that (6.14), (6.16), (6.17) and the two first assertions in (6.15) are satisfied. In order to show the last assertion in (6.15), we use thanks to (6.16), we have

$$\begin{aligned}&\Big|\frac{du^0_\varepsilon}{dx_1}(0)\Big|^2+\big|Z_\varepsilon(0)\big|^2\leq C\fint_{B_{N-1}(0,r\varepsilon)}|w_\varepsilon(0,x'')|^2dx''\\&\leq\frac{C}{\varepsilon}\int_0^\varepsilon\fint_{B_{N-1}(0,r\varepsilon)}|w_\varepsilon|^2dx''dx_1+C\varepsilon\int_0^\varepsilon\fint_{B_{N-1}(0,r\varepsilon)}|Dw_\varepsilon|^2dx''dx_1,\end{aligned}\tag{6.26}$$

where the second inequality just follows by using the change of variables $y=x/\varepsilon$ which transforms the open set $(0,\varepsilon)\times B_{N-1}(0,r\varepsilon)$ into the fixed open set $(0,1)\times B_{N-1}(0,r)$ and then the continuity of the trace operator. Using now (6.17) in (6.26) we conclude the last inequality in (6.15).

The case where u_ε also vanishes at $x_1=1$ follows similarly by defining Z_ε, $v_{\varepsilon,1}$ and w_ε as above and

$$u^0_\varepsilon(x_1)=\hat u^0_\varepsilon(x_1)+\varepsilon b''_\varepsilon+\varepsilon\fint_{B_{N-1}(0,r\varepsilon)}\Big((1-x_1)\hat v''_\varepsilon(0,y'')+x_1\hat v_\varepsilon(1,y'')\Big)dy''+\varepsilon\lambda''_\varepsilon x_1$$

$$v''_\varepsilon=\fint_{B_{N-1}(0,r\varepsilon)}\Big(\hat v''_\varepsilon(x_1,y'')-(1-x_1)\hat v''_\varepsilon(0,y'')-x_1\hat v''_\varepsilon(1,y'')\Big)dy''.$$

□

Remark 6.5 *In order to apply Lemma 6.4 to problem (6.6) we just need the case u_ε vanishing at $x_1=0$ and $x_1=1$. However, we have preferred to also state the case where u_ε vanishes only on one basis of the beam. It is the interesting case when we assume Dirichlet conditions in one of the basis and Neumann conditions in the other one.*

Observe that Lemma 6.4 does not allow to take u_ε and Z_ε satisfying the boundary conditions $\frac{du^0_\varepsilon}{dx_1}(0)=0$, $Z_\varepsilon(0)=0$. This is related with the apparition of boundary layer terms when we *deal with the asymptotic behavior of a thin structure (see e.g. [10], [22]). In fact, as a simple example of sequence u_ε in the conditions of Lemma 6.4 we can think in $N=2$ and*

$$u_{\varepsilon,1}(x)=-\frac{x_2}{\sqrt{\varepsilon}}\left(1-\varphi\Big(\frac{x_1}{\varepsilon}\Big)\right),\quad u_{\varepsilon,2}(x)=\frac{x_1}{\sqrt{\varepsilon}},$$

with $\varphi\in C^\infty([0,\infty))$ with compact support and satisfying $\varphi(0)=1$. In this case the sequences v_ε, Z_ε and w''_ε vanish while $u^0_\varepsilon(x)=\sqrt{\varepsilon}x_1$ and $w_{\varepsilon,1}$ is the boundary layer term

$$w_{\varepsilon,1}(x)=\varphi\Big(\frac{x_1}{\varepsilon}\Big)\frac{x_2}{\sqrt{\varepsilon}}.$$

Proof of Theorem 6.2. Using u_ε as test function in (6.6) and taking into account decomposition (6.14) of u_ε and (6.17), we deduce the estimate

$$\frac{1}{\varepsilon^{N-1}}\int_{\Omega_\varepsilon}|e(u_\varepsilon)|^2dx\leq C, \tag{6.27}$$

which combined with (6.15), (6.15), (6.17), (6.19) and (6.20) proves the existence of a subsequence of ε still denoted by ε and functions $u^0\in H^2_0(0,1)^{N-1}$, $Z_0\in H^1_0(0,1)^{(N-1)^2}_a$ and $v\in H^1_0(0,1)^N$ such that

$$u_\varepsilon\rightharpoonup u\ \text{ in } H^2(0,1)^{N-1},\quad Z_\varepsilon\rightharpoonup Z_0\ \text{ in } H^1(0,1)^{(N-1)^2}_a,\quad v_\varepsilon\rightharpoonup v\ \text{ in } H^1(0,1)^N. \tag{6.28}$$

Moreover, using the estimates for w_ε provided by (6.17) and defining W_ε by

$$W_\varepsilon(y)=w_\varepsilon\Big(y_1,\varepsilon y''\Big),$$

we also deduce the existence of $w\in L^2(0,1;H^1(\vartheta))^N$ such that

$$\frac{1}{\varepsilon}D_{y''}W_\varepsilon\rightharpoonup D_{y''}w\ \text{ in } L^2\Big(\{y\in\Omega:\ \text{dist}(y,\partial\Omega\setminus\Gamma)>\delta\}\Big)^{N(N-1)},\ \forall\,\delta>0. \tag{6.29}$$

Now, we consider functions $\bar u^0\in H^2_0(0,1)^{N-1}$, $\bar v_1\in H^1_0(0,1)$, $\bar Z\in H^1_0(0,1)^{(N-1)^2}_a$ and $\bar w\in H^1((0,1)\times\mathbb{R}^{N-1})^N$ such that $\bar w=0$ on $\{0,1\}\times\mathbb{R}^{N-1}$ and we define $\bar u_\varepsilon\in H^1(\Omega_\varepsilon)^N$ by

$$\left\{\begin{aligned}&\bar u_{\varepsilon,1}(x)=\bar v_1(x_1)-\frac{d\bar u^0}{dy_1}(x_1)\cdot\frac{x''}{\varepsilon}+\varepsilon\bar w_1\left(x_1,\frac{x''}{\varepsilon}\right)\\&\bar u''_\varepsilon(x)=\frac{1}{\varepsilon}\bar u^0(x_1)+\bar Z(x_1)\frac{x''}{\varepsilon}+\varepsilon\bar w''\left(x_1,\frac{x''}{\varepsilon}\right)\end{aligned}\right.\qquad\text{a.e. } x\in\Omega_\varepsilon.$$

Taking $\bar u_\varepsilon$ as test function in (6.6) and dividing by ε^{N-1} we get

$$\frac{1}{\varepsilon^{N-1}}\int_{\Omega_\varepsilon}A\left(x_1,\frac{x'}{\varepsilon}\right)e(u_\varepsilon):e(\bar u_\varepsilon)\,dx=\frac{1}{\varepsilon^{N-1}}\int_{\Omega_\varepsilon}\Big(f_\varepsilon\cdot\bar u_\varepsilon+G_\varepsilon:e(\bar u_\varepsilon)\Big)dx. \tag{6.30}$$

In order to pass to the limit in the left-hand side of (6.30) we use the decompositon

$$\Omega_\varepsilon=\Big((0,1)\times(\varepsilon\vartheta^\delta)\Big)\cup\Big(\Omega_\varepsilon\setminus\big((0,1)\times(\varepsilon\vartheta^\delta)\big)\Big)$$

with

$$\vartheta^\delta=\big\{y''\in\vartheta:\ \text{dist}(y,\partial\vartheta)>\delta\big\}.$$

Using the change of variables $y_1 = x_1$, $y'' = x''/\varepsilon$, (6.14), (6.28) and (6.29) we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{(0,1)\times(\varepsilon\vartheta^\delta)} Ae(u_\varepsilon) : e(\bar u_\varepsilon)\, dx = \int_{(0,1)\times\vartheta^\delta} AE(u^0, v_1, Z, w) : E(\bar u^0, \bar v_1, \bar Z, \bar w)\, dy. \quad (6.31)$$

On the other hand, taking into account that $e(\bar u_\varepsilon)$ is bounded in $L^\infty((0,1)\times\mathbb{R}^{N-1})_s^{N^2}$ we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_\varepsilon\setminus((0,1)\times(\varepsilon\vartheta^\delta))} Ae(u_\varepsilon) : e(\bar u_\varepsilon)\, dx \right| \\ &\le C \limsup_{\varepsilon\to 0} \left(\frac{1}{\varepsilon^{N-1}} \int_{\Omega_\varepsilon} \left| e(u_\varepsilon)\right|^2 dx \right)^{\frac12} \frac{\left|\Omega_\varepsilon \setminus \left((0,1)\times(\varepsilon\vartheta^\delta)\right)\right|^{\frac12}}{\varepsilon^{\frac{N-1}{2}}},\end{aligned}$$

where thanks to the definition of Ω_ε, the last factor in this inequality tends to zero if ε and then δ tend to zero. Using then (6.27) and (6.31) and passing to the limit first in ε and then in δ we conclude

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Omega_\varepsilon} A\left(x_1, \frac{x'}{\varepsilon}\right) e(u_\varepsilon) : e(\bar u_\varepsilon)\, dx = \int_\Omega AE(u^0, v_1, Z, w) : E(\bar u^0, \bar v_1, \bar Z, \bar w)\, dy.$$

Analogously,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Omega_\varepsilon} \Big(f_\varepsilon \cdot \bar u_\varepsilon + G_\varepsilon : e(\bar u_\varepsilon)\Big) dx \\ &= \int_\Omega \left(f_1 \left(\bar v_1 - \frac{d\bar u^0}{dy_1} \cdot y'' \right) + f'' \cdot \bar u^0 + G : E(\bar u^0, \bar v_1, \bar Z, \bar w) \right) dy.\end{aligned}$$

So, passing to the limit in (6.30) we have proved that the variational equation in (6.12) holds for every $\bar u^0 \in H_0^2(0,1)^{N-1}$, $\bar v_1 \in H_0^1(0,1)$, $\bar Z \in H_0^1(0,1)_a^{(N-1)^2}$ and $\bar w \in H^1((0,1)\times\mathbb{R}^{N-1})^N$ such that $w = 0$ on $\{0,1\}\times\mathbb{R}^{N-1}$. By density, we then deduce that it holds for every $(u^0, v_1, Z, w) \in \mathcal{E}$.

Now, we take u_ε as test function in (6.6) and we divide by ε^{N-1}. This gives

$$\frac{1}{\varepsilon^{N-1}} \int_{\Omega_\varepsilon} A\left(x_1, \frac{x'}{\varepsilon}\right) e(u_\varepsilon) : e(u_\varepsilon)\, dx = \frac{1}{\varepsilon^{N-1}} \int_{\Omega_\varepsilon} \Big(f_\varepsilon \cdot u_\varepsilon + G_\varepsilon : e(u_\varepsilon)\Big) dx.$$

Reasoning as above we can easily pass to the limit in the right-hand side of this equality, which thanks to (6.12) allows us to prove the convergence of the energies

$$\lim_{\varepsilon\to 0} \fint_{\Omega_\varepsilon} Ae(u_\varepsilon) : e(u_\varepsilon)\, dx = \fint_\Omega AE(u^0, v_1, Z, w) : E(u^0, v_1, Z, w)\, dy.$$

From this equality and (6.14) is now simple to check (6.10). □

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References

[1] G. Allaire. *Homogenization and two-scale convergence.* SIAM J. Math Anal. 23 (1992), 1482–151.

[2] G. Allaire, C. Conca. *Bloch-wave homogenization for a spectral problem in fluid-solid structures.* Arch. Rational Mech. Anal. 135 (1996), 196-257.

[3] T. Arbogast, J. Douglas, U. Hornung. *Derivation of the double porosity model of single phase flow via homogenization theory*. SIAM J. Math. Anal. 21 (1990), 823-836.

[4] G. Bayada, M. Chambat. *Homogenization of the Stokes system in a thin film flow with rapidly varying thickness*. RAIRO Model. Math. Anal. Numer. 23 (1989), 205-234.

[5] N. Benhaboucha, M. Chambat, I. Ciuperca. *Asymptotic behaviour of pressure and stresses in a thin film flow with a rough boundary.* Quart. of Appl. Math. 63 (2005), 369-400.

[6] D. Blanchard, G. Griso. *Decomposition of deformations of thin rods. Application to non-linear elasticity.* Anal. Appl. 7 (2009), 21–71.

[7] D. Bresch, C. Choquet, L. Chupin, T. Colin, M. Glisclon. *Roughness-Induced Effect at Main order on the Reynolds Approximation.* Multiscale Model. Simul. 8 (2010), 997-1017.

[8] J. Casado-Díaz. *Two-scale convergence for nonlinear Dirichlet problems in perforated domains.* Proc. Roy. Soc. Edimburgh 130 A (2000), 249–276.

[9] J. Casado-Díaz, M. Luna-Laynez. *Homogenization of the anisotropic heterogeneous linearized elasticity system in thin reticulated structures.* Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), 1041–1083.

[10] J. Casado-Díaz, M. Luna-Laynez, F. Murat. *The behavior of a beam fixed on small sets of one of its extremities.* Discrete Contin. Dyn. Syst. 34 (2014), 4039–4070.

[11] J. Casado-Díaz, M. Luna-Laynez, F.J. Suárez-Grau. *Asymptotic behavior of the Navier-Stokes system in a thin domain with Navier condition on a slightly rough boundary.* SIAM J. Math. Anal. 45 (2013), 1641–1674.

[12] P.G. Ciarlet. *Mathematical Elasticity, Volume II: Theory of Plates.* North-Holland, Amsterdam, 1998.

[13] D. Cioranescu, A. Damlamian, G. Griso. *Periodic unfolding and homogenization.* C.R. Acad. Sci. Paris I 335 (2002), 99–104.

[14] D. Cioranescu, A. Damlamian, G. Griso. *The Periodic Unfolding Method in Homogenization.* SIAM J. Math. Anal. 40 (2008), 1585–1620.

[15] C. Conca, M. Vanninathan. *A spectral problem arising in fluid-solid structures.* Comput. Methods Appl. Mech. Engrg. 69 (1988), 215-242.

[16] H. Le Dret. *Convergence of displacements and stresses in linearly elastic slender rods as the thickness goes to zero.* Asympt. Anal. 10 (1995), 367-402.

[17] G.P. Galdi. *An introduction to the Mathematical theory of the Navier-Stokes equations, Volume I.* Springer, New York, 1994.

[18] G. Griso. *Decompositions of displacements of thin structures.* J. Math. Pures Appl. 89 (2008), 199–223.

[19] M. Lenczner. *Homogénéisation d'un circuit électrique.* C. R. Acad. Sci. Paris II B 324 (1997), 537–542.

[20] F. Murat, A. Sili. *Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces.* C. R. Acad. Sci. Paris I, 328 (1999), 17–184.

[21] G. Nguetseng. *A general convergence result for a functional related to the theory of homogenization.* SIAM J. Math. Anal. 20 (1989), 608–623.

[22] G. Panasenko. *Multi-Scale Modelling for Structures and Composites.* Springer, Dordrecht, 2005.

[23] L. Tartar. *Topics in nonlinear analysis.* Publications Mathématiques d'Orsay 78, 1978.

[24] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* North-Holland, Amsterdam, 1984.

[25] L. Trabucho, J.M. Viaño. *Mathematical modelling of rods. Handbook of Numerical Analysis, Volume IV.* North-Holland, Amsterdam, 1996.