Mathematical modeling of micropolar fluid flows through a thin porous medium

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Abstract

We study the flow of a micropolar fluid in a thin domain with microstructure, i.e. a thin domain with thickness ε which is perforated by periodically distributed solid cylinders of size a_ε. A main feature of this study is the dependence of the characteristic length of the micropolar fluid on the small parameters describing the geometry of the thin porous medium under consideration. Depending on the ratio of a_ε with respect to ε, we derive three different generalized Darcy equations where the interaction between the velocity and the microrotation fields is preserved.

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1 Introduction

Based on the micropolar fluid theory [27, 28], which takes into account the effects of solid particles additive in a Newtonian fluid, we study flows of micropolar fluids in a thin domain which is perforated by periodically distributed solid cylinders (microstructure) which is called thin porous medium (**TPM**). This type of domains include two small parameters: one called ε is connected to the fluid film thickness and the other called a_ε to the microstructure representing the size of the cylinders and the interspatial distance between them. The behavior of fluid flows through **TPM** has been studied extensively, mainly because of its importance in many industrial processes, see [30, 31, 35, 41, 43, 44]. However, the literature on non-Newtonian micropolar fluid flows in this type of domains is far less complete, although these problems have now become of great practical relevance. Therefore, the objetive of this paper is to derive generalized micropolar Darcy equations for the pressure depending on the magnitude of the parameters involving the **TPM**.

Definition of the TPM. A periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = (-1/2, 1/2)^2$ which is made of two complementary parts: the fluid part Y'_f, and the solid part Y'_s ($Y'_f \cup Y'_s = Y'$ and $Y'_f \cap Y'_s = \emptyset$). More precisely, we assume that ω is a smooth, bounded, connected set in $\mathbb{R}^2$ and that Y'_s is an open connected subset of Y' with a smooth boundary $\partial Y'_s$, such that $\overline{Y'_s}$ is strictly included in Y'.

The microscale of a porous medium is given by a small positive number a_ε. The domain ω is covered by a regular mesh of size a_ε: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',a_\varepsilon} = a_\varepsilon k' + a_\varepsilon Y'$ is divided in a fluid part $Y'_{f_{k'},a_\varepsilon}$ and a solid part $Y'_{s_{k'},a_\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size a_ε. We define $Y = Y' \times (0,1) \subset \mathbb{R}^3$, which is divided in a fluid part Y_f and a solid part Y_s, and consequently $Y_{k',a_\varepsilon} = Y'_{k',a_\varepsilon} \times (0,1) \subset \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k'},a_\varepsilon}$, and a solid part $Y_{s_{k'},a_\varepsilon}$.

We denote by $\tau(\overline{Y}'_{s_{k'},a_\varepsilon})$ the set of all translated images of $\overline{Y}'_{s_{k'},a_\varepsilon}$. The set $\tau(\overline{Y}'_{s_{k'},a_\varepsilon})$ represents the solids in $\mathbb{R}^2$. The fluid part of the bottom $\omega_\varepsilon \subset \mathbb{R}^2$ of the porous medium is defined by $\omega_\varepsilon = \omega \setminus \bigcup_{k' \in \mathcal{K}_\varepsilon} \overline{Y}_{s_{k'},a_\varepsilon}$, where $\mathcal{K}_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',a_\varepsilon} \cap \omega \neq \emptyset\}$. The whole fluid part $\Omega_\varepsilon \subset \mathbb{R}^3$ is defined by

$$\Omega_\varepsilon = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega_\varepsilon,\ 0 < x_3 < \varepsilon\}. \tag{1.1}$$

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We make the assumption that the solids $\tau(\overline{Y}'_{s_{k'},a_\varepsilon})$ do not intersect the boundary $\partial\omega$. We define $Y^\varepsilon_{s_{k'},a_\varepsilon} = Y'_{s_{k'},a_\varepsilon} \times (0,\varepsilon)$. Denote by S_ε the set of the solids contained in Ω_ε. Then, S_ε is a finite union of solids, i.e. $S_\varepsilon = \bigcup_{k'\in\mathcal{K}_\varepsilon} \overline{Y}^\varepsilon_{s_{k'},a_\varepsilon}$. We define $\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1)$, $\Omega = \omega \times (0,1)$, and $Q_\varepsilon = \omega \times (0,\varepsilon)$. We observe that $\widetilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k'\in\mathcal{K}_\varepsilon} \overline{Y}_{s_{k'},a_\varepsilon}$, and we define $T_\varepsilon = \bigcup_{k'\in\mathcal{K}_\varepsilon} \overline{Y}_{s_{k'},a_\varepsilon}$ as the set of the solids contained in $\widetilde{\Omega}_\varepsilon$. We remark that along this paper, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

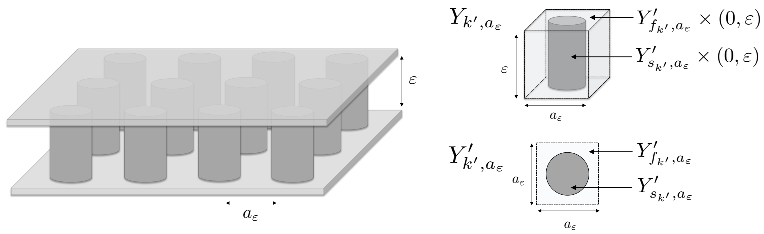

Figure 1: View of the domain Ω_ε (left) and periodic cells Y_{k',a_ε} and Y'_{k',a_ε} (right).

A direct numerical treatment of fluid flows through **TPM** becomes very difficult due to the rapid variations on the microscale level, so it would be necessary to obtain macroscopic laws to describe the fluid flows in such a domain. Thus, due to the description of the domain by using the parameters ε and a_ε, it is possible to describe the macroscopic behavior by means of the homogenization theory, developed in the studies of partial differential equations for strongly heterogeneous problems. In this sense, for Newtonian fluids, this problem has been addressed in [29] proving the existence of three types of **TPM** depending on the relation between the parameters ε and a_ε:

- The proportionally thin porous medium (**PTPM**), corresponding to the critical case when the cylinder height is proportional to the interspatial distance, with λ the proportionality constant, that is $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$.
- The homogeneously thin porous medium (**HTPM**), corresponding to the case when the cylinder height is much larger than interspatial distance, i.e. $a_\varepsilon \ll \varepsilon$ which is equivalent to $\lambda = 0$.
- The very thin porous medium (**VTPM**), corresponding to the case when the cylinder height is much smaller than the interspatial distance, i.e. $a_\varepsilon \gg \varepsilon$ which is equivalent to $\lambda = +\infty$.

In particular, denoting the velocity and the pressure in the **TPM** by u_ε and p_ε, respectively, and starting from the following Stokes system

$$\left\{\begin{aligned} -\Delta u_\varepsilon + \nabla p_\varepsilon &= f && \text{in} \quad \Omega_\varepsilon,\\ \operatorname{div} u_\varepsilon &= 0 && \text{in} \quad \Omega_\varepsilon,\\ u_\varepsilon &= 0 && \text{on} \quad \partial Q_\varepsilon \cup \partial S_\varepsilon, \end{aligned}\right.$$

it can be deduced, when ε tends to zero, that the flow is governed by the following 2D Darcy equation depending on $\lambda \in [0,+\infty]$,

$$\left\{\begin{array}{l} u'(x') = K_\lambda(f'(x') - \nabla_{x'} p(x')), \quad u_3(x') = 0 \quad \text{in } \omega,\\ \operatorname{div}_{x'} u' = 0 \quad \text{in } \omega, \quad u'\cdot n = 0 \quad \text{on } \partial\omega, \end{array}\right. \tag{1.2}$$

where $u = (u', u_3)$ is the velocity, p is the pressure and $K_\lambda \in \mathbb{R}^{2\times 2}$ is a macroscopic quantity known as flow factor which takes into account the microstructure of the **TPM**:

- In the **PTPM**, the flow factor K_λ, $0 < \lambda < +\infty$, is calculated by solving 3D Stokes local problems depending on the parameter λ. More precisely, $(K_\lambda)_{ij} = \int_{Y_f} D_\lambda u^i(y) : D_\lambda u^j(y)\, dy$, $i,j = 1,2$, where the function $u^i(y)$, $i = 1,2$, is the solution of the local 3D Stokes local problem posed in Y

$$\begin{cases} -\Delta_\lambda u^i + \nabla_\lambda \pi^i = e_i & \text{in} \quad Y_f, \\ \operatorname{div}_\lambda u^i = 0 & \text{in} \quad Y_f, \\ u^i = 0 & \text{in} \quad Y_s, \\ u^i(y), \pi^i(y) & Y' - \text{periodic}, \end{cases}$$

where $D_\lambda = D_{y'} + \lambda \partial_{y_3}$, $\Delta_\lambda = \Delta_{y'} + \lambda^2 \partial^2_{y_3}$, $\nabla_\lambda = (\nabla_{y'}, \lambda \partial_{y_3})^t$, $\operatorname{div}_\lambda = \operatorname{div}_{y'} + \lambda \partial_{y_3}$ and $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

- In the **HTPM**, the flow factor K_0 is calculated by solving 2D Stokes local problems. More precisely, $(K_0)_{ij} = \int_{Y'_f} D_{y'} u^i(y') : D_{y'} u^j(y')\, dy'$, $i,j = 1,2$, where the function $u^i(y')$, $i = 1,2$, is the solution of the 2D Stokes local problem posed in Y'

$$\begin{cases} -\Delta_{y'} u^i + \nabla_{y'} \pi^i = e_i & \text{in} \quad Y'_f, \\ \operatorname{div}_{y'} u^i = 0 & \text{in} \quad Y'_f, \\ u^i = 0 & \text{in} \quad Y'_s, \\ u^i(y'), \pi^i(y') & Y' - \text{periodic}. \end{cases}$$

- In the **VTPM**, the flow factor K_∞ is calculated by solving 2D Hele-Shaw local problems. More precisely, $(K_\infty)_{ij} = \int_{Y'_f} \left(e_i + \nabla_{y'} \pi^i\right) e_j\, dy'$, $i,j = 1,2$, where the function $\pi^i(y')$, $i = 1,2$, is the solution of the 2D Hele-Shaw local problem posed in Y'

$$\begin{cases} \Delta_{y'} \pi^i = 0 & \text{in} \quad Y'_f, \\ \left(\nabla_{y'} \pi^i + e_i\right) \cdot n = 0 & \text{in} \quad Y'_s, \\ \pi^i(y') & Y' - \text{periodic}. \end{cases}$$

From the above, it is obtained that the model problem considered as an average problem could be solved by using the following homogenization procedure:

1. Solve the local problem numerically corresponding to the value of $\lambda \in [0, +\infty]$.
2. Use the solution to compute the components of the flow factor K_λ.
3. Find p by solving the homogenized problem $(1.2)_2$ numerically.
4. Compute u by means of $(1.2)_1$.

Remark that in the intermediate case **PTPM**, the local problems are three-dimensional and the coefficient of proportionality λ appears as a parameter in the equations. In the extreme cases **HTPM** and **VTPM**, the local problems are two-dimensional which, from the numerical point of view, represents a considerable simplification compared with the intermediate case.

These results were proved in [29] by using the multiscale expansion method, which is a formal but powerful tool to analyze homogenization problems, and later rigorously developed in [14] by using an adaptation of the periodic unfolding method [18, 23, 24]. This adaptation consists of a combination of the unfolding method in the horizontal variables with a rescaling in the height variable, in order to work with a domain of fixed height, and then to use suitable compactness results to pass to the limit when the geometrical parameters ε and a_ε tend to zero. We remark that this adaptation was developed in [13] to study the case of non-Newtonian power law fluids in the **TPM** and it was recently applied to the case of non-Newtonian Bingham fluids in [11, 12]. For other studies concerning **TPM**, see [7, 8, 9, 10, 15, 16, 17].

On the other hand, micropolar fluids are very important in industrial and engineering applications, see for instance [1, 2, 3, 4, 5, 33, 38, 39, 40]. In view of that, in this paper we consider a non-Newtonian micropolar fluid flow in **TPM**, denoted by Ω_ε, governed by the linearized micropolar equations, with body forces f_ε and body torque g_ε, written in a non-dimensional form (see [34] for more details)

$$\left\{\begin{array}{rl} -\Delta u_\varepsilon + \nabla p_\varepsilon = 2N^2 \mathrm{rot}\, w_\varepsilon + f_\varepsilon & \text{in} \quad \Omega_\varepsilon,\\ \mathrm{div}\, u_\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon,\\ -R_M \Delta w_\varepsilon + 4N^2 w_\varepsilon = 2N^2 \mathrm{rot}\, u_\varepsilon + g_\varepsilon & \text{in} \quad \Omega_\varepsilon,\\ u_\varepsilon = w_\varepsilon = 0 & \text{on} \quad \partial Q_\varepsilon \cup \partial S_\varepsilon. \end{array}\right. \tag{1.3}$$

In the system (1.3), the velocity u_ε, the pressure p_ε and the microrotation w_ε (i.e. the angular velocity field of rotation of particles) are unknown. Observe that equation of the linear momentum $(1.3)_1$ has the familiar form of the Stokes equation but it is coupled with equation of the angular momentum $(1.3)_3$, which esentially describes the motion of the particles inside the microvolume as they undergo microrotational effects represented by the microrotational vector w_ε. Dimensionless (non-Newtonian) parameter N^2 with $0 < N < 1$ is called coupling number and it characterizes the coupling of the linear and angular momentum equations. When N is identically zero, the equations are decoupled and equation of the linear momentum reduces to the classical Stokes equations which represent Newtonian fluids. The parameter R_M is called characteristic length and it characterizes the interaction between the micropolar fluid and the microgeometry of the domain. It is small and has to be related to the other small geometrical parameters depending on the type of **TPM**, i.e. we will assume that $R_M = a_\varepsilon^2 R_c$ in the **PTPM** and **HTPM**, and $R_M = \varepsilon^2 R_c$ in the **VTPM**, where $R_c = \mathcal{O}(1)$ (see Section 3 for more details).

By using the homogenization techniques developed in [13], when ε tends to zero, we derive that the flow is governed by a generalized 2D Darcy equation depending on the $\lambda \in [0, +\infty]$,

$$\left\{\begin{array}{ll} u'(x') = K_\lambda^{(1)}\left(f'(x') - \nabla_{x'} p(x')\right) + K_\lambda^{(2)} g(x'), & u_3(x') = 0 \quad \text{in } \omega,\\ w'(x') = L_\lambda^{(1)}\left(f'(x') - \nabla_{x'} p(x')\right) + L_\lambda^{(2)} g(x'), & w_3(x') = 0 \quad \text{in } \omega,\\ \mathrm{div}_{x'} u' = 0 \quad \text{in } \omega, \quad u'\cdot n = 0 \quad \text{on } \partial\omega, & \end{array}\right. \tag{1.4}$$

where the flow factors $K_\lambda^{(k)}, L_\lambda^{(k)} \in \mathbb{R}^{2\times 2}$, $k = 1, 2$, are calculated depending on the microstructure of the **TPM**:

- In the **PTPM**, the flow factors $K_\lambda^{(k)}, L_\lambda^{(k)}$, $0 < \lambda < +\infty$ are calculated by solving 3D micropolar local problems posed in the 3D unit cell Y and depending on the parameter λ, the coupling number and the characteristic length (see Theorem 4.3). More precisely, $(K_\lambda^{(k)})_{ij} = \int_{Y_f} u_j^{i,k}(y)\,dy$, $(L_\lambda^{(k)})_{ij} = \int_{Y_f} w_j^{i,k}(y)\,dy$, $i, j, k = 1, 2$, where $u^{i,k}(y)$, $w^{i,k}(y)$, $i, k = 1, 2$, are the solution of the 3D micropolar local problems posed in Y
$$\left\{\begin{array}{rl} -\Delta_\lambda u^{i,k} + \nabla_\lambda \pi^{i,k} - 2N^2 \mathrm{rot}_\lambda w^{i,k} = e_i \delta_{1k} & \text{in } Y_f,\\ \mathrm{div}_\lambda u^{i,k} = 0 & \text{in } Y_f,\\ -R_c \Delta_\lambda w^{i,k} + 4N^2 w^{i,k} - 2N^2\, \mathrm{rot}_\lambda u^{i,k} = \ e_i \delta_{2k} & \text{in } Y_f,\\ u^{i,k} = w^{i,k} = 0 & \text{in } Y_s,\\ u^{i,k}(y), w^{i,k}(y), \pi^{i,k}(y) \quad Y' - \text{periodic}, & \end{array}\right.$$
where $\mathrm{rot}_\lambda v = (\mathrm{rot}_{y'} v_3 + \lambda \mathrm{rot}_{y_3} v', \mathrm{Rot}_{y'} v')$, with $\mathrm{rot}_{y'}$, rot_{y_3} and $\mathrm{Rot}_{y'}$ defined in (2.9).

- In the **HTPM**, the flow factors $K_0^{(k)}, L_0^{(k)}$ are calculated by solving 3D micropolar local problems posed in the 2D unit cell Y' and depending on the coupling number and the characteristic length (see Theorem 5.3). More precisely, $(K_0^{(k)})_{ij} = \int_{Y_f'} u_j^{i,k}(y')\,dy', (L_0^{(k)})_{ij} = \int_{Y_f'} w_j^{i,k}(y')\,dy'$, $i, j, k = 1, 2$, where $u^{i,k}(y')$, $w^{i,k}(y')$,

$i,k=1,2$, are the solution of the 3D micropolar local problems posed in Y'

$$\begin{cases} -\Delta_{y'}(u^{i,k})' + \nabla_{y'}\pi^{i,k} - 2N^2 \mathrm{rot}_{y'} w_3^{i,k} = e_i\delta_{1k} & \text{in } Y'_f,\\ -\Delta_{y'}u_3^{i,k} - 2N^2 \mathrm{Rot}_{y'}(w^{i,k})' = 0 & \text{in } Y'_f,\\ \mathrm{div}_{y'}(u^{i,k})' = 0 & \text{in } Y_f,\\ -R_c\Delta_{y'}(w^{i,k})' + 4N^2(w^{i,k})' - 2N^2\,\mathrm{rot}_{y'}u_3^{i,k} = e_i\delta_{2k} & \text{in } Y'_f,\\ -R_c\Delta_{y'}w_3^{i,k} + 4N^2 w_3^{i,k} - 2N^2\,\mathrm{Rot}_{y'}(u^{i,k})' = 0 & \text{in } Y'_f,\\ u^{i,k} = w^{i,k} = 0 & \text{in } Y'_s,\\ u^{i,k}(y'), w^{i,k}(y'), \pi^{i,k}(y') \quad Y'-\text{periodic.} \end{cases}$$

- In the **VTPM**, the flow factors $K_\infty^{(k)}, L_\infty^{(k)}$ are calculated by solving 2D local micropolar Reynolds problems posed in the 2D unit cell Y' and depending on the coupling number and the characteristic length, see Theorem 6.3. More precisely, $(K_\infty^{(k)})_{ij} = \frac{1}{1-N^2}\int_{Y'_f}\Phi(N,R_c)\left(\partial_{y_i}\pi^{j,k}(y') + \delta_{ij}\delta_{1k}\right)dy'$, $L_\infty^{(1)} = 0$, $(L_\infty^{(2)})_{ij} = -\frac{1}{4N^3}\sqrt{\frac{R_c}{1-N^2}}\left(\int_{Y'_f}\Psi(N,R_c)\,dy'\right)\delta_{ij}$, $i,j,k=1,2$, where $\pi^{i,k}(y')$, $i,k=1,2$, is the unique solutions of the local problems

$$\begin{cases} -\mathrm{div}_{y'}\left(\dfrac{1}{1-N^2}\Phi(N,R_c)\left(\nabla_{y'}\pi^{i,k}(y') + e_i\delta_{1k}\right)\right) = 0 & \text{in } Y'_f\,,\\ \left(\frac{1}{1-N^2}\Phi(N,R_c)\left(\nabla_{y'}\pi^{i,k}(y') + e_i\delta_{1k}\right)\right)\cdot n = 0 & \text{in } \partial Y'_s\,, \end{cases}$$

with Φ and Ψ defined by (6.79) and (6.80) respectively.

Therefore, the average problem (1.4) could be solved by using a procedure similar to the one described above for the Newtonian case:

1. Solve the local problem numerically corresponding to the value of $\lambda \in [0,+\infty]$.
2. Use the solution to compute the components of the flow factors K_λ and L_λ.
3. Find p by solving the homogenized problem $(1.4)_3$ numerically.
4. Compute u by means of $(1.4)_1$ and w by means of $(1.4)_2$.

We also observe that in the intermediate case **PTPM**, the local problems are three-dimensional and the coefficient of proportionality λ appears as a parameter in the equations. Moreover, in the extreme cases **HTPM** and **VTPM**, the local problems are simpler, which represents a considerable simplification from the numerical point of view. As far as the author knows, this is the first attempt to carry out such a theoretical analysis for micropolar fluids, which could be instrumental for understanding the effects on the flows of micropolar fluids and the microstructure of the domain. In view of that, more efficient numerical algorithms could be developed improving, hopefully, the known engineering practice.

The paper is organized as follows. In Section 2, we introduce the problem under consideration. In Section 3, we give some a priori estimates for the dilated velocity, the microrotation and the pressure, we introduce the extension of the unknowns to the whole domain Ω, and finally we recall the version of the unfolding method necessary to pass to the limit in the next sections. Namely, we analyze the case **PTPM** in Section 4, the case **HTPM** in Section 5 and the case **VTPM** in Section 6. The paper ends with a conclusion section and with a list of references.

2 Statement of the problem

In this section, we introduce the problem under consideration and also the rescaled problem posed in a domain of fixed height. We finish this section giving the equivalent weak variational formulations. To do this, let us first define some functional spaces which are necesary along the paper. Let $C^\infty_\#(Y)$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are Y'-periodic. By $L^2_\#(Y)$ (resp. $H^1_\#(Y)$) we denote its completion in the norm $L^2(Y)$ (resp. $H^1(Y)$) and by $L^2_{0,\#}(Y)$ the space of functions in $L^2_\#(Y)$ with zero mean value.

When the distance between two surfaces becomes very small, the experimental results from the tribology literature (see e.g. [32, 36, 37]) suggest that the fluid's internal structure should be taken into account as well. Among various non-Newtonian models, the model of micropolar fluid (proposed by Eringen [28] in 60's) turns out to be the most appropriate since it acknowledges the effects of the local structure and micro-motions of the fluid elements. Physically, micropolar fluids consist in a large number of small spherical particles uniformly dispersed in a viscous medium. Assuming that the particles are rigid and ignoring their deformations, the related mathematical model expresses the balance of momentum, mass and angular momentum. A new unknown function called microrotation (i.e. the angular velocity field of rotation of particles) is added to the usual velocity and pressure fields. Consequently, Navier-Stokes equations become coupled with a new vector equation coming from the conservation of angular momentum with four microrotation viscosities introduced (see [34] for more details). Being able to describe numerous real fluids better than the classical (Newtonian) model, micropolar fluid models have been extensively studied in recent years (see e.g. [21, 22, 25, 42]).

Taking into account the application we want to model, it is reasonable to assume a small Reynolds number and omit the inertial terms in momentum equations of the micropolar system. Also, it has been observed that the magnitude of the viscosity coefficients appearing in the micropolar equations may influence the effective flow. Thus, it is reasonable to work with the system written in a non-dimensional form (see e.g. [20] for more details). Thus, we consider the stationary flow of an incompressible micropolar fluid in Ω_ε which is governed by the following linearized micropolar system formulated in a non-dimensional form

$$\left\{\begin{array}{rll} -\mathrm{div}(Du_\varepsilon)+\nabla p_\varepsilon = 2N^2\mathrm{rot}\,w_\varepsilon + f_\varepsilon & \text{in} & \Omega_\varepsilon,\\ \mathrm{div}\,u_\varepsilon = 0 & \text{in} & \Omega_\varepsilon,\\ -R_M\mathrm{div}(Dw_\varepsilon)+4N^2 w_\varepsilon = 2N^2\mathrm{rot}\,u_\varepsilon + g_\varepsilon & \text{in} & \Omega_\varepsilon,\end{array}\right. \tag{2.5}$$

with homogeneous boundary conditions (it does not alter the generality of the problem under consideration),

$$u_\varepsilon = w_\varepsilon = 0 \quad \text{on} \quad \partial Q_\varepsilon \cup \partial S_\varepsilon. \tag{2.6}$$

Under assumptions that $f_\varepsilon, g_\varepsilon \in L^2(\Omega_\varepsilon)^3$, it is well known that problem (2.5)-(2.6) has a unique weak solution $(u_\varepsilon, w_\varepsilon, p_\varepsilon) \in H^1_0(\Omega_\varepsilon)^3 \times H^1_0(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ (see [34]), where the space $L^2_0(\Omega_\varepsilon)$ is the space of functions of $L^2(\Omega_\varepsilon)$ with null integral.

Our aim is to study the asymptotic behavior of u_ε, w_ε and p_ε when ε and a_ε tend to zero and identify homogenized models coupling the effects of the thickness of the domain and its microgeometry. For this purpose, as usual when we deal with thin domains, we use the dilatation in the variable x_3 given by

$$y_3 = \frac{x_3}{\varepsilon}\,, \tag{2.7}$$

in order to have the functions defined in the open set with fixed height $\widetilde{\Omega}_\varepsilon$.

Namely, we define $\tilde u_\varepsilon, \tilde w_\varepsilon \in H^1_0(\widetilde{\Omega}_\varepsilon)^3$ and $\tilde p_\varepsilon \in L^2_0(\widetilde{\Omega}_\varepsilon)$ by

$$\tilde u_\varepsilon(x',y_3) = u_\varepsilon(x',\varepsilon y_3), \quad \tilde w_\varepsilon(x',y_3) = w_\varepsilon(x',\varepsilon y_3), \quad \tilde p_\varepsilon(x',y_3) = p_\varepsilon(x',\varepsilon y_3), \quad \text{a.e. } (x',y_3)\in\widetilde{\Omega}_\varepsilon\,. \tag{2.8}$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v=(v',v_3)$ and a scalar function w, we introduce the operators D_ε, ∇_ε and rot_ε by

$$(D_\varepsilon v)_{ij} = \partial_{x_j} v_i \text{ for } i=1,2,3,\ j=1,2, \quad (D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\partial_{y_3} v_i \text{ for } i=1,2,3,$$

$$\nabla_\varepsilon w = \left(\nabla_{x'} w, \frac{1}{\varepsilon}\partial_{y_3} w\right)^t, \quad \mathrm{div}_\varepsilon v = \mathrm{div}_{x'} v' + \frac{1}{\varepsilon}\partial_{y_3} v_3, \quad \mathrm{rot}_\varepsilon v = \left(\mathrm{rot}_{x'} v_3 + \frac{1}{\varepsilon}\mathrm{rot}_{y_3} v', \mathrm{Rot}_{x'} v'\right)^t,$$

where, denoting $(v')^\perp = (-v_2, v_1)^t$, we define

$$\mathrm{rot}_{x'} v_3 = (\partial_{x_2} v_3, -\partial_{x_1} v_3)^t, \quad \mathrm{rot}_{y_3} v' = (\partial_{y_3} v')^\perp, \quad \mathrm{Rot}_{x'} v' = \partial_{x_1} v_2 - \partial_{x_2} v_1. \tag{2.9}$$

Using the transformation (2.7), the rescaled system (2.5)-(2.6) can be rewritten as

$$\left\{\begin{array}{rl} -\mathrm{div}_\varepsilon(D_\varepsilon \tilde{u}_\varepsilon) + \nabla_\varepsilon \tilde{p}_\varepsilon = 2N^2 \mathrm{rot}_\varepsilon\, \tilde{w}_\varepsilon + \tilde{f}_\varepsilon & \text{in} \quad \widetilde{\Omega}_\varepsilon, \\ \mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0 & \text{in} \quad \widetilde{\Omega}_\varepsilon, \\ -R_M \mathrm{div}_\varepsilon(D_\varepsilon \tilde{w}_\varepsilon) + 4N^2 \tilde{w}_\varepsilon = 2N^2 \mathrm{rot}_\varepsilon \tilde{u}_\varepsilon + \tilde{g}_\varepsilon & \text{in} \quad \widetilde{\Omega}_\varepsilon\,, \end{array}\right. \tag{2.10}$$

with homogeneous boundary conditions

$$\tilde{u}_\varepsilon = \tilde{w}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \cup \partial T_\varepsilon\,, \tag{2.11}$$

where $\tilde{f}_\varepsilon$ and $\tilde{g}_\varepsilon$ are defined similarly as in (2.8).

Our goal then is to describe the asymptotic behavior of this new sequences $\tilde{u}_\varepsilon$, $\tilde{w}_\varepsilon$ and $\tilde{p}_\varepsilon$ when ε and a_ε tend to zero. For this, it will be useful to use the the equivalent weak variational formulation of system (2.5)-(2.6) and the rescaled system (2.10)-(2.11).

Weak variational formulations. For problem (2.5)-(2.6), the weak variational formulation is to find $u_\varepsilon, w_\varepsilon \in H^1_0(\Omega_\varepsilon)^3$ and $p_\varepsilon \in L^2_0(\Omega_\varepsilon)$ such that

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi\, dx - \int_{\Omega_\varepsilon} p_\varepsilon \,\mathrm{div}\, \varphi\, dx = 2N^2 \int_{\Omega_\varepsilon} \mathrm{rot}\, w_\varepsilon \cdot \varphi\, dx + \int_{\Omega_\varepsilon} f_\varepsilon \cdot \varphi\, dx, \\ \displaystyle R_M \int_{\Omega_\varepsilon} Dw_\varepsilon : D\psi\, dx + 4N^2 \int_{\Omega_\varepsilon} w_\varepsilon \cdot \psi\, dx = 2N^2 \int_{\Omega_\varepsilon} \mathrm{rot}\, u_\varepsilon \cdot \psi\, dx + \int_{\Omega_\varepsilon} g_\varepsilon \cdot \psi\, dx\,, \end{array}\right. \tag{2.12}$$

for every $\varphi, \psi \in H^1_0(\Omega_\varepsilon)^3$, and the equivalent weak variational formulation for the rescaled system (2.10)-(2.11) is to find $\tilde{u}_\varepsilon, \tilde{w}_\varepsilon \in H^1_0(\widetilde{\Omega}_\varepsilon)^3$ and $\tilde{p}_\varepsilon \in L^2_0(\widetilde{\Omega}_\varepsilon)$ such that

$$\left\{\begin{array}{l} \displaystyle\int_{\widetilde{\Omega}_\varepsilon} D_\varepsilon \tilde{u}_\varepsilon : D_\varepsilon \varphi\, dx'dy_3 - \int_{\widetilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \,\mathrm{div}_\varepsilon \varphi\, dx'dy_3 = 2N^2 \int_{\widetilde{\Omega}_\varepsilon} \mathrm{rot}_\varepsilon \tilde{w}_\varepsilon \cdot \varphi\, dx'dy_3 + \int_{\widetilde{\Omega}_\varepsilon} \tilde{f}_\varepsilon \cdot \varphi\, dx'dy_3\,, \\ \displaystyle R_M \int_{\widetilde{\Omega}_\varepsilon} D_\varepsilon \tilde{w}_\varepsilon : D_\varepsilon \psi\, dx'dy_3 + 4N^2 \int_{\widetilde{\Omega}_\varepsilon} \tilde{w}_\varepsilon \cdot \psi\, dx'dy_3 = 2N^2 \int_{\widetilde{\Omega}_\varepsilon} \mathrm{rot}_\varepsilon \tilde{u}_\varepsilon \cdot \psi\, dx'dy_3 + \int_{\widetilde{\Omega}_\varepsilon} \tilde{g}_\varepsilon \cdot \psi\, dx'dy_3\,, \end{array}\right. \tag{2.13}$$

for every $\varphi, \psi \in H^1_0(\widetilde{\Omega}_\varepsilon)^3$. We recall that : denotes the full contraction of two matrices; for $A = (a_{ij})_{1\le i,j\le 3}$ and $B = (b_{ij})_{1\le i,j\le 3}$, we have $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$.

3 *A priori* estimates

In the sequel we make the following assumptions concerning f_ε, g_ε, R_M and N:

i) in the cases **PTPM** and **HTPM**, we assume

$$f_\varepsilon(x) = (f'(x'), 0), \quad g_\varepsilon(x) = (a_\varepsilon g'(x'), 0), \quad \text{a.e. } x \in \Omega_\varepsilon, \quad \text{where } f', g' \in L^2(\omega)^2, \tag{3.14}$$

$$N^2 = \mathcal{O}(1), \quad R_M = a_\varepsilon^2 R_c \quad \text{with } R_c = \mathcal{O}(1)\,, \tag{3.15}$$

ii) in the case **VTPM**, we assume

$$f_\varepsilon(x) = (f'(x'), 0), \quad g_\varepsilon(x) = (\varepsilon g'(x'), 0), \quad \text{a.e. } x \in \Omega_\varepsilon, \quad \text{where } f', g' \in L^2(\omega)^2, \tag{3.16}$$

$$N^2 = \mathcal{O}(1), \quad R_M = \varepsilon^2 R_c \quad \text{with } R_c = \mathcal{O}(1)\,. \tag{3.17}$$

Along the paper, we denote by C a generic constant which can change from line to line.

Remark 3.1. *We point out that in* **PTPM** *and* **HTPM** *the parameter R_M is compared with the size of the obstacles while in the case* **VTPM** *with the film thickness, which are the most challenging ones and they answer to the question addressed in the paper, all preserve in the limit a strong coupling between velocity and microrotation. This choice is justified by many studies, for example in the selected applications chapter in [34] (see also [19, 20]).*

We also observe that due to the thickness of the domain, it is usual to assume that the vertical components of f and g can be neglected and, moreover they can be considered independent of the vertical variable. The parameters for g_ε are chosen to obtain appropriate estimates in each case.

First, we recall the Poincaré inequality in a thin porous medium domain Ω_ε (see [13]).

Lemma 3.2. *There exists a constant C independent of ε, such that,*

i) in the cases **PTPM** *and* **HTPM***, then*

$$\|v\|_{L^2(\Omega_\varepsilon)^3} \le Ca_\varepsilon \|Dv\|_{L^2(\Omega_\varepsilon)^{3\times 3}}, \quad \forall v \in H^1_0(\Omega_\varepsilon)^3, \tag{3.18}$$

ii) in the case **VTPM***, then*

$$\|v\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon \|Dv\|_{L^2(\Omega_\varepsilon)^{3\times 3}}, \quad \forall v \in H^1_0(\Omega_\varepsilon)^3. \tag{3.19}$$

Next, we give the following results relating the derivative and the rotational (see [26]).

Lemma 3.3. *The following inequality holds*

$$\|\mathrm{rot}\, v\|_{L^2(\Omega_\varepsilon)^3} \le \|Dv\|_{L^2(\Omega_\varepsilon)^{3\times 3}}, \quad \forall v \in H^1_0(\Omega_\varepsilon)^3. \tag{3.20}$$

Moreover, if $\mathrm{div}\, v = 0$ *in* Ω_ε*, then it holds*

$$\|\mathrm{rot}\, v\|_{L^2(\Omega_\varepsilon)^3} = \|Dv\|_{L^2(\Omega_\varepsilon)^{3\times 3}}. \tag{3.21}$$

We start by obtaining some *a priori* estimates for $\tilde u_\varepsilon$ and $\tilde w_\varepsilon$.

Lemma 3.4. *There exists a constant C independent of ε, such that the rescaled solution $(\tilde u_\varepsilon, \tilde w_\varepsilon)$ of the problem (2.10)-(2.11) satisfies*

i) in the cases **PTPM** *and* **HTPM***,*

$$\|\tilde u_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le Ca_\varepsilon^2, \qquad \|D_\varepsilon \tilde u_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le Ca_\varepsilon\,, \tag{3.22}$$

$$\|\tilde w_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le Ca_\varepsilon, \qquad \|D_\varepsilon \tilde w_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\,. \tag{3.23}$$

ii) in the case **VTPM***,*

$$\|\tilde u_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\varepsilon^2, \qquad \|D_\varepsilon \tilde u_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\varepsilon\,, \tag{3.24}$$

$$\|\tilde w_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\varepsilon, \qquad \|D_\varepsilon \tilde w_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\,. \tag{3.25}$$

Proof. We analyze the different cases.

i) Cases **PTPM** and **HTPM**. We first obtain the estimates for the velocity. Taking $\varphi = u_\varepsilon$ as test function in the first equation of (2.12), taking into account $\int_{\Omega_\varepsilon} \mathrm{rot}\, w_\varepsilon \cdot u_\varepsilon\, dx = \int_{\Omega_\varepsilon} \mathrm{rot}\, u_\varepsilon \cdot w_\varepsilon\, dx$, applying Cauchy-Schwarz's inequality and from Lemma 3.2 and (3.21), we have

$$\begin{aligned}
\|Du_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} &= 2N^2\int_{\Omega_\varepsilon} \mathrm{rot}\, w_\varepsilon \cdot u_\varepsilon\, dx + \int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon\, dx \\
&= 2N^2\int_{\Omega_\varepsilon} w_\varepsilon \cdot \mathrm{rot}\, u_\varepsilon\, dx + \int_{\Omega_\varepsilon} f'(x') \cdot u'_\varepsilon\, dx \\
&\le 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \varepsilon^{\frac{1}{2}} a_\varepsilon C \|f'\|_{L^2(\omega)^2} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\,,
\end{aligned} \tag{3.26}$$

which implies

$$\varepsilon^{-\frac{1}{2}} a_\varepsilon^{-1} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le \varepsilon^{-\frac{1}{2}} a_\varepsilon^{-1} 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} + C\|f'\|_{L^2(\omega)^2}\,. \tag{3.27}$$

Taking now $\psi = w_\varepsilon$ as test function in the second equation of (2.12), applying Cauchy-Schwarz's inequality and taking into account (3.14) and (3.15), we have

$$\begin{aligned} &a_\varepsilon^2 R_c \|Dw_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} + 4N^2\|w_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^3} \\ &\quad = 2N^2 \int_{\Omega_\varepsilon} \operatorname{rot} u_\varepsilon \cdot w_\varepsilon\, dx + a_\varepsilon \int_{\Omega_\varepsilon} g'(x')\cdot w'_\varepsilon\, dx \\ &\quad \le 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \varepsilon^{\frac{1}{2}} a_\varepsilon \|g'\|_{L^2(\omega)^2} \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\,, \end{aligned} \tag{3.28}$$

which implies

$$\varepsilon^{-\frac{1}{2}} a_\varepsilon^{-1} 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le \varepsilon^{-\frac{1}{2}} a_\varepsilon^{-1} N^2 \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \frac{1}{2}\|g'\|_{L^2(\omega)^2}\,. \tag{3.29}$$

Then, from (3.27) and (3.29), we conclude that

$$\varepsilon^{-\frac{1}{2}} a_\varepsilon^{-1} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le \frac{C}{1-N^2}\|f'\|_{L^2(\omega)^2} + \frac{1}{2(1-N^2)}\|g'\|_{L^2(\omega)^2}\,,$$

which gives

$$\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C a_\varepsilon \varepsilon^{\frac{1}{2}}.$$

This together with Lemma 3.2 gives

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C a_\varepsilon^2 \varepsilon^{\frac{1}{2}}, \tag{3.30}$$

and by means of the dilatation (2.7) we get (3.22).

Finally, we obtain the estimates for the microrotation. We use $\int_{\Omega_\varepsilon} \operatorname{rot} u_\varepsilon \cdot w_\varepsilon\, dx = \int_{\Omega_\varepsilon} \operatorname{rot} w_\varepsilon \cdot u_\varepsilon\, dx$ in (3.28), Lemma 3.2 and (3.20), and proceeding as above we obtain

$$\begin{aligned} &a_\varepsilon^2 R_c \|Dw_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} + 4N^2\|w_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^3} \\ &\quad \le 2N^2 \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \varepsilon^{\frac{1}{2}} a_\varepsilon^2 C \|g'\|_{L^2(\omega)^2} \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\,, \end{aligned} \tag{3.31}$$

which, by using the estimate of u_ε given in (3.30), provides

$$a_\varepsilon^2 R_c \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\left(2N^2 \varepsilon^{\frac{1}{2}} a_\varepsilon^2 + \varepsilon^{\frac{1}{2}} a_\varepsilon^2 \|g'\|_{L^2(\omega)^2}\right).$$

This implies

$$\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C a_\varepsilon \varepsilon^{\frac{1}{2}}, \quad \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{1}{2}},$$

and by means of the dilatation, we get (3.23).

ii) Case **VTPM**. For the velocity, proceeding as above by taking into account (3.16) and (3.17), we conclude

$$\varepsilon^{-\frac{3}{2}} \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le \varepsilon^{-\frac{3}{2}} 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} + C\|f'\|_{L^2(\omega)^2}\,,$$

and

$$\varepsilon^{-\frac{3}{2}} 2N^2 \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le \varepsilon^{-\frac{3}{2}} N^2 \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \frac{1}{2}\|g'\|_{L^2(\omega)^2}\,.$$

Then, we deduce estimate

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{5}{2}}, \quad \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{3}{2}},$$

and by means of the dilatation, we get (3.24).

For the microrotation, similarly as the previous case, we get that

$$\varepsilon^2 R_c \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\left(2N^2 \varepsilon^{\frac{5}{2}} + \varepsilon^{\frac{5}{2}} \|g'\|_{L^2(\omega)^2}\right),$$

which implies

$$\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{3}{2}}, \quad \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{1}{2}},$$

and by means of the dilatation we get (3.25).

□

3.1 The extension of $(\tilde{u}_\varepsilon, \tilde{w}_\varepsilon, \tilde{p}_\varepsilon)$ to the whole domain Ω

We extend the velocity $\tilde{u}_\varepsilon$ and the microrotation $\tilde{w}_\varepsilon$ by zero to the $\Omega\setminus\widetilde{\Omega}_\varepsilon$, and denote the extension by the same symbol. Obviously, estimates (3.22)-(3.25) remain valid and the extension of $\tilde{u}_\varepsilon$ is divergence free too.

In order to extend the pressure to the whole domain Ω, we use the mapping R^ε, defined in Lemma 4.5 in [13] as R_2^ε, which allows us to extend the pressure p_ε from Ω_ε to Q_ε by introducing F_ε in $H^{-1}(Q_\varepsilon)^3$ in the following way (brackets are for the duality products between H^{-1} and H_0^1):

$$\langle F_\varepsilon, \varphi\rangle_{Q_\varepsilon} = \langle \nabla p_\varepsilon, R^\varepsilon\varphi\rangle_{\Omega_\varepsilon}\,, \quad \text{for any } \varphi\in H_0^1(Q_\varepsilon)^3\,. \tag{3.32}$$

Using Lemma 3.4 for fixed ε and a_ε, we see that it is a bounded functional on $H_0^1(Q_\varepsilon)$ (see the proof of Lemma 3.5 below), and in fact $F_\varepsilon\in H^{-1}(Q_\varepsilon)^3$. Moreover, $\mathrm{div}\varphi=0$ implies $\langle F_\varepsilon,\varphi\rangle_{Q_\varepsilon}=0$, and the DeRham theorem gives the existence of P_ε in $L_0^2(Q_\varepsilon)$ with $F_\varepsilon=\nabla P_\varepsilon$.

We calcule the right hand side of (3.32) by using the first equation of (2.12) and we have

$$\langle F_\varepsilon,\varphi\rangle_{Q_\varepsilon} = \quad -\int_{\Omega_\varepsilon} Du_\varepsilon : DR^\varepsilon\varphi\,dx + 2N^2\int_{\Omega_\varepsilon}\mathrm{rot}\,w_\varepsilon\cdot R^\varepsilon\varphi\,dx + \int_{\Omega_\varepsilon} f'(x')\cdot(R^\varepsilon\varphi)'\,dx\,. \tag{3.33}$$

We get for any $\tilde{\varphi}\in H_0^1(\Omega)^3$ where $\tilde{\varphi}(x',y_3)=\varphi(x',\varepsilon y_3)$, using the change of variables (2.7), that

$$\langle \nabla_\varepsilon \tilde{P}_\varepsilon,\tilde{\varphi}\rangle_\Omega \quad = -\int_\Omega \tilde{P}_\varepsilon\,\mathrm{div}_\varepsilon\,\tilde{\varphi}\,dx'dy_3 = -\varepsilon^{-1}\int_{Q_\varepsilon} P_\varepsilon\,\mathrm{div}\,\varphi\,dx = \varepsilon^{-1}\langle\nabla P_\varepsilon,\varphi\rangle_{Q_\varepsilon}\,.$$

Then, using the identification (3.33) of F_ε, we have

$$\langle \nabla_\varepsilon \tilde{P}_\varepsilon,\tilde{\varphi}\rangle_\Omega = \varepsilon^{-1}\Big(-\int_{\Omega_\varepsilon} Du_\varepsilon : DR^\varepsilon\varphi\,dx + 2N^2\int_{\Omega_\varepsilon}\mathrm{rot}\,w_\varepsilon\cdot R^\varepsilon\varphi\,dx + \int_{\Omega_\varepsilon} f'(x')\cdot(R^\varepsilon\varphi)'\,dx\Big)\,,$$

and applying the change of variables (2.7), we obtain

$$\langle \nabla_\varepsilon \tilde{P}_\varepsilon,\tilde{\varphi}\rangle_\Omega = -\int_{\widetilde{\Omega}_\varepsilon} D_\varepsilon\tilde{u}_\varepsilon : D_\varepsilon\tilde{R}^\varepsilon\tilde{\varphi}\,dx'dy_3 + 2N^2\int_{\widetilde{\Omega}_\varepsilon}\mathrm{rot}_\varepsilon\tilde{w}_\varepsilon\cdot\tilde{R}^\varepsilon\tilde{\varphi}\,dx'dy_3 + \int_{\widetilde{\Omega}_\varepsilon} f(x')\cdot(\tilde{R}^\varepsilon\tilde{\varphi})'\,dx'dy_3\,, \tag{3.34}$$

where $\tilde{R}^\varepsilon\tilde{\varphi}=R^\varepsilon\varphi$ for any $\tilde{\varphi}\in H_0^1(\Omega)^3$.

Now, we estimate the right-hand side of (3.34).

Lemma 3.5. *There exists a constant $C>0$ independent of ε, such that the extension $\tilde{P}_\varepsilon\in L_0^2(\Omega)$ of the pressure $\tilde{p}_\varepsilon$ satisfies*

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)}\le C. \tag{3.35}$$

Proof. From the proof of Lemma 4.6-(i) in [13], we have that $\tilde{R}^\varepsilon\tilde{\varphi}$ satisfies the following estimates

$$\left\{\begin{array}{l}
\|\tilde{R}^\varepsilon(\tilde{\varphi})\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}\le C\left(\dfrac{1}{a_\varepsilon^2}\|\tilde{\varphi}\|^2_{L^2(\Omega)^3}+\|D_{x'}\tilde{\varphi}\|^2_{L^2(\Omega)^{3\times 2}}+\dfrac{1}{a_\varepsilon^2}\|\partial_{y_3}\tilde{\varphi}\|^2_{L^2(\Omega)^3}\right),\\[2mm]
\|D_{x'}\tilde{R}^\varepsilon\tilde{\varphi}\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 2}}\le C\left(\dfrac{1}{a_\varepsilon^2}\|\tilde{\varphi}\|^2_{L^2(\Omega)^3}+\|D_{x'}\tilde{\varphi}\|^2_{L^2(\Omega)^{3\times 2}}+\dfrac{1}{a_\varepsilon^2}\|\partial_{y_3}\tilde{\varphi}\|^2_{L^2(\Omega)^3}\right),\\[2mm]
\|\partial_{y_3}\tilde{R}^\varepsilon\tilde{\varphi}\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^3}\le C\left(\|\tilde{\varphi}\|^2_{L^2(\Omega)^3}+a_\varepsilon^2\|D_{x'}\tilde{\varphi}\|^2_{L^2(\Omega)^{2\times 3}}+\|\partial_{y_3}\tilde{\varphi}\|^2_{L^2(\Omega)^3}\right).
\end{array}\right.$$

This implies that

$$\|\tilde{R}^\varepsilon(\tilde{\varphi})\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\le C\left(\|\tilde{\varphi}\|^2_{L^2(\Omega)^3}+a_\varepsilon\|D_{x'}\tilde{\varphi}\|_{L^2(\Omega)^{3\times 2}}+\|\partial_{y_3}\tilde{\varphi}\|^2_{L^2(\Omega)^3}\right)\le C\|\tilde{\varphi}\|_{H_0^1(\Omega)^3}.$$

Moreover, in the case **PTPM** and **HTPM**,

$$\|D_\varepsilon \tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\left(\frac{1}{a_\varepsilon}\|\tilde\varphi\|_{L^2(\Omega)^3} + \|D_{x'}\tilde\varphi\|_{L^2(\Omega)^{3\times 2}} + \frac{1}{a_\varepsilon}\|\partial_{y_3}\tilde\varphi\|_{L^2(\Omega)^3}\right) \le \frac{C}{a_\varepsilon}\|\tilde\varphi\|_{H^1_0(\Omega)^3},$$

and in the case **VTPM**,

$$\|D_\varepsilon \tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\left(\frac{1}{\varepsilon}\|\tilde\varphi\|_{L^2(\Omega)^3} + \frac{a_\varepsilon}{\varepsilon}\|D_{x'}\tilde\varphi\|_{L^2(\Omega)^{3\times 3}} + \frac{1}{\varepsilon}\|\partial_{y_3}\tilde\varphi\|_{L^2(\Omega)^3}\right) \le \frac{C}{\varepsilon}\|\tilde\varphi\|_{H^1_0(\Omega)^3}.$$

Thus, in the cases **PTPM** and **HTPM**, by using estimates for $D_\varepsilon \tilde u_\varepsilon$ in (3.22), for $D_\varepsilon w_\varepsilon$ in (3.23) and $f' \in L^2(\omega)^2$, we respectively obtain

$$\left|\int_{\widetilde\Omega_\varepsilon} D_\varepsilon \tilde u_\varepsilon : D_\varepsilon \tilde R^\varepsilon \tilde\varphi\, dx'dy_3\right| \le C a_\varepsilon \|D_\varepsilon \tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}} \le C\|\tilde\varphi\|_{H^1_0(\Omega)^3},$$
$$\left|\int_{\widetilde\Omega_\varepsilon} \mathrm{rot}_\varepsilon w_\varepsilon \cdot \tilde R^\varepsilon \tilde\varphi\, dx'dy_3\right| \le \|D_\varepsilon \tilde w_\varepsilon\|_{L^2(\widetilde\Omega_\varepsilon)^{3\times 3}}\|\tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C a_\varepsilon^2 \varepsilon^{-2}\|\tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\|\tilde\varphi\|_{H^1_0(\Omega)^3},$$
$$\left|\int_{\widetilde\Omega_\varepsilon} f' \cdot \tilde R^\varepsilon \tilde\varphi\, dx'dy_3\right| \le C\|\tilde R^\varepsilon \tilde\varphi\|_{L^2(\widetilde\Omega_\varepsilon)^3} \le C\|\tilde\varphi\|_{H^1_0(\Omega)^3}\,,$$

which together with (3.34) gives $\|\nabla_\varepsilon \tilde P_\varepsilon\|_{L^2(\Omega)^3} \le C$. By using the Nečas inequality there exists a representative $\tilde P_\varepsilon \in L^2_0(\Omega)$ such that

$$\|\tilde P_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla \tilde P_\varepsilon\|_{H^{-1}(\Omega)^3} \le C\|\nabla_\varepsilon \tilde P_\varepsilon\|_{H^{-1}(\Omega)^3},$$

which implies (3.35).

Finally, proceeding similarly in the case **VTPM**, by using now estimates for $D_\varepsilon \tilde u_\varepsilon$ in (3.24) and for $D_\varepsilon w_\varepsilon$ in (3.25), we also obtain (3.35).

□

3.2 Adaptation of the unfolding method

The change of variables (2.7) does not provide the information we need about the behavior of $\tilde u_\varepsilon$ and $\tilde w_\varepsilon$ in the microstructure associated to $\widetilde\Omega_\varepsilon$. To solve this difficulty, we use an adaptation of the unfolding method (see [18, 23, 24] for more details) introduced in [13].

Let us recall that this adaptation of the unfolding method divides the domain $\widetilde\Omega_\varepsilon$ in cubes of lateral length a_ε and vertical length 1. Thus, given $(\tilde u_\varepsilon, \tilde w_\varepsilon, \tilde P_\varepsilon) \in H^1_0(\Omega)^3 \times H^1_0(\Omega)^3 \times L^2_0(\Omega)$, we define $(\hat u_\varepsilon,\, \hat w_\varepsilon, \hat P_\varepsilon)$ by

$$\hat u_\varepsilon(x', y) = \tilde u_\varepsilon\left(a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \quad \hat w_\varepsilon(x', y) = \tilde w_\varepsilon\left(a_\varepsilon \kappa\left(\frac{x'}{a_\varepsilon}\right) + a_\varepsilon y', y_3\right), \tag{3.36}$$

$$\hat P_\varepsilon(x', y) = \tilde P_\varepsilon\left(a_\varepsilon \kappa\left(\frac{x'}{\varepsilon}\right) + a_\varepsilon y', y_3\right), \tag{3.37}$$

a.e. $(x', y) \in \omega \times Y$, where $\tilde u_\varepsilon$, $\tilde w_\varepsilon$ and $\tilde P_\varepsilon$ are extended by zero outside Ω. For $k' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \Longleftrightarrow x' \in Y'_{k',1}\,. \tag{3.38}$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k'\in\mathbb{Z}^2}\partial Y'_{k',1}$). Moreover, for every $a_\varepsilon > 0$, we have

$$\kappa\left(\frac{x'}{a_\varepsilon}\right) = k' \Longleftrightarrow x' \in Y'_{k',a_\varepsilon}\,.$$

Remark 3.6. *For $k'\in\mathcal{K}_\varepsilon$, the restrictions of $(\hat u_\varepsilon,\hat w_\varepsilon,\hat P_\varepsilon)$ to $Y'_{k',a_\varepsilon}\times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde u_\varepsilon,\tilde w_\varepsilon,\tilde P_\varepsilon)$ by using the change of variables $y'=\frac{x'-a_\varepsilon k'}{a_\varepsilon}$, which transforms Y_{k',a_ε} into Y.*

Now, we get the estimates for the sequences $(\hat u_\varepsilon,\hat w_\varepsilon,\hat P_\varepsilon)$ similarly as in the proof of Lemma 4.9 in [13].

Lemma 3.7. *There exists a constant $C>0$ independent of ε, such that $(\hat u_\varepsilon,\hat w_\varepsilon,\hat P_\varepsilon)$ defined by (3.36)-(3.37) satisfies*

i) in the cases **PTPM** *and* **HTPM**,

$$\|\hat u_\varepsilon\|_{L^2(\omega\times Y)^3}\le Ca_\varepsilon^2,\quad \|D_{y'}\hat u_\varepsilon\|_{L^2(\omega\times Y)^{3\times3}}\le Ca_\varepsilon^2,\quad \|\partial_{y_3}\hat u_\varepsilon\|_{L^2(\omega\times Y)^3}\le Ca_\varepsilon\varepsilon, \tag{3.39}$$

$$\|\hat w_\varepsilon\|_{L^2(\omega\times Y)^3}\le Ca_\varepsilon,\quad \|D_{y'}\hat w_\varepsilon\|_{L^2(\omega\times Y)^{3\times3}}\le Ca_\varepsilon,\quad \|\partial_{y_3}\hat w_\varepsilon\|_{L^2(\omega\times Y)^3}\le C\varepsilon, \tag{3.40}$$

ii) in the case **VTPM**,

$$\|\hat u_\varepsilon\|_{L^2(\omega\times Y)^3}\le C\varepsilon^2,\quad \|D_{y'}\hat u_\varepsilon\|_{L^2(\omega\times Y)^{3\times3}}\le Ca_\varepsilon\varepsilon,\quad \|\partial_{y_3}\hat u_\varepsilon\|_{L^2(\omega\times Y)^3}\le C\varepsilon^2, \tag{3.41}$$

$$\|\hat w_\varepsilon\|_{L^2(\omega\times Y)^3}\le C\varepsilon,\quad \|D_{y'}\hat w_\varepsilon\|_{L^2(\omega\times Y)^{3\times3}}\le Ca_\varepsilon,\quad \|\partial_{y_3}\hat w_\varepsilon\|_{L^2(\omega\times Y)^3}\le C\varepsilon, \tag{3.42}$$

and, moreover, in every cases,

$$\|\hat P_\varepsilon\|_{L^2(\omega\times Y)}\le C.$$

Weak variational formulation. To finish this section, we will give the variational formulation satisfied by the functions $(\hat u_\varepsilon,\hat w_\varepsilon,\hat P_\varepsilon)$, which will be useful in the following sections.

We consider $\varphi_\varepsilon(x',y_3)=\varphi(x',x'/\varepsilon,y_3)$ and $\psi_\varepsilon(x',y_3)=\psi(x',x'/\varepsilon,y_3)$ as test function in (2.13) where $\varphi(x',y),\ \psi(x',y)\in\mathcal{D}(\omega;C^\infty_\#(Y)^3)$, and taking into account the extension of the pressure, we have

$$\int_{\widetilde\Omega_\varepsilon}\nabla_\varepsilon\tilde p_\varepsilon\cdot\varphi_\varepsilon\,dx'dy_3=\int_\Omega\nabla_\varepsilon\tilde P_\varepsilon\cdot\varphi_\varepsilon\,dx'dy_3\,,$$

and the extension of $(\tilde u_\varepsilon,\tilde w_\varepsilon)$, we get

$$\left\{\begin{array}{l}\displaystyle\int_{\widetilde\Omega_\varepsilon}D_\varepsilon\tilde u_\varepsilon:D_\varepsilon\varphi_\varepsilon\,dx'dy_3-\int_\Omega\tilde P_\varepsilon\,\mathrm{div}_\varepsilon\varphi_\varepsilon\,dx'dy_3=2N^2\int_{\widetilde\Omega_\varepsilon}\mathrm{rot}_\varepsilon\tilde w_\varepsilon\cdot\varphi_\varepsilon\,dx'dy_3+\int_{\widetilde\Omega_\varepsilon}f'\cdot\varphi'_\varepsilon\,dx'dy_3\,,\\[2ex]\displaystyle R_M\int_{\widetilde\Omega_\varepsilon}D_\varepsilon\tilde w_\varepsilon:D_\varepsilon\psi_\varepsilon\,dx'dy_3+4N^2\int_{\widetilde\Omega_\varepsilon}\tilde w_\varepsilon\cdot\psi_\varepsilon\,dx'dy_3=2N^2\int_{\widetilde\Omega_\varepsilon}\mathrm{rot}_\varepsilon\tilde u_\varepsilon\cdot\psi_\varepsilon\,dx'dy_3+\int_{\widetilde\Omega_\varepsilon}g'_\varepsilon\cdot\psi'_\varepsilon\,dx'dy_3\,,\end{array}\right. \tag{3.43}$$

where R_M and g'_ε depend on the case, see assumptions (3.14)-(3.17).

Now, by the change of variables given in Remark 3.6 (see [13] for more details), we obtain

$$\left\{\begin{array}{l}\displaystyle\frac{1}{a_\varepsilon^2}\int_{\omega\times Y_f}D_{y'}\hat u'_\varepsilon:D_{y'}\varphi'\,dx'dy+\frac{1}{\varepsilon^2}\int_{\omega\times Y_f}\partial_{y_3}\hat u'_\varepsilon:\partial_{y_3}\varphi'\,dx'dy\\[2ex]\displaystyle-\int_{\omega\times Y_f}\hat P_\varepsilon\mathrm{div}_{x'}\varphi'\,dx'dy-\frac{1}{a_\varepsilon}\int_{\omega\times Y_f}\hat P_\varepsilon\mathrm{div}_{y'}\varphi'\,dx'dy\\[2ex]\displaystyle=\frac{2N^2}{a_\varepsilon}\int_{\omega\times Y_f}\mathrm{rot}_{y'}\hat w_{\varepsilon,3}\cdot\varphi'\,dx'dy+\frac{2N^2}{\varepsilon}\int_{\omega\times Y_f}\mathrm{rot}_{y_3}\hat w'_\varepsilon\cdot\varphi'\,dx'dy+\int_{\omega\times Y_f}f'\cdot\varphi'\,dx'dy+O_\varepsilon\,,\\[4ex]\displaystyle\frac{1}{a_\varepsilon^2}\int_{\omega\times Y_f}\nabla_{y'}\hat u_{\varepsilon,3}\cdot\nabla_{y'}\varphi_3\,dx'dy+\frac{1}{\varepsilon^2}\int_{\omega\times Y_f}\partial_{y_3}\hat u_{\varepsilon,3}\cdot\partial_{y_3}\varphi_3\,dx'dy-\frac{1}{\varepsilon}\int_{\omega\times Y_f}\hat P_\varepsilon\partial_{y_3}\varphi_3\,dx'dy\\[2ex]\displaystyle=\frac{2N^2}{a_\varepsilon}\int_{\omega\times Y_f}\mathrm{Rot}_{y'}\hat w'_\varepsilon\,\varphi_3\,dx'dy+O_\varepsilon\,,\end{array}\right. \tag{3.44}$$

and

$$\begin{cases} \dfrac{R_M}{a_\varepsilon^2}\displaystyle\int_{\omega\times Y_f} D_{y'}\hat{w}'_\varepsilon : D_{y'}\psi'\,dx'dy + \dfrac{R_M}{\varepsilon^2}\int_{\omega\times Y_f}\partial_{y_3}\hat{w}'_\varepsilon : \partial_{y_3}\psi'\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}'_\varepsilon\cdot\psi'\,dx'dy \\ = \dfrac{2N^2}{a_\varepsilon}\displaystyle\int_{\omega\times Y_f}\mathrm{rot}_{y'}\hat{u}_{\varepsilon,3}\cdot\psi'\,dx'dy + \dfrac{2N^2}{\varepsilon}\int_{\omega\times Y_f}\mathrm{rot}_{y_3}\hat{u}'_\varepsilon\cdot\psi'\,dx'dy + \int_{\omega\times Y_f} g'_\varepsilon\cdot\psi'\,dx'dy + O_\varepsilon\,, \\ \\ \dfrac{R_M}{a_\varepsilon^2}\displaystyle\int_{\omega\times Y_f}\nabla_{y'}\hat{w}_{\varepsilon,3}\cdot\nabla_{y'}\psi_3\,dx'dy + \dfrac{R_M}{\varepsilon^2}\int_{\omega\times Y_f}\partial_{y_3}\hat{w}_{\varepsilon,3} : \partial_{y_3}\psi_3\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}_{\varepsilon,3}\cdot\psi_3\,dx'dy \\ = \dfrac{2N^2}{a_\varepsilon}\displaystyle\int_{\omega\times Y_f}\mathrm{Rot}_{y'}\hat{u}'_\varepsilon\,\psi_3\,dx'dy + O_\varepsilon\,. \end{cases} \tag{3.45}$$

Along the paper, we denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line.

When ε tends to zero, we obtain for $(\hat{u}_\varepsilon, \hat{w}_\varepsilon, \hat{P}_\varepsilon)$ different asymptotic behaviors depending on the cases **PTPM**, **HTPM** and **VTPM**. We will analyze them in the next sections.

4 Proportionally Thin Porous Medium (PTPM)

It corresponds to the critical case when the cylinder height is proportional to the interspatial distance, with λ the proportionality constant, that is $a_\varepsilon \approx \varepsilon$, with $a_\varepsilon/\varepsilon \to \lambda$, $0 < \lambda < +\infty$.

Let us introduce some notation which will be useful along this section. For a vectorial function $v = (v', v_3)$ and a scalar function w, we introduce the operators D_λ, ∇_λ, div_λ and rot_λ by

$$(D_\lambda v)_{ij} = \partial_{x_j} v_i \text{ for } i = 1,2,3,\ j = 1,2, \quad (D_\lambda v)_{i,3} = \lambda\partial_{y_3} v_i \text{ for } i = 1,2,3,$$

$$\Delta_\lambda v = \Delta_{y'} v + \lambda^2\partial^2_{y_3} v, \quad \nabla_\lambda w = (\nabla_{y'} w, \lambda\partial_{y_3} w)^t,$$

$$\mathrm{div}_\lambda v = \mathrm{div}_{y'} v' + \lambda\partial_{y_3} v_3, \quad \mathrm{rot}_\lambda v = (\mathrm{rot}_{y'} v_3 + \lambda\mathrm{rot}_{y_3} v', \mathrm{Rot}_{y'} v')\,,$$

where $\mathrm{rot}_{y'}$, rot_{y_3} and $\mathrm{Rot}_{y'}$ are defined in (2.9). Next, we give some compactness results about the behavior of the extended sequences $(\tilde{u}_\varepsilon, \tilde{w}_\varepsilon, \tilde{P}_\varepsilon)$ and the unfolding functions $(\hat{u}_\varepsilon, \hat{w}_\varepsilon, \hat{P}_\varepsilon)$ satisfying the *a priori* estimates given in Lemmas 3.4, 3.5 and 3.7 respectively.

Lemma 4.1. *For a subsequence of ε still denote by ε, we have that*

i) (Velocity) there exist $\tilde{u} \in H^1_0(0,1;L^2(\omega)^3)$ with $\tilde{u}_3 = 0$ and $\hat{u} \in L^2(\omega; H^1_{0,\#}(Y))^3$ with $\hat{u} = 0$ on $\omega\times Y_s$, such that $\int_Y \hat{u}(x',y)dy = \int_0^1 \tilde{u}(x',y_3)\,dy_3$ with $\int_Y \hat{u}_3\,dy = 0$ and moreover

$$a_\varepsilon^{-2}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ in } H^1(0,1;L^2(\omega)^3), \quad a_\varepsilon^{-2}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\omega; H^1(Y)^3), \tag{4.46}$$

$$\mathrm{div}_{x'}\left(\int_0^1 \tilde{u}'(x',y_3)\,dy_3\right) = 0 \text{ in } \omega, \quad \left(\int_0^1 \tilde{u}'(x',y_3)\,dy_3\right)\cdot n = 0 \text{ in } \partial\omega\,, \tag{4.47}$$

$$\mathrm{div}_\lambda\hat{u} = 0 \text{ in } \omega\times Y, \quad \mathrm{div}_{x'}\left(\int_Y \hat{u}'(x',y)\,dy\right) = 0 \text{ in } \omega, \quad \left(\int_Y \hat{u}'(x',y)\,dy\right)\cdot n = 0 \text{ in } \partial\omega\,, \tag{4.48}$$

ii) (Microrotation) there exist $\tilde{w} \in H^1_0(0,1;L^2(\omega)^3)$ with $\tilde{w}_3 = 0$ and $\hat{w} \in L^2(\omega; H^1_{0,\#}(Y))^3$ with $\hat{w} = 0$ on $\omega\times Y_s$, such that $\int_Y \hat{w}(x',y)dy = \int_0^1 \tilde{w}(x',y_3)\,dy_3$ with $\int_Y \hat{w}_3\,dy = 0$ and moreover

$$a_\varepsilon^{-1}\tilde{w}_\varepsilon \rightharpoonup (\tilde{w}', 0) \text{ in } H^1(0,1;L^2(\omega)^3), \quad a_\varepsilon^{-1}\hat{w}_\varepsilon \rightharpoonup \hat{w} \text{ in } L^2(\omega; H^1(Y)^3), \tag{4.49}$$

iii) (Pressure) there exists a function $\tilde{P} \in L^2_0(\Omega)$, independent of y_3, such that

$$\tilde{P}_\varepsilon \to \tilde{P} \text{ in } L^2(\Omega), \quad \hat{P}_\varepsilon \to \tilde{P} \text{ in } L^2(\omega \times Y). \tag{4.50}$$

Proof. The proof of this result for the velocity is obtained by arguing similarly to Section 5 in [13].

The proof of the results for the microrotation is analogous to the ones of the velocity, except to prove that $\tilde{w}_3 = 0$. To do this, we consider as test function $\psi_\varepsilon(x', y_3) = (0, 0, a_\varepsilon^{-1}\psi_3)$ in the variational formulation (3.43), and we get

$$\begin{aligned} & a_\varepsilon R_c \int_\Omega \nabla_{x'} \tilde{w}_{\varepsilon,3} \cdot \nabla_{x'} \psi_3 \, dx'dy_3 + a_\varepsilon \varepsilon^{-2} R_c \int_\Omega \partial_{y_3} \tilde{w}_{\varepsilon,3} \, \partial_{y_3} \psi_3 \, dx'dy_3 + 4N^2 a_\varepsilon^{-1} \int_\Omega \tilde{w}_{\varepsilon,3} \psi_3 \, dx'dy_3 \\ & = 2N^2 a_\varepsilon^{-1} \int_\Omega \mathrm{Rot}_{x'} \tilde{u}'_\varepsilon \, \psi_3 \, dx'dy_3 \, . \end{aligned}$$

Passing to the limit by using convergences of $\tilde{u}_\varepsilon$ and $\tilde{w}_\varepsilon$ given in (4.46) and (4.49), we get

$$\lambda^2 R_c \int_\Omega \partial_{y_3} \tilde{w}_3 \, \partial_{y_3} \psi_3 \, dx'dy_3 + 4N^2 \int_\Omega \tilde{w}_3 \, \psi_3 \, dx'dy_3 = 0 \, ,$$

and taking into account that $\tilde{w}_3 = 0$ on $y_3 = \{0, 1\}$, it is easily deduced that $\tilde{w}_3 = 0$ a.e. in Ω.

We finish with the proof for the pressure. Estimate (3.7) implies, up to a subsequence, the existence of $\tilde{P} \in L^2_0(\Omega)$ such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \quad \text{in } L^2(\Omega). \tag{4.51}$$

Also, from $\|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{L^2(\Omega)^3} \le C$, by noting that $\partial_{y_3} \tilde{P}_\varepsilon / \varepsilon$ also converges weakly in $H^{-1}(\Omega)$, we obtain $\partial_{y_3} \tilde{P} = 0$ and so $\tilde{P}$ is independent of y_3. Next, following [45], it can be proved that the convergence of the pressure is in fact strong giving the first convergence of (4.50). Finally, we remark that the strong convergence of sequence $\hat{P}_\varepsilon$ to $\tilde{P}$ is a consequence of the strong convergence of $\tilde{P}_\varepsilon$ to $\tilde{P}$ (see [24, Proposition 2.9]).

□

Unsing previous convergences, in the following theorem we give the homogenized system satisfied by $(\hat{u}, \hat{w}, \tilde{P})$.

Theorem 4.2. *In the case* **PTPM**, *the sequence $(a_\varepsilon^{-2}\hat{u}_\varepsilon, a_\varepsilon^{-1}\hat{w}_\varepsilon)$ converges weakly to $(\hat{u}, \hat{w})$ in $L^2(\omega; H^1(Y)^3) \times L^2(\omega; H^1(Y)^3)$ and $\hat{P}_\varepsilon$ converges strongly to $\tilde{P}$ in $L^2(\omega)$, where $(\hat{u}, \hat{w}, \tilde{P}) \in L^2(\omega; H^1_{0,\#}(Y)^3) \times L^2(\omega; H^1_{0,\#}(Y)^3) \times (L^2_0(\omega) \cap H^1(\omega))$, with $\int_Y \hat{u}_3 \, dy = \int_Y \hat{w}_3 \, dy = 0$, is the unique solution of the following homogenized system*

$$\left\{ \begin{aligned} -\Delta_\lambda \hat{u} + \nabla_\lambda \hat{q} = 2N^2 \mathrm{rot}_\lambda \hat{w} + f'(x') - \nabla_{x'} \tilde{P}(x') & \quad \text{in } \omega \times Y_f, \\ \mathrm{div}_\lambda \hat{u} = 0 & \quad \text{in } \omega \times Y_f, \\ -R_c \Delta_\lambda \hat{w} + 4N^2 \hat{w} = 2N^2 \, \mathrm{rot}_\lambda \hat{u} + g'(x') & \quad \text{in } \omega \times Y_f, \\ \hat{u} = \hat{w} = 0 & \quad \text{in } \omega \times Y_s, \\ \mathrm{div}_{x'} \left(\int_Y \hat{u}'(x', y) \, dy \right) = 0 & \quad \text{in } \omega, \\ \left(\int_Y \hat{u}'(x', y) \, dy \right) \cdot n = 0 & \quad \text{on } \partial\omega, \\ \hat{u}(x', y), \hat{w}(x', y), \hat{q}(x', y) \quad Y' - \text{periodic}. & \end{aligned} \right. \tag{4.52}$$

Proof. For every $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $\mathrm{div}_\lambda \varphi = 0$ in $\omega \times Y$ and $\mathrm{div}_{x'}(\int_Y \varphi' \, dy) = 0$ in ω, we choose $\varphi_\varepsilon = (\varphi', \lambda(\varepsilon/a_\varepsilon)\varphi_3)$ in (3.44). Taking into account that thanks to $\mathrm{div}_\lambda \varphi = 0$ in $\omega \times Y$, we have that

$$\frac{1}{a_\varepsilon} \int_{\omega \times Y} \hat{P}_\varepsilon (\mathrm{div}_{y'} \varphi' + \lambda \partial_{y_3} \varphi_3) \, dx'dy = 0 \, .$$

Thus, passing to the limit using the convergences (4.46), (4.49), (4.50) and $\lambda(\varepsilon/\eta_\varepsilon)\to 1$, we obtain

$$\begin{aligned}&\int_{\omega\times Y_f} D_\lambda\hat u : D_\lambda\varphi\,dx'dy-\int_{\omega\times Y}\tilde P\,\mathrm{div}_{x'}\varphi'\,dx'dy\\&=2N^2\int_{\omega\times Y_f}(\mathrm{rot}_{y'}\hat w_3\cdot\varphi'+\lambda\mathrm{rot}_{y_3}\hat w'\cdot\varphi'+\mathrm{Rot}_{y'}\hat w'\,\varphi_3)\,dx'dy+\int_{\omega\times Y_f} f'\cdot\varphi'\,dx'dy\,.\end{aligned}\tag{4.53}$$

Since $\tilde P$ does not depend on y and $\mathrm{div}_{x'}\int_Y\varphi'\,dy=0$ in ω, we have that the second term is zero, and so we get

$$\int_{\omega\times Y_f} D_\lambda\hat u : D_\lambda\varphi\,dx'dy=2N^2\int_{\omega\times Y_f}\mathrm{rot}_\lambda\hat w\cdot\varphi\,dx'dy+\int_{\omega\times Y_f} f'\cdot\varphi'\,dx'dy\,.\tag{4.54}$$

Next, for every $\psi\in\mathcal D(\omega;C^\infty_\#(Y)^3)$, we choose $\psi_\varepsilon=a_\varepsilon^{-1}\psi$ in (3.45) with g_ε and R_M satisfying (3.14) and (3.15). Then, passing to the limit using convergences (4.46) and (4.49), we get

$$R_c\int_{\omega\times Y_f} D_\lambda\hat w : D_\lambda\psi\,dx'dy+4N^2\int_{\omega\times Y_f}\hat w\cdot\psi\,dx'dy=2N^2\int_{\omega\times Y_f}\mathrm{rot}_\lambda\hat u\cdot\psi\,dx'dy+\int_{\omega\times Y_f} g'\cdot\psi'\,dx'dy\,.\tag{4.55}$$

By density (4.54) holds for every function φ in the Hilbert space V defined by

$$V=\left\{\begin{array}{l}\varphi(x',y)\in L^2(\omega;H^1_{0,\#}(Y)^3)\text{ such that}\\[2mm] \mathrm{div}_{x'}\left(\displaystyle\int_{Y_f}\varphi(x',y)\,dy\right)=0\text{ in }\omega,\quad\left(\displaystyle\int_{Y_f}\varphi(x',y)\,dy\right)\cdot n=0\text{ on }\partial\omega\\[2mm] \mathrm{div}_\lambda\varphi(x',y)=0\text{ in }\omega\times Y_f,\quad\varphi(x',y)=0\text{ in }\omega\times Y_s\end{array}\right\},$$

and (4.55) in $W=\{\psi(x',y)\in L^2(\omega;H^1_{0,\#}(Y)^3)\,:\,\psi(x',y)=0\text{ in }\omega\times Y_s\}$.

From Theorem 2.4.2 in [34], the variational formulation (4.54)-(4.55) admits a unique solution $(\hat u,\hat w)$ in $V\times W$.

From Lemma 2.4.1 in [34] (see also [6]), the orthogonal of V with respect to the usual scalar product in $L^2(\omega\times Y)$ is made of gradients of the form $\nabla_{x'}q(x')+\nabla_\lambda\hat q(x',y)$, with $q(x')\in L^2_0(\omega)$ and $\hat q(x',y)\in L^2(\omega;H^1_\#(Y))$. Therefore, by integration by parts, the variational formulations (4.54)-(4.55) are equivalent to the homogenized system (4.52). It remains to prove that the pressure $\tilde P(x')$, arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div}_{x'}(\int_Y\hat u(x',y)dy)=0$, is the same as the limit of the pressure $\hat P_\varepsilon$. This can be easily done by considering in equation (3.44) a test function with div_λ equal to zero, and obtain the variational formulation (4.53). Since $2N^2\mathrm{rot}_\lambda\hat w+f'\in L^2(\omega\times Y)^3$ we deduce that $\tilde P\in H^1(\omega)$.

Finally, since from Lemma 2.4.1 in [34] we have that (4.52) admits a unique solution, and then the complete sequence $(a_\varepsilon^{-2}\hat u_\varepsilon,a_\varepsilon^{-1}\hat w_\varepsilon,\hat P_\varepsilon)$ converges to the solution $(\hat u(x',y),\hat w(x',y),\tilde P(x'))$.

□

Let us define the local problems which are useful to eliminate the variable y of the previous homogenized problem and then obtain a Darcy equation for the pressure $\tilde P$.

For every $i,k=1,2$ and $0<\lambda<+\infty$, we consider the following local micropolar problems

$$\left\{\begin{array}{rl}-\Delta_\lambda u^{i,k}+\nabla_\lambda\pi^{i,k}-2N^2\mathrm{rot}_\lambda w^{i,k}=e_i\delta_{1k} & \text{in }Y_f,\\ \mathrm{div}_\lambda u^{i,k}=0 & \text{in }Y_f,\\ -R_c\Delta_\lambda w^{i,k}+4N^2w^{i,k}-2N^2\,\mathrm{rot}_\lambda u^{i,k}=e_i\delta_{2k} & \text{in }Y_f,\\ u^{i,k}=w^{i,k}=0 & \text{in }Y_s,\\ u^{i,k}(y),w^{i,k}(y),\pi^{i,k}(y)\quad Y'-\text{periodic.}\end{array}\right.\tag{4.56}$$

It is known (see Lemma 2.5.1 in [34]) that there exist a unique solution $(u^{i,k},w^{i,k},\pi^{i,k})\in H^1_{0,\#}(Y_f)^3\times H^1_{0,\#}(Y_f)^3\times L^2_{0,\#}(Y_f)$ of problem (4.56), and moreover $\pi^{i,k}\in H^1(Y_f)$.

Now, we give the main result concerning the homogenized flow.

Theorem 4.3. *Let $(\hat{u}, \hat{w}, \tilde{P}) \in L^2(\omega; H^1_{0,\#}(Y)^3) \times L^2(\omega; H^1_{0,\#}(Y)^3) \times (L^2_0(\omega) \cap H^1(\omega))$ be the unique weak solution of problem (4.52). Then, the extensions $(a_\varepsilon^{-2}\tilde{u}_\varepsilon, a_\varepsilon^{-1}\tilde{w}_\varepsilon)$ and $\tilde{P}_\varepsilon$ of the solution of problem (2.10)-(2.11) converge weakly to $(\tilde{u}, \tilde{w})$ in $H^1(0,1; L^2(\omega)^3) \times H^1(0,1; L^2(\omega)^3)$ and strongly to $\tilde{P}$ in $L^2(\omega)$ respectively, with $\tilde{u}_3 = \tilde{w}_3 = 0$. Moreover, defining $\widetilde{U}(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$ and $\widetilde{W}(x') = \int_0^1 \tilde{w}(x', y_3)\, dy_3$, it holds*

$$
\begin{aligned}
&\widetilde{U}'(x') = K_\lambda^{(1)} \left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + K_\lambda^{(2)} g(x'), \qquad \widetilde{U}_3(x') = 0 \quad \text{in } \omega,\\
&\widetilde{W}'(x') = L_\lambda^{(1)} \left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + L_\lambda^{(2)} g(x'), \qquad \widetilde{W}_3(x') = 0 \quad \text{in } \omega,
\end{aligned}
\tag{4.57}
$$

where $K_\lambda^{(k)}$, $L_\lambda^{(k)} \in \mathbb{R}^{2\times 2}$, $k = 1, 2$, are matrices with coefficients

$$
\left(K_\lambda^{(k)}\right)_{ij} = \int_{Y_f} u_j^{i,k}(y)\, dy, \quad \left(L_\lambda^{(k)}\right)_{ij} = \int_{Y_f} w_j^{i,k}(y)\, dy, \quad i, j = 1, 2,
$$

where $u^{i,k}$, $w^{i,k}$ are the solutions of the local micropolar problems defined in (4.56).

Here, $\tilde{P} \in H^1(\omega) \cap L^2_0(\omega)$ is the unique solution of the 2D Darcy equation

$$
\left\{
\begin{aligned}
&\operatorname{div}_{x'}\left(K_\lambda^{(1)}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + K_\lambda^{(2)} g(x')\right) = 0 \quad \text{in } \omega,\\
&\left(K_\lambda^{(1)}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + K_\lambda^{(2)} g(x')\right)\cdot n = 0 \quad \text{in } \partial\omega.
\end{aligned}
\right.
\tag{4.58}
$$

Proof. We eliminate the microscopic variable y in the effective problem (4.52). To do that, we consider the following identification

$$
\begin{aligned}
\hat{u}(x', y) &= \sum_{i=1}^2 \left[\left(f_i(x') - \partial_{x_i}\tilde{P}(x')\right) u^{i,1}(y) + g_i(x') u^{i,2}(y)\right],\\
\hat{w}(x', y) &= \sum_{i=1}^2 \left[\left(f_i(x') - \partial_{x_i}\tilde{P}(x')\right) w^{i,1}(y) + g_i(x') w^{i,2}(y)\right],\\
\hat{q}(x', y) &= \sum_{i=1}^2 \left[\left(f_i(x') - \partial_{x_i}\tilde{P}(x')\right) \pi^{i,1}(y) + g_i(x') \pi^{i,2}(y)\right],
\end{aligned}
\tag{4.59}
$$

and thanks to the identity $\int_{Y_f} \hat{\varphi}(x', y)\, dy = \int_0^1 \tilde{\varphi}(x', y_3)\, dy_3$ with $\int_{Y_f} \hat{\varphi}_3\, dy = 0$ satisfied by velocity and microrotation given in Lemma 4.1, we deduce that $\widetilde{U}$ and $\widetilde{W}$ are given by (4.57).

Finally, the divergence condition with respect to the variable x' given in (4.52) together with the expression of $\widetilde{U}'(x')$ gives (4.58), which has a unique solution since $K_\lambda^{(1)}$ is positive definite and then the whole sequence converges, see Part III - Theorem 2.5.2 in [34].

□

Remark 4.4. *We observe that when N is identically zero, taking into account the linear momentum equations from (4.52), we can deduce that the Darcy equation (4.58) agrees with the ones obtained in [14, 29] in the case* **PTPM**.

5 The homogeneously thin porous medium (HTPM)

It corresponds to the case when the cylinder height is much larger than interspatial distance, i.e. $a_\varepsilon \ll \varepsilon$ which is equivalent to $\lambda = 0$.

Next, we give some compactness results about the behavior of the extended sequences $(\tilde{u}_\varepsilon, \tilde{w}_\varepsilon, \tilde{P}_\varepsilon)$ and the unfolding functions $(\hat{u}_\varepsilon, \hat{w}_\varepsilon, \hat{P}_\varepsilon)$ by using the *a priori* estimates given in Lemmas 3.4 and 3.5, and Lemma 3.7, respectively.

Lemma 5.1. *For a subsequence of ε still denoted by ε, there exist the following functions:*

i) (Velocity) there exist $\tilde u\in L^2(\Omega)^3$, with $\tilde u_3=0$ and $\hat u\in L^2(\Omega;H^1_\#(Y')^3)$ with $\hat u=0$ on $\omega\times Y_s$, such that $\int_{Y_f}\hat u(x',y)dy=\int_0^1\tilde u(x',y_3)dy_3$ with $\int_{Y_f}\hat u_3(x',y)dy=0$, $\hat u_3$ independent of y_3 and moreover

$$a_\varepsilon^{-2}\tilde u_\varepsilon\rightharpoonup(\tilde u',0)\ \text{in } L^2(\Omega)^3,\quad a_\varepsilon^{-2}\hat u_\varepsilon\rightharpoonup\hat u\ \text{in } L^2(\Omega;H^1(Y')^3),\tag{5.60}$$

$$\mathrm{div}_{x'}\left(\int_0^1\tilde u'(x',y_3)\,dy_3\right)=0\ \text{in }\omega,\quad \left(\int_0^1\tilde u'(x',y_3)\,dy_3\right)\cdot n=0\ \text{in }\partial\omega\,,\tag{5.61}$$

$$\mathrm{div}_{y'}\hat u'=0\ \text{in }\omega\times Y_f,\quad \mathrm{div}_{x'}\left(\int_{Y_f}\hat u'(x',y)\,dy\right)=0\ \text{in }\omega,\quad \left(\int_{Y_f}\hat u(x',y)\,dy\right)\cdot n=0\ \text{in }\partial\omega\,,\tag{5.62}$$

ii) (Microrotation) there exist $\tilde w\in L^2(\Omega)^3$ with $\tilde w_3=0$ and $\hat w\in L^2(\Omega;H^1_\#(Y')^3)$ with $\hat w=0$ on $\omega\times Y_s$, such that $\int_{Y_f}\hat w(x',y)dy=\int_0^1\tilde w(x',y_3)\,dy_3$ with $\int_{Y_f}\hat w_3(x',y)\,dy=0$, $\hat u_3$ independent of y_3 and moreover

$$a_\varepsilon^{-1}\tilde w_\varepsilon\rightharpoonup(\tilde w',0)\ \text{in } L^2(\Omega)^3,\quad a_\varepsilon^{-1}\hat w_\varepsilon\rightharpoonup\hat w\ \text{in } L^2(\Omega;H^1(Y')^3),\tag{5.63}$$

iii) (Pressure) there exists $\tilde P\in L^2_0(\Omega)$ independent of y_3, such that

$$\tilde P_\varepsilon\to\tilde P\ \text{in } L^2(\omega),\quad \hat P_\varepsilon\to\tilde P\ \text{in } L^2(\omega).\tag{5.64}$$

Proof. The proof of this result is obtained by arguing similarly to Section 5 in [13] and Lemma 4.1 of the present paper.

□

Using previous convergences, in the following theorem we give the homogenized system satisfied by $(\hat u,\hat w,\tilde P)$.

Theorem 5.2. *In the case* **HTPM**, *the sequence $(a_\varepsilon^{-2}\hat u_\varepsilon,a_\varepsilon^{-1}\hat w_\varepsilon)$ converges weakly to $(\hat u,\hat w)$ in $L^2(\Omega;H^1(Y')^3)\times L^2(\Omega;H^1(Y')^3)$ and $\hat P_\varepsilon$ converges strongly to $\tilde P$ in $L^2(\omega)$, where $(\hat u,\hat w,\tilde P)\in L^2(\Omega;H^1_\#(Y')^3)\times L^2(\Omega;H^1_\#(Y')^3)\times(L^2_0(\omega)\cap H^1(\omega))$ with $\hat u_3$ and $\hat w_3$ independent of y_3 and $\int_{Y'_f}\hat u_3(x',y')\,dy'=\int_{Y'_f}\hat w_3(x',y')\,dy'=0$. Moreover, defining $\hat U=\int_0^1\hat u(x',y)dy_3$, $\hat W=\int_0^1\hat u(x',y)dy_3$, we have that $(\hat U,\hat W)\in L^2(\omega;H^1_\#(Y')^3)\times L^2(\omega;H^1_\#(Y')^3)$ is the unique solution of the following homogenized system*

$$\left\{\begin{array}{rl}
-\Delta_{y'}\hat U'+\nabla_{y'}\hat q=2N^2\mathrm{rot}_{y'}\hat W_3+f'(x')-\nabla_{x'}\tilde P(x') & \text{in }\omega\times Y'_f,\\
-\Delta_{y'}\hat U_3=2N^2\mathrm{Rot}_{y'}\hat W' & \text{in }\omega\times Y'_f,\\
-R_c\Delta_{y'}\hat W'+4N^2\hat W'=2N^2\mathrm{rot}_{y'}\hat U_3+g'(x') & \text{in }\omega\times Y'_f,\\
-R_c\Delta_{y'}\hat W_3+4N^2\hat W_3=2N^2\mathrm{Rot}_{y'}\hat U' & \text{in }\omega\times Y'_f,\\
\mathrm{div}_{y'}\hat U'=0 & \text{in }\omega\times Y'_f,\\
\hat U'=\hat W'=0 & \text{in }\omega\times Y'_s,\\
\mathrm{div}_{x'}\left(\displaystyle\int_{Y'_f}\hat U'(x',y')\,dy'\right)=0 & \text{in }\omega,\\
\left(\displaystyle\int_{Y'_f}\hat U'(x',y')\,dy'\right)\cdot n=0 & \text{on }\partial\omega,\\
\hat U(x',y'),\hat w(x',y'),\hat q(x',y')\quad Y'-\text{periodic.}
\end{array}\right.\tag{5.65}$$

Proof. We choose $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $\text{div}_{y'}\varphi' = 0$ in $\omega\times Y$, $\text{div}_{x'}(\int_Y \varphi'\,dy) = 0$ in ω and φ_3 independent of y_3 in (3.44). Taking into account that thanks to $\text{div}_{y'}\varphi' = 0$ in $\omega \times Y_f$ and φ_3 independent of y_3, we have that

$$\frac{1}{a_\varepsilon}\int_{\omega\times Y} \hat{P}_\varepsilon \text{div}_{y'}\varphi'\,dx'dy = 0 \quad \text{and} \quad \frac{1}{\varepsilon}\int_{\omega\times Y} \partial_{y_3}\hat{P}_\varepsilon \partial_{y_3}\varphi_3\,dx'dy = 0\,.$$

Thus, passing to the limit using the convergences (5.60), (5.63), (5.64), $a_\varepsilon/\varepsilon \to 0$ and using in the limit that $\tilde{P}$ does not depend on y and $\text{div}_{x'}(\int_Y \varphi'\,dy) = 0$, we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{\omega\times Y_f} D_{y'}\hat{u}' : D_{y'}\varphi'\,dx'dy = 2N^2\int_{\omega\times Y_f}\text{rot}_{y'}\hat{w}_3\cdot\varphi'\,dx'dy + \int_{\omega\times Y_f} f'\cdot\varphi'\,dx'dy\\ \displaystyle\int_{\omega\times Y_f}\nabla_{y'}\hat{u}_3 : \nabla_{y'}\varphi_3\,dx'dy = 2N^2\int_{\omega\times Y_f}\text{Rot}_{y'}\hat{w}'\,\varphi_3\,dx'dy\,.\end{array}\right. \tag{5.66}$$

Next, for every $\psi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with ψ_3 independent of y_3, we choose $\psi_\varepsilon = a_\varepsilon^{-1}\psi$ in (3.45) taking into account that g_ε and R_M satisfy (3.14) and (3.15). Then, passing to the limit using convergences (5.60) and (5.63), we get

$$\left\{\begin{array}{l}\displaystyle R_c\int_{\omega\times Y_f} D_{y'}\hat{w}' : D_{y'}\psi'\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}'\cdot\psi'\,dx'dy\\ \displaystyle = 2N^2\int_{\omega\times Y_f}\text{rot}_{y'}\hat{u}_3\cdot\psi'\,dx'dy + \int_{\omega\times Y_f} g'(x')\cdot\psi'\,dx'dy\,,\\ \displaystyle R_c\int_{\omega\times Y_f}\nabla_{y'}\hat{w}_3 : \nabla_{y'}\psi_3\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}_3\cdot\psi_3\,dx'dy = 2N^2\int_{\omega\times Y_f}\text{Rot}_{y'}\hat{u}'\cdot\psi_3\,dx'dy.\end{array}\right. \tag{5.67}$$

We take into account that there is no y_3-dependence in the obtained variational formulation. For that, we can consider φ,ψ independent of y_3, which implies that $(\hat{U},\hat{W})$ satisfies the same variational formulation with integral in $\omega\times Y'_f$. By density, we can deduce that the variational formulation for $(\hat{U},\hat{W})$ is equivalent to problem (5.65).

□

The local problems to eliminate the variable y of the previous homogenized problem can be defined by using the local system (4.56) with $\lambda = 0$. In that case, since there is no y_3 dependence, then it is correct to consider all equations in 2D domain Y'_f instead of Y_f. Therefore, for every $i,k = 1,2$, we consider $(u^{i,k}, w^{i,k}, \pi^{i,k}) \in H^1_\#(Y'_f)^3 \times H^1_\#(Y'_f)^3 \times (H^1(Y'_f)\cap L^2_{0,\#}(Y'_f))$ the unique solutions of the following local micropolar problems

$$\left\{\begin{array}{rl} -\Delta_{y'}(u^{i,k})' + \nabla_{y'}\pi^{i,k} - 2N^2\text{rot}_{y'}w_3^{i,k} = e_i\delta_{1k} & \text{in } Y'_f,\\ -\Delta_{y'}u_3^{i,k} - 2N^2\text{Rot}_{y'}(w^{i,k})' = 0 & \text{in } Y'_f,\\ \text{div}_{y'}(u^{i,k})' = 0 & \text{in } Y_f,\\ -R_c\Delta_{y'}(w^{i,k})' + 4N^2(w^{i,k})' - 2N^2\,\text{rot}_{y'}u_3^{i,k} = e_i\delta_{2k} & \text{in } Y'_f,\\ -R_c\Delta_{y'}w_3^{i,k} + 4N^2 w_3^{i,k} - 2N^2\,\text{Rot}_{y'}(u^{i,k})' = 0 & \text{in } Y'_f,\\ u^{i,k} = w^{i,k} = 0 & \text{in } Y'_s,\\ u^{i,k}(y'), w^{i,k}(y'), \pi^{i,k}(y') \quad Y' - \text{periodic.} & \end{array}\right. \tag{5.68}$$

We give the main result concerning the homogenized flow.

Theorem 5.3. *Let $(\hat{U},\hat{W},\tilde{P}) \in L^2(\omega; H^1_\#(Y')^3)\times L^2(\omega; H^1_\#(Y')^3)\times(L^2_0(\omega)\cap H^1(\omega))$ be the unique weak solution of problem (5.65). Then, the extensions $(a_\varepsilon^{-2}\tilde{u}_\varepsilon, a_\varepsilon^{-1}\tilde{w}_\varepsilon)$ and $\tilde{P}_\varepsilon$ of the solution of problem (2.10)-(2.11) converge weakly to $(\tilde{u},\tilde{w})$ in $L^2(\Omega)^3\times L^2(\Omega)^3$ and strongly to $\tilde{P}$ in $L^2(\omega)$ respectively, with $\tilde{u}_3 = \tilde{w}_3 = 0$. Moreover, defining $\widetilde{U}(x') = \int_0^1 \tilde{u}(x',y_3)\,dy_3$ and $\widetilde{W}(x') = \int_0^1 \tilde{w}(x',y_3)\,dy_3$, it holds*

$$\begin{array}{ll}\widetilde{U}'(x') = K_0^{(1)}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + K_0^{(2)}g(x'), & \widetilde{U}_3(x') = 0 \quad \text{in } \omega,\\ \widetilde{W}'(x') = L_0^{(1)}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right) + L_0^{(2)}g(x'), & \widetilde{W}_3(x') = 0 \quad \text{in } \omega,\end{array} \tag{5.69}$$

where $K_0^{(k)}$, $L_0^{(k)}\in\mathbb{R}^{2\times 2}$, $k=1,2$, *are matrices with coefficients*

$$\left(K_0^{(k)}\right)_{ij}=\int_{Y_f'}u_j^{i,k}(y')\,dy',\quad \left(L_0^{(k)}\right)_{ij}=\int_{Y_f'}w_j^{i,k}(y')\,dy',\quad i,j=1,2,$$

where $u^{i,k}$, $w^{i,k}$ *are the solutions of the local micropolar problems defined in (5.68).*

Here, $\tilde P\in H^1(\omega)\cap L^2_0(\omega)$ *is the unique solution of the 2D Darcy equation*

$$\left\{\begin{array}{l}\mathrm{div}_{x'}\left(K_0^{(1)}\left(f'(x')-\nabla_{x'}\tilde P(x')\right)+K_0^{(2)}g(x')\right)=0\quad \text{in } \omega,\\ \left(K_0^{(1)}\left(f'(x')-\nabla_{x'}\tilde P(x')\right)+K_0^{(2)}g(x')\right)\cdot n=0\quad \text{in } \partial\omega.\end{array}\right.\tag{5.70}$$

Proof. To eliminate the microscopic variable y' in the effective problem (5.65), we proceed as for the critical case by considering the local systems (5.68).

Thanks to the identities for the velocity $\int_{Y_f'}\hat U(x',y')\,dy'=\tilde U(x')$ with $\int_{Y_f'}\hat U_3\,dy'=0$ and the analogous one for the microrotation given in Lemma 5.1, we deduce that $\widetilde U$ and $\widetilde W$ are given by (5.69).

Finally, the divergence condition with respect to the variable x' given in (5.61) together with the expression of $\widetilde U'(x')$ gives (5.70), which has a unique solution since $K_0^{(1)}$ is positive definite and then the whole sequence converges, see Part III - Theorem 2.5.2 in [34].

□

Remark 5.4. *We observe that when* N *is identically zero, taking into account the linear momentum equations from (5.65), we can deduce that the Darcy equation (5.70) agrees with the ones obtained in [14, 29] in the case* **HTPM**.

6 The very thin porous medium (VTPM)

It corresponds to the case when the cylinder height is much smaller than the interspatial distance, i.e. $a_\varepsilon\gg\varepsilon$ which is equivalent to $\lambda=+\infty$.

Next, we give some compactness results about the behavior of the extended sequences $(\tilde u_\varepsilon,\tilde w_\varepsilon,\tilde P_\varepsilon)$ and the unfolding functions $(\hat u_\varepsilon,\hat w_\varepsilon,\hat P_\varepsilon)$ satisfying the *a priori* estimates given in Lemmas 3.4 and 3.5, and Lemma 3.7, respectively.

Lemma 6.1. *For a subsequence of* ε *still denoted by* ε, *there exist the following functions:*

i) (Velocity) there exist $\tilde u\in H^1_0(0,1;L^2(\omega)^3)$ *with* $\tilde u_3=0$ *and* $\hat u\in H^1_0(0,1;L^2_\#(\omega\times Y')^3)$ *with* $\hat u=0$ *in* $\omega\times Y_s$, *such that* $\int_Y\hat u(x',y)dy=\int_0^1\tilde u(x',y_3)\,dy_3$ *with* $\int_Y\hat u_3\,dy=0$, $\hat u_3$ *independent of* y_3 *and moreover*

$$\varepsilon^{-2}\tilde u_\varepsilon\rightharpoonup(\tilde u',0)\ \text{in } H^1(0,1;L^2(\omega)^3),\quad \varepsilon^{-2}\hat u_\varepsilon\rightharpoonup\hat u\ \text{in } H^1(0,1;L^2(\omega\times Y')^3),\tag{6.71}$$

$$\mathrm{div}_{x'}\left(\int_0^1\tilde u'(x',y_3)\,dy_3\right)=0\ \text{in } \omega,\quad \left(\int_0^1\tilde u'(x',y_3)\,dy_3\right)\cdot n=0\ \text{in } \partial\omega,\tag{6.72}$$

$$\mathrm{div}_{y'}\hat u'=0\ \text{in } \omega\times Y_f,\quad \mathrm{div}_{x'}\left(\int_{Y_f}\hat u'(x',y)\,dy\right)=0\ \text{in } \omega,\quad \left(\int_{Y_f}\hat u'(x',y)\,dy\right)\cdot n=0\ \text{in } \partial\omega,\tag{6.73}$$

ii) (Microrotation) there exist $\tilde w\in H^1_0(0,1;L^2(\omega)^3)$ *with* $\tilde w_3=0$ *and* $\hat w\in H^1_0(0,1;L^2_\#(\omega\times Y')^3)$ *with* $\hat w=0$ *in* $\omega\times Y_s$, *such that* $\int_{Y_f}\hat w(x',y)dy=\int_0^1\tilde w(x',y_3)\,dy_3$ *with* $\int_{Y_f}\hat w_3\,dy=0$, $\hat w_3$ *independent of* y_3 *and moreover*

$$\varepsilon^{-1}\tilde w_\varepsilon\rightharpoonup(\tilde w',0)\ \text{in } H^1(0,1;L^2(\omega)^3),\quad \varepsilon^{-1}\hat w_\varepsilon\rightharpoonup\hat w\ \text{in } H^1(0,1;L^2(\omega\times Y')^3),\tag{6.74}$$

iii) (Pressure) there exists $\tilde{P}\in L^2_0(\Omega)$ independent of y_3, such that

$$\tilde{P}_\varepsilon \to \tilde{P} \ \text{in } L^2(\omega), \quad \hat{P}_\varepsilon \to \tilde{P} \ \text{in } L^2(\omega). \tag{6.75}$$

Proof. The proof of this result is obtained by arguing similarly to Section 5 in [13] and Lemma 4.1 of the present paper.

□

Theorem 6.2. *In the case* **VTPM**, *the sequence $(\varepsilon^{-2}\hat{u}_\varepsilon, \varepsilon^{-1}\hat{w}_\varepsilon)$ converges weakly to $(\hat{u},\hat{w})$ in $H^1(0,1;L^2(\omega\times Y')^3)\times H^1(0,1;L^2(\omega\times Y')^3)$ and $\hat{P}_\varepsilon$ converges strongly to $\tilde{P}$ in $L^2(\omega)$, where $(\hat{u},\hat{w},\tilde{P})\in H^1_0(\omega;L^2_\#(\omega\times Y')^3)\times H^1_0(\omega;L^2_\#(\omega\times Y')^3)\times (L^2_0(\omega)\cap H^1(\omega))$ with $\hat{u}_3=\hat{w}_3=0$, is the unique solution of the following homogenized system*

$$\begin{cases} -\partial_{y_3}\hat{u}' + \nabla_{y'}\hat{q} = 2N^2\,\mathrm{rot}_{y_3}\hat{w}' + f'(x') - \nabla_{x'}\tilde{P}(x') & \text{in } \omega\times Y_f,\\ -R_c\partial_{y_3}\hat{w}' + 4N^2\hat{w} = 2N^2\,\mathrm{rot}_{y_3}\hat{u}' + g'(x') & \text{in } \omega\times Y_f,\\ \mathrm{div}_{y'}\hat{u}' = 0 & \text{in } \omega\times Y_f,\\ \hat{u}' = \hat{w}' = 0 & \text{in } \omega\times Y_s,\\ \mathrm{div}_{x'}\left(\displaystyle\int_Y \hat{u}'(x',y)\,dy\right) = 0 & \text{in } \omega,\\ \left(\displaystyle\int_Y \hat{u}'(x',y)\,dy\right)\cdot n = 0 & \text{on } \partial\omega,\\ \hat{u}'(x',y), \hat{w}'(x',y), \hat{q}(x',y') \quad Y'-\text{periodic}. \end{cases} \tag{6.76}$$

Proof. We choose $\varphi\in\mathcal{D}(\omega;C^\infty_\#(Y)^3)$ with $\mathrm{div}_{y'}\varphi'=0$ in $\omega\times Y$, $\mathrm{div}_{x'}(\int_Y\varphi'\,dy)=0$ in ω and φ_3 independent of y_3 in (3.44). Taking into account that thanks to $\mathrm{div}_{y'}\varphi'=0$ in $\omega\times Y$ and φ_3 independent of y_3, we have that

$$\frac{1}{a_\varepsilon}\int_{\omega\times Y}\hat{P}_\varepsilon \mathrm{div}_{y'}\varphi'\,dx'dy = 0 \quad \text{and} \quad \frac{1}{\varepsilon}\int_{\omega\times Y}\partial_{y_3}\hat{P}_\varepsilon\partial_{y_3}\varphi_3\,dx'dy = 0\,.$$

Thus, passing to the limit using the convergences (6.71), (6.74), (6.75), $\varepsilon/a_\varepsilon\to 0$ and using in the limit that $\tilde{P}$ does not depend on y and $\mathrm{div}_{x'}(\int_Y\varphi'\,dy)=0$, we obtain

$$\begin{cases} \displaystyle\int_{\omega\times Y} D_{y'}\hat{u}' : D_{y'}\varphi'\,dx'dy = 2N^2\int_{\omega\times Y}\mathrm{rot}_{y_3}\hat{w}'\cdot\varphi'\,dx'dy + \int_{\omega\times Y} f'\cdot\varphi'\,dx'dy,\\ \displaystyle\int_{\omega\times Y}\nabla_{y'}\hat{u}_3 : \nabla_{y'}\varphi_3\,dx'dy = 0. \end{cases} \tag{6.77}$$

Next, for every $\psi\in\mathcal{D}(\omega;C^\infty_\#(Y)^3)$ with ψ_3 independent of y_3, we choose $\psi_\varepsilon = a_\varepsilon^{-1}\psi$ in (3.45) with g_ε and R_M satisfying (3.16) and (3.17). Then, passing to the limit using convergences (6.71) and (6.74), we get

$$\begin{cases} R_c\displaystyle\int_{\omega\times Y_f}\partial_{y_3}\hat{w}' : \partial_{y_3}\psi'\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}'\cdot\psi'\,dx'dy\\ \displaystyle = 2N^2\int_{\omega\times Y_f}\mathrm{rot}_{y_3}\hat{u}'\cdot\psi'\,dx'dy + \int_{\omega\times Y_f} g'(x')\cdot\psi'\,dx'dy\,,\\ R_c\displaystyle\int_{\omega\times Y_f}\partial_{y_3}\hat{w}_3 : \partial_{y_3}\psi_3\,dx'dy + 4N^2\int_{\omega\times Y_f}\hat{w}_3\cdot\psi_3\,dx'dy = 0. \end{cases} \tag{6.78}$$

The second equations of (6.77) and (6.78) together to the boundary conditions imply $\hat{u}_3=\hat{w}_3=0$. By density, we can deduce that this variational formulation is equivalent to problem (6.76).

□

Let us define the local problems which are useful to eliminate the variable y of the previous homogenized problem and then obtain a Darcy equation for $\tilde P$. We define Φ and Ψ by

$$\Phi(N,R_c)=\frac{1}{12}+\frac{R_c}{4(1-N^2)}-\frac{1}{4}\sqrt{\frac{N^2R_c}{1-N^2}}\coth\left(N\sqrt{\frac{1-N^2}{R_c}}\right), \tag{6.79}$$

$$\Psi(N,R_c)=\frac{\tanh\left(N\sqrt{\frac{1-N^2}{R_c}}\right)}{1-N\sqrt{\frac{1-N^2}{R_c}}\tanh\left(N\sqrt{\frac{1-N^2}{R_c}}\right)}, \tag{6.80}$$

and for every $k=1,2$, we consider the following 2D local micropolar Darcy problems for $\pi^{i,k}$ as follows

$$\left\{\begin{array}{l} -\mathrm{div}_{y'}\left(\dfrac{1}{1-N^2}\Phi(N,R_c)\left(\nabla_{y'}\pi^{i,k}(y')+e_i\delta_{1k}\right)\right)=0 \quad \text{in } Y_f',\\ \left(\frac{1}{1-N^2}\Phi(N,R_c)\left(\nabla_{y'}\pi^{i,k}(y')+e_i\delta_{1k}\right)\right)\cdot n=0 \quad \text{in } \partial Y_s'. \end{array}\right. \tag{6.81}$$

It is known that from the positivity of function Φ, problem (6.85) has a unique solution for $\pi^{i,k}\in H^1_\#(Y')$ (see [19] for more details).

Theorem 6.3. *Let $(\hat u,\hat w,\tilde P)\in H^1_0(\omega;L^2_\#(\omega\times Y')^3)\times H^1_0(\omega;L^2_\#(\omega\times Y')^3)\times(L^2_0(\omega)\cap H^1(\omega))$ be the unique weak solution of problem (6.76). Then, the extensions $(\varepsilon^{-2}\tilde u_\varepsilon,\varepsilon^{-1}\tilde w_\varepsilon)$ and $\tilde P_\varepsilon$ of the solution of problem (2.10)-(2.11) converge weakly to $(\tilde u,\tilde w)$ in $H^1(0,1;L^2(\omega)^3)\times H^1(0,1;L^2(\omega)^3)$ and strongly to $\tilde P$ in $L^2(\omega)$ respectively, with $\tilde u_3=\tilde w_3=0$. Moreover, defining $\widetilde U(x')=\int_0^1\tilde u(x',y_3)\,dy_3$ and $\widetilde W(x')=\int_0^1\tilde w(x',y_3)\,dy_3$, it holds*

$$\begin{array}{ll} \widetilde U'(x')=K^{(1)}_\infty\left(f'(x')-\nabla_{x'}\tilde P(x')\right)+K^{(2)}_\infty g(x'), & \widetilde U_3(x')=0 \quad in\ \omega,\\ \widetilde W'(x')=L^{(2)}_\infty g(x'), & \widetilde W_3(x')=0 \quad in\ \omega, \end{array} \tag{6.82}$$

where the matrices $K^{(k)}_\infty\in\mathbb R^{2\times 2}$, $k=1,2$, and $L^{(2)}_\infty\in\mathbb R^{2\times2}$ are matrices with coefficients

$$\begin{array}{l} \left(K^{(k)}_\infty\right)_{ij}=\dfrac{1}{1-N^2}\displaystyle\int_{Y_f'}\Phi(N,R_c)\left(\partial_{y_i}\pi^{j,k}(y')+\delta_{ij}\delta_{1k}\right)dy',\quad i,j=1,2,\\ \left(L^{(2)}_\infty\right)_{ij}=-\dfrac{1}{4N^3}\sqrt{\dfrac{R_c}{1-N^2}}\left(\displaystyle\int_{Y_f'}\Psi(N,R_c)\,dy'\right)\delta_{ij}, \end{array} \tag{6.83}$$

with Φ and Ψ given by (6.79) and (6.80) respectively, and $\pi^{i,k}\in H^1_\#(Y')$, $i,k=1,2$, the unique solutions of the local problems (6.85). Here, $\tilde P\in H^1(\omega)\cap L^2_0(\omega)$ is the unique solution of the 2D Darcy problem

$$\left\{\begin{array}{l} \mathrm{div}_{x'}\left(K^{(1)}_\infty\left(f'(x')-\nabla_{x'}\tilde P(x')\right)+K^{(2)}_\infty g(x')\right)=0 \quad in\ \omega,\\ \left(K^{(1)}_\infty\left(f'(x')-\nabla_{x'}\tilde P(x')\right)+K^{(2)}_\infty g(x')\right)\cdot n=0 \quad in\ \partial\omega. \end{array}\right. \tag{6.84}$$

Proof. We proceed as in in the proof of Theorem 4.3 in order to obtain (6.82). Thus, by using (4.59) where $(u^{i,k},w^{i,k},\pi^{i,k})\in H^1_{0,\#}(Y_f)^2\times H^1_{0,\#}(Y_f)^2\times L^2_0(Y_f')$, $i,k=1,2$, is the unique solution of

$$\left\{\begin{array}{rl} -\partial_{y_3}u^{i,k}+\nabla_{y'}\pi^{i,k}-2N^2\mathrm{rot}_{y_3}w^{i,k}=-e_i\delta_{1k} & \text{in } Y_f,\\ \mathrm{div}_{y'}u^{i,k}=0 & \text{in } Y_f,\\ -R_c\partial_{y_3}w^{i,k}+4N^2w^{i,k}-2N^2\,\mathrm{rot}_{y_3}u^{i,k}=-e_i\delta_{2k} & \text{in } Y_f,\\ u^{i,k}=w^{i,k}=0 & \text{in } Y_s,\\ u^{i,k}(y),w^{i,k}(y),\pi^{i,k}(y')\quad Y'-\text{periodic}, & \end{array}\right. \tag{6.85}$$

then, thanks to the identities $\int_{Y_f} \hat{u}(x',y)\,dy = \int_0^1 \tilde{u}(x',y_3)\,dy_3$ with $\hat{u}_3 = 0$ and $\int_{Y_f} \hat{w}(x',y)\,dy = \int_0^1 \tilde{w}(x',y_3)\,dy_3$ with $\hat{w}_3 = 0$ given in Lemma 6.1, it holds

$$\begin{aligned}
&\widetilde{U}'(x') = \int_Y \hat{u}'(x',y)\,dy = -K_\infty^{(1)}\left(\nabla_{x'}\tilde{P}(x') - f'(x')\right) + K_\infty^{(2)} g'(x'), \qquad \widetilde{U}_3(x') = \int_Y \hat{u}_3(x',y')\,dy = 0 \quad \text{in } \omega,\\
&\widetilde{W}'(x') = \int_Y \hat{w}'(x',y)\,dy = -L_\infty^{(1)}\left(\nabla_{x'}\tilde{P}(x') - f'(x')\right) + L_\infty^{(2)} g'(x'), \qquad \widetilde{W}_3(x') = \int_Y \hat{w}_3(x',y)\,dy = 0 \quad \text{in } \omega,
\end{aligned} \tag{6.86}$$

where $K_\infty^{(k)}$, $L_\infty^{(k)}$, $k = 1, 2$, are matrices defined by their coefficients

$$\left(K_\infty^{(k)}\right)_{ij} = -\int_Y u_j^{i,k}(y)\,dy, \quad \left(L_\infty^{(k)}\right)_{ij} = -\int_Y w_j^{i,k}(y)\,dy\,, \quad i,j = 1,2\,. \tag{6.87}$$

Then, by the divergence condition in the variable x' given in (6.76), we get the Darcy equation (6.84).

However, we observe that (6.85) can be viewed as a system of ordinary differential equations with constant coefficients, with respect to the variable y_3 and unkowns functions $y_3 \mapsto u_1^{i,k}(y',y_3), w_2^{i,k}(y',y_3), u_2^{i,k}(y',y_3), w_1^{i,k}(y',y_3)$, where y' is a parameter, $y' \in Y'$. Thus, we can give explicit expressions for $u^{i,k}$ and $w^{i,k}$ given in terms of $\pi^{i,k}$ as follows (see [19, 20, 46] for more details):

$$\begin{aligned}
u^{i,k}(y) = \;& \tfrac{1}{2(1-N^2)}\left[y_3^2 - y_3 + \tfrac{N^2}{k}\left(\sinh(ky_3) - (\cosh(ky_3)-1)\coth\left(\tfrac{k}{2}\right)\right)\right]\left(\nabla_{y'}\pi^{i,k}(y') + e_i\delta_{1k}\right)\\
& + \tfrac{1}{N^2}\left[\left(\tfrac{2N^2}{k}\sinh(ky_3) - 2y_3\right)A + \tfrac{2N^2}{k}(\cosh(ky_3)-1)B - y_3\right](e_i\delta_{2k})^{\perp}\,,\\
w^{i,k}(y) = \;& \tfrac{1}{4(1-N^2)}\left[2y_3 + \left(\cosh(ky_3) - 1 - \sinh(ky_3)\coth\left(\tfrac{k}{2}\right)\right)\right]\left(\nabla_{y'}\pi^{i,k}(y') + e_i\delta_{1k}\right)^{\perp}\\
& - \tfrac{1}{2N^2}\Big[\cosh(ky_3)A + \sinh(ky_3)B\Big]e_i\delta_{2k}\,,
\end{aligned} \tag{6.88}$$

where $k = \sqrt{\frac{4N^2(1-N^2)}{R_c}}$ and A, B are given by

$$A(y') = \frac{\sinh(k)}{-2\sinh(k) + \frac{4N^2}{k}(\cosh(k)-1)}, \quad B(y') = \frac{-(\cosh(k)-1)}{-2\sinh(k) + \frac{4N^2}{k}(\cosh(k)-1)}\,.$$

Integrating with respect to y_3, we get

$$\begin{aligned}
&\int_0^1 u^{i,k}(y',y_3)\,dy_3 = -\frac{1}{1-N^2}\Phi(N,R_c)\left(\nabla_{y'}\pi^{i,k}(y') + e_i\delta_{1k}\right)\,,\\
&\int_0^1 w^{i,k}(y',y_3)\,dy_3 = \frac{1}{4N^3}\sqrt{\frac{R_c}{1-N^2}}\Psi(N,R_c)e_i\delta_{2k}\,,
\end{aligned} \tag{6.89}$$

with Φ and Ψ given by (6.79) and (6.80), and so that $\pi^{i,k}$ satisfies the Darcy local problem (6.81). Using the expressions of $u^{i,k}$ and $w^{i,k}$ together with (6.86), (6.87) and (6.89), we easily get (6.82), which has a unique solution since $K_\lambda^{(1)}$ is positive definite and then the whole sequence converges. Observe that, from the second equation in (6.89) with $k = 2$, we have $L_0^{(1)} = 0$, which ends the proof.

□

Remark 6.4. *We observe that then R_c tends to zero, the function Φ given by (6.79) becomes identical to $1/12$. In that case, when N is identically zero, taking into account the linear momentum equations from (6.76), we can deduce that the Darcy equation (6.84) agrees with the ones obtained in [14, 29] in the case* **VTPM**.

7 Conclusion

A micropolar fluid flow has been considered in a thin domain with microstructure, i.e. a thin domain which is perforated by periodically distributed solid cylinders which is called thin porous medium (**TPM**). This type of domains include a parameter ε connected to the fluid film thickness and another a_ε connected to the microstructure representing the size of the cylinders and the interspatial distance between them.

A direct numerical treatment of fluid flows through **TPM** becomes very difficult due to the rapid variations on the microscale level, so it would be necessary to obtain macroscopic laws to describe the fluid flows in such a domain. Thus, due to the description of the domain by using the parameters ε and a_ε, it is possible to describe the macroscopic behavior by means of the homogenization theory. In this sense, by using homogenization techniques, we derive that the flow is governed by a generalized 2D Darcy equation

$$\begin{cases} u'(x') = K_\lambda^{(1)}\left(f'(x') - \nabla_{x'}p(x')\right) + K_\lambda^{(2)}g(x'), \qquad u_3(x') = 0 \quad \text{in } \omega, \\ w'(x') = L_\lambda^{(1)}\left(f'(x') - \nabla_{x'}p(x')\right) + L_\lambda^{(2)}g(x'), \qquad w_3(x') = 0 \quad \text{in } \omega, \\ \mathrm{div}_{x'}u' = 0 \quad \text{in } \omega, \quad u'\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{7.90}$$

where λ is the proportionality constant between a_ε and ε and we remark that the interaction between the velocity and the microrotation fields is preserved. Moreover, we have that the flow factors $K_\lambda^{(k)}$, $L_\lambda^{(k)}$, $k = 1, 2$, are calculated depending on the type of **TPM**:

- In the **PTPM**, i.e. when $0 < \lambda < +\infty$, the flow factors $K_\lambda^{(k)}$, $L_\lambda^{(k)}$ are calculated by solving 3D micropolar local problems posed in a 3D unit cell Y and depending on the parameter λ, the coupling number and the characteristic length.
- In the **HTPM**, i.e. when $\lambda = 0$, the flow factors $K_0^{(k)}$, $L_0^{(k)}$ are calculated by solving 3D micropolar local problems posed in a 2D unit cell Y' and depending on the coupling number and the characteristic length.
- In the **VTPM**, i.e. when $\lambda = +\infty$, the flow factors $K_\infty^{(k)}$, $L_\infty^{(k)}$ are calculated by solving 2D micropolar Reynolds local problems posed in a 2D unit cell Y' and depending on the coupling number and the characteristic length.

From the above, it is obtained that the model problem considered as an average problem could be solved by using the following homogenization procedure:

1. Solve the local problem numerically corresponding to the value of $\lambda \in [0, +\infty]$.
2. Use the solution to compute the components of the flow factors K_λ and L_λ.
3. Find p by solving the homogenized problem $(7.90)_3$ numerically.
4. Compute u by means of $(7.90)_1$ and w by means of $(7.90)_2$.

We remark that in the intermediate case **PTPM**, the local problems are three-dimensional and the coefficient of proportionality λ appears as a parameter in the equations. In the extreme cases **HTPM** and **VTPM**, the local problems are simpler, which represents a considerable simplification, compared with the intermediate case, from the numerical point of view. In view of that, more efficient numerical algorithms could be developed improving, hopefully, the known engineering practice of micropolar flows through **TPM**.

References

[1] S. Abbas, M.Y. Malik and S. Nadeem, Transportation of magnetized micropolar hybrid nanomaterial fluid flow over a Riga curface surface, Comput. Methods Programs Biomed., 185 (2020) 105136.

[2] S. Abbas, S. Nadeem and M.Y. Malik, On extended version of Yamada-Ota and Xue models in micropolar fluid flow under the region of stagnation point, Physica A, 542 (2020) 123512.

[3] S. Abbas, S. Nadeem and M.Y. Malik, Theoretical study of micropolar hybrid nanofluid over Riga channel with slip conditions, Physica A, 551 (2020) 124083.

[4] S. Ahmad and S. Nadeem, Application of CNT-based micropolar hybrid nanofluid flow in the presence of Newtonian heating, Applied Nanoscience, (2020).

[5] S. Ahmad, S. Nadeem, N. Muhammad and M. N. Kahn, Cattaneo?Christov heat flux model for stagnation point flow of micropolar nanofluid toward a nonlinear stretching surface with slip effects, J. Therm. Anal. Calorim., (2020).

[6] G. Allaire, Homogenization of the Stokes flow in a connected porous medium, Asymptotic Analysis, 2 (1989) 203-222.

[7] M. Anguiano, Homogenization of a non-stationary non-Newtonian flow in a porous medium containing a thin fissure, European Journal of Applied Mathematics, 30 (2019) 248-277.

[8] M. Anguiano, Derivation of a quasi-stationary coupled Darcy-Reynolds equation for incompressible viscous fluid flow through a thin porous medium with a fissure, Mathematical Methods in the Applied Sciences, 40 (2017) 4738-4757.

[9] M. Anguiano, On the non-stationary non-Newtonian flow through a thin porous medium, ZAMM - Journal of Applied Mathematics and Mechanics, 97 (2017) 895-915.

[10] M. Anguiano, Darcy's laws for non-stationary viscous fluid flow in a thin porous medium, Mathematical Methods in the Applied Sciences, 40 (2017) 2878-2895.

[11] M. Anguiano and R. Bunoiu, On the flow of a viscoplastic fluid in a thin periodic domain. Constanda C. , Harris P. (eds) Integral Methods in Science and Engineering, Birkäuser, Cham., (2019).

[12] M. Anguiano and R. Bunoiu, Homogenization of Bingham flow in thin porous media, Networks and Heterogeneous Media, 15 (2020).

[13] M. Anguiano and F.J. Suárez-Grau, Homogenization of an incompressible non-Newtonian flow through a thin porous medium, ZAMP-Journal of Applied Mathematics and Physics, (2017) 68:45, DOI: 10.1007/s00033-017-0790-z.

[14] M. Anguiano and F.J. Suárez-Grau, The transition between the Navier-Stokes equations to the Darcy equation in a thin porous medium, Mediterr. J. Math., (2018) 15:45.

[15] M. Anguiano and F.J. Suárez-Grau, Newtonian fluid flow in a thin porous medium with a non-homogeneous slip boundary conditions, Netw. Heterog. Media, 14 (2019) 289-316.

[16] M. Anguiano and F.J. Suárez-Grau, Analysis of the effects of a fissure for a non-Newtonian fluid flow in a porous medium, Commun. Math. Sci., 16 (2018) 273-292.

[17] M. Anguiano and F.J. Suárez-Grau, Derivation of a coupled Darcy-Reynolds equation for a fluid flow in a thin porous medium including a fissure, ZAMP-Journal of Applied Mathematics and Physics, 68 (2017) 68:52

[18] T. Arbogast, J. Douglas J.R. and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal., 21 (1990) 823-836.

[19] G. Bayada, M. Chambat and S.R. Gamouana, About thin film micropolar asymptotic equations, Quart. Appl. Math., 59 (2001) 413-439.

[20] G. Bayada and G. Lukaszewicz, On micropolar fluids in the theory of lubrication. Rigorous derivation of an analogue of the Reynolds equation. Internat. J. Engrg. Sci., 34 (1996) 1477-1490.

[21] Bonnivard, M., Pazanin I., Suárez-Grau, F.J.: Effects of rough boundary and nonzero boundary conditions on the lubrication process with micropolar fluid. Eur. J. Mech. B Fluids 72, 501-518 (2018).

[22] Boukrouche M., Paoli L.: Asymptotic analysis of a micropolar fluid flow in a thin domain with a free and rough boundary. SIAM J. Math. Anal. 44, 1211-1256 (2012).

[23] D. Cioranescu, A. Damlamian and G. Griso, Periodic unfolding and homogenization, C.R. Acad. Sci. Paris Ser. I, 335 (2002) 99-104.

[24] D. Cioranescu, A. Damlamian and G. Griso, The periodic unfolding method in homogenization, SIAM J. Math. Anal., 40 (2008) 1585-1620.

[25] Dupuy, D., Panasenko, G., Stavre, R.: Asymptotic solution for a micropolar flow in a curvilinear channel. ZAMM Z. Angew. Math. Mech. 88, 793-807 (2008).

[26] G. Duvaut and J.L. Lions, Les inequations en mechanique et en physique [The inequations in mechanics and physics], Dunod, Paris (1972).

[27] A.C. Eringen, Simple mocrofluids, Internat. J. Engrg. Sci., 2 (1964) 205-207.

[28] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech., 16 (1966) 1-18.

[29] J. Fabricius, J. Gunnar I. Hellström, T. Staffan Lundström, Elena Miroshnikova and Peter Wall, Darcy's Law for flow in a periodic thin porous medium confined between two parallel plates, Transp. Porous Med., 115 (2016) 473-493.

[30] V. Frishfelds, T.S. Lundström and A. Jakovics, Lattice gas analysis of liquid front in non-crimp fabrics, Transp. Porous Med., 84 (2011) 75-93

[31] W. Jeon and C.B. Shin, Design and simulation of passive mixing in microfluidic systems with geometric variations, Chem. Eng. J., 152 (2009) 575-582.

[32] Johnston, G.J., Wayte, R., Spikes, H.A.: The measurement and study of very thin lubricant films in concentrated contacts. Tribol. Trans. 34, 187-194 (1991).

[33] M. N. Khan, S. Nadeem and N. Muhammad, Micropolar fluid flow with temperature-dependent transport properties, Heat transfer, 49, (2020) 2375-2389.

[34] G. Lukaszewicz, Micropolar fluids, theory and applications, Modeling and Simulation in Science, Engineering and Technology, Birkhaüser, (1999).

[35] T.S. Lundström, S. Toll and J.M. Hakanson, Measurements of the permeability tensor of compressed fibre beds, Transp. Porous Med., 47 (2002) 363-380.

[36] J.B. Luo, P. Huang and S.Z. Wen, Thin film lubrication part I: study on the transition between EHL and thin film lubrication using relative optical interference intensity technique. Wear 194, 107-115 (1996).

[37] J.B. Luo, P. Huang, S.Z. Wen and L. Lawrence, Characteristics of fluid lubricant films at nano-scale, J. Tribol. 121, 872-878 (1999).

[38] S. Nadeem, M.Y. Malik and N. Abbas, Heat transfer of three-dimensional micropolar fluid on a Riga plate, Can. J. Phys., 98 (2020) 32-38.

[39] S. Nadeem, N. Abbas, Y. Elmasry and M.Y. Malik, Numerical analysis of water based CNTs flow of micropolar fluid through rotating frame, Compt. Meth. Prog. Bio., 186 (2020) 105194.

[40] S. Nadeem, M. N. Kiani, A. Saleem and A. Issakhov, Microvascular blood flow with heat transfer in a wavy channel having electroosmotic effects, Electrophoresis, 41 (2020) 1198-1205.

[41] M. Nordlund and T.S. Lundström, Effect of multi-scale porosity in local permeability modelling of non-crimp fabrics, Transp. Porous Med., 73 (2008) 109-124.

[42] Pazanin, I., Suárez-Grau, F.J.: Analysis of the thin film flow in a rough thin domain filled with micropolar fluid. Comput. Math. Appl. 68, 1915-1932 (2014).

[43] F. Singh, B. Stoeber, and S.I. Green, Micro-PIV measurement of flow upstream of papermaking forming fabrics, Transp. Porous Med., 107 (2015) 435-448.

[44] H. Tan and K.M. Pillai, Multiscale modeling of unsaturated flow in dual-scale fiber preforms of liquid composite molding I: Isothermal flows, Compos. Part A Appl. Sci. Manuf., 43 (2012) 1-13.

[45] L. Tartar, Incompressible fluid flow in a porous medium convergence of the homogenization process, Appendix to Lecture Notes in Physics, 127, Berlin, Springer-Velag, 1980.

[46] F. J. Suárez-Grau, Analysis of the Roughness Regimes for Micropolar Fluids via Homogenization, Bull. Malaysian Math. Sci. Soc., (2020), https://doi.org/10.1007/s40840-020-01027-1