Asymptotic behavior of a non-Newtonian flow in a thin domain with Navier law on a rough boundary

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Abstract

We consider a non-Newtonian flow in a thin domain of thickness ε. The flow is described by the 3D incompressible Navier-Stokes (Stokes) system with a nonlinear viscosity, being a power of the shear rate (power law) of flow index p.

The bottom of the domain is irregular by the present of slight roughness of amplitude ε^δ and period ε^β, satisfying the relation $1 < \beta < \delta$. Assuming pure slip or partial slip with a friction coefficient $\varepsilon^{-\gamma}$, with $\gamma > 0$, on the rough boundary, we consider the limit when domain thickness tends to zero and we obtain different models depending on the magnitude δ with respect to $\frac{2p-1}{p}\beta - \frac{p-1}{p}$, and the magnitude γ with respect to $\frac{1}{p-1}$.

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1. Introduction

Roughness of the solid surface as well as the rheological properties of the fluid have influence in the fluid-solid interface condition. For that reason, the choice of boundary conditions of fluid flow is a relevant problem to determine such influence. It is commonly accepted that viscous fluids adhere to rough

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surfaces, and so the no-slip condition at the rough surfaces of a domain is widely used. This does not seem always valid for the non-Newtonian fluids, indeed non-Newtonian fluids melt and solutions slip against the surface. This phenomenon has been related in many mechanical papers concerning non-Newtonian fluids (see [24], [34]).

In this sense, it has been suggested that, in many cases, the velocity field of the fluid u_ε in a domain Ω_ε obeys a Navier condition at the rough surface $\Gamma_\varepsilon \subset \partial\Omega_\varepsilon$:

$$\big[S\,\nu\big]_\tau = -\lambda[u_\varepsilon]_\tau \quad \text{on } \Gamma_\varepsilon, \qquad u_\varepsilon\,\nu = 0, \quad \text{on } \Gamma_\varepsilon,$$

where S is the deviatoric viscous stress tensor, ν denotes the outside unitary normal vector to Ω_ε on Γ_ε, the subscript $_\tau$ denotes the orthogonal projection on the tangent space of Γ_ε, and $\lambda > 0$ is the friction coefficient.

Notice that depending on the value of λ, we shall consider either pure slip, partial slip or no-slip at the boundary. This means that the friction coefficient λ shall be either zero, positive or $+\infty$.

Recently, based on the so called rugosity effect, some results mathematically justify that viscous fluids adhere completely to the boundary of an impermeable domain. More precisely, these results accounts asymptotically for the transformation of pure slip boundary conditions on a rough surface in no-slip boundary conditions, as the amplitude of the roughness vanishes, provided that the energy of the solutions is uniformly bounded and there is enough roughness of the oscillating boundaries. We refer to the pioneering paper [15] for the case of periodic and very smooth boundaries, and to [6], [10], [11], [12] (see also [18], [21], [25], [31]) for having a quite complete understanding for arbitrary boundaries.

In this paper, assuming Navier condition on a slightly rough surface of a thin domain, our goal is to identify, depending on the magnitudes of the amplitude of the roughness and the friction coefficient, the type of boundary condition in which the starting Navier condition is transformed in the limit.

Next, let us set the position of the problem. We consider a smooth bounded open set $\omega \subset \mathbb{R}^2$ and a function Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z' = (-1/2, 1/2)^2$, satisfying

$$\text{Span}(\{\nabla\Psi(z') \,:\, z' \in \mathbb{R}^2\}) = \mathbb{R}^2, \tag{1}$$

which always holds except in the case where Ψ is constant in one direction (i.e. not included roughness of riblets-type: $\Psi = \Psi(z_1)$ or $\Psi = \Psi(z_2)$).

Then, we consider a spatial domain Ω_ε determined through

$$\Omega_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x' \in \omega,\; -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right) < x_3 < \varepsilon\right\}, \tag{2}$$

where ε is a positive parameter devoted to tends to zero representing the characteristic thickness of the domain, and ε^δ and ε^β represent the amplitud and the period, respectively, of the roughness satisfying $1 < \beta < \delta < +\infty$, i.e. relation

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta-\beta} = 0, \quad \lim_{\varepsilon\to 0} \varepsilon^{\beta-1} = 0. \tag{3}$$

By $\Gamma_\varepsilon \subset \partial\Omega_\varepsilon$ we denote the rough boundary of Ω_ε, that is

$$\Gamma_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x' \in \omega,\; x_3 = -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right)\right\}. \tag{4}$$

From the other side it is well-known that for non-Newtonian flow through a thin domain the non linear Poiseuille law is used. For that reason, in this study we deal with the case where the viscosity is not constant. Thus, we consider that the viscosity satisfies the non linear power law, which is widely used for melted polymers, oil, mud,... Denoting the shear rate by $\mathbb{D}[u_\varepsilon] = \frac{1}{2}(Du_\varepsilon + D^t u_\varepsilon)$, the viscosity as a function of the shear rate is given by

$$\eta_p(\mathbb{D}[u_\varepsilon]) = \mu|\mathbb{D}[u_\varepsilon]|^{p-2}, \quad p > 1,$$

where the two material parameters $\mu > 0$ and $1 < p < +\infty$ are called the consistency and the flow index, respectively. For simplicity we suppose $\mu = 1$.

Recall that $p = 2$ yields the Newtonian fluid. For $p < 2$ the fluid is pseudoplastic (shear thinning), which is characteristic of high polymers, polymer solutions, and many suspension, whereas for $p > 2$ the fluid is dilatant (shear thickening), whose behavior is reported for certain slurries, like mud, clay, or cement, and implies an increased resistance to flow with intensified shearing.

Therefore, assuming the fluid incompressible and in a stationary state, and depending on the value of p, the velocity u_ε and the pressure π_ε satisfy:

- For $9/5 \leq p < +\infty$, the non-Newtonian Navier-Stokes system

$$\begin{aligned} -\mathrm{div}\,(\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + (u_\varepsilon \nabla)u_\varepsilon + \nabla \pi_\varepsilon = f_\varepsilon \qquad \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon. \end{aligned} \tag{5}$$

- For $1 < p < 9/5$, due to the known technical difficulties with inertial term for $p < 9/5$, the non-Newtonian Stokes system

$$\begin{aligned} -\mathrm{div}\,(\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla \pi_\varepsilon = f_\varepsilon \qquad \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon, \end{aligned} \tag{6}$$

where the right-hand side f_ε is of the form

$$f_\varepsilon(x) = \widetilde{f}\left(x', \frac{x_3}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega_\varepsilon, \tag{7}$$

with $\widetilde{f}$ assumed in $L^{p'}(\omega \times (-1,1))^3$.

In order to study the roughness effects on the flow, we assume Navier condition on Γ_ε,

$$\Big[\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]\,\nu\Big]_\tau = -\lambda\,\varepsilon^{-\gamma}\big[u_\varepsilon\big]_\tau \quad \text{on } \Gamma_\varepsilon, \quad u_\varepsilon\,\nu = 0 \quad \text{on } \Gamma_\varepsilon, \tag{8}$$

where $\lambda\,\varepsilon^{-\gamma}$ is the friction coefficient expressed in terms of the magnitude of the domain. Here $\lambda = \mathcal{O}(1)$, and to cover interesting cases $\gamma \in \{-\infty\} \cup [0, +\infty)$; namely $\gamma = -\infty$ yields pure slip, and $\gamma \in [0, +\infty)$ yields partial slip. Notice that when $\gamma > \frac{1}{p-1}$ the friction coefficient is so big that we must consider no-slip condition instead of Navier one (see (96) for how to reach that conclusion). For that reason, from now on we assume $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1}]$.

For simplicity, we assume no-slip condition on the rest of the boundary,

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon. \tag{9}$$

The goal of this work is to study the asymptotic behavior of the solutions of the above problems and to know in which way the irregular boundary affects the flow. To do this, it is required to study the asymptotic study of the problem by using monotonicity arguments together with an appropriate combination of two changes of variables: the usual rescaling to transform the thin domain into one with fixed height providing the macroscopic behavior of the fluid, and the unfolding method (see e.g. [2], [13], [16], [20], [26]) to capture the microscopic behavior of the fluid near the rough boundary. Then, denoting by $\widetilde{u}'$ and $\widetilde{\pi}$ the limit velocity and pressure, we recover the simplified non-Newtonian Stokes system in $\Omega = \omega \times (0,1)$:

$$-\partial_{y_3}\left(|\partial_{y_3}\widetilde{u}'|^{p-2}\,\partial_{y_3}\widetilde{u}'\right) = 2^{\frac{p}{2}}\left(\widetilde{f}' - \nabla_{y'}\widetilde{\pi}\right) \quad \text{in } \Omega,$$

The roughness effects apppear into the boundary condition on $\Gamma = \omega \times \{0\}$; we show that the boundary condition on Γ satisfied by $\widetilde{u}'$ includes the roughness effects, which depend on the values related with the amplitude of the roughness and the friction coefficient of the Navier condition, i.e. δ and γ respectively:

- Concerning $\delta \in (\beta, +\infty)$; there is one critical regime, namely $\delta = \widetilde{\beta}_p$, and therefore three different regimes, namely $\delta \in (\beta, \widetilde{\beta}_p)$, $\delta = \widetilde{\beta}_p$, and $\delta > \widetilde{\beta}_p$. The parameter $\widetilde{\beta}_p$ is defined by

$$\widetilde{\beta}_p := \frac{2p-1}{p}\beta - \frac{p-1}{p}, \tag{10}$$

  which satisfies $\widetilde{\beta}_p \in (\beta, +\infty)$, $\forall\, p > 1$.

- Concerning $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1}]$; three regimes are addressed, namely the critical regime $\gamma = \frac{1}{p-1}$, and thus $\gamma \in [0, \frac{1}{p-1})$ and $\gamma = -\infty$.

Below every cases are discussed:

(i) Case $\delta \in (\beta, \widetilde{\beta}_p)$. We show that $\widetilde{u}'$ satisfies no-slip:

$$\widetilde{u}' = 0 \;\; \text{on } \Gamma,$$

for every $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1}]$. This means that no matter if we start with pure or partial slip, actually the roughness is so strong that the limit velocity of the fluid vanishes at the bottom.

(ii) Case $\delta = \widetilde{\beta}_p$. This depends on γ:

- If $\gamma = \frac{1}{p-1}$, then $\widetilde{u}'$ satisfies partial slip:

$$-|\partial_{y_3}\widetilde{u}'(y',0)|^{p-2}\partial_{y_3}\widetilde{u}'(y',0) = -\lambda\, 2^{\frac{p}{2}}\, \widetilde{u}'(y',0) - 2^{\frac{p}{2}}\, R(\widetilde{u}'(y',0)) \text{ on } \Gamma,$$

  which means that, starting from partial slip with friction coefficient of magnitude of order $\varepsilon^{-\frac{1}{p-1}}$, we also obtain partial slip in the limit, but with a higher friction coefficient coming from the effects of the roughness (see Theorem 3.2 for more details of R).

- If $\gamma \in [0, \frac{1}{p-1})$, then $\widetilde{u}'$ satisfies partial slip:

$$-|\partial_{y_3}\widetilde{u}'(y',0)|^{p-2}\partial_{y_3}\widetilde{u}'(y',0) = -2^{\frac{p}{2}}\, R(\widetilde{u}'(y',0)) \text{ on } \Gamma,$$

  which means that, starting from partial slip with a friction coefficient of magnitude of order smaller than $\varepsilon^{-\frac{1}{p-1}}$, we obtain partial slip with only the friction coefficient coming from the roughness.

- If $\gamma = +\infty$, then, starting from pure slip, $\widetilde{u}'$ satisfies partial slip with the same friction coefficient as the previous case.

(iii) Case $\delta > \widetilde{\beta}_p$. This depends on γ:

- If $\gamma = \frac{1}{p-1}$, then $\widetilde{u}'$ satisfies partial slip:

$$-|\partial_{y_3}\widetilde{u}'(y',0)|^{p-2}\partial_{y_3}\widetilde{u}'(y',0) = -\lambda\, 2^{\frac{p}{2}}\, \widetilde{u}'(y',0) \text{ on } \Gamma,$$

which means that, starting from partial slip with friction coefficient of magnitude of order $\varepsilon^{-\frac{1}{p-1}}$, we obtain partial slip with only λ as the friction coefficient. Here the roughness is so slight that no roughness effects appear in the limit.

- If $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$, then $\widetilde{u}'$ satisfies pure slip:

$$-|\partial_{y_3}\widetilde{u}'(y',0)|^{p-2}\partial_{y_3}\widetilde{u}'(y',0) = 0 \quad \text{on} \quad \Gamma,$$

  which means that, starting from pure or partial slip with a friction coefficient of magnitude of order smaller than $\varepsilon^{-\frac{1}{p-1}}$, we obtain pure slip.

To conclude with this discussion, notice that the usual case $\delta = \beta$ (amplitude and period of the same order) is not included in this work, because we have considered $1 < \beta < \delta$ which is necessary to apply the adaptation of the unfolding method. However, as for $\delta \in (\beta, \widetilde{\beta}_p)$ the roughness effects are so strong that we get no-slip in the limit, then for $\delta \leq \beta$ the no-slip condition in the limit must be verified. In this sense, due to the wide variety of choices of parameters δ and β concerning the roughness, and γ relative to the friction coefficient, this study may be instrumental for understanding the effects of the rough boundary and fluid microstructure on the lubrication process.

Moreover, by means of the limit equation, as usual in the asymptotic study of fluids in thin domains, we obtain the corresponding non-linear Reynolds type equation assuming the pure slip, partial slip or no-slip boundary conditions obtained in each case. Related to this, for the rigorous mathematical justification of the linear Reynolds equation for a flow between two plain surfaces, we refer to [3] and [32], whereas for the justification of the non-linear Reynolds type equation we give the references [9] and [29]. Remark that the last reference has been later extended in [7] and [8], assuming stick-slip conditions given by Tresca law on the boundary.

Finally, comment that approximate the behavior of flows in domains containing singular boundaries is critical for the description of a number of phenomena related to the behavior of flows over rough surfaces. In this sense, it worth to refer the following papers concerning fluid flows and domains containing corners, [4], [22] and [30].

2. Notation

The elements $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$.

By Z', we denote the unitary cube of $\mathbb{R}^2$, $Z' = (-\frac{1}{2}, \frac{1}{2})^2$, and by $\widehat{Q}$ the set $\widehat{Q} = Z' \times (0, +\infty)$. For every $M > 0$ we write $\widehat{Q}_M = Z' \times (0, M)$.

We assume p, with $p > 1$, and p' conjugate exponents, i.e. $1/p + 1/p' = 1$. We use the index # to mean periodicity with respect Z', for example $L^p_\#(Z')$, $q > 1$ denotes the space of functions $u \in L^p_{loc}(\mathbb{R}^2)$ which are Z'-periodic, while $L^p_\#(\widehat{Q})$ denotes the space of functions $\widehat{u} \in L^p_{loc}(\mathbb{R}^2 \times (0, +\infty))$ such that

$$\int_{\widehat{Q}} |\widehat{u}|^p dz < +\infty, \quad \widehat{u}(z' + k', z_3) = \widehat{u}(z), \quad \forall k' \in \mathbb{Z}^2, \quad \text{a.e. } z \in \mathbb{R}^2 \times (0, +\infty).$$

For a bounded measurable set $\Theta \subset \mathbb{R}^N$, we denote by $L^p_0(\Theta)$ the space of functions of $L^p(\Theta)$ with null integral.

We denote by ε, ε^β and ε^δ three positive parameters which tend to zero and satisfy $1 < \beta < \delta < +\infty$, i.e.

$$\lim_{\varepsilon \to 0} \varepsilon^{\delta - \beta} = 0, \quad \lim_{\varepsilon \to 0} \varepsilon^{\beta - 1} = 0.$$

Then, for a function $\Psi \in W^{2,\infty}_\#(Z')$, $\Psi \geq 0$ in Z', we define the open set $\Lambda_\varepsilon \subset \mathbb{R}^3$ by

$$\Lambda_\varepsilon = \left\{ x \in \mathbb{R}^3 : \; -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right) < x_3 < \varepsilon \right\}, \tag{11}$$

and for a Lipschitz bounded connected open set $\omega \subset \mathbb{R}^2$, we take

$$\Omega_\varepsilon = \Lambda_\varepsilon \cap (\omega \times \mathbb{R}), \tag{12}$$

$$\Omega^-_\varepsilon = \Omega_\varepsilon \cap (\omega \times (-\infty, 0)), \quad \Omega^+_\varepsilon = \Omega_\varepsilon \cap (\omega \times (0, +\infty)), \tag{13}$$

$$\Gamma_\varepsilon = \left\{ x \in \mathbb{R}^3 : \; x' \in \omega, \, x_3 = -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right) \right\}, \tag{14}$$

$$\widetilde{\Omega}_\varepsilon = \left\{ y \in \mathbb{R}^3 : \; y' \in \omega, \, -\varepsilon^{\delta - 1} \Psi\left(\frac{y'}{\varepsilon^\beta}\right) < y_3 < 1 \right\}, \tag{15}$$

$$\widetilde{\Gamma}_\varepsilon = \left\{ y \in \mathbb{R}^3 : \ y' \in \omega, \, y_3 = -\varepsilon^{\delta-1} \Psi\left(\frac{y'}{\varepsilon^\beta}\right) \right\}, \tag{16}$$

$$\Omega = \omega \times (0,1), \quad \Gamma = \omega \times \{0\}. \tag{17}$$

We denote by ν the outside unitary normal vector to Ω_ε on $\partial\Omega_\varepsilon$.

The orthogonal projection on the tangent space of $\partial\Omega_\varepsilon$ will be denoted by T, i.e.

$$T\xi = \xi - (\xi\nu)\nu, \quad \forall \xi \in \mathbb{R}^3, \ \text{ a.e. on } \partial\Omega_\varepsilon.$$

For $k' \in \mathbb{Z}^2$ and $\rho > 0$, we denote

$$C_\rho^{k'} = \rho k' + \rho Z', \quad Q_\rho^{k'} = \Lambda_\varepsilon \cap (C_\rho^{k'} \times \mathbb{R}).$$

We define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \Leftrightarrow x' \in C_1^{k'}.$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial C_1^{k'}$). Moreover, for every $\rho > 0$, we have

$$\kappa\left(\frac{x'}{\rho}\right) = k' \Leftrightarrow x' \in C_\rho^{k'}.$$

For a.e. $x' \in \mathbb{R}^2$ we define $C_{\varepsilon^\beta}(x')$ as the square $C_{\varepsilon^\beta}^{k'}$ such that x' belongs to $C_{\varepsilon^\beta}^{k'}$.

We denote by $\mathcal{V}$ the space of functions $\widehat{v} : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ such that

$$\widehat{v} \in W_\#^{1,p}(\widehat{Q}_M), \quad \forall M > 0, \quad \nabla \widehat{v} \in L_\#^p(\widehat{Q})^3.$$

It is a Banach space endowed with the norm $\|\cdot\|_\mathcal{V}$ defined by

$$\|\widehat{v}\|_\mathcal{V}^p = \|\widehat{v}\|_{L^p(Z' \times \{0\})}^p + \|\nabla \widehat{v}\|_{L^p(\widehat{Q})^3}^p.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line.

We denote by C a generic positive constant which can change from line to line.

3. Main results

In the present section we describe the asymptotic behavior of the solutions $(u_\varepsilon, \pi_\varepsilon)$ of the non-Newtonian Navier-Stokes system (5) $(9/5 \leq p < +\infty)$ and the unique solution of the non-Newtonian Stokes system (6) $(1 < p < 9/5)$ posed in Ω_ε, assuming Navier condition (8) on the rough part of the boundary Γ_ε and no-slip (9) on the rest of the boundary, where Ω_ε and Γ_ε are defined by (2) and (4), respectively.

The existence of solution for system (5) with (8)-(9), the existence and uniqueness of solution for system (6) with (8)-(9), and a priori estimates are given by the following result.

Theorem 3.1. *(i) Problem (5) $(9/5 \leq p < +\infty)$ together with boundary conditions (8)-(9) admits at least one weak solution $(u_\varepsilon, \pi_\varepsilon) \in W^{1,p}(\Omega_\varepsilon)^3 \times L^{p'}_0(\Omega_\varepsilon)$,*

(ii) Problem (6) $(1 < p < 9/5)$ together with boundary conditions (8)-(9) admits a unique weak solution $(u_\varepsilon, \pi_\varepsilon) \in W^{1,p}(\Omega_\varepsilon)^3 \times L^{p'}_0(\Omega_\varepsilon)$,

where $L^{p'}_0(\Omega_\varepsilon)$ is the space of functions of $L^{p'}(\Omega_\varepsilon)$ with null integral, and $p' = p/(p-1)$ is the exponent conjugate to p.

Moreover, there exists $C > 0$, which does not depend on ε, such that every solutions of the above problems satisfy

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{2p-1}{p(p-1)}+1}, \quad \|Du_\varepsilon\|_{L^p(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{2p-1}{p(p-1)}}, \quad \|\pi_\varepsilon\|_{L^{p'}_0(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{p'}}. \tag{18}$$

As usual when we deal with thin domains, we use the dilatation

$$y' = x', \quad y_3 = \frac{x_3}{\varepsilon}, \tag{19}$$

which transforms Ω_ε in the sequence of open sets with fixed height, $\widetilde{\Omega}_\varepsilon$, defined by (15). Thus, we introduce $\widetilde{u}_\varepsilon \in W^{1,p}(\widetilde{\Omega}_\varepsilon)^3$ and $\widetilde{\pi}_\varepsilon \in L^{p'}_0(\widetilde{\Omega}_\varepsilon)$ by

$$\widetilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \widetilde{\pi}_\varepsilon(y) = \pi_\varepsilon(y', \varepsilon y_3), \quad \text{a.e. } y \in \widetilde{\Omega}_\varepsilon. \tag{20}$$

Our goal then becomes in describing the asymptotic behavior of these new sequences $\widetilde{u}_\varepsilon$, $\widetilde{\pi}_\varepsilon$. As set in the introduction, that behavior depends on the values of parameters δ and γ:

- Concering $\delta \in (\beta, +\infty)$; there is one critical regime, namely $\delta = \widetilde{\beta}_p$, and therefore three different regimes, namely $\delta \in (\beta, \widetilde{\beta}_p)$, $\delta = \widetilde{\beta}_p$, and $\delta > \widetilde{\beta}_p$. Recall that $\widetilde{\beta}_p$ is defined by (10).

- Concerning $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1}]$; three regimes are addressed, namely the critical regime $\gamma = \frac{1}{p-1}$, $\gamma = -\infty$ and $\gamma \in [0, \frac{1}{p-1})$.

There are therefore many cases, so for that reason the results are presented in three theorems depending on the value of δ, namely Theorem 3.2 corresponds to the critical regime $\delta = \widetilde{\beta}_p$, Theorem 3.3 corresponds to $\delta > \widetilde{\beta}_p$, and the case $\delta \in (\beta, \widetilde{\beta}_p)$ is addressed in Theorem 3.4. In each theorem, the regimes concerning γ are discussed.

Theorem 3.2 (Case $\delta = \widetilde{\beta}_p$). *Let $(u_\varepsilon, \pi_\varepsilon) \in W^{1,p}(\Omega_\varepsilon)^3 \times L_0^{p'}(\Omega_\varepsilon)$, $p' = p/(p-1)$, be a solution of (5) with (8)-(9) $(9/5 \le p < +\infty)$ or the unique solution of (6) with (8)-(9) $(1 < p < 9/5)$, and let $\widetilde{u}_\varepsilon, \widetilde{\pi}_\varepsilon$ the corresponding rescaled functions given by (20). Then, we have*

$$\begin{gathered}\varepsilon^{-\frac{1}{p-1}} \widetilde{u}_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(\Omega)^3, \quad \varepsilon^{-\frac{p}{p-1}} \widetilde{u}_\varepsilon \rightharpoonup (\widetilde{u}', 0) \text{ in } W^{1,p}(0,1; L^p(\omega))^3, \\ \varepsilon^{-\frac{2p-1}{p-1}} \widetilde{u}_{\varepsilon,3} \rightharpoonup \widetilde{w} \text{ in } W^{2,p}(0,1; W^{-1,p}(\omega)),\end{gathered} \tag{21}$$

$$\widetilde{\pi}_\varepsilon \to \widetilde{\pi} \text{ in } L^{p'}(\Omega), \tag{22}$$

where $\widetilde{u}' \in W^{1,p}(0,1; L^p(\omega))^2$, $\widetilde{w} \in W^{2,p}(0,1; W^{-1,p}(\omega))$ and $\widetilde{\pi} \in W^{1,p'}(\omega) \cap L_0^{p'}(\omega)$, are the unique solutions of the system

$$\left\{\begin{array}{l} -\partial_{y_3}\left(|\partial_{y_3}\widetilde{u}'|^{p-2}\,\partial_{y_3}\widetilde{u}'\right) = 2^{\frac{p}{2}}\left(\widetilde{f}' - \nabla_{y'}\widetilde{\pi}\right) \quad \text{in } \Omega, \\ \mathrm{div}_{y'}\widetilde{u}' + \partial_{y_3}\widetilde{w} = 0 \quad \text{in } \Omega \\ \widetilde{u}'(y',1) = \widetilde{w}(y',0) = \widetilde{w}(y',1) = 0 \ \text{ in } \omega, \ \displaystyle\int_0^1 \widetilde{u}'(y',s)\,ds\,\nu = 0 \quad \text{on } \partial\omega \end{array}\right. \tag{23}$$

and the boundary condition on Γ below, which depends on γ.

Defining $(\widehat{\phi}^{\xi'}, \widehat{q}^{\xi'}) \in \mathcal{V}^3 \times L^{p'}_{\#}(\widehat{Q})$, for every $\xi' \in \mathbb{R}^2$, as a solution of

$$\begin{cases} -\operatorname{div}_z(|\mathbb{D}_z[\widehat{\phi}^{\xi'}]|^{p-2}\mathbb{D}_z[\widehat{\phi}^{\xi'}]) + \nabla_z \widehat{q}^{\xi'} = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ \operatorname{div}_z \widehat{\phi}^{\xi'} = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ \widehat{\phi}_3^{\xi'}(z', 0) = -\nabla\Psi(z')\xi' \quad \text{on } \mathbb{R}^2 \times \{0\}, \\ (|\mathbb{D}_z[\widehat{\phi}^{\xi'}]|^{p-2}\mathbb{D}_z[\widehat{\phi}^{\xi'}])_{i,3} = 0,\ i = 1, 2, \quad \text{on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{24}$$

and $R = (R_1, R_2)^T \in \mathbb{R}^2$ by

$$R(\xi')_i = \int_{\widehat{Q}} |\mathbb{D}_z[\widehat{\phi}^{\xi'}]|^{p-2}\mathbb{D}_z[\widehat{\phi}^{\xi'}] : D_z \widehat{\phi}^{e_i} dz \quad \forall\, \xi' \in \mathbb{R}^2, \quad i = 1, 2, \tag{25}$$

we have

- *Case $\gamma = \frac{1}{p-1}$:*

$$\widetilde{\tau}(y') = -\lambda\, 2^{\frac{p}{2}}\, \widetilde{u}'(y', 0) - 2^{\frac{p}{2}}\, R(\widetilde{u}'(y', 0)) \quad \text{on} \quad \Gamma. \tag{26}$$

- *Case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$:*

$$\widetilde{\tau}(y') = -2^{\frac{p}{2}}\, R(\widetilde{u}'(y', 0)) \quad \text{on} \quad \Gamma, \tag{27}$$

where

$$\widetilde{\tau}(y') = -|\partial_{y_3}\widetilde{u}'(y', 0)|^{p-2}\partial_{y_3}\widetilde{u}'(y', 0). \tag{28}$$

Moreover, we prove the following corrector result for u_ε and π_ε in the strong topologies of $L^p(\Omega_\varepsilon)$ and $L^{p'}(\Omega_\varepsilon)$ respectively (the sets Ω_ε^- and Ω_ε^+ are defined in (13))

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\Omega_\varepsilon^-} |u_\varepsilon|^p\, dx = 0, \\ &\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\Omega_\varepsilon^+} \left(\left| u'_\varepsilon - \varepsilon^{\frac{p}{p-1}} \widetilde{u}'(x', \frac{x_3}{\varepsilon}) \right|^p + |u_{\varepsilon,3}|^p \right) dx = 0, \end{aligned} \tag{29}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\int_{\Omega_\varepsilon^-} |\pi_\varepsilon|^{p'}\,dx = 0,\quad \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_{\Omega_\varepsilon^+} |\pi_\varepsilon - \widetilde{\pi}(x')|^{p'}\,dx = 0, \tag{30}$$

whereas the corrector for $\mathbb{D}[u_\varepsilon]$ *in the strong topology of* $L^p(\Omega_\varepsilon)$*, defining* $\widehat{u} : \omega\times(\mathbb{R}^2\times\mathbb{R}^+)\to\mathbb{R}^3$ *by*

$$\widehat{u}(x',z) = \widehat{\phi}^{\,\widetilde{u}'(x',0)}(z),\qquad \text{for a.e. } (x',z)\in\omega\times(\mathbb{R}^2\times\mathbb{R}^+),$$

is given by

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^-} |\mathbb{D}[u_\varepsilon]|^p\,dx = 0,\\
&\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+} \left|\mathbb{D}[u_\varepsilon] - \frac{\varepsilon^{\frac{1}{p-1}}}{2}\sum_{i=1}^2 \partial_{y_3}\widetilde{u}_i(x',\frac{x_3}{\varepsilon})(e_i\otimes e_3 + e_3\otimes e_i)\right. \\
&\qquad\qquad \left. -\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}} \fint_{C_{\varepsilon^\beta(x')}} \mathbb{D}_z[\widehat{u}](s',\frac{x}{\varepsilon^\beta})ds'\right|^p dx = 0,
\end{aligned} \tag{31}$$

Theorem 3.3 (Case $\delta > \widetilde{\beta}_p$). *Under the assumptions of Theorem 3.2, we have the existence of*

$$\widetilde{u}'\in W^{1,p}(0,1;L^p(\omega))^2,\quad \widetilde{w}\in W^{2,p}(0,1;W^{-1,p}(\omega)),\quad \widetilde{\pi}\in W^{1,p'}(\omega)\cap L_0^{p'}(\omega),$$

which satisfy convergences (21)-(22), and are the unique solutions of the system (23), together with the boundary condition on Γ *below.*

- *Case* $\gamma = \frac{1}{p-1}$*:*

$$\widetilde{\tau}(y') = -\lambda\, 2^{\frac{p}{2}}\, \widetilde{u}'(y',0) \quad \text{on} \quad \Gamma. \tag{32}$$

- *Case* $\gamma\in\{-\infty\}\cup[0,\frac{1}{p-1})$*:*

$$\widetilde{\tau}(y') = 0 \quad \text{on} \quad \Gamma, \tag{33}$$

where $\widetilde{\tau}$ *is defined by (28).*

Moreover, the corrector results for u_ε and π_ε are given by (29) and (30), respectively, whereas for $\mathbb{D}[u_\varepsilon]$ is given by

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon^-} |\mathbb{D}[u_\varepsilon]|^p \, dx = 0,$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon^+} \left| \mathbb{D}[u_\varepsilon] - \frac{\varepsilon^{\frac{1}{p-1}}}{2} \sum_{i=1}^{2} \partial_{y_3} \widetilde{u}_i(x', \frac{x_3}{\varepsilon})(e_i \otimes e_3 + e_3 \otimes e_i) \right|^p dx = 0. \tag{34}$$

Theorem 3.4 (Case $\delta \in (\beta, \widetilde{\beta}_p)$). *Under the assumptions of Theorem 3.2, we have the existence of*

$$\widetilde{u}' \in W^{1,p}(0,1; L^p(\omega))^2, \quad \widetilde{w} \in W^{2,p}(0,1; W^{-1,p}(\omega)), \quad \widetilde{\pi} \in W^{1,p'}(\omega) \cap L_0^{p'}(\omega),$$

which satisfy convergences (21)-(22), and are the unique solutions of the system (23), together with the boundary condition on Γ

$$\widetilde{u}' = 0 \quad on \quad \Gamma. \tag{35}$$

Moreover, the corrector results for u_ε and π_ε are given by (29) and (30), respectively, whereas the corrector for $\mathbb{D}[u_\varepsilon]$ is given by (34).

Remark 3.5. *Theorems 3.2, 3.3 and 3.4 generalize the result proved in [19] for a Newtonian fluid, case $p = 2$. In [19] the critical size is $\delta = 3\beta/2 - 1/2$, which agrees with the critical size in the present paper $\delta = \widetilde{\beta}_p$ when $p = 2$.*

Additionally also generalize the result proved in [35] for a non-Newtonian fluid trough a domain with fixed height, in which the critical size is $\delta = \beta(2p-1)/p$. Indeed, the functions $\widehat{\phi}^{\xi'}$ and $\widehat{q}^{\xi'}$, for every $\xi' \in \mathbb{R}^2$, are the same functions which appear in [35] to describe the behavior of the velocity and the pressure near the rough boundary.

From the limit problem (23), and the limit boundary conditions on Γ obtained in Theorems 3.2, 3.3 and 3.4 depending on the values of δ and γ, we get the corresponding non-linear averaged momentum equations. For sake of simplicity, we just consider the case where $\widetilde{f}'$ does not depend on the variable

y_3. Note that this assumption usually holds in applications because Ω_ε is very thin, and so the variations in height of the exterior forces can be neglected.

Theorem 3.6. *Under the assumptions of Theorem 3.2, then $\widetilde{\pi}(y')$, $\widetilde{\tau}(y')$, and $\widetilde{\mathcal{U}}(y') = \int_0^1 \widetilde{u}'(y', y_3)dy_3$ satisfy*

$$\begin{cases} \operatorname{div}_{y'} \widetilde{\mathcal{U}}(y') = 0 \quad \text{in } \omega, \\ \widetilde{\mathcal{U}}(y')\,\nu = 0 \quad \text{on } \partial\omega, \end{cases} \tag{36}$$

where $\widetilde{\mathcal{U}}$ is given below depending on the values of δ and γ.

- *Case $\delta = \widetilde{\beta}_p$. Depends on γ:*

  - *Case $\gamma = \frac{1}{p-1}$:*

$$\begin{aligned} \widetilde{\mathcal{U}}(y') =& \int_0^1 \left(\int_0^{y_3} |\widetilde{\tau}(y') - \widetilde{g}(y')\xi|^{p'-2}(\widetilde{\tau}(y') - \widetilde{g}(y')\xi)d\xi \right) dy_3 \\ &+ \int_0^1 |\widetilde{\tau}(y') - \widetilde{g}(y')\xi|^{p'-2}(\widetilde{\tau}(y') - \widetilde{g}(y')\xi)d\xi, \end{aligned} \tag{37}$$

  where $\widetilde{\tau}(y')$ is given by (26), and $\widetilde{g}(y') = 2^{\frac{p}{2}}(\widetilde{f}'(y') - \nabla_{y'}\widetilde{\pi}(y'))$.

  - *Case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$: $\widetilde{\mathcal{U}}$ satisfies (37) with $\widetilde{\tau}(y')$ given by (27).*

- *Case $\delta > \widetilde{\beta}_\varepsilon$. Depends on γ:*

  - *Case $\gamma = \frac{1}{p-1}$: $\widetilde{\mathcal{U}}$ satisfies (37) with $\widetilde{\tau}(y')$ given by (32).*
  - *Case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$: $\widetilde{\mathcal{U}}$ satisfies (37) with $\widetilde{\tau}(y')$ given by (33).*

- *Case $\delta \in (\beta, \widetilde{\beta}_\varepsilon)$:*

$$\widetilde{\mathcal{U}}(y') = \frac{1}{2^{\frac{p'}{2}}(p'+1)} |\widetilde{f}'(y') - \nabla_{y'}\widetilde{\pi}(y')|^{p'-2}(\widetilde{f}'(y') - \nabla_{y'}\widetilde{\pi}(y')).$$

4. Useful inequalities and proof of Theorem 3.1

Our goal in this section is the proof of Theorem 3.1. For this purpose, we need some previous estimates which are given in Proposition 4.1, 4.2 and 4.3 below.

Let us introduce some notation which will be useful in the following. Associated to the change of variables (19), we introduce operators: $D_{y'}$, $\mathrm{div}_{y'}$, and $D_{\varepsilon,y}$, $\mathbb{D}_{\varepsilon,y}$, $\mathrm{div}_{\varepsilon,y}$ by

$$
\begin{aligned}
&(D_{\varepsilon,y}\widetilde{v})_{i,j} = (D_{y'}\widetilde{v})_{i,j} = \partial_{y_j}\widetilde{v}_i \quad \text{ for } i=1,2,3,\ j=1,2,\\
&(D_{\varepsilon,y}\widetilde{v})_{i,3} = \frac{1}{\varepsilon}\partial_{y_3}\widetilde{v}_i \quad \text{ for } i=1,2,3.\\
&\mathbb{D}_{\varepsilon,y}[\widetilde{v}] = \frac{1}{2}(D_{\varepsilon,y}\widetilde{v} + D^t_{\varepsilon,y}\widetilde{v})\\
&\mathrm{div}_{\varepsilon,y}\widetilde{v} = \mathrm{div}_{y'}\widetilde{v}' + \frac{1}{\varepsilon}\partial_{y_3}\widetilde{v}_3 = \sum_{i=1}^{2}\partial_{y_i}\widetilde{v}_i + \frac{1}{\varepsilon}\partial_{y_3}\widetilde{v}_3.
\end{aligned}
$$

Taking into account the definition of rescaled functions $\widetilde{u}_\varepsilon, \widetilde{\pi}_\varepsilon$, it is easily observed that the Navier-Stokes problem (5) and the Stokes problem (6) are equivalent, after rescaling, to

$$
\left.
\begin{aligned}
-\mathrm{div}_\varepsilon\left(|\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]|^{p-2}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]\right) + (\widetilde{u}_\varepsilon\nabla_{\varepsilon,y})\widetilde{u}_\varepsilon + \nabla_{\varepsilon,y}\widetilde{p}_\varepsilon = \widetilde{f} & \quad \text{ in } \widetilde{\Omega}_\varepsilon,\\
\mathrm{div}_{\varepsilon,y}\widetilde{u}_\varepsilon = 0 & \quad \text{ in } \widetilde{\Omega}_\varepsilon,
\end{aligned}
\right. \tag{38}
$$

$$
\left.
\begin{aligned}
-\mathrm{div}_\varepsilon\left(|\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]|^{p-2}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]\right) + \nabla_{\varepsilon,y}\widetilde{p}_\varepsilon = \widetilde{f} & \quad \text{ in } \widetilde{\Omega}_\varepsilon,\\
\mathrm{div}_{\varepsilon,y}\widetilde{u}_\varepsilon = 0 & \quad \text{ in } \widetilde{\Omega}_\varepsilon.
\end{aligned}
\right. \tag{39}
$$

Proposition 4.1. *There exists $C > 0$, such that for every $w_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$, $1 < p < +\infty$, with $w_\varepsilon = 0$ on $\omega \times \{\varepsilon\}$, we have the following:*

(i) (Poincare's inequality)

$$
\|w_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon\|\partial_{x_3}w_\varepsilon\|_{L^p(\Omega_\varepsilon)}. \tag{40}
$$

(ii) (Korn's inequality)

$$
\|Dw_\varepsilon\|_{L^p(\Omega_\varepsilon)^{3\times 3}} \le C\|\mathbb{D}[w_\varepsilon]\|_{L^p(\Omega_\varepsilon)^{3\times 3}}. \tag{41}
$$

Proof. Statement (i) follows using that

$$
w_\varepsilon(x) = -\int_{x_3}^{\varepsilon}\partial_{x_3}w_\varepsilon(x',t)\,dt, \text{ a.e. } x \in \Omega_\varepsilon.
$$

In order to prove (41), we extend $w_\varepsilon(x)$ by zero for $x_3 > \varepsilon$. Then w_ε belongs to $W^{1,p}(\Omega^*_\varepsilon)$, with

$$\Omega^*_\varepsilon = \left\{ x \in \mathbb{R}^3 \,:\, x' \in \omega,\ -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right) < x_3 < 1 \right\}.$$

and thus the result follows from Sec. 5.2 in[11].

■

Proposition 4.2 (Estimates for velocity). *Let u_ε be a solution of (5) ($9/5 \leq p < +\infty$) or the unique solution of (6) ($1 < p < 9/5$), and $\widetilde{u}_\varepsilon$ given by (20) the corresponding rescaled solutions of (38) or (39). Then we have*

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{2p-1}{p(p-1)}+1}, \quad \|Du_\varepsilon\|_{L^p(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{2p-1}{p(p-1)}}, \quad \|\mathbb{D}[u_\varepsilon]\|_{L^p(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{2p-1}{p(p-1)}}, \tag{42}$$

$$\|\widetilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{\frac{p}{p-1}}, \quad \|D_{\varepsilon,y}u_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{p-1}}, \quad \|\mathbb{D}_{\varepsilon,y}[u_\varepsilon]\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{p-1}},$$
$$\|D_{y'}\widetilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^{3\times 2}} \leq C\varepsilon^{\frac{1}{p-1}}, \quad \|\partial_{y_3}\widetilde{u}_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{\frac{p}{p-1}} \tag{43}$$

Proof. Using energy equality corresponding to the momentum equation (5) and (6), hypothesis (7), Korn inequality (41), and Poincaré inequality (40) give (42).

Finally, estimates (42) combined to the change of variables (19) easily imply that $\widetilde{u}_\varepsilon$ satisfies (43).

■

Proposition 4.3 (Estimates for pressure). *Let π_ε be a solution of (5) ($9/5 \leq p < +\infty$) or the unique solution of (6) ($1 < p < 9/5$), and $\widetilde{\pi}_\varepsilon$ given by (20) the corresponding rescaled solutions of (38) or (39). Then we have*

$$\|\pi_\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{p'}}, \quad \|\widetilde{\pi}_\varepsilon\|_{L^{p'}(\widetilde{\Omega}_\varepsilon)} \leq C, \tag{44}$$

$$\|\nabla_{\varepsilon,y}\widetilde{\pi}_\varepsilon\|_{W_0^{-1,p'}(\widetilde{\Omega}_\varepsilon)^3} \leq C, \quad \|\nabla_{y'}\widetilde{\pi}_\varepsilon\|_{W_0^{-1,p'}(\widetilde{\Omega}_\varepsilon)^3} \leq C, \quad \|\partial_{y_3}\widetilde{\pi}_\varepsilon\|_{W_0^{-1,p'}(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon. \tag{45}$$

Proof. Rescaled momentum equation (38) gives

$$
\begin{aligned}
\langle \nabla_\varepsilon \widetilde{\pi}_\varepsilon, \widetilde{\varphi}\rangle_{W^{-1,p'}(\widetilde{\Omega}_\varepsilon), W_0^{1,p}(\widetilde{\Omega}_\varepsilon)} &= \langle \nabla_{y'} \widetilde{\pi}_\varepsilon, \widetilde{\varphi}'\rangle + \frac{1}{\varepsilon}\langle \partial_{y_3}\widetilde{\pi}_\varepsilon, \widetilde{\varphi}_3\rangle \\
&= -\int_{\widetilde{\Omega}_\varepsilon} |D_\varepsilon \widetilde{u}_\varepsilon|^{p-2} D_\varepsilon \widetilde{u}_\varepsilon : D_\varepsilon \widetilde{\varphi}\, dy \\
&\quad + \int_{\widetilde{\Omega}_\varepsilon} \widetilde{f}\widetilde{\varphi}\, dy - \int_{\widetilde{\Omega}_\varepsilon} (\widetilde{u}_\varepsilon \nabla_\varepsilon)\widetilde{u}_\varepsilon\, \widetilde{\varphi}\, dy,
\end{aligned} \tag{46}
$$

for every $\widetilde{\varphi} \in W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3$. By (43) we have that for every $p > 1$,

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} |\mathbb{D}_\varepsilon[u_\varepsilon]|^{p-2}\mathbb{D}_\varepsilon[u_\varepsilon] : D_\varepsilon\varphi\, dy\right| &\le \|\mathbb{D}_\varepsilon[u_\varepsilon]\|^{p-1}_{L^{p'}(\Omega_\varepsilon)^{3\times 3}} \|D_\varepsilon \widetilde{\varphi}\|_{L^p(\widetilde{\Omega}_\varepsilon)} \\
&\le \frac{1}{\varepsilon}\|\mathbb{D}_\varepsilon[u_\varepsilon]\|^{p-1}_{L^{p'}(\Omega_\varepsilon)^{3\times 3}} \|\widetilde{\varphi}\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)} \\
&\le C\|\widetilde{\varphi}\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)}, \\
\left|\int_{\widetilde{\Omega}_\varepsilon} \widetilde{f}\,\widetilde{\varphi}\, dy\right| \le C\|\varphi\|_{W_0^{1,p}}.
\end{aligned} \tag{47}
$$

Hence, to derive estimates for pressure $\widetilde{\pi}_\varepsilon$ from (46), we just need to consider the initial terms. It can be written

$$
\begin{aligned}
\int_{\widetilde{\Omega}_\varepsilon} (\widetilde{u}_\varepsilon \nabla_{\varepsilon,y})\widetilde{u}_\varepsilon\, \widetilde{\varphi}\, dy = \;& -\int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_\varepsilon \widetilde{\otimes} \widetilde{u}_\varepsilon \nabla_{y'}\widetilde{\varphi}\, dy \\
& + \frac{1}{\varepsilon}\left(\int_{\widetilde{\Omega}_\varepsilon} \partial_{y_3}\widetilde{u}_{\varepsilon,3}\widetilde{u}_\varepsilon \varphi\, dy + \int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_{\varepsilon,3}\partial_{y_3}\widetilde{u}_\varepsilon\varphi\, dx\right),
\end{aligned} \tag{48}
$$

where

$$
(u\widetilde{\otimes} v)_{i,j} = u_i v_j, \quad i = 1,2; \quad j = 1,2,3.
$$

We consider separately the two terms in the right-hand side of (48).

(i) First term of (48): Applying Holder's inequality, we have

$$
\left|\int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_\varepsilon \widetilde{\otimes}\widetilde{u}_\varepsilon \nabla_{y'}\widetilde{\varphi}\, dy\right| \le C\|u_\varepsilon\|^2_{L^{q'}(\widetilde{\Omega}_\varepsilon)^3} \|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3},
$$

with $2/q' + 1/p = 1$, i.e. $q' = 2p/(p-1)$. The Sobolev embedding theorem and (43) imply that

$$
\|\widetilde{u}_\varepsilon\|_{L^{p*}(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{1}{p-1}}, \tag{49}
$$

where $p^* = 3p/(3-p)$ if $9/5 \leq p < 3$, $p^* \in [p, +\infty)$ if $p = 3$ and $p^* = [p, +\infty]$ if $p > 3$.

In order to interpolate between L^p and $W^{1,p}$ we introduce interpolation parameter θ such that

$$\frac{1}{q'} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$$

in order to have the interpolation (use (43) and (49))

$$\|\widetilde{u}_\varepsilon\|_{L^{q'}(\widetilde{\Omega}_\varepsilon)^3} \leq \|\widetilde{u}_\varepsilon\|^{\theta}_{L^p(\widetilde{\Omega}_\varepsilon)^3} \|\widetilde{u}_\varepsilon\|^{1-\theta}_{L^{p^*}(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{\theta\frac{p}{p-1} + (1-\theta)\frac{1}{p-1}}.$$

We choose θ from inequality

$$\theta\frac{p}{p-1} + (1-\theta)\frac{1}{p-1} \geq 0,$$

that is

$$1 \geq \theta \geq \theta_0 = \max\left\{0, \frac{-1}{p-1}\right\}.$$

Then, $\theta_0 = 0$, and we have

$$\left|\int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_\varepsilon \widetilde{\otimes} \widetilde{u}_\varepsilon \nabla_{y'} \widetilde{\varphi}\, dy\right| \leq C\|u_\varepsilon\|^2_{L^{q'}(\widetilde{\Omega}_\varepsilon)^3} \|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{2(\theta + \frac{1}{p-1})} \|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3},$$

which implies

$$\left|\int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_\varepsilon \widetilde{\otimes} \widetilde{u}_\varepsilon \nabla_{y'} \widetilde{\varphi}\, dy\right| \leq C\varepsilon^{\alpha} \|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3}, \tag{50}$$

wich $\alpha > 0$.

(ii) Estimate of the second part of the right-hand side of (48) has the form

$$\frac{C}{\varepsilon}\|\partial_{y_3}\widetilde{u}_{\varepsilon,3}\|_{L^p(\widetilde{\Omega}_\varepsilon)} \|\widetilde{u}_\varepsilon\|_{L^{q'}(\widetilde{\Omega}_\varepsilon)^3} \|\varphi\|_{L^{q'}(\widetilde{\Omega}_\varepsilon)^3}.$$

Working as in item (i), and taking into account last estimate in (43), we get

$$\frac{1}{\varepsilon}\left(\int_{\widetilde{\Omega}_\varepsilon} \partial_{y_3}\widetilde{u}_{\varepsilon,3}\widetilde{u}_\varepsilon \varphi\, dy + \int_{\widetilde{\Omega}_\varepsilon} \widetilde{u}_{\varepsilon,3}\partial_{y_3}\widetilde{u}_\varepsilon \varphi\, dy\right) \leq C\varepsilon^{\alpha}\|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3}, \tag{51}$$

with $\alpha = \theta + 2/(p-1) > 0$.

Putting together (50) and (51) we obtain the estimate

$$\int_{\widetilde{\Omega}_\varepsilon} (\widetilde{u}_\varepsilon \nabla_{\varepsilon,y})\widetilde{u}_\varepsilon\, \widetilde{\varphi}\, dy \leq C\varepsilon^{\alpha}\|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3},$$

with $\alpha > 0$. Now, previous estimate and (47) give estimate (45), i.e.

$$\langle \nabla_{\varepsilon,y}\widetilde{\pi}_\varepsilon, \widetilde{\varphi}\rangle_{W^{-1,p'}(\widetilde{\Omega}_\varepsilon), W_0^{1,p}(\widetilde{\Omega}_\varepsilon)} \leq C\|\varphi\|_{W_0^{1,p}(\widetilde{\Omega}_\varepsilon)^3},$$

which, using the Bogovskii operator (see [5]), gives the second estimate in (44). Applying the rescaling $x_3 = \varepsilon y_3$, this estimate implies the first one.

For the Stokes case, proceeding similarly by using the rescaled momentum equation (39), the result follows from (46) and (47).

■

Proof of Theorem 3.1. The classical theory (see e.g.[23], [27], [36]) gives the existence of at least one weak solution $(u_\varepsilon, \pi_\varepsilon) \in W^{1,p}(\Omega_\varepsilon)^3 \times L_0^{p'}(\Omega_\varepsilon)$, $p' = p/(p-1)$ for (5), under assumption $9/5 \leq p < +\infty$, and the existence of a unique one for (6), under assumption $1 < p < 9/5$. The estimates for velocity and pressure given in (18) follow from Proposition 4.2 and 4.3.

■

5. Some compactness results

In this section we obtain some compactness results about the behavior of a sequence $(u_\varepsilon, \pi_\varepsilon)$ satisfying a priori estimates (18) together with boundary conditions $u_\varepsilon = 0$ on $\omega \times \{\varepsilon\}$, $u_\varepsilon \nu = 0$ on $\partial\Omega_\varepsilon \setminus (\omega \times \{\varepsilon\})$, but where $(u_\varepsilon, \pi_\varepsilon)$ is not necessarily the solution of any PDE.

Lemma 5.1. *Let u_ε be in $W^{1,p}(\Omega_\varepsilon)^3$, $p > 1$, with $u_\varepsilon = 0$ on $\omega \times \{\varepsilon\}$, $u_\varepsilon \nu = 0$ on $\partial\Omega_\varepsilon \setminus (\omega \times \{\varepsilon\})$, $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, and such that there exists a constant C independent of ε satisfying*

$$\int_{\Omega_\varepsilon} |Du_\varepsilon|^p dx \leq C\varepsilon^{\frac{2p-1}{p-1}}. \tag{52}$$

Let us define $\widetilde{u}_\varepsilon \in W^{1,p}(\widetilde{\Omega}_\varepsilon)^3$ by (20). Then, for a subsequence of ε still denoted by ε, there exist $\widetilde{u}' \in W^{1,p}(0,1;L^p(\omega))^2$ and $\widetilde{w} \in W^{2,p}(0,1;W^{-1,p}(\omega))$ such that

$$\widetilde{u}'(1) = 0 \ \textit{in} \ L^p(\omega), \quad \widetilde{w}(0) = \widetilde{w}(1) = 0 \ \textit{in} \ W^{-1,p}(\omega), \tag{53}$$

$$\operatorname{div}_{y'} \widetilde{u}' + \partial_{y_3} \widetilde{w} = 0 \ \textit{in} \ W^{1,p}(0,1; W^{-1,p}(\omega)), \tag{54}$$

$$\operatorname{div}_{y'} \int_0^1 \widetilde{u}'(y', t)\, dt = 0 \quad \textit{in} \ L^p(\omega), \tag{55}$$

$$\int_0^1 \widetilde{u}'(y', t)\, dt\, \nu = 0 \quad \textit{in} \ W^{-\frac{1}{p},p}(\partial\omega), \tag{56}$$

$$\varepsilon^{-\frac{1}{p-1}} \widetilde{u}_\varepsilon \rightharpoonup 0 \quad \textit{in} \ W^{1,p}(\Omega)^3, \tag{57}$$

$$\varepsilon^{-\frac{p}{p-1}} \tilde{u}_\varepsilon \rightharpoonup (\widetilde{u}', 0) \quad \textit{in} \ W^{1,p}(0,1; L^p(\omega))^3, \tag{58}$$

$$\varepsilon^{-\frac{2p-1}{p-1}} \tilde{u}_{\varepsilon,3} \rightharpoonup \tilde{w} \ \textit{in} \ W^{2,p}(0,1; W^{-1,p}(\omega)), \tag{59}$$

$$\varepsilon^{-\frac{p}{p-1}} \operatorname{div}_{y'} \widetilde{u}'_\varepsilon + \varepsilon^{-\frac{2p-1}{p-1}} \partial_{y_3} \widetilde{u}_{\varepsilon,3} \rightharpoonup 0 \ \textit{in} \ W^{1,p}(0,1; W^{-1,p}(\omega)). \tag{60}$$

Proof. Since u_ε vanishes on $\omega \times \{\varepsilon\}$, estimates (40) and (52) imply that u_ε also satisfies

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^p dx \leq C\varepsilon^{\frac{2p-1}{p-1}+p}.$$

This inequality combined to the change of variables (19) and inequality (52) imply that $\widetilde{u}_\varepsilon$ satisfies

$$\int_{\widetilde{\Omega}_\varepsilon} |\widetilde{u}_\varepsilon|^p dy \leq C\varepsilon^{\frac{p^2}{p-1}}, \quad \int_{\widetilde{\Omega}_\varepsilon} \left(|\nabla_{y'} \widetilde{u}_\varepsilon|^p + \frac{1}{\varepsilon^p} |\partial_{y_3} \widetilde{u}_\varepsilon|^p \right) dy \leq C\varepsilon^{\frac{p}{p-1}}, \tag{61}$$

Therefore, up to a subsequence, there exist $\widetilde{u} \in W^{1,p}(0,1; L^p(\omega))^3$, with $\widetilde{u}(1) = 0$, such that

$$\varepsilon^{-\frac{p}{p-1}} \widetilde{u}_\varepsilon \rightharpoonup \widetilde{u} \ \text{ in } W^{1,p}(0,1; L^p(\omega))^3, \tag{62}$$

such that (58) and (60) hold. By (62), we also have that

$$\varepsilon^{-\frac{p}{p-1}}\operatorname{div}_{y'}\widetilde{u}'_\varepsilon \rightharpoonup \operatorname{div}_{y'}\widetilde{u}', \quad \text{ in } W^{1,p}(0,1;W^{-1,p}(\omega)), \tag{63}$$

and together with $\operatorname{div} u_\varepsilon = 0$ implies that $\partial_{y_3}\widetilde{u}_{\varepsilon,3}/\varepsilon^{\frac{2p-1}{p-1}}$ is bounded in $W^{1,p}(0,1;W^{-1,p}(\omega))$. Using then that $\widetilde{u}_{\varepsilon,3} = 0$ on $\omega \times \{1\}$, we deduce that $\widetilde{u}_{\varepsilon,3}/\varepsilon^{\frac{2p-1}{p-1}}$ is bounded in $W^{2,p}(0,1;W^{-1,p}(\omega))$ and therefore, up to a subsequence, there exists $\widetilde{w} \in W^{2,p}(0,1;W^{-1,p}(\omega))$, with $\tilde{w}(1) = 0$ in $W^{-1,p}(\omega)$, such that (59) holds. By (62), we get that $\widetilde{u}_3 = 0$ which finishes the proof of (58). From (59), (60) and (63), we also deduce (54).

Now, we consider $\eta \in C^\infty(\omega)$. Integrating by parts in $\widetilde{\Omega}_\varepsilon$ and taking into account that $u_\varepsilon\nu = 0$ on $\partial\Omega_\varepsilon$, we get

$$0 = \int_{\widetilde{\Omega}_\varepsilon}\left(\varepsilon^{-\frac{p}{p-1}}\operatorname{div}_{y'}\widetilde{u}'_\varepsilon + \varepsilon^{-\frac{2p-1}{p-1}}\partial_{y_3}\widetilde{u}_{\varepsilon,3}\right)\eta(y')dy = -\int_{\widetilde{\Omega}_\varepsilon}\varepsilon^{-\frac{p}{p-1}}\widetilde{u}'_\varepsilon(y)\nabla_{y'}\eta(y')dy.$$

Since (61) implies

$$\int_{\widetilde{\Omega}_\varepsilon\setminus\Omega}\left|\varepsilon^{-\frac{p}{p-1}}\widetilde{u}'_\varepsilon\right| dy \to 0,$$

we can write the previous equality as

$$\int_\Omega \varepsilon^{-\frac{p}{p-1}}\widetilde{u}'_\varepsilon(y)\nabla_{y'}\eta(y')dy + O_\varepsilon = 0.$$

Passing to the limit in this equality by means of (58), we get

$$\int_\omega\int_0^1 \widetilde{u}'(y',y_3)\,dy_3\nabla_{y'}\eta(y')\,dy' = 0,$$

which implies (55) and (56). Integrating (54) with respect to y_3 in $(0,1)$, we now deduce that $\widetilde{w}(0) = 0$, which concludes the proof of (53).

■

The change of variables (19) does not provide the information we need about the behavior of u_ε in the part of Ω_ε close to Γ_ε. To solve this difficulty, we introduce an adaptation of the unfolding method (see e.g. [2], [13], [16], [20], [26]), which is strongly related to the two-scale convergence method ([1], [33]).

For this purpose, given $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)^3$, $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, and assuming u_ε extended by zero to the set Λ_ε given by (11), we define $\widehat{u}_\varepsilon$ by

$$\widehat{u}_\varepsilon(x', z) = u_\varepsilon\left(\varepsilon^\beta \kappa\left(\frac{x'}{\varepsilon^\beta}\right) + \varepsilon^\beta z', \varepsilon^\beta z_3\right), \quad \text{a.e. } (x', z') \in \mathbb{R}^2 \times \widehat{Z}_\varepsilon, \tag{64}$$

with

$$\widehat{Z}_\varepsilon = \left\{z \in Z' \times \mathbb{R} : -\varepsilon^{\delta-\beta}\Psi(z') < z_3 < \varepsilon^{1-\beta}\right\}.$$

Remark 5.2. *For $k' \in \mathbb{Z}^2$ the restriction of $\widehat{u}_\varepsilon$ to $C^{k'}_{\varepsilon^\beta} \times \widehat{Z}_\varepsilon$ does not depend on x', while as function of z it is obtained from u_ε by using the change of variables*

$$z' = \frac{x' - \varepsilon^\beta k'}{\varepsilon^\beta}, \qquad z_3 = \frac{x_3}{\varepsilon^\beta}, \tag{65}$$

which transforms $Q^{k'}_{\varepsilon^\beta}$ into $\widehat{Z}_\varepsilon$. Therefore, the idea in the definition of the function $\widehat{u}_\varepsilon$ is to apply a dilatation in order to study the behavior of u_ε at a very small distance of Γ_ε. In addition, we observe that the change of variables (65), with x' fixed, transforms Γ_ε into the surface $\{z_3 = -\varepsilon^{\delta-\beta}\,\Psi(z')\}$ which, thanks to the assumption $\varepsilon^{\delta-\beta}$ converging to zero, almost agrees with the flat boundary $\{z_3 = 0\}$.

We will use the following lemma, whose proof is elementary and thus omitted.

Lemma 5.3. *Let $v_\varepsilon \in L^p(\mathbb{R}^2)$, $p > 1$, be a sequence which converges weakly to a function v in $L^p(\mathbb{R}^2)$. We define $\bar{v}_\varepsilon \in L^p(\mathbb{R}^2)$ by*

$$\bar{v}_\varepsilon(x') = \fint_{C_{\varepsilon^\beta}(x')} v_\varepsilon(\eta')\, d\eta', \quad \textit{a.e. } x' \in \mathbb{R}^2.$$

Then we have:

(i) The sequence $\bar{v}_\varepsilon$ converges weakly to v in $L^p(\mathbb{R}^2)$. Moreover, if the convergence of v_ε is strong in $L^p(\mathbb{R}^2)$, then the convergence of $\bar{v}_\varepsilon$ is also strong in $L^p(\mathbb{R}^2)$.

(ii) *For every $\tau' \in \mathbb{R}^2$, we have*

$$\frac{\bar{v}_\varepsilon(x'+\varepsilon^\beta\tau')-\bar{v}_\varepsilon(x')}{\varepsilon^\beta} \rightharpoonup \nabla v\,\tau' \quad \text{in } W^{-1,p}(\mathbb{R}^2).$$

Lemma 5.4. *We consider a sequence $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)^3$, $p>1$, satisfying (52), $u_\varepsilon = 0$ on $\omega\times\{\varepsilon\}$, $u_\varepsilon\nu = 0$ on $\partial\Omega_\varepsilon\backslash(\omega\times\{\varepsilon\})$. We define $\widetilde{u}_\varepsilon \in W^{1,p}(\widetilde{\Omega}_\varepsilon)^3$ by (20) and suppose there exists $\widetilde{u}' \in W^{1,p}(0,1;L^p(\omega))^2$ such that (58) holds. Taking into account the definition of $\widetilde{\beta}_p$ given in (10), then we have the following:*

(i) *If $\delta \in (\beta, \widetilde{\beta}_p)$, then*

$$\widetilde{u}'(x',0)\nabla\Psi(z') = 0 \quad a.e.\ (x',z') \in \omega\times Z'. \tag{66}$$

(ii) *If $\delta = \widetilde{\beta}_p$, then there exists $\widehat{u} \in L^p(\omega;\mathcal{V}^3)$ with*

$$\widehat{u}_3(x',z',0) = -\nabla\Psi(z')\widetilde{u}'(x',0), \quad a.e.\ (x',z') \in \omega\times Z', \tag{67}$$

such that for every $M>0$, the sequence $\widehat{u}_\varepsilon$ defined by (64) satisfies

$$\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}} D_z\widehat{u}_\varepsilon \rightharpoonup D_z\widehat{u} \quad \text{in } L^p(\omega\times\widehat{Q}_M)^{3\times3}. \tag{68}$$

Besides, if $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, then

$$\operatorname{div}_z\widehat{u} = 0 \quad \text{in } \omega\times\widehat{Q}. \tag{69}$$

Proof. We divide the proof in four steps.

Step 1. Let us obtain some estimates for the sequence $\widehat{u}_\varepsilon$ defined by (64).

For $M>0$, definition of (64) of $\widehat{u}_\varepsilon$ and (52) prove that for every $\varepsilon>0$ small enough (depending on M), we have

$$\begin{aligned}\int_{\mathbb{R}^2\times\widehat{Q}_M}|D_z\widehat{u}_\varepsilon(x',z)|^p dx'dz &\le \varepsilon^{\beta(p+2)}\sum_{k'\in\mathbb{Z}^2}\int_{\widehat{Q}_M}|Du_\varepsilon(\varepsilon^\beta(k'+z'),\varepsilon^\beta z_3)|^p dz' \\ &\le \sum_{k'\in\mathbb{Z}^2}\varepsilon^{\beta(p-1)}\int_{Q^{k'}_{\varepsilon^\beta}}|Du_\varepsilon|^p dx \le \varepsilon^{\beta(p-1)}\int_{\Omega_\varepsilon}|Du_\varepsilon|^p dx \le C\varepsilon^{\beta(p-1)+\frac{2p-1}{p-1}}.\end{aligned} \tag{70}$$

On the other hand, defining

$$\bar{u}_\varepsilon(x') = \fint_{C_{\varepsilon^\beta}(x')} u_\varepsilon(\tau',0)\,d\tau = \fint_{C_{\varepsilon^\beta}(x')}\widetilde{u}_\varepsilon(\tau',0)\,d\tau = \int_{Z'}\widehat{u}_\varepsilon(x',z',0)\,dz', \tag{71}$$

a.e. $x' \in \mathbb{R}^2$, using the inequality

$$\int_\omega \int_{\widehat{Q}_M} |\widehat{u}_\varepsilon(x',z) - \bar{u}_\varepsilon(x')|^p \, dzdx' \leq C_M \int_\omega \int_{\widehat{Q}_M} |D_z \widehat{u}_\varepsilon|^p \, dzdx', \tag{72}$$

where C_M does not depend on ε, and taking into account (70), we deduce that

$$\widehat{U}_\varepsilon = \frac{\widehat{u}_\varepsilon(x',y) - \bar{u}_\varepsilon}{\varepsilon^{\beta\frac{p-1}{p} + \frac{2p-1}{p(p-1)}}} \quad \text{is bounded in } L^p(\mathbb{R}^2; W^{1,p}(\widehat{Q}_M))^3 \quad \forall M > 0. \tag{73}$$

Thus, there exists $\widehat{u} \in L^p(\omega; W^{1,p}(\widehat{Q}_M))^3$ for every $M > 0$, such that, up to a subsequence,

$$\widehat{U}_\varepsilon \rightharpoonup \widehat{u} \quad \text{in } L^p(\omega; W^{1,p}(\widehat{Q}_M))^3 \quad \forall\, M > 0, \tag{74}$$

and then

$$\varepsilon^{-\beta\frac{p-1}{p} - \frac{2p-1}{p(p-1)}} D_z \widehat{u}_\varepsilon \rightharpoonup D_z \widehat{u} \quad \text{in } L^p(\omega \times Q_M)^{3\times 3}, \quad \forall\, M > 0. \tag{75}$$

By semicontinuity, inequality (70) proves

$$\int_{\omega \times \widehat{Q}_M} |D_z \widehat{u}|^p dx' dz \leq C \quad \forall\, M > 0.$$

Once we prove the Z'-periodicity of $\widehat{u}$ in z' (Step 2), the arbitrariness of M will then imply that $\widehat{u}$ belongs to $L^p(\omega; \mathcal{V}^3)$.

Moreover, if we also assume that $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, then by definition (64) of $\widehat{u}_\varepsilon$, we have $\operatorname{div}_z \widehat{u}_\varepsilon = 0$ in $\mathbb{R}^2 \times \widehat{Q}_M$, which combined with (75) proves (69).

Step 2. Let us prove that $\widehat{u}$ is Z'-periodic in the variable z'.

We observe that by definition (64) of $\widehat{u}_\varepsilon$, for every $M > 0$, we have

$$\widehat{u}_\varepsilon \left(x_1 + \varepsilon^\beta, x_2, -\frac{1}{2}, z_2, z_3 \right) = \widehat{u}_\varepsilon \left(x_1, x_2, \frac{1}{2}, z_2, z_3 \right),$$

a.e. $(x', z_2, z_3) \in \mathbb{R}^2 \times (-\frac{1}{2}, \frac{1}{2}) \times (0, M)$, which implies

$$\widehat{U}_\varepsilon \left(x_1 + \varepsilon^\beta, x_2, -\frac{1}{2}, z_2, z_3 \right) - \widehat{U}_\varepsilon \left(x', \frac{1}{2}, z_2, z_3 \right) = \frac{-\bar{u}_\varepsilon(x_1 + \varepsilon^\beta, x_2) + \bar{u}_\varepsilon(x')}{\varepsilon^{\beta\frac{p-1}{p} + \frac{2p-1}{p(p-1)}}} \tag{76}$$

Since $\varepsilon^{-\frac{p}{p-1}} \widetilde{u}_\varepsilon(x', 0)$ is bounded in $L^p(\mathbb{R}^2)^3$, we can apply Lemma 5.3-(ii) to deduce that the right-hand side of this equality tends to zero in $W^{-1,p}(\mathbb{R}^2)$.

Therefore, passing to the limit in the previous equation by (74), and taking into account the arbitrariness of M we get

$$\widehat{u}\left(x', -\frac{1}{2}, z_2, z_3\right) - \widehat{u}\left(x', \frac{1}{2}, z_2, z_3\right) = 0 \quad \text{a.e. } (x', z_2, z_3) \in \omega \times (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R}.$$

Analogously, we can prove

$$\widehat{u}\left(x', z_1, -\frac{1}{2}, z_3\right) - \widehat{u}\left(x', z_1, \frac{1}{2}, z_3\right) = 0 \quad \text{a.e. } (x', z_1, z_3) \in \omega \times (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R}.$$

These equalities prove the periodicity of $\widehat{u}$.

Step 3. Using the continuous embedding of $W^{1,p}(0,1;L^p(\omega))$ into $L^p(\Gamma)$ and Lemma 5.3-(i), we deduce from (58) that $\varepsilon^{-\frac{p}{p-1}}\overline{u}_\varepsilon$ converges weakly to $(\widetilde{u}'(x',0),0)$ in $L^p(\omega)^3$. Thus, by (3) and (73), we get

$$\varepsilon^{-\frac{p}{p-1}}\widehat{u}_\varepsilon(x',z) \to (\widetilde{u}'(x',0),0) \quad \text{in } L^p(\omega; W^{1,p}(\widehat{Q}_M))^3 \quad \forall\, M>0. \tag{77}$$

Step 4. Using the change of variables (65) in the equality $u_\varepsilon \nu = 0$ on Γ_ε, we get

$$-\varepsilon^{\delta-\beta}\nabla\Psi(z')\widehat{u}'_\varepsilon\left(x', z', -\varepsilon^{\delta-\beta}\Psi(z')\right) - \widehat{u}_{\varepsilon,3}\left(x', z', -\varepsilon^{\delta-\beta}\Psi(z')\right) = 0, \quad \text{a.e. in } \mathbb{R}^2 \times Z'. \tag{78}$$

Thanks to (78), we then have

$$\begin{gathered}\left|\varepsilon^{\delta-\beta}\nabla\Psi(z')\widehat{u}'_\varepsilon(x',z',0) + \widehat{u}_{\varepsilon,3}(x',z',0)\right| = \\ \int_{-\varepsilon^{\delta-\beta}\Psi(z')}^{0} \left|\varepsilon^{\delta-\beta}\nabla\Psi(z')\partial_{z_3}\widehat{u}'_\varepsilon(x',z',t) + \partial_{z_3}\widehat{u}_{\varepsilon,3}(x',z',t)\right| dt \\ \le C\left(\varepsilon^{\delta-\beta}\right)^{\frac{1}{p'}} \left(\int_{-\varepsilon^{\delta-\beta}\Psi(z')}^{0} |\partial_{z_3}\widehat{u}_\varepsilon(x',z',t)|^p\,dt\right)^{\frac{1}{p}} \quad \text{a.e. } (x',z') \in \mathbb{R}^2 \times Z'.\end{gathered}$$

Taking the power p, integrating in $\mathbb{R}^2 \times Z'$ and using (70) we then deduce

$$\int_{\mathbb{R}^2\times Z'} \left|\varepsilon^{\delta-\beta}\nabla\Psi(z')\widehat{u}'_\varepsilon(x',z',0) + \widehat{u}_{\varepsilon,3}(x',z',0)\right|^p \,dx'dz' \le C\varepsilon^{\delta(p-1)+\frac{2p-1}{p-1}},$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^2\times Z'} \Bigg| &\varepsilon^{\delta-\beta}\nabla\Psi(z')\,\widehat{u}'_\varepsilon(x',z',0) + \widehat{u}_{\varepsilon,3}(x',z',0) \\ &- \int_{Z'} \left(\varepsilon^{\delta-\beta}\nabla\Psi(z')\widehat{u}'_\varepsilon(x',z',0) + \widehat{u}_{\varepsilon,3}(x',z',0)\right) dz' \Bigg|^p dx'dz' \le C\varepsilon^{\delta(p-1)+\frac{2p-1}{p-1}}.\end{aligned}$$

Dividing by $\varepsilon^{\beta(p-1)+\frac{2p-1}{p-1}}$, using definition (73) of $\widehat{U}_\varepsilon$, and taking into account that $\nabla\Psi$ has null integral in Z' and (3), we get

$$\int_{\mathbb{R}^2\times Z'}\left|\varepsilon^{\delta+(\frac{p-1}{p}-\beta\frac{2p-1}{p})}\nabla\Psi(z')\frac{\widehat{u}'_\varepsilon(x',z',0)}{\varepsilon^{\frac{p}{p-1}}} - \varepsilon^{\delta-\beta}\int_{Z'}\nabla\Psi(z')\left(\frac{\widehat{u}'_\varepsilon(x',z',0)-\bar{u}'_\varepsilon(x')}{\varepsilon^{\beta\frac{p-1}{p}+\frac{2p-1}{p(p-1)}}}\right)dz' \right.$$
$$\left. + \frac{\widehat{u}_{\varepsilon,3}(x',z',0)-\bar{u}_{\varepsilon,3}(x')}{\varepsilon^{\beta\frac{p-1}{p}+\frac{2p-1}{p(p-1)}}}\right|^p dx'dz' \le C\varepsilon^{(\delta-\beta)(p-1)}\to 0. \tag{79}$$

and then, by (74), and defining $\widetilde{\beta}_p = \varepsilon^{\beta\frac{2p-1}{p}-\frac{p-1}{p}}$,

$$\varepsilon^{\delta-\widetilde{\beta}_p}\nabla\Psi(z')\frac{\widehat{u}'_\varepsilon(x',z',0)}{\varepsilon^{\frac{p}{p-1}}}\to -\widehat{u}_3(x',z',0) \ \text{ in } L^p(\omega\times Z').$$

This convergence and (77) imply (66) or (67), depending on whether $\delta\in(\beta,\,\widetilde{\beta}_p)$ or $\delta=\widetilde{\beta}_p$.

■

6. Obtaining the limit system and corrector result

In this section we use the results of previous sections to prove Theorems 3.2, 3.3, 3.4 and 3.6 describing the asymptotic behavior of a solution $(u_\varepsilon,\pi_\varepsilon)$ of the non-Newtonian Navier-Stokes system (5) with (8)-(9) ($9/5\le p<+\infty$) or the unique solution of the non-Newtonian Stokes system (6) with (8)-(9) ($1<p<9/5$).

Notice that the case $\delta=\widetilde{\beta}_p$ is the most difficult case to analyze. In fact, to Lemma 5.4-(ii) gives a compactness result of u_ε providing a relation of the limit function $\widetilde{u}'$, which gives the macroscopic behavior of the fluid, and the limit function $\widehat{u}$, which represents the microscopic behavior capturing the effects of the roughness. So, taking into account that relation, this case requires an appropriate combination of the asymptotic study of the problem using monotonicity arguments together with two changes of variables: the usual rescaling to study the behavior of the fluid away from the rough boundary, and the unfolding method to study its behavior near the rough boundary. For that reason the proof of Theorem 3.2 is more technical than those of Theorem 3.3 and 3.4 and so, it will be developed in more detail.

Proof of Theorem 3.2. Along this proof, we will work with Navier-Stokes system (5) ($9/5 \le p < +\infty$), giving the corresponding computations for the case of Stokes system (6) ($1 < p < 9/5$) when necessary.

From (18) and $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, Lemma 5.1 assures, up to a subsequence, the existence of $\widetilde{u}' \in W^{1,p}(0,1;L^p(\omega))^2$ and $\widetilde{w} \in W^{2,p}(0,1;W^{-1,p}(\omega))$ satisfying (21) and the two last lines of (23).

Moreover, taking into account (18), we deduce that, up to a subsequence, there exists $\widetilde{\pi} \in L^{p'}(\Omega)$ such that

$$\widetilde{\pi}_\varepsilon \rightharpoonup \widetilde{\pi} \quad \text{in} \quad L^{p'}(\Omega). \tag{80}$$

where by (45), the function $\widetilde{\pi}$ does not depend on the variable y_3, and has null mean value in ω (since π_ε has null mean value in Ω_ε).

On the other hand, we remark that $(u_\varepsilon, \pi_\varepsilon)$ satisfies the variational equation

$$\left\{\begin{array}{l}
\displaystyle\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]) : D\varphi_\varepsilon \, dx + \int_{\Omega_\varepsilon} \nabla \pi_\varepsilon \, \varphi_\varepsilon \, dx + \int_{\Omega_\varepsilon} (u_\varepsilon \nabla) u_\varepsilon \, \varphi_\varepsilon \, dx \\[2ex]
\displaystyle\qquad\qquad + \lambda \varepsilon^{-\gamma} \int_{\Gamma_\varepsilon} u_\varepsilon \, \varphi_\varepsilon \, d\sigma = \int_{\Omega_\varepsilon} f \, \varphi_\varepsilon \, dx, \\[2ex]
\forall \, \varphi_\varepsilon \in W^{1,p}(\Omega_\varepsilon)^3, \quad \varphi_\varepsilon \, \nu = 0 \quad \text{on } \Gamma_\varepsilon, \quad \varphi_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon,
\end{array}\right. \tag{81}$$

where $\gamma \in \{-\infty\} \cup [0, +\infty)$ and, in order to simplify the notation, we define S as the p-Laplace operator

$$S(\xi) = |\xi|^{p-2}\xi \quad \forall \, \xi \in \mathbb{R}^{3\times 3}_{\text{sym}}.$$

The proof of Theorem 3.2 will be carried out using suitable test functions φ_ε in (81), and it will be split in six steps.

Step 1. Obtaining the limit system. First, we remark that thanks to (18), $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, and (18), we can apply Lemma 5.4 to deduce the existence of a function $\widehat{u} \in L^p(\omega; \mathcal{V}^3)$, which satisfies (67) and (69), such that defining $\widehat{u}_\varepsilon$ by (64), covergence (68) holds, up to a subsequence.

For $\widetilde{\varphi}' \in C^1_c(\omega \times (-1,1))^2$, $\widehat{\varphi} \in C^1_c(\omega; C^1_\sharp(\widehat{Q})^3)$ such that

$$\begin{cases} D_z\widehat{\varphi}(x',z) = 0 \ \text{ a.e. in } \{z_3 > M\} \text{ for some constant } M > 0, \\ \widetilde{\varphi}'(y', y_3) = \widetilde{\varphi}'(y',0) \ \text{ if } y_3 \leq 0, \qquad \widehat{\varphi}(x',z',z_3) = \widehat{\varphi}(x',z',0) \ \text{ if } z_3 \leq 0, \\ \nabla\Psi(z')\widetilde{\varphi}'(y',0) + \widehat{\varphi}_3(y',z',0) = 0, \\ \operatorname{div}_z \widehat{\varphi}(x',z) = 0 \end{cases} \tag{82}$$

and $\zeta \in C^\infty(\mathbb{R})$ satisfying

$$\zeta(s) = 1 \text{ if } s < \frac{1}{3}, \qquad \zeta(s) = 0 \text{ if } s > \frac{2}{3}, \tag{83}$$

we define $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)^3$ by

$$\begin{cases} \varphi'_\varepsilon(x) = \dfrac{1}{\varepsilon}\widetilde{\varphi}'\left(x', \dfrac{x_3}{\varepsilon}\right) + \dfrac{\varepsilon^{\delta-\beta}}{\varepsilon}\,\widehat{\varphi}'\left(x', \dfrac{x}{\varepsilon^\beta}\right)\zeta\left(\dfrac{x_3}{\varepsilon}\right) \\ \varphi_{\varepsilon,3}(x) = \qquad \dfrac{\varepsilon^{\delta-\beta}}{\varepsilon}\,\widehat{\varphi}_3\left(x', \dfrac{x}{\varepsilon^\beta}\right)\zeta\left(\dfrac{x_3}{\varepsilon}\right) - \dfrac{\varepsilon^{2(\delta-\beta)}}{\varepsilon}\,\widehat{\varphi}'\left(x', \dfrac{x}{\varepsilon^\beta}\right)\nabla\Psi\left(\dfrac{x'}{\varepsilon^\beta}\right)\zeta\left(\dfrac{x_3}{\varepsilon^\beta}\right). \end{cases}$$

Thanks to $\widetilde{\varphi}'(x)$ and $\widehat{\varphi}(x',z)$ equal zero for x' outside a compact subset of ω and (82), the sequence φ_ε satisfies that

$$\varphi_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \quad \varphi_\varepsilon \nu = 0 \text{ on } \Gamma_\varepsilon.$$

Thus, we can take such φ_ε in (81). The problem is to pass to the limit in the different terms which appear in (81). Before, we remark that since $D_z\widehat{\varphi} = 0$ a.e in $\{z_3 > M\}$ and (83), we have

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\left(\widetilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right), 0\right) + g_\varepsilon(x) \ \text{ in } \overline{\Omega}_\varepsilon, \tag{84}$$

$$D\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\left(x', \frac{x_3}{\varepsilon}\right)e_i \otimes e_3 + \frac{\varepsilon^{\delta-2\beta}}{\varepsilon}D_z\widehat{\varphi}\left(x', \frac{x}{r_\varepsilon}\right) + h_\varepsilon(x), \quad \text{in } \Omega_\varepsilon, \tag{85}$$

with $g_\varepsilon \in C^0(\bar{\Omega}_\varepsilon)^3$, $h_\varepsilon \in C^0(\bar{\Omega}_\varepsilon)^{3\times 3}$ satisfying (thanks to (3) and $\delta = \widetilde{\beta}_p$)

$$\varepsilon^{p-1}\int_{\Omega_\varepsilon} |g_\varepsilon|^p dx \leq C\left(\varepsilon^{p(\delta-\beta)} + \frac{\varepsilon^{2p(\delta-\beta)}}{\varepsilon^{1-2\beta}}\right) = O_\varepsilon. \tag{86}$$

$$\varepsilon^{\frac{p}{p-1}}\int_{\Gamma_\varepsilon} |g_\varepsilon|^{p'} dx \leq C\varepsilon^{p'(\delta-\beta)} = O_\varepsilon. \tag{87}$$

$$\varepsilon^{2p-1}\int_{\Omega_\varepsilon} |h_\varepsilon|^p dx \leq C\varepsilon^{2p-1}\left(\frac{1}{\varepsilon^{p-1}} + \frac{\varepsilon^{p(\delta-\beta)}}{\varepsilon^{p-1}} + \frac{\varepsilon^{p(\delta-\beta)}}{\varepsilon^{2p-1}} + \frac{\varepsilon^{2p(\delta-\beta)}}{\varepsilon^{p}\varepsilon^{\beta(p-1)}}\right) = O_\varepsilon. \tag{88}$$

• *First term in (81).* Thanks to (18), (85) and (88), we easily have

$$\begin{aligned}
&\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon](x)) : D\varphi_\varepsilon(x)\,dx \\
&= \varepsilon^{-2}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon](x)) : \left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\left(x', \frac{x_3}{\varepsilon}\right) e_i\otimes e_3\right) dx \qquad (89)\\
&+\frac{\varepsilon^{\delta-\beta}}{\varepsilon^{\beta+1}}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon](x)) : D_z\widehat{\varphi}\left(x', \frac{x}{r_\varepsilon}\right) dx + O_\varepsilon.
\end{aligned}$$

445 Thus, using the change of variables (19) and (43), we have that $S(\varepsilon^{-\frac{1}{p-1}}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon])$ is bounded in $L^{p'}(\widetilde{\Omega}_\varepsilon)^{3\times 3}$, and so there exists $\widetilde{\xi} \in L^{p'}(\Omega)^{3\times 3}$ such that

$$S(\varepsilon^{-\frac{1}{p-1}}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]) \rightharpoonup \widetilde{\xi} \text{ in } L^{p'}(\Omega)^{3\times 3}. \tag{90}$$

Then, the first term on the right-hand side of (89) reads

$$\begin{aligned}
&\varepsilon^{-2}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]) : \left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\left(x', \frac{x_3}{\varepsilon}\right) e_i\otimes e_3\right) dx \\
&= \int_\Omega S(\varepsilon^{-\frac{1}{p-1}}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]) : \left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i(y)\, e_i\otimes e_3\right) dy \\
&= \int_\Omega \widetilde{\xi} : \left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i(y)\, e_i\otimes e_3\right) dy + O_\varepsilon.
\end{aligned}$$

In the second term of the right-hand side of (89), we introduce the sequence $\widehat{u}_\varepsilon$ defined by (64). By (18) and Lemma 5.4, we have that $S(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon])$ is bounded in $L^{p'}(\omega\times\widehat{Q}_M)^{3\times 3}$ $\forall\, M>0$, and so there exists $\widehat{\xi} \in L^{p'}(\omega\times\widehat{Q})^{3\times 3}$
450 such that

$$S(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon]) \rightharpoonup \widehat{\xi} \text{ in } L^{p'}(\omega\times\widehat{Q})^{3\times 3}. \tag{91}$$

Analogously, using the change of variables (65), assumptions on the support of $D_z\widehat{\varphi}$, (91), and the fact that $\delta = \widetilde{\beta}_p$, we get

$$
\begin{aligned}
&\frac{\varepsilon^{\delta-\beta}}{\varepsilon^{\beta+1}}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]) : D_z\widehat{\varphi}\left(x', \frac{x}{r_\varepsilon}\right)\, dx \\
&\qquad = \frac{\varepsilon^{\delta}}{\varepsilon^{\beta\frac{2p-1}{p}-\frac{p-1}{p}}}\int_{\omega\times\widehat{Q}_M} S(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon]) : D_z\widehat{\varphi}\, dx'dz \\
&\qquad = \int_{\omega\times\widehat{Q}} \widehat{\xi} : D_z\widehat{\varphi}\, dx'dz + O_\varepsilon.
\end{aligned}
$$

Therefore, (89) can be written as

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]) : D\varphi_\varepsilon\, dx \\
&\qquad = \int_{\Omega} \widetilde{\xi} : \left(\sum_{i=1}^{2} \partial_{y_3}\widetilde{\varphi}_i(y)\, e_i \otimes e_3\right)\, dy + \int_{\omega\times\widehat{Q}} \widehat{\xi} : D_z\widehat{\varphi}\, dx'dz + O_\varepsilon.
\end{aligned}
\tag{92}
$$

• *Second term in (81).* Thanks to the first estimate in (45), (84), (86), (19) and (80) we get

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \nabla\pi_\varepsilon(x)\varphi_\varepsilon(x)\, dx &= \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^+} \nabla_{x'}\pi_\varepsilon(x)\widetilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\, dx + O_\varepsilon \\
&= \int_{\Omega} \nabla_{y'}\widetilde{\pi}(y')\widetilde{\varphi}'(y)\, dy + O_\varepsilon.
\end{aligned}
\tag{93}
$$

• *Third term in (81).* Reasoning similarly the proof of Proposition 4.3, we get

$$
\int_{\Omega_\varepsilon} (u_\varepsilon\nabla)u_\varepsilon\varphi_\varepsilon\, dx = O_\varepsilon. \tag{94}
$$

• *Fourth term in (81).* Thanks to $u_\varepsilon(x', \varepsilon) = 0$ in ω and (18), we have that

$$
\int_{\Gamma_\varepsilon} |u_\varepsilon|^p d\sigma \le C\varepsilon^{p-1}\int_{\Omega_\varepsilon} |Du_\varepsilon|^p dx \le C\varepsilon^{\frac{p^2}{p-1}}.
$$

So, taking into account (84), (87) and $\tilde{\varphi}'(y) = \tilde{\varphi}'(y', 0)$ a.e. in $\{y_3 \le 0\}$, we have

$$
\begin{aligned}
&\lambda\varepsilon^{-\gamma}\int_{\Gamma_\varepsilon} u_\varepsilon\varphi_\varepsilon d\sigma \\
&= \lambda\varepsilon^{-\gamma-1}\int_{\omega} u'_\varepsilon\left(x', -\varepsilon^{\delta}\Psi\left(\frac{x'}{\varepsilon^{\beta}}\right)\right)\widetilde{\varphi}'(x', 0)\sqrt{1+\varepsilon^{2(\delta-\beta)}\left|\nabla\Psi\left(\frac{x'}{\varepsilon^{\beta}}\right)\right|^2}dx' + O_\varepsilon \\
&= \lambda\varepsilon^{-\gamma-1}\int_{\omega} u'_\varepsilon\left(x', -\varepsilon^{\delta}\Psi\left(\frac{x'}{\varepsilon^{\beta}}\right)\right)\widetilde{\varphi}'(x', 0)dx' + O_\varepsilon.
\end{aligned}
$$

However, integrating in the x_3 variable, we have

$$\int_\omega \left| u_\varepsilon \left(x', -\delta_\varepsilon \Psi \left(\frac{x'}{r_\varepsilon} \right) \right) - u_\varepsilon(x',0) \right|^p dx' \leq C\varepsilon^{\delta(p-1)} \int_{\Omega_\varepsilon} |Du_\varepsilon|^p dx = O_\varepsilon, \quad (95)$$

and so, using that $u_\varepsilon(x',0) = \widetilde{u}_\varepsilon(x',0)$ in ω, we get

$$\lambda\varepsilon^{-\gamma}\int_{\Gamma_\varepsilon} u_\varepsilon \varphi_\varepsilon d\sigma = \lambda\varepsilon^{-\gamma-1} \int_\omega \widetilde{u}'_\varepsilon(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon.$$

By convergence (58), in order to pass to the limit in this term it must be verified the following relation

$$-\gamma - 1 = -\frac{p}{p-1},$$

which is true for $\gamma = \frac{1}{p-1}$. Let us discuss the pass of the limit of this term depending on the parameter γ:

- If $\gamma = -\infty$, then pure slip is considered, and so this term is zero.

- If $\gamma \in [0, \frac{1}{p-1})$, then

$$\begin{aligned}\lambda\varepsilon^{-\gamma}\int_{\Gamma_\varepsilon} u_\varepsilon \varphi_\varepsilon d\sigma &= \lambda\varepsilon^{-\gamma-1} \int_\omega \widetilde{u}'_\varepsilon(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon \\ &= \lambda\varepsilon^{\frac{1}{p-1}-\gamma} \int_\omega \varepsilon^{-\frac{p}{p-1}} \widetilde{u}'_\varepsilon(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon \\ &= O_\varepsilon,\end{aligned}$$

  because $\frac{1}{p-1} - \gamma > 0$.

- If $\gamma = \frac{1}{p-1}$, then

$$\begin{aligned}\lambda\varepsilon^{-\frac{1}{p-1}}\int_{\Gamma_\varepsilon} u_\varepsilon \varphi_\varepsilon d\sigma &= \lambda \int_\omega \varepsilon^{-\frac{p}{p-1}} \widetilde{u}'_\varepsilon(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon \\ &= \lambda \int_\omega \widetilde{u}'(y',0)\widetilde{\varphi}'(y',0)dy'.\end{aligned} \quad (96)$$

- If $\gamma \in (\frac{1}{p-1}, +\infty)$, then

$$\begin{aligned}\lambda\varepsilon^{-\gamma}\int_{\Gamma_\varepsilon} u_\varepsilon \varphi_\varepsilon d\sigma &= \lambda\varepsilon^{-\gamma-1} \int_\omega \widetilde{u}'_\varepsilon(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon \\ &= \lambda\varepsilon^{\frac{1}{p-1}-\gamma} \int_\omega \varepsilon^{-\frac{p}{p-1}} \widetilde{u}'(y',0)\widetilde{\varphi}'(y',0)dy' + O_\varepsilon,\end{aligned}$$

and this goes to infinity, because $\frac{1}{p-1} - \gamma < 0$. For that reason, we drop this case; it must be required no-slip condition on Γ_ε, instead of Navier condition.

- *Fifth term in (81).* Thanks to (84), (86) and the change of variables (19) we get

$$\begin{aligned} \int_{\Omega_\varepsilon} f_\varepsilon(x)\varphi_\varepsilon(x)\,dx \ &= \varepsilon^{-1}\int_{\Omega_\varepsilon^+} \widetilde{f}'\left(x', \frac{x_3}{\varepsilon}\right)\widetilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx + O_\varepsilon \\ &= \int_\Omega \widetilde{f}'(y)\widetilde{\varphi}'(y)dy + O_\varepsilon. \end{aligned} \tag{97}$$

From (92)-(97), we then deduce that the limit system reads as

$$\begin{aligned} &\int_\Omega \widetilde{\xi} : \left(\sum_{i=1}^2 \partial_{y_3}\widetilde{\varphi}_i(y)(e_i \otimes e_3)\right)dy + \int_\Omega \nabla_{y'}\widetilde{\pi}(y')\widetilde{\varphi}'(y)dy \\ &\qquad + \int_\omega\int_{\widehat{Q}} \widehat{\xi} : D_z\widehat{\varphi}(x', z)\,dzdx' + \lambda\int_\Gamma \widetilde{u}'\widetilde{\varphi}'d\sigma = \int_\Omega \widetilde{f}'(y)\widetilde{\varphi}'(y)\,dy, \end{aligned} \tag{98}$$

for every $\widetilde{\varphi}' \in C_c^1(\omega \times (-1,1))^2$, $\widehat{\varphi} \in C_c^1(\omega; C_\sharp^1(\widehat{Q}))^3$ such that (82) is satisfied, with $\lambda = 0$ in the case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$ and $\lambda \neq 0$ in the case $\gamma = \frac{1}{p-1}$. By density, this holds true for every $\widetilde{\varphi}' \in W^{1,p}(0,1; L^2(\omega))^2$, and every $\widehat{\varphi} \in L^p(\omega; \mathcal{V})^3$ such that

$$\widetilde{\varphi}'(x', 1) = 0 \ \text{ a.e. } x' \in \omega, \quad \nabla\Psi(z')\widetilde{\varphi}'(x', 0) + \widehat{\varphi}_3(x', z', 0) = 0 \ \text{ a.e. } (x', z') \in \omega \times Z'.$$

Step 2. Previous property. Let us prove

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]) : Du_\varepsilon\,dx + \lambda\varepsilon^{-\frac{2p}{p-1}}\int_{\Gamma_\varepsilon}|u_\varepsilon|^2\,d\sigma\right) \\ &= \int_\Omega \widetilde{\xi} : \left(\sum_{i=1}^2 \partial_{y_3}\widetilde{u}_i\, e_i \otimes e_3\right)dy + \lambda\int_\Gamma |\widetilde{u}'|^2 d\sigma + \int_\omega\int_{\widehat{Q}} \widehat{\xi} : D_z\widehat{u}(x', z)dzdx', \end{aligned} \tag{99}$$

with $\lambda = 0$ in the case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$ and $\lambda \neq 0$ in the case $\gamma = \frac{1}{p-1}$.

For this purpose we take $\varepsilon^{-\frac{2p-1}{p-1}}u_\varepsilon$ as test function in (5). Using $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, this gives

$$\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]) : Du_\varepsilon\,dx + \lambda\varepsilon^{-\frac{2p}{p-1}}\int_{\Gamma_\varepsilon}|u_\varepsilon|^2\,d\sigma = \varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx. \tag{100}$$

In order to pass to the limit for the last term we use the change of variables (19) and (58), which easily gives

$$\varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon} f_\varepsilon \, u_\varepsilon \, dx = \varepsilon^{-\frac{p}{p-1}} \int_{\widetilde{\Omega}_\varepsilon} \widetilde{f} \, \widetilde{u}_\varepsilon \, dy = \int_\Omega \widetilde{f}' \widetilde{u}' \, dy + O_\varepsilon.$$

So, (100) can be written as

$$\varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]) : Du_\varepsilon \, dx + \lambda \varepsilon^{-\frac{2p}{p-1}} \int_{\Gamma_\varepsilon} |u_\varepsilon|^2 \, d\sigma = \int_\Omega \widetilde{f}' \widetilde{u}' \, dy + O_\varepsilon \tag{101}$$

Taking now $\widetilde{u}'$ and $\widehat{u}$ as test functions in (98), and taking into account that $\mathrm{div}_{y'} \widetilde{u}' = 0$ in Ω and $\mathrm{div}_z \widehat{u} = 0$ in $\omega \times \widehat{Q}$, we then deduce (99) from (101).

Step 3. Corrector result for $\mathbb{D}[u_\varepsilon]$. In order to prove (31), we define

$$\widehat{\vartheta}_\varepsilon(x', z) = \fint_{C_{\varepsilon^\beta(x')}} \mathbb{D}_z[\widehat{u}(s', z)] ds', \quad \text{a.e. } (x', z) \in \omega \times \widehat{Q}.$$

It is well known that it holds

$$\widehat{\vartheta}_\varepsilon \to \mathbb{D}_z[\widehat{u}] \quad \text{ in } L^p(\omega \times \widehat{Q})^{3\times 3}, \tag{102}$$

and that using the properties of the p-Laplace operator S we get

$$S(\widehat{\vartheta}_\varepsilon) \to S(\mathbb{D}_z[\widehat{u}]) \quad \text{ in } L^{p'}(\omega \times \widehat{Q})^{3\times 3}. \tag{103}$$

We also define

$$\widetilde{\vartheta}(y) = \frac{1}{2} \sum_{i=1}^{2} \partial_{y_3} \widetilde{u}_i(y) \, (e_i \otimes e_3 + e_3 \otimes e_i), \quad \text{a.e. } y \in \widetilde{\Omega}_\varepsilon. \tag{104}$$

We consider a new parameter $\varepsilon^s > 0$, with $1 < s < \beta$, which implies

$$\lim_{\varepsilon \to 0} \varepsilon^{s-1} = 0, \quad \lim_{\varepsilon \to 0} \varepsilon^{s-\beta} = +\infty. \tag{105}$$

Using the Hölder inequality and applying the change of variables $y = x/\varepsilon$, we prove

$$\begin{aligned} &\int_{\{x_3 > \varepsilon^s\}} |\varepsilon^{-\frac{\beta}{p}} \widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})|^p dx \\ &\qquad = \varepsilon^{2\beta} \sum_{k' \in \mathbb{Z}^2} \int_{Z'} \int_{\varepsilon^{s-\beta}}^{\varepsilon^{1-\beta}} \left| \fint_{C^{k'}_{\varepsilon^\beta}} \mathbb{D}_z[\widehat{u}](s', z) \, ds' \right|^p dz_3 \, dz' \\ &\qquad \le \int_\omega \int_{Z'} \int_{\varepsilon^{s-\beta}}^{\infty} |\mathbb{D}_z[\widehat{u}](s', z)|^2 dz_3 dz' \, ds' = O_\varepsilon. \end{aligned} \tag{106}$$

and on the contrary, we get

$$\varepsilon^{-1}\int_{\{x_3<s_\varepsilon\}}|\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})|^p dx=\int_{\{0<y_3<\frac{s_\varepsilon}{\varepsilon}\}}|\widetilde{\vartheta}(y)|^p dy=O_\varepsilon. \tag{107}$$

Since properties of the p-Laplace operator S (i.e. p-coercivity, $(p-1)$-growth and monotonicity) , proving the strong convergence (31) is equivalent to prove the following statement

$$\begin{aligned}
E_\varepsilon := \Bigg[&\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^-} S(\mathbb{D}[u_\varepsilon]):\mathbb{D}[u_\varepsilon]\,dx\\
&+\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+}\left(S(\mathbb{D}[u_\varepsilon])-S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right)\right)\\
&\qquad :\left(\mathbb{D}[u_\varepsilon]-\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})-\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{r_\varepsilon})\right)dx\\
&+\varepsilon^{-\frac{2p}{p-1}}\lambda\int_{\Gamma_\varepsilon}|u_\varepsilon-\varepsilon^{\frac{p}{p-1}}(\tilde{u}'(x',0),0)|^2 d\sigma\Bigg]\to 0,
\end{aligned}$$

with $\lambda=0$ in the case $\gamma\in\{-\infty\}\cup[0,\frac{1}{p-1})$ and $\lambda\neq 0$ in the case $\gamma=\frac{1}{p-1}$.

Developing the expression of E_ε, we have

$$\begin{aligned}
E_\varepsilon =\ &\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]):\mathbb{D}[u_\varepsilon]\,dx+\varepsilon^{-\frac{2p}{p-1}}\lambda\int_{\Gamma_\varepsilon}|u_\varepsilon|^2 d\sigma\\
&-\varepsilon^{-2}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]):\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\,dx-\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\Omega} S(\mathbb{D}[u_\varepsilon]):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx\\
&-\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx\\
&+\varepsilon^{-2}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\,dx\\
&+\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx\\
&-2\varepsilon^{-\frac{p}{p-1}}\lambda\int_{\Gamma_\varepsilon} u'_\varepsilon\,\widetilde{u}'(x',0)d\sigma+\lambda\int_{\Gamma_\varepsilon}|u(x',0)|^2\,d\sigma.
\end{aligned} \tag{108}$$

Let us pass to the limit in the different terms of this expression.

- *First and second terms.* They can be estimated using the previous property

(99) given in *Step 2*, which gives

$$
\begin{aligned}
&\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon} S(\mathbb{D}[u_\varepsilon]):\mathbb{D}[u_\varepsilon]\,dx+\varepsilon^{-\frac{2p}{p-1}}\int_{\Gamma_\varepsilon}|u_\varepsilon|^2 d\sigma\\
&=\int_\Omega \widetilde{\xi}:\widetilde{\vartheta}(y)\,dy+\lambda\int_\Gamma |\widetilde{u}'|^2 d\sigma+\int_{\omega\times\widehat{Q}}\widehat{\xi}:\mathbb{D}_z[\widehat{u}]\,dx'dz+O_\varepsilon.
\end{aligned}
$$

- *Third term.* We use the change of variables (19) together with (90) prove

$$
\begin{aligned}
-\varepsilon^{-2}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]):\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\,dx \quad &= -\int_{\widetilde{\Omega}_\varepsilon^+} S(\varepsilon^{-\frac{1}{p-1}}\mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon]):\widetilde{\vartheta}(y)\,dy\\
&= -\int_\Omega \widetilde{\xi}\,:\,\widetilde{\vartheta}(y)\,dy+O_\varepsilon.
\end{aligned}
$$

- *Fourth term.* Using the sequence s_ε given in (105), we get

$$
\begin{aligned}
&-\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx\\
&\qquad= -\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\{x_3>\varepsilon^s\}} S(\mathbb{D}[u_\varepsilon]):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx\\
&\qquad\quad -\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\{0<x_3<\varepsilon^s\}} S(\mathbb{D}[u_\varepsilon]):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx
\end{aligned}
$$

485 Taking into account property (106), the first integral converges to zero, and by applying the change of variables (65) in the second one together with the weak convergence of $S(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon])$ to $\widehat{\xi}$ in $L^{p'}(\omega\times\widehat{Q})^{3\times 3}$ and (102), we have

$$
\begin{aligned}
&-\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]):\widehat{\vartheta}_\varepsilon(x',\frac{x}{r_\varepsilon})\,dx\\
&\qquad= -\int_{\omega\times\widehat{Q}_{\varepsilon^{1-\beta}}} S(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon]):\widehat{\vartheta}_\varepsilon\,dx'dz\\
&\qquad= -\int_{\omega\times\widehat{Q}}\widehat{\xi}\,:\,\mathbb{D}_z[\widehat{u}]\,dx'dz+O_\varepsilon.
\end{aligned}
$$

- *Fifth term.* We use the sequence ε^s defined in (105) to split this term as follows

$$
\begin{aligned}
&-\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx\\
&\quad= -\varepsilon^{-\frac{2p-1}{p-1}}\int_{\{x_3>\varepsilon^s\}} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx\\
&\qquad -\varepsilon^{-\frac{2p-1}{p-1}}\int_{\{0<x_3<\varepsilon^s\}} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx
\end{aligned}
$$

By using property (106) and (18), we prove for $p > 2$:

$$\left|\varepsilon^{-\frac{2p-1}{p-1}} \int_{\{x_3>\varepsilon^s\}} \left(S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right) - S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon})\right)\right) : \mathbb{D}[u_\varepsilon]dx\right|$$

$$\leq \varepsilon^{-\frac{2p-1}{p-1}} \int_{\{x_3>\varepsilon^s\}} \left(\left|\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right|^{p-2} + \left|\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon})\right|^{p-2}\right) \cdot$$

$$\cdot \left|\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right| |\mathbb{D}[u_\varepsilon]|\, dx$$

$$\leq C\varepsilon^{-\frac{2p-1}{p-1}} \left[\left(\int_{\{x_3>\varepsilon^s\}} \left|\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right|^p dx\right)^{\frac{p-2}{p}}\right.$$

$$\left. + \left(\int_{\{x_3>\varepsilon^s\}} \left|\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon})\right|^p dx\right)^{\frac{p-2}{p}}\right] \cdot$$

$$\cdot \left(\int_{\{x_3>\varepsilon^s\}} \left|\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right|^p dx\right)^{\frac{1}{p}} \left(\int_{\{x_3>\varepsilon^s\}} |\mathbb{D}[u_\varepsilon]|^p\, dx\right)^{\frac{1}{p}} = O_\varepsilon,$$

and for $9/5 \leq p < 2$ (Navier-Stokes case), and for $1 < p < 9/5$ (Stokes case):

$$\left|\varepsilon^{-\frac{2p-1}{p-1}} \int_{\{x_3>\varepsilon^s\}} \left(S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right) - S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon})\right)\right) : \mathbb{D}[u_\varepsilon]dx\right|$$

$$\leq C\varepsilon^{-\frac{2p-1}{p-1}} \int_{\{x_3>\varepsilon^s\}} \left|\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right|^{p-1} |\mathbb{D}[u_\varepsilon]|\, dx$$

$$\leq C\varepsilon^{-\frac{2p-1}{p-1}} \left(\int_{\{x_3>\varepsilon^s\}} \left|\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right|^p dx\right)^{\frac{1}{p}} \left(\int_{\{x_3>\varepsilon^s\}} |\mathbb{D}[u_\varepsilon]|^p\, dx\right)^{\frac{1}{p}} = O_\varepsilon.$$

Analogously, by using (107), we can prove

$$\varepsilon^{-\frac{2p-1}{p-1}} \int_{\{0<x_3<\varepsilon^s\}} \left(S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right)\right.$$

$$\left. - S\left(\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x', \frac{x}{\varepsilon^\beta})\right)\right) : \mathbb{D}[u_\varepsilon]dx = O_\varepsilon.$$

Then, taking into account the above, (58), (68), and (103), we get

$$
\begin{aligned}
&-\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx\\
&\quad=-\varepsilon^{-\frac{2p-1}{p-1}}\int_{\{x_3>\varepsilon^s\}} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\right):\mathbb{D}[u_\varepsilon]\,dx\\
&\qquad-\varepsilon^{-\frac{2p-1}{p-1}}\int_{\{0<x_3<\varepsilon^s\}} S\left(\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\mathbb{D}[u_\varepsilon]\,dx+O_\varepsilon\\
&\quad=-\int_{\{y_3>\varepsilon^{s-1}\}} S\left(\widetilde{\vartheta}(y)\right):\left(\frac{\varepsilon^{-\frac{p}{p-1}}}{2}\sum_{i=1}^{2}\partial_{y_3}\widetilde{u}_{\varepsilon,i}(y)(e_i\otimes e_3+e_3\otimes e_i)\right)dy\\
&\qquad-\int_{\omega\times\widehat{Q}_{\varepsilon^{s-\beta}}} S\left(\widehat{\vartheta}_\varepsilon(x',z)\right):\left(\varepsilon^{-\beta\frac{p-1}{p}-\frac{2p-1}{p(p-1)}}\mathbb{D}_z[\widehat{u}_\varepsilon]\right)dx'dz+O_\varepsilon\\
&\quad=-\int_{\Omega} S\left(\widetilde{\vartheta}(y)\right):\widetilde{\vartheta}(y)\,dy-\int_{\omega\times\widehat{Q}} S\left(\mathbb{D}_z[\widehat{u}]\right):\left(\mathbb{D}_z[\widehat{u}]\right)dx'dz+O_\varepsilon.
\end{aligned}
\tag{109}
$$

- *Sixth and seventh terms.* Reasoning as in the fifth term, we have

$$
\begin{aligned}
&\varepsilon^{-2}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\,dx\\
&\quad=\int_{\Omega} S\left(\widetilde{\vartheta}(y)\right):\widetilde{\vartheta}(y)\,dy+O_\varepsilon,
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
&\varepsilon^{-\frac{\beta}{p}-\frac{2p-1}{p}}\int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})+\varepsilon^{-\frac{\beta}{p}+\frac{2p-1}{p(p-1)}}\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\right):\widehat{\vartheta}_\varepsilon(x',\frac{x}{\varepsilon^\beta})\,dx\\
&\quad=\int_{\omega\times\widehat{Q}} S\left(\mathbb{D}_z[\widehat{u}]\right):\mathbb{D}_z[\widehat{u}]\,dx'dz+O_\varepsilon.
\end{aligned}
\tag{111}
$$

- *Eighth term.* Using

$$\int_{\Gamma_\varepsilon}|u_\varepsilon-u_\varepsilon(x',0)|^p dx=O_\varepsilon,$$

and (21), we have

$$-2\lambda\varepsilon^{-\frac{p}{p-1}}\int_{\Gamma_\varepsilon}u'_\varepsilon\,\tilde{u}'(x',0)d\sigma=-2\lambda\int_{\Gamma}\varepsilon^{-\frac{p}{p-1}}\tilde{u}'_\varepsilon\,\tilde{u}'d\sigma+O_\varepsilon=-2\lambda\int_{\Gamma}|\tilde{u}'|^2d\sigma+O_\varepsilon.$$

- *Ninth term.* We have

$$\lambda\int_{\Gamma_\varepsilon}|u(x',0)|^2\,d\sigma=\lambda\int_{\Gamma}|u(x',0)|^2\,d\sigma+O_\varepsilon.$$

The estimates obtained for the different terms on the right-hand side of (108) prove that E_ε tends to zero and then (31).

Step 4. Identification of $\widetilde{\xi}$ and $\widehat{\xi}$. By using the corrector result of $\mathbb{D}[u_\varepsilon]$, obtained in *Step 3*, we will prove that

$$\widetilde{\xi} = 2^{-\frac{p}{2}} |\partial_{y_3} \widetilde{u}'|^{p-2} \sum_{i=1}^{2} \partial_{y_3} \widetilde{u}_i (e_i \otimes e_3 + e_3 \otimes e_i) \text{ in } \Omega, \tag{112}$$

$$\widehat{\xi} = S(\mathbb{D}_z[\widehat{u}]) \quad \text{in } \omega \times \widehat{Q}. \tag{113}$$

On the one hand, using a sequence ε^s satisfying (105), the weak convergences of $S(\varepsilon^{\frac{-1}{p-1}} \mathbb{D}_{\varepsilon,y}[\widetilde{u}_\varepsilon])$ to $\tilde{\xi}$ in $L^{p'}(\Omega)^{3\times 3}$ and $S(\varepsilon^{-\beta\frac{p-1}{p} - \frac{2p-1}{p(p-1)}} \mathbb{D}_z[\widehat{u}_\varepsilon])$ to $\widehat{\xi}$ in $L^{p'}(\omega \times \widehat{Q}_M)^{3\times 3}$, for some $M > 0$, then for every $\widetilde{\phi} \in C_c^1(\omega \times (-1,1))^3$ and $\widehat{\phi} \in C_c^1(\omega; C_\#^1(\widehat{Q})^3)$, with $D_z \phi_2(x', y) = 0$ a.e. in $\{z_3 > M\}$, for some $M > 0$, we prove that

$$\begin{aligned}
&\varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]) : \left(\varepsilon^{\frac{1}{p-1}} \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i (x', \frac{x_3}{\varepsilon}) e_i \otimes e_3 \right) + \varepsilon^{-\frac{\beta}{p} + \frac{2p-1}{p(p-1)}} D_z \widehat{\phi}(x', \frac{x}{\varepsilon^\beta}) \right) dx \\
&\quad = \varepsilon^{-2} \int_{\{x_3 > \varepsilon^s\}} S(\mathbb{D}[u_\varepsilon]) : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i (x', \frac{x_3}{\varepsilon}) e_i \otimes e_3 \right) dx \\
&\qquad + \int_{\{0 < x_3 < \varepsilon^s\}} \varepsilon^{-\frac{\beta}{p} - \frac{2p-1}{p}} S(\mathbb{D}[u_\varepsilon]) : D_z \widehat{\phi}(x', \frac{x}{\varepsilon^\beta}) \, dx \\
&\quad = \int_{\{y_3 > \varepsilon^{s-1}\}} S(\varepsilon^{-\frac{1}{p-1}} \mathbb{D}_{y,\varepsilon}[\widetilde{u}_\varepsilon]) : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i (y) e_i \otimes e_3 \right) dy \\
&\qquad + \int_{\omega \times \widehat{Q}_{\varepsilon^{s-r}}} S(\varepsilon^{-\beta\frac{p-1}{p} - \frac{2p-1}{p(p-1)}} \mathbb{D}_z[\widehat{u}_\varepsilon]) : D_z \widehat{\phi}(x', z) \, dx' dz \\
&\quad = \int_{\Omega} \widetilde{\xi} : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i (y) e_i \otimes e_3 \right) dy + \int_{\omega \times \widehat{Q}} \widehat{\xi} : D_z \widehat{\phi}(x', z) \, dx' dz + O_\varepsilon
\end{aligned} \tag{114}$$

On the other hand, taking into account the sequence ε^s satisfying (105), using the corrector result (31) for $\mathbb{D}[u_\varepsilon]$, properties (106), (107), and proceeding as in

(109), we prove

$$
\begin{aligned}
&\varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon^+} S(\mathbb{D}[u_\varepsilon]) : \left(\varepsilon^{\frac{1}{p-1}} \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i(x', \frac{x_3}{\varepsilon}) e_i \otimes e_3 \right) + \varepsilon^{-\frac{\beta}{p} + \frac{2p-1}{p(p-1)}} D_z \widehat{\phi}(x', \frac{x}{\varepsilon^\beta}) \right) dx \\
&\quad = \varepsilon^{-\frac{2p-1}{p-1}} \int_{\Omega_\varepsilon^+} S\left(\varepsilon^{\frac{1}{p-1}} \widetilde{\vartheta}(x', \frac{x_3}{\varepsilon}) + \varepsilon^{-\frac{\beta}{p} + \frac{2p-1}{p(p-1)}} \widehat{\vartheta}(x', \frac{x}{\varepsilon^\beta}) \right) \\
&\qquad\qquad : \left(\varepsilon^{\frac{1}{p-1}} \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i(x', \frac{x_3}{\varepsilon}) e_i \otimes e_3 \right) + \varepsilon^{-\frac{\beta}{p} + \frac{2p-1}{p(p-1)}} D_z \widehat{\phi}(x', \frac{x}{\varepsilon^\beta}) \right) dx + O_\varepsilon \\
&\quad = \int_\Omega S\left(\widetilde{\vartheta}(y) \right) : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i(x', \frac{x_3}{\varepsilon}) e_i \otimes e_3 \right) dy \\
&\qquad\qquad + \int_{\omega \times \widehat{Q}} S(\mathbb{D}_z[\widehat{u}]) : D_z \widehat{\phi}(x', z)\, dx' dz + O_\varepsilon.
\end{aligned}
\tag{115}
$$

Thus, (114) and (115) implies

$$
\int_\Omega \tilde{\xi} : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i(y) e_i \otimes e_3 \right) dy = \int_\Omega S\left(\widetilde{\vartheta}(y) \right) : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\phi}_i(y) e_i \otimes e_3 \right) dy,
$$

$$
\int_{\omega \times \widehat{Q}} \widehat{\xi} : D_z \widehat{\phi}(x', z)\, dx' dz = \int_{\omega \times \widehat{Q}} S(\mathbb{D}_z[\widehat{u}]) : D_z \widehat{\phi}(x', z)\, dx' dz
$$

for every $\widetilde{\phi}$ and $\widehat{\phi}$ as above. By density this implies (113). To get (112), it remains to prove

$$
S(\widetilde{\vartheta}(y)) = 2^{-\frac{p}{2}} |\partial_{y_3} \widetilde{u}'|^{p-2} \sum_{i=1}^{2} \partial_{y_3} \widetilde{u}_i(y)(e_i \otimes e_3 + e_3 \otimes e_i) \quad \text{in } \Omega.
$$

This follows just taking into account that $S(\widetilde{\vartheta})$ can be expressed in term of the second invariant of the strain tensor $D_{II}(\widetilde{\vartheta}) = D(\widetilde{\vartheta}) D(\widetilde{\vartheta})^t$ by

$$
|D(\widetilde{\vartheta})|^{p-2} = |D_{II}(\widetilde{\vartheta})|^{p/2-1}.
$$

Step 5. Boundary layer system satisfied by $\widehat{u}$. From the previous steps, the limit problem (98) reads as

$$
\begin{aligned}
&\int_\Omega 2^{-\frac{p}{2}} |\partial_{y_3} \widetilde{u}'|^{p-2} \sum_{i=1}^{2} \partial_{y_3} \widetilde{u}_i(y)(e_i \otimes e_3 + e_3 \otimes e_i) : \left(\sum_{i=1}^{2} \partial_{y_3} \widetilde{\varphi}_i(y)\, e_i \otimes e_3 \right) dy \\
&+ \int_\Omega \nabla_{y'} \widetilde{\pi}\, \widetilde{\varphi}' dy + \int_{\omega \times \widehat{Q}} S(\mathbb{D}_z[\widehat{u}]) : D_z \widehat{\varphi}\, dz dx' + \lambda \int_\Gamma \tilde{u}' \widetilde{\varphi}' d\sigma = \int_\Omega \widetilde{f}' \widetilde{\varphi}'\, dy,
\end{aligned}
\tag{116}
$$

for every $\varphi \in W^{1,p}(\Omega)^3$, and every $\widehat{\varphi} \in L^p(\omega; \mathcal{V}^3)$ such that

$$\widetilde{\varphi} = 0 \text{ on } \partial\Omega \setminus \Gamma,$$

$$\widetilde{\varphi}_3(x', 0) = 0, \quad \widehat{\varphi}_3(x', z', 0) = -\lambda \nabla\Psi(z')\widetilde{\varphi}'(x', 0), \quad \text{a.e. } (x', y') \in \omega \times Z',$$

with $\lambda = 0$ in the case $\gamma \in \{-\infty\} \cup [0, \frac{1}{p-1})$ and $\lambda \neq 0$ in the case $\gamma = \frac{1}{p-1}$.

Now, we will obtain an equation for $(\widetilde{u}', \widetilde{\pi})$ eliminating $\widehat{u}$ in (116). For this purpose, we take $\widetilde{\varphi}' = 0$ in (116) we prove that $\widehat{u} \in \mathcal{V}^3$ satisfies

$$\begin{cases} \displaystyle\int_{\omega \times \widehat{Q}} S(\mathbb{D}_z[\widehat{u}]) : D_z \widehat{\varphi}\, dx' dz = 0 \\ \text{div}_z \widehat{u} = 0, \\ \widehat{u}_3(x', z', 0) = -\nabla\Psi(z')\widetilde{u}'(x', 0) \;\text{ on } \mathbb{R}^2 \times \{0\}, \\ S(\mathbb{D}_z \widehat{u})_{i,3} = 0,\; i = 1, 2, \;\text{ on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{117}$$

a.e. in ω. Defining $(\widehat{\phi}^{\xi'}, \widehat{q}^{\xi'})$, for every $\xi' \in \mathbb{R}^2$, by (24), we deduce that

$$\widehat{u}(x', z) = \widehat{\phi}^{\,\widetilde{u}'(x',0)}(z), \quad \text{a.e. } (x', z) \in \omega \times \widehat{Q}. \tag{118}$$

Now, for $\varphi \in W^{1,p}(\Omega)^3$, with $\varphi = 0$ on $\partial\Omega \setminus \Gamma$, $\varphi_3 = 0$ on Γ, we take $\widetilde{\varphi}'$ and $\widehat{\varphi}(x', z) = \widetilde{\varphi}_1(x', 0)\widehat{\phi}^{e_1}(z) + \widetilde{\varphi}_2(x', 0)\widehat{\phi}^{e_2}(z)$, as test functions in (116). Taking into account (118) and definition (25) we get

$$\begin{aligned} &\int_\Omega 2^{-\frac{p}{2}} |\partial_{y_3}\widetilde{u}'|^{p-2} \sum_{i=1}^2 \partial_{y_3}\widetilde{u}_i(y)(e_i \otimes e_3 + e_3 \otimes e_i) : \left(\sum_{i=1}^2 \partial_{y_3}\widetilde{\varphi}_i(y)\, e_i \otimes e_3\right) dy \\ &\quad - \int_\Omega \widetilde{\pi}\, \text{div}_{y'} \widetilde{\varphi}'\, dy + \lambda \int_\Gamma \widetilde{u}'\, \tilde{\varphi}'\, d\sigma + \int_\omega R(\tilde{u}'(y', 0))\tilde{\varphi}'(y', 0) dy' = \int_\Omega \widetilde{f}'\, \tilde{\varphi}'\, dy. \end{aligned} \tag{119}$$

By the arbitrariness of φ, this proves that $(\widetilde{u}', \widetilde{\pi})$ is a solution of (23)-(26) with $\lambda \neq 0$ (case $\gamma = \frac{1}{p-1}$), or a solution of (23)-(27) with $\lambda = 0$ (case $\gamma = \{-\infty\} \cup [0, \frac{1}{p-1})$).

Finally, since $\widetilde{u}' \in W^{1,p}(0, 1; L^p(\omega))^2$, using variational formulation of problem (23), we have $|\partial_{y_3}\widetilde{u}'|^{p-2}\partial_{y_3}\widetilde{u}' \in L^{p'}(\Omega)^2$. Since $\widetilde{f}' \in L^{p'}(\Omega)^2$, by using standard arguments (see [29]), this gives that $\nabla_{y'}\widetilde{\pi} \in L^{p'}(\omega)^2$, which implies

$\widetilde{\pi} \in W^{1,p'}(\omega) \times L_0^{p'}(\omega)$ and so, the strong convergence (22) and (30).

Step 6. Corrector result for u_ε. We consider a sequence ε^s satisfying (105). Using that u_ε and $\tilde{u}'$ vanish on $\omega \times \{\varepsilon\}$ and $\omega \times \{1\}$ respectively, and taking into account (18), (31) and (106), we easily get

$$
\begin{aligned}
&\varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\Omega_\varepsilon^-} |u_\varepsilon|^p dx + \varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\Omega_\varepsilon^+} \left(|u_{\varepsilon,3}|^p + \left| u'_\varepsilon - \varepsilon^{\frac{p}{p-1}} \tilde{u}'\left(x', \frac{x_3}{\varepsilon}\right)\right|^p \right) dx \\
&\le 2^{p-1}\varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\Omega_\varepsilon \cap \{x_3 < \varepsilon^s\}} |u_\varepsilon|^p dx + 2^{p-1}\varepsilon^{-1} \int_{\Omega \cap \{x_3 < \varepsilon^s\}} \left| \tilde{u}'\left(x', \frac{x_3}{\varepsilon}\right)\right|^p dx \\
&\quad + \varepsilon^{-\frac{2p-1}{p-1}-p} \int_{\{x_3 > \varepsilon^s\}} \left(|u_{\varepsilon,3}|^2 + \left| u'_\varepsilon - \varepsilon^{\frac{p}{p-1}} \tilde{u}'\left(x', \frac{x_3}{\varepsilon}\right)\right|^p \right) dx \\
&\le 2^{p-1}\varepsilon^{-\frac{2p-1}{p-1}-p} \int_\omega \int_{-\varepsilon^\delta \psi(\frac{x'}{\varepsilon^\beta})}^{\varepsilon^s} \left| \int_{x_3}^{\varepsilon} \partial_{x_3} u_\varepsilon(x',t)dt \right|^p dx_3 dx' \\
&\qquad\qquad + 2^{p-1}\varepsilon^{-p-1} \int_\omega \int_0^{\varepsilon^s} \left| \int_{x_3}^{\varepsilon} \partial_{y_3} \tilde{u}'\left(x', \frac{t}{\varepsilon}\right) dt \right|^p dx_3 dx' \\
&\quad + \varepsilon^{-\frac{2p-1}{p-1}-p} \int_\omega \int_{\varepsilon^s}^{\varepsilon} \Bigg(\left| \int_{x_3}^{\varepsilon} \partial_{x_3} u_{\varepsilon,3}(x',t)dt \right|^p \\
&\qquad\qquad + \left| \int_{x_3}^{\varepsilon} \left(\partial_{x_3} u'_\varepsilon(x',t) - \varepsilon^{\frac{1}{p-1}} \partial_{y_3} \tilde{u}'\left(x', \frac{t}{\varepsilon}\right)\right) dt \right|^p \Bigg) dx_3 dx' \\
&\le 2^{p-1}(\varepsilon^s + C\varepsilon^\delta)\varepsilon^{-\frac{3p-2}{p-1}} \int_{\Omega_\varepsilon} |\partial_{x_3} u_\varepsilon|^p dx + 2^{p-1}\varepsilon^{s-1} \int_\Omega |\partial_{y_3} \tilde{u}'|^p dy \\
&\quad + \varepsilon^{-\frac{2p-1}{p-1}} \int_{\{x_3 > \varepsilon^s\}} \left(|\partial_{x_3} u_{\varepsilon,3}|^p + \left| \partial_{x_3} u'_\varepsilon - \varepsilon^{\frac{1}{p-1}} \partial_{y_3} \tilde{u}'\left(x', \frac{x_3}{\varepsilon}\right)\right|^p \right) dx = O_\varepsilon.
\end{aligned}
$$

This proves (29).

■

Proof of Theorem 3.3. As in the proof of Theorem 3.2, we consider $\widetilde{\varphi}' \in C_c^1(\omega \times (-1,1))^2$, with $\widetilde{\varphi}'(y) = \widetilde{\varphi}'(y', 0)$ if $y_3 \le 0$. Then, for $\zeta \in C^\infty(\mathbb{R})$ satisfying (83), we define $\varphi_\varepsilon \in W^{1,p}(\Omega_\varepsilon)^3$ by

$$
\varphi'_\varepsilon(x) = \frac{1}{\varepsilon}\widetilde{\varphi}\left(x', \frac{x_3}{\varepsilon}\right), \quad \varphi_{\varepsilon,3} = -\frac{\varepsilon^{\delta-\beta}}{\varepsilon}\eta\left(\frac{x_3}{\varepsilon^\beta}\right) \nabla\Psi\left(\frac{x'}{\varepsilon^\beta}\right).
$$

For every $\varepsilon > 0$, the function φ_ε satisfies $\varphi_\varepsilon = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, $\varphi_\varepsilon \nu = 0$ on Γ_ε. So, we can choose such φ_ε in (81). Taking into account that, thanks to $\delta > \widetilde{\beta}_p$, we have

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon}\left|D\varphi_\varepsilon-\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\left(x',\frac{x_3}{\varepsilon}\right)e_i\otimes e_3\right|^p dx\right)=0,$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon}\varepsilon^{p-1}|\varphi_{\varepsilon,3}(x)|^p dx\right)=0,$$

and (21), (80), we can pass to the limit in (81) to get

$$\int_{\Omega}\widetilde{\xi}:\left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\, e_i\otimes e_3\right)dy+\int_{\Omega}\nabla_{y'}\widetilde{\pi}\,\widetilde{\varphi}'\,dy+\lambda\int_{\Gamma}\widetilde{u}'\widetilde{\varphi}'d\sigma=\int_{\Omega}\widetilde{f}'\,\widetilde{\varphi}'\,dy, \tag{120}$$

for every $\widetilde{\varphi}'$ as above, where $\lambda=0$ in the case $\gamma\in\{-\infty\}\cup[0,\frac{1}{p-1})$ and $\lambda\neq 0$ in the case $\gamma=\frac{1}{p-1}$. Proceeding as in *Step 2* of the previous proof, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon}S(\mathbb{D}[u_\varepsilon]):Du_\varepsilon\,dx+\lambda\varepsilon^{-\frac{2p}{p-1}}\int_{\Gamma_\varepsilon}|u_\varepsilon|^2\,d\sigma\right)\\&=\int_{\Omega}\widetilde{\xi}:\left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{u}_i\, e_i\otimes e_3\right)dy+\lambda\int_{\Gamma}|\widetilde{u}'|^2 d\sigma dx',\end{aligned}\tag{121}$$

which in particular implies

$$\begin{aligned}E_\varepsilon:=\Bigg[&\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^-}S(\mathbb{D}[u_\varepsilon]):\mathbb{D}[u_\varepsilon]\,dx\\&+\varepsilon^{-\frac{2p-1}{p-1}}\int_{\Omega_\varepsilon^+}\left(S(\mathbb{D}[u_\varepsilon])-S\left(\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\right)\right):\left(\mathbb{D}[u_\varepsilon]-\varepsilon^{\frac{1}{p-1}}\widetilde{\vartheta}(x',\frac{x_3}{\varepsilon})\right)dx\\&+\varepsilon^{-\frac{2p}{p-1}}\lambda\int_{\Gamma_\varepsilon}|u_\varepsilon-\varepsilon^{\frac{p}{p-1}}(\tilde{u}'(x',0),0)|^2 d\sigma\Bigg]\to 0,\end{aligned}\tag{122}$$

with $\widetilde{\vartheta}$ defined by (104). Since properties of the p-Laplace operator S, relation (122) implies the strong convergence of $\mathbb{D}[u_\varepsilon]$ given in (34). As consequence, we have that $\widetilde{\xi}$ is given by (112). Finally, reasoning by density we obtain

$$\begin{aligned}&\int_{\Omega}2^{-\frac{p}{2}}|\partial_{y_3}\widetilde{u}'|^{p-2}\sum_{i=1}^{2}\partial_{y_3}\widetilde{u}_i(e_i\otimes e_3+e_3\otimes e_i):\left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\, e_i\otimes e_3\right)dy\\&-\int_{\Omega}\widetilde{\pi}\,\mathrm{div}_{y'}\widetilde{\varphi}'\,dy+\lambda\int_{\Gamma}\widetilde{u}'\,\widetilde{\varphi}'\,d\sigma=\int_{\Omega}\tilde{f}'\,\widetilde{\varphi}'\,dy,\end{aligned}\tag{123}$$

for every $\widetilde{\varphi}'$ as above. This is equivalent to problems (23)-(32) or (23)-(33) depending if $\lambda\neq 0$ or $\lambda=0$ respectively. The corrector result given in (29) is obtained in a similar manner to Step 6 in the proof of Theorem 3.2.

$\blacksquare$

Proof of Theorem 3.4. From Lemma 5.4-(i), we have that $\widetilde{u}$ satisfies condition

$$\widetilde{u}'(y',0)\nabla\Psi(z')=0,\quad \text{a.e. } y'\in\omega.$$

Since property (1), then it holds $\widetilde{u}'(y',0)=0$. According to this, following the proof of Theorem 3.2, we now consider $\widetilde{\varphi}'\in C_c^\infty(\omega\times(-1,1))^2$, with $\widetilde{\varphi}'(y)=\widetilde{\varphi}'(y',0)$ if $y_3\leq 0$ and satisfying the boundary condition

$$\widetilde{\varphi}'(y',0)=0,\quad \text{a.e. } y'\in\omega. \tag{124}$$

Observe that this choice of $\widetilde{\varphi}'$ implies that φ_ε defined by

$$\varphi'_\varepsilon(x)=\frac{1}{\varepsilon}\widetilde{\varphi}'\left(x',\frac{x_3}{\varepsilon}\right),\quad \varphi_{\varepsilon,3}(x)=0,$$

satisfies $\varphi_\varepsilon=0$ on $\partial\Omega_\varepsilon\setminus\Gamma_\varepsilon$, $\varphi_\varepsilon\nu=0$ on Γ_ε. Taking such φ_ε in (81) and reasoning as above, we can pass to the limit to deduce that it holds (120) holds for $\widetilde{\varphi}'$ with $\lambda=0$ (because $\widetilde{\varphi}(y',0)=\widetilde{u}(y',0)=0$). We can reason by density and prove property (122), which implies that $\widetilde{\xi}$ is given by (112), and then that $\widetilde{u}'$ satisfies

$$\int_\Omega 2^{-\frac{p}{2}}|\partial_{y_3}\widetilde{u}'|^{p-2}\sum_{i=1}^{2}\partial_{y_3}\widetilde{u}_i(e_i\otimes e_3+e_3\otimes e_i):\left(\sum_{i=1}^{2}\partial_{y_3}\widetilde{\varphi}_i\,e_i\otimes e_3\right)dy$$
$$-\int_\Omega\widetilde{\pi}\,\mathrm{div}_{y'}\widetilde{\varphi}'\,dy=\int_\Omega\widetilde{f}'\,\widetilde{\varphi}'\,dy,$$

for every $\widetilde{\varphi}'$ as above, which is equivalent to problem (23)-(35). The corrector result given in (29) is obtained in a similar manner to Step 6 in the proof of Theorem 3.2.

$\blacksquare$

Proof of Theorem 3.6. To simplify the exposition let us only consider the case $\{\delta=\widetilde{\beta}_p,\,\gamma=\frac{1}{p-1}\}$. For this, we will follow the reasoning given in [7]. The case $\delta>\widetilde{\beta}_p$ is obtained by proceeding similarly. We refer to [29] for the case $\delta \in (\beta, \widetilde{\beta}_p)$ (i.e. the case with no-slip condition on the bottom and the top).

Taking into account that $\widetilde{f}'$ and $\nabla_{y'}\widetilde{\pi}$ do not depend on the variable y_3, integrating equation (23) twice with respect to y_3, we obtain

$$\widetilde{u}'(y', y_3) = \int_0^{y_3} |\widetilde{\tau}(y') - g(x')\xi|^{p-2}(\widetilde{\tau}(y') - g(x')\xi)d\xi + \widetilde{u}'(x', 0).$$

with $\widetilde{\tau}(y')$ given by (26). Since $\widetilde{u}'(y', 1) = 0$, we get

$$\widetilde{u}'(y', 0) = -\int_0^1 |\widetilde{\tau}(y') - g(x')\xi|^{p-2}(\widetilde{\tau}(y') - g(x')\xi)d\xi,$$

which gives (37). Finally, substituting (37) into the second equation in (23) and integrating in $(0, 1)$ with respect to y_3, we deduce (36). The boundary condition in (36) just follows from $\int_0^1 \widetilde{u}'(y)dy_3\, \nu = 0$.

■

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