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ORIGINAL RESEARCH

Allocation rules for communication situations with incompatibilities

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Abstract

In this article we analyze certain situations with restricted cooperation. To do this we introduce a model that combines two types of games well studied in the literature: graph-restricted games and games with incompatible players. In particular, our model extends Myerson's model for communication situations and Bergantiños' model for incompatible relationships. Our approach is based on the concept of profit measure, which allows us to deal simultaneously with both types of bilateral relationships. We show that in the situations considered there are multiple possible definitions of the profit achievable for each coalition. This leads us to introduce different allocation rules for these cooperative situations.

Keywords Cooperative games · Signed graphs · Communication structures · Incompatibilities · Myerson value

1 Introduction

Cooperative game theory provides mathematical models for situations in which a group of players work together to achieve a common profit. A cooperative game (with transferable utility) is given by a characteristic function that assigns to each subset of players (coalition) the profit generated by these players when they cooperate. One of the main applications of cooperative games is to provide allocation rules for distributing the joint profit generated

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by the grand coalition. The best-known allocation rule for cooperative games is the Shapley value (Shapley, 1953).

Cooperative game theory also provides models for situations in which there are restrictions on cooperation. These restrictions can be of different types. Myerson (1977) introduced a model to analyze cooperative situations with restricted communication. In this model the players are represented by the nodes of a graph and there is a link between two nodes if and only if the players that they represent can communicate directly with each other. By using the Shapley value, Myerson obtained an allocation rule, the Myerson value, applicable to these situations. Once fixed a cooperative game in the set of players, the Myerson value assigns a payoff vector to each communication situation, that is, to each graph with vertex set equal to the set of players. Multiple extensions and variations of the Myerson value have been introduced in the literature, providing more complex models to accurately represent communication between players. In some of these extensions, structures more general than simple graphs have been used, such as hypergraphs Myerson (1980), probabilistic graphs Calvo et al. (1999), fuzzy graphs Jiménez-Losada et al. (2013) or directed graphs Li and Shan (2020). In other cases, variations in the game have been considered, such as games with fuzzy coalitions Xu et al. (2017) or players located on the edges of the graph Alarcón et al. (2022). Other variations have been obtained by imposing properties different from those satisfied by the Myerson value, such as efficiency Beál et al. (2015) or marginality Manuel et al. (2020).

Bergantiños et al. (1993) introduced a model to analyze cooperative situations with a different type of cooperation restrictions: bilateral incompatibilities. In this model the players are represented also by the nodes of a graph, but in this case a link between two nodes indicates that the players that they represent are incompatible, that is, they cannot cooperate. Although initially it could be thought that this model is dual to that of Myerson, this is not the case. This model for bilateral incompatibilities has been applied to various types of situations with restricted cooperation Alonso-Meijide et al. (2009); Gallardo et al. (2020). See et al. (2014) proposed a model for voting games with bilateral incompatibilities but their approach is essentially different from that considered in Bergantiños et al. (1993).

In the models proposed in Bergantiños et al. (1993) and See et al. (2014) it is assumed that if two players are not incompatible, then they can communicate directly with each other. Therefore, it remained to study situations with both bilateral incompatibilities and communication restrictions. Skibski et al. (2022) used signed graphs (Zaslavsky, 1982) to model situations which combine both types of constraints. They consider that, for any two players, one (and only one) of the following scenarios will occur: (1) the players are incompatible, in which case there will be a negative edge between the corresponding nodes, (2) the players are not incompatible and can communicate directly with each other, which is described by a positive edge between them, (3) the players are not incompatible and cannot communicate directly with each other, in which case there is no link between them. Skibski et al. propose an allocation rule for these cooperative situations. This rule is, roughly speaking, a combination of the value of Myerson (1977) for games with communication restrictions and the solution proposed in See et al. (2014) for games with bilateral incompatibilities. Signed graphs were recently used by Li and Morse (2022) to analyze non-cooperative games.

Motivation

We aim to study cooperative situations with both types of constraints: communication restrictions and bilateral incompatibilities. The key difference with Skibski et al. (2022) will be

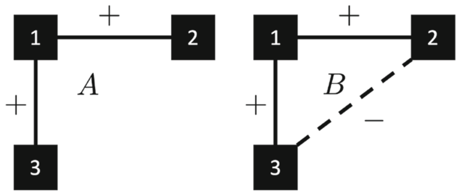


Fig. 1 Scenarios (A) and (B)

that, while they use the model proposed by See et al. to deal with incompatibilities, we will use the model introduced by Bergantiños et al., so we will find allocation rules that extend the Myerson value Myerson (1977) and the incompatibility value Bergantiños et al. (1993). Let us provide an example to illustrate the motivation behind our main goal. Consider a software developer consisting of several teams. Each of these teams is represented by a node in a graph. Given two nodes i and j in this graph,

(i) there is a positive edge between i and j if the corresponding teams can collaborate, regardless of the actions of the other teams,
(ii) there is no edge between i and j if the corresponding teams can collaborate only with the help or intermediation of other teams (which connect i and j),
(iii) there is a negative edge between i and j if both teams do the same work in any project and, therefore, cannot work together.

The developer aims to obtain the most valuable software possible and distribute the profit among the teams. For instance, we will consider three teams 1, 2 and 3. For each $S \subseteq \{1, 2, 3\}$, the real number $v(S)$ describes the market value of the software that the teams in S would create if the other teams do not help them in any way. In a first scenario (A), see Fig. 1, we consider positive connections $\{1, 2\}$ and $\{1, 3\}$, and no connection between 2 and 3. If team 1 were not present, teams 2 and 3 would work separately and achieve a collective profit of $v(\{2\}) + v(\{3\})$. But thanks to 1, the developer can sell a new software obtaining a profit of $v(\{1, 2, 3\})$. Now we consider a second scenario (B) with a negative connection between teams 2 and 3. The model in Skibski et al. (2022) is based on the one in See et al. (2014) (which is focused on power allocation problems) and attributes no profit to the developer in this case. However, the developer could obtain profit, for example, by having teams 1 and 2 create a software jointly, while team 3 creates another software separately. If we assume that the profit function v is superadditive, then, according to our model, the profit obtained by the developer would be $\max\{v(\{1, 2\}) + v(\{3\}), v(\{1, 3\}) + v(\{2\})\}$, which coincides with the profit proposed in Bergantiños et al. (1993). Therefore, the model that we introduce is different from the one in Skibski et al. (2022).

The main difficulty that will arise lies in the fact that whereas in the models proposed by Myerson and Bergantiños et al. there is only one reasonable way to define the profit achievable by each coalition, in our mixed model there exists a family (which is infinite if the number of agents is greater than 3) of reasonable profit measures, each one of them generating a different allocation rule. The study of the mathematical structure of this family of profit measures will help us to show the interest of the allocation rules found.

Structure

The paper is organized as follows. Section 2 is intended to make the paper as self-contained as possible. Some preliminaries regarding partially ordered sets, convex sets, partitions, graphs and cooperative games are given. In Sect. 3, we present the model proposed by Myerson

(1977) for communication situations and the model proposed by Bergantiños et al. (1993) for situations with incompatibilities. In order to construct a model for communication situations with incompatibilities that extend the model of Myerson and that of Bergantiños et al., we introduce, inspired by Jackson's network games Jackson (2005, 2008), the concept of profit measure, which allows us to deal with both types of relationships. In Sect. 4 we study communication situations with incompatibilities using signed graphs introducing new concepts and notations. We propose a motivating example to explain the possibility to introduce a new model. In Sect. 5 we obtain a family of allocation rules for cooperative games with these situations. In Sect. 6, two particular profit measures are studied. In Sect. 7 we analyze the mathematical structure of the family of profit measures for communication situations with incompatibilities. Finally, in Sect. 8 some conclusions are drawn.

2 Preliminaries

2.1 Posets and convex sets

A partially ordered set (*poset*) is a pair $(X, \leq)$ where X is a set and $\leq$ is a partial order relation on X. When there is no ambiguity we will write X instead $(X, \leq)$. Let X be a poset. If $x, y \in X$ and $x \leq y$ we denote $[x, y]_X = \{z \in X : x \leq z \leq y\}$. The poset X is said to have a *bottom* if there exist $\perp \in X$ such that $\perp \leq x$ for every $x \in X$. And X is said to have a *top* if there exists $\top \in X$ such that $x \leq \top$ for every $x \in X$. We write $x \geq y$ as equivalent to $y \leq x$ and $x < y$ to mean that $x \leq y$ and $x \neq y$. If $x > y$ and there is no $z \in X$ such that $x > z > y$ we will say that x *covers* y (or y is covered by x), and it will be denoted $x \rhd y$. If $Y \subset X$ then we can consider Y as a poset with the inherited order (the restriction of $\leq$ to Y). If $A \subseteq X$, then $y \in X$ is an upper bound (resp. lower bound) of A if $x \leq y$ $(x \geq y)$ for every $x \in A$. Given $x, y \in X$, if the set of upper bounds (resp. lower bounds) of $\{x, y\}$ is nonempty and has a bottom (resp. a top) then such element is called the *supremum* (resp. *infimum*) of x, y and it is denoted by $x \vee y$ (resp. $x \wedge y$). The poset X is said to be a lattice if for every $x, y \in X$ there exist $x \vee y$ and $x \wedge y$. If X is a finite lattice then X has a top and a bottom.

Let X be a real vector space. For each $x, y \in X$ the *segment* between x and y is the set $\overline{xy}^X = \{tx + (1-t)y : t \in [0, 1]\}$. If $Y \subseteq X$, then Y is *convex* if $\overline{xy}^X \subseteq Y$ for every $x, y \in Y$. If X is a poset, we say that $\leq$ is compatible with the vector space structure if the following two properties are satisfied: a) $x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in X$, and b) $x \leq y$ implies $\lambda x \leq \lambda y$, for every $x, y \in X$ and every $\lambda \geq 0$. If $\leq$ is compatible with the vector space structure we say that X is a partially ordered vector space. If X is a partially ordered vector space, $x, y \in X$ and $x \leq y$ then $\overline{xy}^X \subseteq [x, y]_X$.

For further information on the aspects of posets see Stanley (1986).

2.2 Partitions and graphs

Let N be a finite set. We denote by 2^N the family of all the subsets of N. The set of partitions of N is

$$\Pi_N = \left\{ P \subset 2^N \setminus \{\emptyset\} : \bigcup_{U \in P} U = N \text{ and } U \cap W = \emptyset \ \forall U, W \in P, U \neq W \right\}.$$

If $P, P' \in \Pi_N$ we say that P is *finer* than P' if for every $U \in P$ there exists $U' \in P'$ such that $U \subseteq U'$. This finer-than relation is a partial order and it will be denoted by $\leq$. The poset $(\Pi_N, \leq)$ is a lattice.

Consider the set $L^N = \{ij : i, j \in N \text{ and } i \neq j\}$, where ij denotes the unordered pair $\{i, j\}$. A (simple) *graph* on N is a pair $g = (V, L)$ where $V \subseteq N$, $L \subseteq L^N$ and $i, j \in V$ for every $ij \in L$. The set V (resp. L) is called the vertex set (resp. edge set) of g. We denote by $\mathcal{G}^N$ the family of graphs on N endowed with the following partial order relation: if $g = (V, L), \widehat{g} = (\widehat{V}, \widehat{L}) \in \mathcal{G}^N$ then $g \leq \widehat{g}$ if and only if $V \subseteq \widehat{V}$ and $L \subseteq \widehat{L}$. It is easy to check that $\mathcal{G}^N$ is a lattice. Notice that if $V = \widehat{V}$ then $g \rhd \widehat{g}$ if and only if, $L \supset \widehat{L}$ and g has exactly one more edge that $\widehat{g}$. Let $g = (V, L) \in \mathcal{G}^N$. If $V = \emptyset$ then also $L = \emptyset$ and the graph is called the null graph, which is the bottom of $\mathcal{G}^N$ and is denoted by g_0. Graph g is *connected* if for every $i, j \in V$ with $i \neq j$ there exist $\{i_1, \ldots, i_m\}$ such that $i_1 = i, i_m = j$ and $i_{k-1}i_k \in L$ for every $k = 2, \ldots, m$; g is *complete* if $L = L^V$; and g is *independent* if $L = \emptyset$. If $|V| = 1$ then g is connected, complete and independent. The graph g_0 is considered to be neither connected nor complete nor independent. If $T \subseteq N$ the *subgraph induced* in g by T is $g_T = (V \cap T, L \cap L^T)$. A set $T \subseteq V$ is said to be *connected* (resp. *complete*) (resp. *independent*) in g if g_T is connected (resp. complete) (resp. independent). A maximal connected set $T \subseteq V$ is called a connected component of g. The family of connected components of g is denoted by N/g. It is clear that $N/g \in \Pi_V$. The *complement* of g is $g^* = (V, L^V \setminus L) \in \mathcal{G}^N$. If $ij \in L$ we will denote $g_{-ij} = (V, L \setminus \{ij\})$.

More information on graphs, that have been briefly described above, in Graphs (2005).

2.3 Cooperative games

Let N be a finite set of agents (hereinafter called players) that cooperate to achieve a joint profit. One problem that arises is how to distribute this profit among the players. Each *payoff vector* $x \in \mathbb{R}^N$ represents a profit distribution proposal. If $i \in N$, then x_i is the payoff assigned to player i (according to x). In order to obtain reasonable payoff vectors, cooperative games were introduced. A *cooperative game* (with transferable utility) on N is given by a mapping $v : 2^N \to \mathbb{R}$, called characteristic function, that satisfies $v(\emptyset) = 0$. For each subset (coalition) of players $S \subseteq N$, the number $v(S)$ is the profit that the players in S could jointly achieve when they cooperate. In a cooperative game it is assumed that eventually all players (the grand coalition) will cooperate and, therefore, the amount to be distributed is equal to $v(N)$. A game v is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for every $S, T \subseteq N$ with $S \cap T = \emptyset$. If a game models a profit-sharing situation usually it is a superadditive game. In the present paper we will consider only superadditive games. The *Shapley value* (Shapley, 1953) is a mapping that assigns to each game v the payoff vector $\phi^v \in \mathbb{R}^N$ defined by

$$\phi_i^v = \sum_{\{S \subseteq N : i \in S\}} \gamma_{|S|}^{|N|} \left[v(S) - v(S \setminus \{i\}) \right] \quad \text{for every } i \in N, \tag{1}$$

where $\gamma_s^n = \frac{(s-1)!(n-s)!}{n!}$. The Shapley value satisfies the following properties: S1) *efficiency*, it provides a distribution of the total joint profit, that is, $\sum_{i \in N} \phi_i^v = v(N)$; S2) *linearity*, if $a, b \in \mathbb{R}$ and v, w are games on N then $\phi^{av+bw} = a\phi^v + b\phi^w$; S3) *null player*, if $i \in N$ is a null player in a game v, that is, $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N \setminus \{i\}$, then $\phi_i^v = 0$; and S4) *symmetry*, if $i, j \in N$ are symmetric players in v, that is, $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$, then $\phi_i^v = \phi_j^v$. Moreover, the Shapley value is the unique allocation rule satisfying these properties. If $T \in 2^N \setminus \{\emptyset\}$, the *unanimity game* u_T is defined by $u_T(S) = 1$

if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The Shapley value of the unanimity game u_T is given by $\phi_i^{u_T} = \frac{1}{|T|}$ if $i \in T$ and $\phi_i^{u_T} = 0$ if $i \in N \setminus T$.

An excellent reference on cooperative games is Curiel (1997).

3 Communication versus incompatibility

In this section, we review the models for graph-restricted games introduced by Myerson (1977) and Bergantiños et al. (1993). In both models, for a given game, the characteristic function is modified taking into account the interpretation of the bilateral relationships described by a graph. In order to view both models within the same framework, we will draw inspiration from the concept of network game, introduced by Jackson (2005) in 2005. A network game is a characteristic function defined on the family of subgraphs of a complete graph. The motivations, both economic and social, for studying these games are explained in Jackson (2008). Now, considering a classic cooperative game and choosing either the Myerson or the Bergantiños model can be seen as embedding the game in different ways into the family of network games. The different ways of embedding classic games as network games will be referred to as profit measures. Characterizing those models within this common context will allow us, in the next section, to provide models for games with both communication restrictions and incompatibilities.

3.1 Communication

Let v be a fixed superadditive cooperative game[1] on a set of players N. The amount to be distributed is $v(N)$ and the Shapley value ϕ^v is a payoff vector for v.

Myerson (1977) introduced games with communication situations, in which there are restrictions on communication between players. He represented a *communication situation* through a graph $g = (V, L) \in \mathcal{G}^N$ where V is the set of active players[2] and where $ij \in L$ if and only if players i and j can communicate directly with each other. An *allocation rule for communication situations* with underlying game v is a mapping $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns a payoff vector $(\psi_i(g))_{i \in N}$ to each communication situation $g \in \mathcal{G}^N$. Myerson proposed a method to obtain allocation rules for communication situations. In general terms, this method consists of, firstly, defining a new game (the graph-restricted game) that indicates the profit that each coalition can obtain if the communication restrictions are taken into account, and, secondly, applying a classical allocation rule to this new game. In order to obtain the graph-restricted game, it is necessary to identify the feasible coalitions, that is, the coalitions in which all the members can cooperate with each other, thus achieving the same profit that they would obtain without communication restrictions. These *communication feasible coalitions* will be the connected coalitions. Therefore, in order to evaluate the graph-restricted game at a coalition S, we will find the connected coalitions of g_S. In fact, since we are assuming that

[1] The choice of a superadditive game is because it allows us to unify Myerson's and Bergantiños' models, since the former employs maximal partitions of feasible coalitions while the latter uses arbitrary partitions of feasible coalitions.

[2] Myerson considered only graphs with vertex set equal to N, but we will follow the approach in Jiménez-Losada et al. (2013) and consider any graph in $\mathcal{G}^N$.

the underlying game v is superadditive, it suffices to evaluate v at the connected components of g_S. And all that remains is to add these values. Hence, given $g \in \mathcal{G}^N$, the graph-restricted game v_g^C is defined by

$$v_g^C(S) = \sum_{T \in N/g_S} v(T) \tag{2}$$

for every $S \subseteq N$. The *communication value* or Myerson value is the allocation rule $\mu^C : \mathcal{G}^N \to \mathbb{R}^N$ defined by

$$\mu^C(g) = \phi^{v_g^C} \tag{3}$$

for every $g \in \mathcal{G}^N$. The communication value satisfies: C1) *component efficiency*, the worth of each connected component of the graph is distributed among its members, that is, $\sum_{i \in T} \mu_i^C(g) = v(T)$ for every $T \in N/g$; C2) *inactive player*, if $g = (V, L)$ and $i \in N \setminus V$, then $\mu_i^C(g) = 0$; and C3) *fairness*, removing a link of the graph changes the payoffs of the players that form this link in the same amount, that is, if $g = (V, L)$ and $ij \in L$, then $\mu_i^C(g) - \mu_i^C(g_{-ij}) = \mu_j^C(g) - \mu_j^C(g_{-ij})$. Moreover, the Myerson value is the unique allocation rule for communication situations satisfying these properties. In addition, the communication value satisfies C4) *stability*, two players always benefit from reaching a bilateral agreement (if the underlying game is superadditive, as we are assuming), that is, $\mu_i^C(g) \geq \mu_i^C(g_{-ij})$ for every $ij \in L$.

The definition of the Myerson value is based on a measure of the profit achievable for each coalition. It is possible to determine such profit measure by certain reasonable conditions.

Definition 1 A profit measure for communication situations is any mapping $r : \mathcal{G}^N \to \mathbb{R}$ that satisfies the following conditions:

1. If $g = (V, L) \in \mathcal{G}^N$ is connected, then $r(g) = v(V)$.
2. For every $g \in \mathcal{G}^N$, $r(g) = \sum_{T \in N/g} r(g_T)$

The following proposition states the obvious fact that there is a unique profit measure for communication situations.

Proposition 1 *There is a unique profit measure for communication situations and is given by*

$$r^C(g) = \sum_{T \in N/g} v(T).$$

Moreover, if $g \rhd \widehat{g}$*, then* $r^C(g) \geq r^C(\widehat{g})$.

Notice that r^C allows to define the graph-restricted game, since $v_g^C(S) = r^C(g_S)$.

3.2 Incompatibility

Bergantiños et al. (1993) considered a different model of games with graph-restricted communication. They introduced cooperative games with incompatibilities. A *situation with incompatibilities* is given by a graph $g = (V, L) \in \mathcal{G}^N$ where V is the set of active players and where $ij \in L$ if and only if i and j are incompatible player. The authors showed that their model is not dual to Myerson's. Indeed, in this model a link between i and j means that these two players cannot cooperate in any way, and, therefore, no coalition containing i and j can be formed. In Myerson's model, the fact that the link ij is not in the graph does not necessarily imply that i and j cannot cooperate, but just that i and j cannot communicate directly, and, consequently, the formation of a coalition containing i and j might be possible, provided that all players in the coalition can communicate, directly or indirectly, with each other. An *allocation rule for situations with incompatibilities* with underlying game v is a mapping $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns a payoff vector $(\psi_i(g))_{i\in N}$ to each situation with incompatibilities $g \in \mathcal{G}^N$. In order to define an allocation rule for situations with incompatibilities, Bergantiños et al. followed an approach similar to that considered by Myerson to obtain allocation rules for communication situations. Firstly it is necessary to identify the *compatibility feasible coalitions*, that is, the coalitions which do not contain any pair of incompatible players. Notice that these are the independent coalitions. The goal is to calculate the profit that can be achieved by a coalition S, taking into account the incompatibility relations. Therefore, we must determine the partitions of S into independent coalitions. Since the underlying game is superadditive, it is enough to obtain the coarsest of such partitions. If we consider the family of partitions of S into independent coalitions, we will denote by $\mathcal{P}_g(S)$ the subset of partitions which are maximal in that family.[3] Notice that each partition in $\mathcal{P}_g(S)$ represents a possible organization of the players in S to generate a profit. Reasonably, the players within the coalition will choose the most profitable of such organizations. This leads to define the graph-restricted game as

$$v_g^I(S) = \max_{P\in\mathcal{P}_g(S)} \sum_{U\in P} v(U) \tag{4}$$

for every $S \subseteq N$. The *incompatibility value* is the allocation rule $\mu^I : \mathcal{G}^N \to \mathbb{R}^N$ defined by

$$\mu^I(g) = \phi^{v_g^I} \tag{5}$$

for every $g \in \mathcal{G}^N$. The incompatibility value satisfies: I1) *complement component efficiency*, the players of each connected component of the complement graph will distribute among themselves the largest amount that they can generate, that is, if $T \in N/g^*$ then $\sum_{i\in T} \mu_i^I(g) = v_g^I(T)$; I2) *inactive player*, if $g = (V, L)$ and $i \in N \setminus V$ then $\mu_i^I(g) = 0$; and I3) *fairness*, if $g = (V, L)$ and $ij \in L$, then $\mu_i^I(g) - \mu_i^I(g_{-ij}) = \mu_j^I(g) - \mu_j^I(g_{-ij})$, that is, removing a link of the graph changes the payoffs of the players that form this link in the same amount. Moreover, this allocation rule for situations with incompatibilities is the only one satisfying these properties. In addition, the incompatibility value satisfies *stability*, two players always benefit from becoming compatible (if the underlying game is superadditive, as we are assuming), that is, $\mu_i^I(g) \le \mu_i^I(g_{-ij})$ for every $ij \in L$.

We introduce also the concept of profit measure for incompatibilities.

Definition 2 A profit measure for situations with incompatibilities is any mapping over the graphs, $r : \mathcal{G}^N \to \mathbb{R}$, that satisfies the following conditions:

1. For every $g = (V, L) \in \mathcal{G}^N$, there exists a partition $P \in \mathcal{P}_g(V)$ such that $r(g) = \sum_{U\in P} v(U)$.
2. If $g, \widehat{g} \in \mathcal{G}^N$ and $g \rhd \widehat{g}$, then $r(g) \le r(\widehat{g})$.

[3] Bergantiños et al considered all the partitions into independent coalitions, since they did not restrict themselves to superadditive games.

Proposition 2 *There is a unique profit measure for situations with incompatibilities and is given by*

$$r^I(g) = \max_{P \in \mathcal{P}_g(V)} \sum_{U \in P} v(U)$$

for every $g \in \mathcal{G}^N \setminus \{g_0\}$ *and* $r^I(g_0) = 0$. *Moreover,* $r^I(g) = \sum_{T \in N/g^*} r^I(g_T)$ *for every* $g \in \mathcal{G}^N$.

Proof It is easy to check that r^I is a profit measure for situations with incompatibilities. Obviously, r^I satisfies condition 1. In order to see that it satisfies condition 2, notice that any independent set in g is an independent set in $\widehat{g}$.

It remains to prove the uniqueness. Let r be a profit measure for situations with incompatibilities. Let $g = (V, L) \in \mathcal{G}^N$. By condition 1 there exists a partition $P \in \mathcal{P}_g(V)$ such that $r(g) = \sum_{U \in P} v(U)$. Then,

$$r(g) = \sum_{U \in P} v(U) \leq \max_{P \in \mathcal{P}_g(V)} \sum_{U \in P} v(U) = r^I(g).$$

Now we will prove that $r(g) \geqslant r^I(g)$. Let $P' \in \mathcal{P}_g(V)$ a partition for which the maximum in the definition of $r^I(g)$ is attained, that is, $r^I(g) = \sum_{U \in P'} v(U)$. Let $P' = \{U_1, U_2, \ldots U_m\}$. Consider $g' = \big(V, L'\big)$, where

$$L' = L \cup \{ij \in L^V : \{i, j\} \nsubseteq U_k \text{ for all } k = 1, \ldots, m\}.$$

It is clear that P' is the only element in $\mathcal{P}_{g'}(V)$, since every independent set in g' must be contained in some U_k. Since r is a profit measure for situations with incompatibilities, we have that $r(g') = \sum_{U \in \widehat{P}} v(U) = r^I(g)$. Besides, successively applying the second condition in Definition 2 we obtain that $r(g) \geq r(g')$. Therefore, we conclude that $r(g) \geqslant r^I(g)$. □

Notice that r^I allows to define the graph-restricted game, since $v_g^I(S) = r^I(g_S)$.

4 Communication situations with incompatibilities

4.1 Communication situations with incompatibilities as signed graphs

Let N be a finite set of players. Henceforth, we consider a fixed superadditive game v. In this section, our goal is to present the concept of communication situation with incompatibilities among the players, and to study how the profit achievable by each coalition in v is modified by the situation.

The model of communication by Myerson considers bilateral relations among the players with two options, communication or non-communication. The model of incompatibilities by Bergantiños et al. also considers two options: communication or incompatibility. Since non-communication and incompatibility are not the same we propose communication situations with incompatibilities, thus considering three options for the bilateral relations among the players: communication, incompatibility or neither (non-communication). Following Skibski et al. (2022), we will describe a communication situation with incompatibilities by means of a signed graph (Zaslavsky, 1982) with two possible values on the edges: positive edges and negative ones.

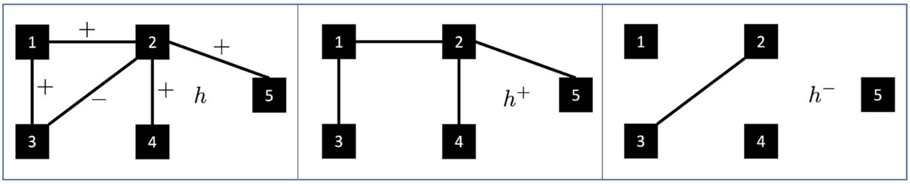


Fig. 2 Communication situation with incompatibilities: positive and negative graphs

Definition 3 A communication situation with incompatibilities is a triplet $h = \left(V, L^+, L^-\right)$ where $V \subseteq N$, $L^+, L^- \subseteq L^V$ and $L^+ \cap L^- = \emptyset$. The family of communication situations with incompatibilities will be denoted by $\mathcal{H}^N$. If $h = (V, L^+, L^-) \in \mathcal{H}^N$ we will denote $h^+ = (V, L^+) \in \mathcal{G}^N$ and $h^- = (V, L^-) \in \mathcal{G}^N$.

Given $h = (V, L^+, L^-) \in \mathcal{H}^N$, an edge $\{i, j\}$ is contained in L^+ if and only if i and j can communicate directly with each other. And $\{i, j\}$ is contained in L^- if and only if i and j are mutually incompatible. Therefore, h^+ and h^- describe the direct communications and the mutual incompatibilities, respectively. As a graphical representation of h we will use signed graph.[4] The edges in L^+ are named positive, and the edges in L^- negative. Besides, the graphs L^+ and L^- will sometimes be called the positive graph and the negative graph, respectively. We denote by h_0 the only communication situation with incompatibilities such that its set of vertices is the empty set.

Example 1 Given $N = \{1, 2, 3, 4, 5, 6\}$, consider the communication situation with incompatibilities $h = (V, L^+, L^-)$ with $V = \{1, 2, 3, 4, 5\}$, $L^+ = \{12, 13, 24, 25\}$ and $L^- = \{23\}$. In Fig. 2 we represent h, h^+ and h^-.

The positive graph h^+ represents the feasible direct communications between players and the negative graph h^- represents the mutual incompatibilities.

Remark 1 (1) If $g = (V, L) \in \mathcal{G}^N$ describes a communication situation, then g can be identified with $h = (V, L, \emptyset) \in \mathcal{H}^N$. In this way, the family of communication situations is identified with $\mathcal{H}_C^N = \{h \in \mathcal{H}^N : h = (V, L^+, \emptyset)\}$.
(2) If $g = (V, L) \in \mathcal{G}^N$ describes a situation with incompatibilities, then g can be identified with $h = (V, L^V \setminus L, L) \in \mathcal{H}^N$. Therefore, the family of situations with incompatibilities is identified with $\mathcal{H}_I^N = \{h \in \mathcal{H}^N : h = (V, L^+, L^-), L^+ \cup L^- = L^V\}$.

Example 2 In Fig. 3 we represent, on the left, a graph g, and, on the right, the communication situations with incompatibilities that are identified with g if we consider that g describes a communication situation (above) or if we consider that g describes a situation with incompatibilities (below).

Let $h \in \mathcal{H}^N$ with $h = (V, L^+, L^-)$. If $T \subseteq N$ we denote by h_T the restriction of h to T, that is,

$$h_T = (T \cap V, \{ij \in L^+ : i, j \in T\}, \{ij \in L^- : i, j \in T\}).$$

Notice that $(h_T)^+ = (h^+)_T$ and $(h_T)^- = (h^-)_T$. These graphs will be denoted by h_T^+ and h_T^-, respectively.
If $ij \in L^+ \cup L^-$ then we denote by h_{-ij} the element in $\mathcal{H}^N$ obtained when we remove the edge ij from either L^+ (if $ij \in L^+$) or from L^- (if $ij \in L^-$).

[4] Sometimes in the literature, positive and negative edges are differentiate by using symbols $+$ and $-$.

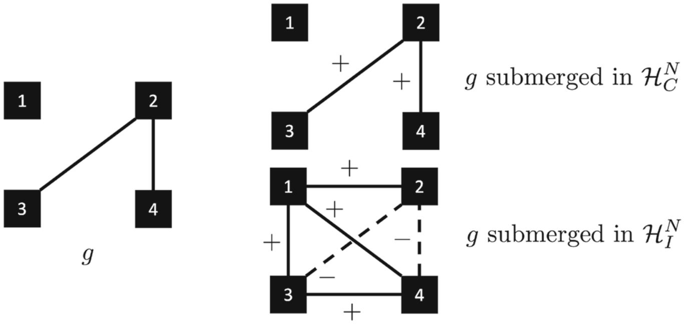


Fig. 3 Immersion of communication structures and incompatibility structures in $\mathcal{H}^N$

Next, we will endow $\mathcal{H}^N$ with a partial order.[5] If $h, \widehat{h} \in \mathcal{H}^N$ with $h = (V, L^+, L^-)$ and $\widehat{h} = (V, \widehat{L}^+, \widehat{L}^-)$, then $\widehat{h} \leq h$ if and only if $L^+ \supseteq \widehat{L}^+$ and $L^- \subseteq \widehat{L}^-$. Notice that $h \rhd \widehat{h}$ if and only if either $\widehat{h} = (V, L^+ \setminus \{ij\}, L^-)$ or $\widehat{h} = (V, L^+, L^- \cup \{ij\})$. In both cases we will denote $h - \widehat{h} = \{ij\}$.

4.2 Profit measures for communication situations with incompatibilities

Next, we introduce the concept of profit measure for communication situations with incompatibilities as a mixture of the two measures above.

Definition 4 A mapping $r : \mathcal{H}^N \to \mathbb{R}$ is said to be a profit measure on $\mathcal{H}^N$ if it satisfies the following conditions:

1. If $h \in \mathcal{H}_C^N$ and h^+ is connected, then $r(h) = r^C(h^+)$.
2. If $h \in \mathcal{H}_I^N$, then $r(h) = r^I(h^-)$.
3. $r(h) = \sum_{T \in N/h^+} r(h_T)$ for every $h \in \mathcal{H}^N$.
4. If $h, \widehat{h} \in \mathcal{H}^N$ and $h \rhd \widehat{h}$, then $r(h) \geq r(\widehat{h})$.

The family of all profit measures on $\mathcal{H}^N$ will be denoted by $\mathcal{B}\left(\mathcal{H}^N\right)$.

Notice that from conditions 1 and 3 it follows that if $h \in \mathcal{H}_C^N$ then $r(h) = r^C(h^+)$. In Sect. 5 we will show that $\mathcal{B}(\mathcal{H}^N) \neq \emptyset$ and, contrary to what happens in cases of communication situations or incompatibilities, there is more than one profit measure.

The profit measures determine an equivalence relation on the set $\mathcal{H}^N$. The negative edges connecting different positive connected components do not affect the profit.

Definition 5 Given $h = (V, L^+, L^-) \in \mathcal{H}^N$ and $ij \in L^-$, it will be said that ij is superfluous in h if i and j are not connected in h^+.

Definition 6 It will be said that $h = (V, L^+, L^-), \widehat{h} = (\widehat{V}, \widehat{L}^+, \widehat{L}^-) \in \mathcal{H}^N$ are equivalent if the following conditions are satisfied:

- $V = \widehat{V}$,

[5] Two situations with different vertex sets will not be comparable with this partial order, since we do not intend to compare situations in those cases.

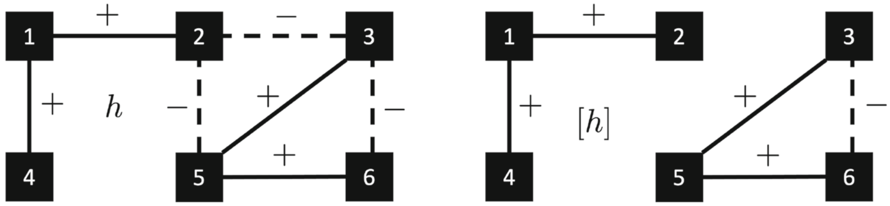


Fig. 4 Superfluous edges

- $L^+ = \widehat{L}^+$,
- If $ij \in L^- \setminus \widehat{L}^-$ then ij is superfluous in h,
- If $ij \in \widehat{L}^- \setminus L^-$ then ij is superfluous in $\widehat{h}$.

The notation $h \sim \widehat{h}$ will be used to denote that h and $\widehat{h}$ are equivalent. It is clear that $\sim$ is an equivalence relation.

If $h \in \mathcal{H}^N$, we will denote by $[h]$ the unique element in $\mathcal{H}^N$ such that $h \sim [h]$ and $[h]$ does not have any superfluous edges. Notice that $[h]$ is the top of $\{\widehat{h} \in \mathcal{H}^N : \widehat{h} \sim h\}$.

Example 3 Consider $h = (N, L^+, L^-)$ where $N = \{1, 2, 3, 4, 5, 6\}$, $L^+ = \{12, 14, 35, 56\}$ and $L^- = \{23, 25, 36\}$. Notice that h has two superfluous edges, 23 and 25. In Fig. 4 we represent h and $[h]$.

Proposition 3 *If $h \in \mathcal{H}^N$ and $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, then $r(h) = r([h])$.*

Proof The proof follows easily from property 3 in Definition 4, taking into account that $N/h^+ = N/[h]^+$ and $h_T = [h]_T$ for every $T \in N/h^+$. □

5 Allocation rules on $\mathcal{H}^N$

Our goal is to introduce and find allocation rules for communication situations with incompatibilities.

Definition 7 An allocation rule (for v) on $\mathcal{H}^N$ is a mapping $\Psi : \mathcal{H}^N \to \mathbb{R}^N$.

Next, we introduce some reasonable properties for an allocation rule on $\mathcal{H}^N$. They are inspired by the properties satisfied by the communication value introduced by Myerson.
Component r- efficiency. Given $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, an allocation rule Ψ on $\mathcal{H}^N$ satisfies the property of component r-efficiency if $\sum_{i \in T} \Psi_i(h) = r(h_T)$ for every $h \in \mathcal{H}^N$ and for every $T \in N/h^+$.
Inactive player. An allocation rule Ψ on $\mathcal{H}^N$ satisfies the property of inactive player if $\Psi_i(h) = 0$ for every $h = (V, L^+, L^-) \in \mathcal{H}^N$ and for every $i \in N \setminus V$.
Fairness. An allocation rule Ψ on $\mathcal{H}^N$ satisfies the fairness property if for every $h = (V, L^+, L^-) \in \mathcal{H}^N$ and for every $ij \in L^+ \cup L^-$ the following equality holds:

$$\Psi_i(h) - \Psi_i(h_{-ij}) = \Psi_j(h) - \Psi_j(h_{-ij}).$$

Stability. An allocation rule Ψ on $\mathcal{H}^N$ satisfies stability if for every $h = (V, L^+, L^-) \in \mathcal{H}^N$ the following inequalities hold:

- $\Psi_i(h) \geq \Psi_i(h_{-ij})$ for every $ij \in L^+$,

- $\Psi_i(h) \leq \Psi_i(h_{-ij})$ for every $ij \in L^-$.

We showed in Sect. 3 that, in order to obtain allocation rules for communication situations and for situations with incompatibilities, Myerson and, respectively, Bergantiños et al., defined the graph-restricted games v_g^C and v_g^I for each $g \in \mathcal{G}^N$. Notice that $v_g^C(S) = r^C(g_S)$ and $v_g^I(S) = r^I(g_S)$ for every $S \subseteq N$. We will proceed in a similar way in the case of communication situations with incompatibilities.

Definition 8 For each $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ and each $h \in \mathcal{H}^N$ we define the game v_h^r as

$$v_h^r(S) = r(h_S) \qquad \text{for every } S \subseteq N.$$

Definition 9 For each $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, the Myerson $r-$value is defined as

$$\mu^r(h) = \phi^{v_h^r} \qquad \text{for every } h \in \mathcal{H}^N.$$

Theorem 4 *For each $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, the Myerson $r-$value is the unique allocation rule on $\mathcal{H}^N$ that satisfies component $r-$efficiency, inactive player and fairness.*

Proof We will follow a similar reasoning to that used by Myerson (1977).

Firstly we will prove that the Myerson r-value satisfies the properties mentioned in the theorem.

- *Component r-efficiency.*
  Let $h \in \mathcal{H}^N$. For each $\widehat{T} \in N/h^+$ we consider the game $u^{\widehat{T}}$ defined as $u^{\widehat{T}}(S) = r(h_{S\cap\widehat{T}}) = v_h^r(S \cap \widehat{T})$, for every $S \subseteq N$. Notice that

$$v_h^r = \sum_{\widehat{T}\in N/h^+} u^{\widehat{T}} \tag{6}$$

  It is clear that any player in $N \setminus \widehat{T}$ is a null player in $u^{\widehat{T}}$. Since the Shapley value satisfies the null player property, we conclude that

$$\phi_i^{u^{\widehat{T}}} = 0 \quad \text{for every} \quad i \in N \setminus \widehat{T}. \tag{7}$$

  Take $T \in N/h^+$. We have that

$$\sum_{i\in T} \phi_i^{u^T} = \sum_{i\in N} \phi_i^{u^T} = u^T(N) = r(h_T), \tag{8}$$

  where we have used (7) and the efficiency of the Shapley value. By (6), the linearity of the Shapley value, (7) and (8) we obtain

$$\sum_{i\in T} \mu_i^r(h) = \sum_{i\in T} \phi_i^{v_h^r} = \sum_{i\in T} \phi_i^{\sum_{\widehat{T}\in N/h^+} u^{\widehat{T}}} = \sum_{i\in T} \sum_{\widehat{T}\in N/h^+} \phi_i^{u^{\widehat{T}}} = \sum_{i\in T} \phi_i^{u^T} = r(h_T).$$

- *Inactive player.*
  Let $h = (V, L^+, L^-) \in \mathcal{H}^N$ and $i \in N \backslash V$. Notice that $h_S = h_{S\cup\{i\}}$ for every $S \subseteq N$. This implies that i is a null player in v_h^r. By the property of null player of the Shapley value, $\phi_i^{v_h^r} = 0$.

- *Fairness.*
  Let $h = (V, L^+, L^-) \in \mathcal{H}^N$ and $ij \in L^+ \cup L^-$. Notice that, for every $S \subseteq N$ with $\{i, j\} \nsubseteq S$, we have that $h_S = \left(h_{-ij}\right)_S$ and, therefore, $v_h^r(S) - v_{h_{-ij}}^r(S) = 0$. This implies that i and j are symmetric in $v_h^r - v_{h_{-ij}}^r$. By the property of symmetry of the Shapley value, $\phi_i^{v_h^r - v_{h_{-ij}}^r} = \phi_j^{v_h^r - v_{h_{-ij}}^r}$, which leads to $\mu_i^r(h) - \mu_i^r(h_{-ij}) = \mu_j^r(h) - \mu_j^r(h_{-ij})$.

Finally we will check the uniqueness. Let Ψ^1 and Ψ^2 be allocation rules on $\mathcal{H}^N$ that satisfy component r-efficiency, inactive player and fairness. We will prove that $\Psi^1(h) = \Psi^2(h)$ for every $h = (V, L^+, L^-) \in \mathcal{H}^N$. This will be proved by induction on the cardinality of L^+.

- Base case.
  If $\left|L^+\right| = 0$, then $N/h^+ = \{\{i\} : i \in V\}$. Since Ψ^1 and Ψ^2 satisfy component r-efficiency, $\Psi_i^1(h) = \Psi_i^2(h) = r(h_{\{i\}})$, for every $i \in V$. Besides, by the property of inactive player, $\Psi_i^1(h) = \Psi_i^2(h)$ for every $i \in N \backslash V$. We conclude that $\Psi^1(h) = \Psi^2(h)$.
- Induction step.

Suppose that $\Psi^1(h) = \Psi^2(h)$ for every $h \in \mathcal{H}^N$ with $\left|L^+\right| = k$. Let $h \in \mathcal{H}^N$ be such that $\left|L^+\right| = k + 1$. Take $ij \in L^+$. By the fairness property,

$$\Psi_i^1(h) - \Psi_i^1(h_{-ij}) = \Psi_j^1(h) - \Psi_j^1(h_{-ij}),$$
$$\Psi_i^2(h) - \Psi_i^2(h_{-ij}) = \Psi_j^2(h) - \Psi_j^2(h_{-ij}).$$

By induction hypothesis, $\Psi^1(h_{-ij}) = \Psi^2(h_{-ij})$. From this and the equalities above it follows that $\Psi_i^1(h) - \Psi_i^2(h) = \Psi_j^1(h) - \Psi_j^2(h)$. Successively applying this reasoning we can obtain that if $T \in N/h^+$, then $\Psi_i^1(h) - \Psi_i^2(h) = \Psi_j^1(h) - \Psi_j^2(h)$ for every $i, j \in T$. Let us denote $d_T(h) = \Psi_i^1(h) - \Psi_i^2(h)$, for any $i \in T$. By the property of component r-efficiency, if $T \in N/h^+$, then

$$\sum_{i \in T} \Psi_i^1(h) = r(h_T) = \sum_{i \in T} \Psi_i^2(h).$$

It follows that

$$0 = \sum_{i \in T} \left(\Psi_i^1(h) - \Psi_i^2(h)\right) = |T|\, d_T(h) = 0,$$

whence $d_T(h) = 0$. We conclude that $\Psi^1(h) = \Psi^2(h)$. □

Proposition 5 *The Myerson r-value satisfies stability.*

Proof Let $h = (V, L^+, L^-) \in \mathcal{H}^N$ and $ij \in L^+ \cup L^-$. We know that $w = v_h^r - v_{h_{-ij}}^r$ satisfies that $w(S) = 0$ for every $S \subseteq N$ with $\{i, j\} \nsubseteq S$. Suppose now that $\{i, j\} \subseteq S \subseteq N$. Notice that if $ij \in L^+$, then $r\left(h_S\right) \geq r\left(\left(h_S\right)_{-ij}\right)$, and, consequently, $w(S) \geq 0$. And if $ij \in L^-$, then $r\left(h_S\right) \leq r\left(\left(h_S\right)_{-ij}\right)$, which leads to $w(S) \leq 0$. Taking into account that

$$\phi_i^w = \sum_{\{S \subseteq N, i \in S\}} \gamma_{|S|}^{|N|} \left(w(S) - w(S \setminus i)\right) = \sum_{\{S \subseteq N, i \in S\}} \gamma_{|S|}^{|N|} w(S),$$

we conclude that if $ij \in L^+$ then $\phi_i^w \geq 0$ (and, therefore, $\mu_i^r(h) \geq \mu_i^r(h_{-ij})$) and if $ij \in L^-$ then $\phi_i^w \leq 0$ (and, therefore, $\mu_i^r(h) \leq \mu_i^r(h_{-ij})$). □

6 Two examples of benefit measures on $\mathcal{H}^N$

In this section we will present two profit measures on $\mathcal{H}^N$.

6.1 The profit measure $r^{\perp}$

Definition 10 Let $h = (V, L^+, L^-) \in \mathcal{H}^N$ and $U \subseteq N$. We will say that U is bottom-feasible (we will write $\perp-$ feasible) for h if U is connected in h^+ and U does not contain any incompatible pairs, that is, $ij \notin L^-$ for every $\{i, j\} \subseteq U$. We denote by $\mathcal{P}_h^{\perp}$ the family of maximal partitions of V made up of $\perp-$feasible sets for h. We define $r^{\perp} : \mathcal{H}^N \to \mathbb{R}$ as

$$r^{\perp}(h) = \max_{P \in \mathcal{P}_h^{\perp}} \sum_{U \in P} v(U), \text{ for every } h \in \mathcal{H}^N.$$

Starting from v and using h^+ as a communication structure, we derive the graph-restricted game $v_{h^+}^C$. Subsequently, using h^- as an incompatibility structure, we can construct the graph-restricted game $(v_{h^+}^C)_{h^-}^I$. The following proposition shows that this game obtained by combining both concepts of graph-restricted game, is equal to $v_h^{r^{\perp}}$.

Proposition 6 *If $h = (V, L^+, L^-) \in \mathcal{H}^N$, then $r^{\perp}(h) = \left(v_{h^+}^C\right)_{h^-}^I (V)$ and $v_h^{r^{\perp}} = \left(v_{h^+}^C\right)_{h^-}^I$.*

Proof Firstly we will prove that $r^{\perp}(h) \leq \left(v_{h^+}^C\right)_{h^-}^I (V)$. Let $\bar{P} \in \mathcal{P}_h^{\perp}$ be such that $r^{\perp}(h) = \sum_{U \in \bar{P}} v(U)$. Notice that U is connected in h^+ for every $U \in \bar{P}$, and, therefore, $v(U) = v_{h^+}^C(U)$. We have that $r^{\perp}(h) = \sum_{U \in \bar{P}} v_{h^+}^C(U)$. Observe that the coalitions in $\bar{P}$ are feasible in h^-. Let $P' \in \mathcal{P}_{h^-}$ be such that $P' \geq \bar{P}$. We have that

$$r^{\perp}(h) = \sum_{U \in \bar{P}} v_{h^+}^C(U) \leq \sum_{U \in P'} v_{h^+}^C(U) \leq \max_{P \in \mathcal{P}_{h^-}} \sum_{U \in P} v_{h^+}^C(U) = \left(v_{h^+}^C\right)_{h^-}^I (V).$$

Next we will show that $r^{\perp}(h) \geq \left(v_{h^+}^C\right)_{h^-}^I (V)$. Recall that if w is a game on N then $(w)_{h^-}^I (V) = \max_{P \in \mathcal{P}_{h^-}} \sum_{U \in P} w(U)$. Let $\widehat{P} \in \mathcal{P}_{h^-}$ be a partition in which this maximum is attained for $w = v_{h^+}^C$, that is

$$\left(v_{h^+}^C\right)_{h^-}^I (V) = \sum_{U \in \widehat{P}} v_{h^+}^C(U).$$

Notice that each $U \in \widehat{P}$ is feasible for h^- but it is not necessarily $\perp-$feasible for h because it is not necessarily connected in h^+. For each $U \in \widehat{P}$ we will consider the partition U/h^+. Notice that $\bigcup_{U \in \widehat{P}} U/h^+$ is a partition of V into $\perp-$feasible sets for h, but it is not necessarily maximal. Let $\widetilde{P} \in \mathcal{P}_h^{\perp}$ be such that $\widetilde{P} \geq \bigcup_{U \in \widehat{P}} U/h^+$. We have that

$$r^{\perp}(h) \geq \sum_{\widetilde{U} \in \widetilde{P}} v(\widetilde{U}) \geq \sum_{U \in \widehat{P}} \sum_{W \in U/h^+} v(W) = \sum_{U \in \widehat{P}} v_{h^+}^C(U) = \left(v_{h^+}^C\right)_{h^-}^I (V).$$

Finally, if $S \subseteq N$, then

$$v_h^{r^{\perp}}(S) = r^{\perp}(h_S) = \left(v_{h^+}^C\right)_{h^-}^I (S).$$

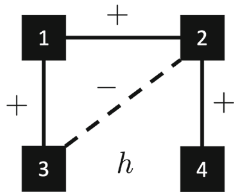


Fig. 5 Communication situation with incompatibilities for Example 4

□

Proposition 7 *The mapping* $r^{\perp} : \mathcal{H}^N \to \mathbb{R}$ *is a profit measure on* $\mathcal{H}^N$.

Proof Let us prove that $r^{\perp}$ satisfies the properties stated in Definition 4:

1. If $h = (V, L^+, L^-) \in \mathcal{H}_C^N$ and h^+ is connected, then $\mathcal{P}_{h^-}(V) = \{V\}$ and, consequently,
$$r^{\perp}(h) = v(V) = r^C(h^+).$$
2. If $h = (V, L^+, L^-) \in \mathcal{H}_I^N$, then $L^+ = L^V \backslash L^-$. Therefore, if $U \subseteq V$ is feasible for h^- then it is connected for h^+ and, consequently, it is $\perp-$feasible. We conclude that $\mathcal{P}_{h^-} = \mathcal{P}_h^{\perp}$, which leads to $r^{\perp}(h) = r^I(h^-)$.
3. Let $h = (V, L^+, L^-) \in \mathcal{H}^N$. Notice that each $\perp-$feasible set for h is contained in one connected component of h^+. It is clear that
$$r^{\perp}(h) = \max_{P \in \mathcal{P}_h^{\perp}} \sum_{U \in P} v(U) = \sum_{T \in N/h^+} \max_{P \in \mathcal{P}_{h_T}^{\perp}} \sum_{U \in P} v(U) = \sum_{T \in N/h^+} r^{\perp}(h_T).$$
4. Let $h = (V, L^+, L^-), \widehat{h} = (\widehat{V}, \widehat{L}^+, \widehat{L}^-) \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. Then, $L^+ \supseteq \widehat{L}^+$ and $L^- \subseteq \widehat{L}^-$. Therefore, every $\perp-$feasible set for $\widehat{h}$ is $\perp-$feasible for h. It follows that
$$r^{\perp}(\widehat{h}) = \max_{P \in \mathcal{P}_{\widehat{h}}^{\perp}} \sum_{U \in P} v(U) \leq \max_{P \in \mathcal{P}_h^{\perp}} \sum_{U \in P} v(U) = r^{\perp}(h).$$

□

Example 4 Let $N = \{1, 2, 3, 4\}$. Consider the communication situation with incompatibilities represented in Fig. 5 and the game $v \in \mathcal{G}^N$ defined as $v(S) = |S|^2$ if $3 \in S$ and $v(S) = |S|$ otherwise. Let us calculate $r^{\perp}(h)$.
Notice that
$$\mathcal{P}_h^{\perp} = \{\{\{1, 3\}, \{2, 4\}\}, \{\{1, 2, 4\}, \{3\}\}\}.$$
Hence,
$$r^{\perp}(h) = \max\{v(\{1, 3\}) + v(\{2, 4\}), v(\{1, 2, 4\}) + v(\{3\})\} = \max\{6, 4\} = 6.$$

Definition 11 We say that a profit measure $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ satisfies the positive connection property if for every $h = (V, L^+, L^-) \in \mathcal{H}^N$ there exists a partition $P \in \Pi_V$ made up of sets which are connected in h^+ and independent in h^- such that
$$r(h) = \sum_{U \in P} v(U).$$

Theorem 8 *The unique profit measure on* $\mathcal{H}^N$ *that satisfies the property of positive connection is* $r^{\perp}$. *Furthermore, if* $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, *then* $r^{\perp} \leq r$.

Proof Firstly we will prove that $r^{\perp} \leq r$ for every $r \in \mathcal{B}\left(\mathcal{H}^N\right)$. Let $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ and $h = (V, L^+, L^-) \in \mathcal{H}(N)$. Let $\widehat{P} \in \mathcal{P}_h^{\perp}$ be such that $r^{\perp}(h) = \sum_{U \in \widehat{P}} v(U)$. Consider

$$\widehat{h} = (V, L^+ \setminus \{ij : \{i, j\} \nsubseteq U \text{ for every } U \in \widehat{P}\}, L^-)$$

By properties 4, 3 and 1 in Definition 4, we obtain

$$r(h) \geq r(\widehat{h}) = \sum_{U \in \widehat{P}} r(h_U) = \sum_{U \in \widehat{P}} r^C(h_U^+) = \sum_{U \in \widehat{P}} v(U) = r^{\perp}(h).$$

From the definition of $r^{\perp}$, it is clear that it satisfies the property of positive connection. Let us see the uniqueness. Let $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ be such that r satisfies the property of positive connection. Let $h = (V, L^+, L^-) \in \mathcal{H}^N$. There exists a partition $\widehat{P} \in \Pi_V$ made up of sets which are connected in h^+ and independent in h^- such that $r(h) = \sum_{U \in \widehat{P}} v(U)$. Let $P' \in \mathcal{P}_h^{\perp}$ be such that $\widehat{P} \leq P'$. Then,

$$r\,(h) = \sum_{U \in \widehat{P}} v(U) \leq \sum_{U \in P'} v(U) \leq \max_{P \in \mathcal{P}_h^{\perp}} \sum_{U \in P} v(U) = r^{\perp}(h).$$

Therefore, $r(h) \leq r^{\perp}(h)$. Since we know that $r(h) \geq r^{\perp}(h)$, we conclude the uniqueness. □

6.2 The profit measure $r^{\top}$

Definition 12 Let $h \in \mathcal{H}^N$ and $U \subseteq N$. We will say that U is top-feasible (we will write $\top-$ feasible) for h if U is contained in a connected component of h^+ and U is independent in h^-. We denote by $\mathcal{P}_h^{\top}$ the family of maximal partitions of $V(h)$ made up of $\top-$feasible sets for h. We define $r^{\top} : \mathcal{H}^N \to \mathbb{R}$ as

$$r^{\top}(h) = \max_{P \in \mathcal{P}_h^{\top}} \sum_{U \in P} v(U), \text{ for every } h \in \mathcal{H}^N.$$

The following proposition shows that if we subsequently apply the concepts of graph-restricted game used in the model of Bergantiños et al. and then in Myerson's model, the resulting game, denoted as $(v_{h^-}^I)_{h^+}^C$, is equal to $v_h^{r^{\top}}$.

Proposition 9 *If* $h = (V, L^+, L^-) \in \mathcal{H}^N$, *then* $r^{\top}(h) = \left(v_{h^-}^I\right)_{h^+}^C (V)$ *and* $v_h^{r^{\top}} = \left(v_{h^-}^I\right)_{h^+}^C$.

Proof Firstly we will prove that $r^{\top}(h) \leq \left(v_{h^-}^I\right)_{h^+}^C (V)$. Let $\widehat{P} \in \mathcal{P}_h^{\top}$ be such that $r^{\top}(h) = \sum_{U \in \widehat{P}} v(U)$. Notice that U is independent in h^- for every $U \in \widehat{P}$, and, therefore, $v(U) = v_{h^-}^I(U)$. We have that $r^{\top}(h) = \sum_{U \in \widehat{P}} v_{h^-}^I(U)$. Observe that the coalitions in $\widehat{P}$ are connected in h^+. Therefore, $\widehat{P} \leq N/h^+$. We have that

$$r^{\top}(h) = \sum_{U \in \widehat{P}} v_{h^-}^I(U) \leq \sum_{U \in N/h^+} v_{h^-}^I(U) = \left(v_{h^-}^I\right)_{h^+}^C (V).$$

Next we will show that $r^{\top}(h) \geq \left(v^I_{h^-}\right)^C_{h^+}(V)$. We have that

$$\left(v^I_{h^-}\right)^C_{h^+}(V) = \sum_{U \in N/h^+} v^I_{h^-}(U).$$

For each $U \in N/h^+$ let $P_U \in \mathcal{P}_{h^-_U}(U)$ be such that $v^I_{h^-}(U) = \sum_{W \in P_U} v(W)$. Notice that $\bigcup_{U \in N/h^+} P_U$ is a partition of V into $\top-$feasible sets for h, but it is not necessarily maximal. Let $P' \in \mathcal{P}^{\top}_h$ be such that $P' \geq \bigcup_{U \in N/h^+} P_U$. We have that

$$r^{\top}(h) \geq \sum_{U' \in P'} v(U') \geq \sum_{U \in N/h^+} \sum_{W \in P_U} v(W) = \sum_{U \in N/h^+} v^I_{h^-}(U) = \left(v^I_{h^-}\right)^C_{h^+}(V).$$

Finally, if $S \subseteq N$, then

$$v^{r^{\top}}_h(S) = r^{\top}(h_S) = \left(v^I_{h^-}\right)^C_{h^+}(S).$$

□

Proposition 10 *The mapping $r^{\top} : \mathcal{H}^N \to \mathbb{R}$ is a profit measure on $\mathcal{H}^N$.*

Proof Let us prove that $r^{\top}$ satisfies the properties stated in Definition 4.

1. If $h = (V, L^+, L^-) \in \mathcal{H}^N_C$ and h^+ is connected, then
$$r^{\top}(h) = v(V) = r^C(h^+).$$
2. If $h = (V, L^+, L^-) \in \mathcal{H}^N_I$, then $L^+ = L^V \setminus L^-$. Therefore, if $U \subseteq V$ is feasible for h^- then it is connected for h^+ and, consequently, it is $\top-$feasible. We conclude that $\mathcal{P}_{h^-} = \mathcal{P}^{\top}_h$, which leads to $r^{\top}(h) = r^I(h^-)$.
3. Let $h = (V, L^+, L^-) \in \mathcal{H}^N$. We have that
$$\begin{aligned} r^{\top}(h) &= \left(v^I_{h^-}\right)^C_{h^+}(V) \\ &= \sum_{T \in N/h^+} \max_{P \in \mathcal{P}_{h^-_T}} \sum_{U \in P} v(U) = \sum_{T \in N/h^+} \max_{P \in \mathcal{P}^{\top}_{h_T}} \sum_{U \in P} v(U) = \sum_{T \in N/h^+} r^{\top}(h_T). \end{aligned}$$
4. Let $h = (V, L^+, L^-), \widehat{h} = (\widehat{V}, \widehat{L}^+, \widehat{L}^-) \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. Then, $L^+ \supseteq \widehat{L}^+$ and $L^- \subseteq \widehat{L}^-$. Therefore, every $\top-$feasible set for $\widehat{h}$ is $\top-$feasible for h. It follows that
$$r^{\top}(\widehat{h}) = \max_{P \in \mathcal{P}^{\top}_{\widehat{h}}} \sum_{U \in P} v(U) \leq \max_{P \in \mathcal{P}^{\top}_h} \sum_{U \in P} v(U) = r^{\top}(h).$$

□

Example 5 Let us calculate $r^{\top}(h)$ using the signed graph h (Fig. 5) and the game v in Example 4. We have that

$$\mathcal{P}^{\top}_h = \{\{\{1,3\},\{2,4\}\},\{\{1,2,4\},\{3\}\},\{\{1,3,4\},\{2\}\},\{\{1,2\},\{3,4\}\}\}.$$

Therefore,

$$\begin{aligned} r^{\top}(h) &= \max\{v(\{1,3\}) + v(\{2,4\}), v(\{1,2,4\}) + v(\{3\}), \\ &\quad v(\{1,3,4\}) + v(\{2\}), v(\{1,2\}) + v(\{3,4\})\} \\ &= \max\{6, 4, 10, 4\} = 10. \end{aligned}$$

Remark 2 Observe that if we use $r^\perp$ to measure the profit attainable by each coalition, we are assuming that if a player i in a coalition does not actively cooperate with some other players in the coalition, then i will not facilitate the communication between those players either. On the contrary, when we use $r^\top$ we assume that even if i does not cooperate with some players in the coalition, it can facilitate the communication between them.

Definition 13 Let $h = (V, L^+, L^-) \in \mathcal{H}^N$. The *complete hull* of h is the communication situation with incompatibilities $\widetilde{h} = (\widetilde{V}, \widetilde{L}^+, \widetilde{L}^-) \in \mathcal{H}^N$ defined as

$$\begin{aligned}\widetilde{V} &= V,\\ \widetilde{L}^+ &= \bigcup_{T \in N/h^+} \{ij : i, j \in T, ij \notin L^-\},\\ \widetilde{L}^- &= L^-.\end{aligned}$$

We will say that a profit measure $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ satisfies the completeness property if $r(h) = r\left(\widetilde{h}\right)$ for every $h \in \mathcal{H}^N$.

Theorem 11 *The unique profit measure on $\mathcal{H}^N$ that satisfies the completeness property is $r^\top$. Furthermore, if $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, then $r \le r^\top$.*

Proof Let $h \in \mathcal{H}^N$. Notice that $h^- = \widetilde{h}^-$ and $N/h^+ = N/\widetilde{h}^+$. By definition of $r^\top$, this implies that $r^\top(h) = r^\top\left(\widetilde{h}\right)$. Therefore, $r^\top$ satisfies the completeness property.

Let us prove that $r \le r^\top$ for every $r \in \mathcal{B}\left(\mathcal{H}^N\right)$. Let $r \in \mathcal{B}\left(\mathcal{H}^N\right)$ and $h \in \mathcal{H}^N$. By properties 4, 3 and 2 in Definition 4,

$$r(h) \le r\left(\widetilde{h}\right) = \sum_{T \in N/\widetilde{h}^+} r\left(\widetilde{h}_T\right) = \sum_{T \in N/\widetilde{h}^+} r^I\left(\widetilde{h}_T^-\right) = \sum_{T \in N/\widetilde{h}^+} r^\top\left(\widetilde{h}_T\right) = r^\top\left(\widetilde{h}\right) = r^\top(h).$$

Furthermore, if r satisfies completeness then $r(h) = r\left(\widetilde{h}\right)$, and the equalities above would lead to $r(h) = r^\top(h)$. Consequently, $r^\top$ is the unique profit measure on $\mathcal{H}^N$ that satisfies the completeness property. □

7 The structure of $\mathcal{B}\left(\mathcal{H}^N\right)$

In this section we will study structural properties of $\mathcal{B}\left(\mathcal{H}^N\right)$. We denote by $\mathcal{F}\left(\mathcal{H}^N\right)$ the family of functions $r : \mathcal{H}^N \to \mathbb{R}$. Notice that $\mathcal{F}\left(\mathcal{H}^N\right)$ is a vector space with the usual operations of addition and scalar multiplication. If we also consider the usual relation $\le$, then $\mathcal{F}\left(\mathcal{H}^N\right)$ is an ordered vector space. In fact, it is a lattice, in which the infimum and the supremum of $r, \widehat{r} \in \mathcal{F}\left(\mathcal{H}^N\right)$ are given, respectively, by $(r \wedge \widehat{r})(h) = \min\{r(h), \widehat{r}(h)\}$, $(r \vee \widehat{r})(h) = \max\{r(h), \widehat{r}(h)\}$ for every $h \in \mathcal{H}^N$. Evidently, $\mathcal{B}\left(\mathcal{H}^N\right) \subseteq \mathcal{F}\left(\mathcal{H}^N\right)$. We will study the structure of $\mathcal{B}\left(\mathcal{H}^N\right)$ within the ordered vector space $\mathcal{F}\left(\mathcal{H}^N\right)$.

Remark 3 In the previous section we have seen that:

- $\mathcal{B}\left(\mathcal{H}^N\right) \neq \emptyset$.
- If $r \in \mathcal{B}\left(\mathcal{H}^N\right)$, then $r^\perp \le r \le r^\top$, that is, $\mathcal{B}\left(\mathcal{H}^N\right) \subseteq [r^\perp, r^\top]_{\mathcal{F}(\mathcal{H}^N)}$.

The following example shows that the inclusion above is strict, that is, $\mathcal{B}\left(\mathcal{H}^N\right) \subsetneq [r^\perp, r^\top]_{\mathcal{F}(\mathcal{H}^N)}$.

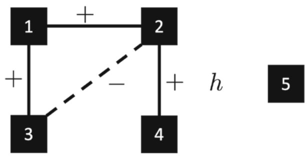


Fig. 6 Structure in Example 6

Example 6 Consider $r \in \mathcal{F}\left(\mathcal{H}^N\right)$ defined as

$$r(h) = \begin{cases} r^{\perp}(h), & \text{if } V = N, \\ r^{\top}(h), & \text{if } V \neq N, \end{cases}$$

for each $h = (V, L^+, L^-)$. It is clear that $r \in [r^{\perp}, r^{\top}]_{\mathcal{F}(\mathcal{H}^N)}$. Now suppose that $N = \{1, 2, 3, 4, 5\}$ and $v = u_{\{1,3,4\}}$. Let us see that $r \notin \mathcal{B}\left(\mathcal{H}^N\right)$, since it does not satisfy property 3 in Definition 4. Take the situation $h \in \mathcal{H}^N$ represented in Fig. 6.
We have that $N/h^+ = \{T_1 = \{1, 2, 3, 4\}, T_2 = \{5\}\}$. Since $V(h) = N$, it follows that $r(h) = r^{\perp}(h) = 0$. Finally, notice that $r(h_{T_1}) = r^{\top}(h_{T_1}) = 1$ and $r(h_{T_2}) = r^{\top}(h_{T_2}) = 0$. Therefore, $r(h) \neq r(h_{T_1}) + r(h_{T_2})$.

Theorem 12 *The family $\mathcal{B}\left(\mathcal{H}^N\right)$ is convex in $\mathcal{F}\left(\mathcal{H}^N\right)$. In particular, $\overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)} \subseteq \mathcal{B}\left(\mathcal{H}^N\right)$.*

Proof Let $r_1, r_2 \in \mathcal{B}\left(\mathcal{H}^N\right)$ and $\alpha \in [0, 1]$. We aim to prove that $r = (1-\alpha)\, r_1 + \alpha r_2 \in \mathcal{B}\left(\mathcal{H}^N\right)$.

1. If $h = (V, L^+, L^-) \in \mathcal{H}^N_C$ and h^+ is connected, then

$$r(h) = (1-\alpha)\, r_1(h) + \alpha r_2(h) = (1-\alpha)\, r^C(h^+) + \alpha r^C(h^+) = r^C(h^+).$$

2. If $h = (V, L^+, L^-) \in \mathcal{H}^N_I$, then

$$r(h) = (1-\alpha)\, r_1(h) + \alpha r_2(h) = (1-\alpha)\, r^I(h^-) + \alpha r^I(h^-) = r^I(h^-).$$

3. If $h \in \mathcal{H}^N$, then

$$r(h) = (1-\alpha)\, r_1(h) + \alpha r_2(h) = (1-\alpha) \sum_{T \in N/h^+} r_1(h_T) + \alpha \sum_{T \in N/h^+} r_2(h_T) = \sum_{T \in N/h^+} r(h_T).$$

4. Let $h, \widehat{h} \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. Then,

$$r(h) = (1-\alpha)\, r_1(h) + \alpha r_2(h) \geq (1-\alpha)\, r_1(\widehat{h}) + \alpha r_2(\widehat{h}) = r(\widehat{h}).$$

We conclude that $r \in \mathcal{B}(\mathcal{H}^N)$. □

Remark 4 In the previous section we showed that, unlike $r^{\perp}$, the profit measure $r^{\top}$ allows a player to facilitate communication between other players with whom it is not actively cooperating. Observe that the profit measure

$$r^{\alpha} = (1-\alpha) r^{\perp} + \alpha r^{\top} \in \overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)}$$

can be used when we aim to follow an intermediate approach.

The following example shows that the inclusion stated in the previous theorem is strict, that is, $\overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)} \subsetneq \mathcal{B}\left(\mathcal{H}^N\right)$.

Example 7 Consider the mapping $r : \mathcal{H}^N \to \mathbb{R}$ defined as

$$r(h) = \begin{cases} r^{\top}(h), & \text{if } V = N \text{ and } h^{+} \text{ is connected} \\ r^{\perp}(h), & \text{otherwise,} \end{cases}$$

for every $h = (V, L^+, L^-) \in \mathcal{H}^N$. Firstly we will prove that $r \in \mathcal{B}\left(\mathcal{H}^N\right)$. Let us check that r satisfies the properties in Definition 4.

1. If $h \in \mathcal{H}^N_C$ and h^+ is connected, then $r(h) = r^{\perp}(h) = r^{\top}(h) = r^C(h^+)$.
2. If $h \in \mathcal{H}^N_I$, then $r(h) = r^{\perp}(h) = r^{\top}(h) = r^I(h^-)$.
3. Let $h \in \mathcal{H}^N$. If h^+ is connected then there is nothing to prove. If h^+ is not connected, then $r(h) = r^{\perp}(h)$ and $r(h_T) = r^{\perp}(h_T)$ for every $T \in N/h^+$. It suffices to use that $r^{\perp}$ satisfies property 3.
4. Let $h, \widehat{h} \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. In particular, $V(h) = V(\widehat{h})$. We consider the following cases:
    - If $V(h) \neq N$, then $r(h) = r^{\perp}(h) \geq r^{\perp}(\widehat{h}) = r^{\perp}(\widehat{h})$.
    - If $V(h) = N$ and $\widehat{h}^+$ is connected. Then, h^+ is also connected and $r(h) = r^{\top}(h) \geq r^{\top}(\widehat{h}) = r(\widehat{h})$.
    - If $V(h) = N$ and h^+ is not connected. Then, $\widehat{h}^+$ is also not connected and $r(h) = r^{\perp}(h) \geq r^{\perp}(\widehat{h}) = r(\widehat{h})$.
    - If $V(h) = N$, h^+ is connected and $\widehat{h}^+$ is not connected, then $r(h) = r^{\top}(h) \geq r^{\perp}(h) \geq r^{\perp}(\widehat{h}) = r(\widehat{h})$.

Let us prove that, in general, $r \notin \overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)}$. Take $N = \{1, 2, 3, 4, 5\}$ and $v = u_{\{1,3,4\}}$. Let $h, \widehat{h} \in \mathcal{H}^N$ be the situations represented in Figs. 2 and 5, respectively. Suppose that $r \in \overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)}$. Then, there would exist $\alpha \in [0, 1]$ such that $r = (1-\alpha)r^{\perp} + \alpha r^{\top}$. Notice that $r(h) = r^{\top}(h) = 1$ and $r^{\perp}(h) = 0$. Therefore, it should be $\alpha = 1$. But observe that $r(\widehat{h}) = r^{\perp}(\widehat{h}) = 0$ and $r^{\top}(\widehat{h}) = 1$. Consequently, it should be $\alpha = 0$. We conclude that $r \notin \overline{r^{\perp}r^{\top}}^{\mathcal{F}(\mathcal{H}^N)}$.

The poset $\left(\mathcal{B}\left(\mathcal{H}^N\right), \leq\right)$ is not a lattice with the operations $\wedge$ and $\vee$ inherited from $\mathcal{F}\left(\mathcal{H}^N\right)$, since $\mathcal{B}\left(\mathcal{H}^N\right)$ is not closed under these operations, as we will show in the following example.

Example 8 Let $E = \{1, 2, 3, 4\}$, $F = \{5, 6, 7, 8\}$ and $N = E \cup F$. Consider the game $v_E = u_{\{134\}}$ on E. Let $r_E^{\perp}, r_E^{\top} \in \mathcal{B}\left(\mathcal{H}^E\right)$ be the bottom and the top measures, respectively, for the game v_E. Now consider the game $v_F = u_{\{6\}} + u_{\{5,7,8\}}$ on F. Let $r_F^{\perp}, r_F^{\top} \in \mathcal{B}\left(\mathcal{H}^F\right)$ be the bottom and the top measures, respectively, for the game v_F. Consider the game v on N defined as

$$v(S) = v_E(S \cap E) + v_F(S \cap F),$$

for every $S \subseteq N$. Let $r^{\perp}, r^{\top} \in \mathcal{B}\left(\mathcal{H}^N\right)$ be the bottom and the top measures, respectively, for the game v. Let $r, \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$ be measures for the game v defined as

$$r(h) = \begin{cases} r_E^{\perp}(h_E) + r_F^{\top}(h_F), & \text{if } h \in H, \\ r^{\top}(h), & \text{if } h \in \mathcal{H}^N \setminus H, \end{cases} \quad \text{and } \widehat{r}(h) = \begin{cases} r_E^{\top}(h_E) + r_F^{\perp}(h_F), & \text{if } h \in H, \\ r^{\top}(h) & \text{if } h \in \mathcal{H}^N \setminus H, \end{cases}$$

where $H = \{h = \left(V, L^+, L^-\right) \in \mathcal{H}^N : ij \notin L^+ \text{ for every } (i, j) \in E \times F\}$. Notice that if $h \in H$ then $N/h^+ = N/h_E^B \cup N/h_F^B$.

Let us check that $r, r' \in \mathcal{B}\left(\mathcal{H}^N\right)$.

1. Let $h = \left(V, L^+, L^-\right) \in \mathcal{H}^N$ be such that h^+ is connected and $L^- = \emptyset$. It is clear that $h \notin H$, whence $r(h) = r^\top(h) = r^C(h^+)$.
2. Let $h = (V, L^+, L^-) \in \mathcal{H}_I^N$. If $h \notin H$, then $r(h) = r^\top(h) = r^I(h^-)$. If $h \in H$, then $h_E^B = \left(h_E^R\right)^*$ and $h_F^B = \left(h_F^R\right)^*$. From this fact and property 2 in Definition 4, it follows that

$$r(h) = r_E^\perp(h_E) + r_F^\top(h_F) = r^I(h_E^R) + r^I(h_F^R) = r^I(h^-).$$

3. Let $h \in \mathcal{H}^N$. We consider two cases:

- If $h \notin H$, then $r(h) = r^\top(h)$. Furthermore,

$$\begin{aligned}\sum_{T\in N/h^+} r(h_T) &= \sum_{\{T\in N/h^+:\, T\subset E\}} r(h_T) + \sum_{\{T\in N/h^+:\, T\subset F\}} r(h_T) \\ &\quad + \sum_{\{T\in N/h^+:\, T\cap F\neq\emptyset, T\cap E\neq\emptyset\}} r(h_T) \\ &= \sum_{\{T\in N/h^+:\, T\subsetneq E\}} (r_E^\perp(h_T) + r_F^\top(\emptyset)) \\ &\quad + \sum_{\{T\in N/h^+:\, T\subsetneq F\}} (r_E^\perp(\emptyset) + r_F^\top(h_T)) \\ &\quad + \sum_{\{T\in N/h^+:\, T\cap F\neq\emptyset, T\cap E\neq\emptyset\}} r^\top(h_T) \\ &= \sum_{T\in N/h^+} r^\top(h_T) = r^\top(h) = r(h).\end{aligned}$$

  where we have used the fact that for all communication situations with incompatibilities with cardinality of vertices less or equal than 3 both $r^\top$ and $r^\perp$ coincide.
- If $h \in H$, then $r(h) = r_E^\perp(h_E) + r_F^\top(h_F)$. Moreover, since $N/h^+ = N/h_E^B \cup N/h_F^B$ we have that

$$\begin{aligned}\sum_{T\in N/h^+} r(h_T) &= \sum_{T\in E/h_E^B} r(h_T) + \sum_{T\in F/h_F^B} r(h_T) = \sum_{T\in E/h_E^B} r_E^\perp(h_T) + \sum_{T\in F/h_F^B} r_F^\top(h_T) \\ &= r_E^\perp(h_E) + r_F^\top(h_F) = r(h).\end{aligned}$$

4. Let $h = (V, L^+, L^-), \widehat{h} = (V, \widehat{L}^+, \widehat{L}^-) \in \mathcal{H}^N$, be such that $h \triangleright \widehat{h}$. Let us consider the following cases:

- If $h, \widehat{h} \notin H$, then $r(h) = r^\top(h) \geq r^\top(\widehat{h}) = r(\widehat{h})$.
- If $h, \widehat{h} \in H$, then

$$r(h) = r_E^\perp(h_E) + r_F^\top(h_F) \geq r_E^\perp(\widehat{h}_E) + r_F^\top(\widehat{h}_F) = r(\widehat{h}).$$

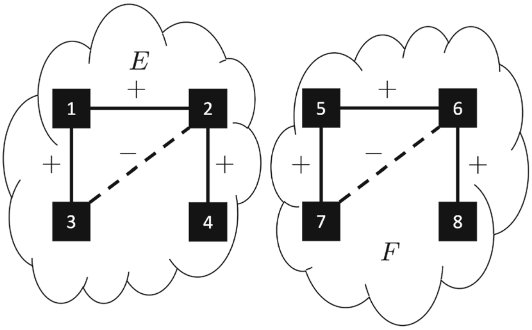
- If $h \notin H$, $\widehat{h} \in H$, then

$$r(\widehat{h}) = r_E^\perp(\widehat{h}_E) + r_F^\top(\widehat{h}_F) \leq r_E^\top(\widehat{h}_E) + r_F^\top(\widehat{h}_F) = r^\top(\widehat{h}) \leq r^\top(h) = r(h).$$

In a similar way it can be proved that $\widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$. Nevertheless $r \wedge \widehat{r}$ is not a profit measure[6] on $\mathcal{H}^N$. It suffices to take the communication situation with incompatibilities $h \in H$ such

[6] Simply by exchanging $\top$ and $\perp$ in the definitions of $r, \widehat{r}$ it can be proved that $\mathcal{B}\left(\mathcal{H}^N\right)$ is also not closed under the operation $\vee$.

Fig. 7 Zones E and F in h



that $h_E = (E, \{12, 13, 24\}, \{23\})$, and $h_F = (F, \{56, 57, 68\}, \{67\})$. The situation h is represented in Fig. 7.

Let us show that property 3 in Definition 4 is not satisfied. On the one hand,

$$(r \wedge \widehat{r})(h) = r(h) \wedge \widehat{r}(h) = \left(r_E^{\perp}(h_E) + r_F^{\top}(h_F)\right) \wedge \left(r_E^{\top}(h_E) + r_F^{\perp}(h_F)\right) = 2 \wedge 2 = 2.$$

And, on the other hand,

$$\sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) = (r \wedge \widehat{r})(h_E) + (r \wedge \widehat{r})(h_F) = (0 \wedge 1) + (2 \wedge 1) = 1.$$

However $\left(\mathcal{B}\left(\mathcal{H}^N\right), \leq\right)$ is indeed a lattice. Next we will define two new operations $\curlyvee$ and $\curlywedge$ in $\mathcal{B}\left(\mathcal{H}^N\right)$ and we will prove that these operations return the supremum and the infimum, respectively, of any two measures in $\mathcal{B}\left(\mathcal{H}^N\right)$.

Definition 14 If $r, \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$, the profit measures $r \curlyvee \widehat{r}$ and $r \curlywedge \widehat{r}$ are defined as

$$(r \curlyvee \widehat{r})(h) = \sum_{T \in N/h^+} (r \vee \widehat{r})(h_T),$$
$$(r \curlywedge \widehat{r})(h) = \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T),$$

for every $h \in \mathcal{H}^N$.

Theorem 13 *The poset $\left(\mathcal{B}\left(\mathcal{H}^N\right), \leq\right)$ is a lattice in which the supremum and the infimum of any profit measures $r, \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$ are equal to $r \curlyvee \widehat{r}$ and $r \curlywedge \widehat{r}$, respectively.*

Proof Let $r, \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$. Firstly we will prove that $r \curlywedge \widehat{r} \leq r \wedge \widehat{r}$. If $h \in \mathcal{H}^N$, then

$$(r \curlywedge \widehat{r})(h) = \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) \leq \sum_{T \in N/h^+} r(h_T) = r(h).$$

In a similar way it can be proved that $r \curlywedge \widehat{r} \leq \widehat{r}$. We conclude that $r \curlywedge \widehat{r} \leq r \wedge \widehat{r}$.

Let us see that $\mathcal{B}\left(\mathcal{H}^N\right)$ is closed under the operation $\curlywedge$. Let $r, \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$.

1. If $h = (V, L^+, L^-) \in \mathcal{H}_C^N$ and h^+ is connected, then $N/h^+ = \{V\}$ and $(r \curlywedge \widehat{r})(h) = (r \wedge \widehat{r})(h) = r^C(h^+)$.

2. If $h \in \mathcal{H}_I^N$, then, by Proposition 2,

$$(r \curlywedge \widehat{r})(h) = \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) = \sum_{T \in N/(h^-)^*} r^I(h_T^-) = r^I(h^-).$$

3. If $h \in \mathcal{H}^N$, then, for each $T \in N/h^+$, the graph h_T^+ is connected, and, consequently, $(r \curlywedge \widehat{r})(h_T) = (r \wedge \widehat{r})(h_T)$. So,

$$(r \curlywedge \widehat{r})(h) = \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) = \sum_{T \in N/h^+} (r \curlywedge \widehat{r})(h_T).$$

4. Let $h = (V, L^+, L^-), \widehat{h} = (V, \widehat{L}^+, \widehat{L}^-) \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. Notice that $h_T \geq \widehat{h}_T$ for every $T \in N/h^+$.
   - If $N/h^+ = N/\widehat{h}^+$, then

$$(r \curlywedge \widehat{r})(h) = \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) \geq \sum_{T \in N/\widehat{h}^+} (r \wedge \widehat{r})(\widehat{h}_T) = (r \curlywedge \widehat{r})(\widehat{h}),$$

   where we have used that $(r \wedge \widehat{r})(h_T) \geq (r \wedge \widehat{r})(\widehat{h}_T)$, since r and $\widehat{r}$ satisfy property 4.
   - If $N/h^+ \neq N/\widehat{h}^+$ then there exist $\hat{T} \in N/h^+$ and $T_1, T_2 \in N/\widehat{h}^+$ such that $N/\widehat{h}^+ = ((N/h^+)\backslash\{\hat{T}\}) \cup \{T_1, T_2\}$. Therefore, $N/(\widehat{h}_{\hat{T}})^+ = \{T_1, T_2\}$. In this case we have that

$$\begin{aligned}
(r \curlywedge \widehat{r})(h) &= \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) \geq \sum_{T \in N/h^+} (r \wedge \widehat{r})(\widehat{h}_T) \\
&= (r \wedge \widehat{r})(\widehat{h}_{\hat{T}}) + \sum_{\{T \in N/h^+ : T \neq \hat{T}\}} (r \wedge \widehat{r})(\widehat{h}_T) \\
&\geq (r \curlywedge \widehat{r})(\widehat{h}_{\hat{T}}) + \sum_{\{T \in N/\widehat{h}^+ : T \neq T_1, T_2\}} (r \wedge \widehat{r})(\widehat{h}_T) \\
&= (r \wedge \widehat{r})(\widehat{h}_{T_1}) + (r \wedge \widehat{r})(\widehat{h}_{T_2}) + \sum_{\{T \in N/\widehat{h}^+ : T \neq T_1, T_2\}} (r \wedge \widehat{r})(\widehat{h}_T) \\
&= \sum_{T \in N/\widehat{h}^+} (r \wedge \widehat{r})(\widehat{h}_T) = (r \curlywedge \widehat{r})(\widehat{h}),
\end{aligned}$$

   where we have used that $(r \wedge \widehat{r})(h_T) \geq (r \wedge \widehat{r})(\widehat{h}_T)$ and $r \curlywedge \widehat{r} \leq r \wedge \widehat{r}$.

From $r \curlywedge \widehat{r} \leq r \wedge \widehat{r}$ and $r \curlywedge \widehat{r} \in \mathcal{B}\left(\mathcal{H}^N\right)$ it follows that $r \curlywedge \widehat{r}$ is a lower bound of $\{r, \widehat{r}\}$ in $\mathcal{B}\left(\mathcal{H}^N\right)$. Let us check that it is the maximum lower bound. Let $r' \in \mathcal{B}\left(\mathcal{H}^N\right)$ be such that $r' \leq r$ and $r' \leq \widehat{r}$. Then, for each $h \in \mathcal{H}^N$

$$r'(h) = \sum_{T \in N/h^+} r'(h_T) \leq \sum_{T \in N/h^+} (r \wedge \widehat{r})(h_T) = (r \curlywedge \widehat{r})(h).$$

In a similar way it can be proved that the supremum of r and $\widehat{r}$ exists and is equal to $r \curlyvee \widehat{r}$. □

8 Conclusions

In this paper we have dealt with a family of cooperative situations with restricted cooperation: communication situations with incompatibilities. Our goal was to find a model for such

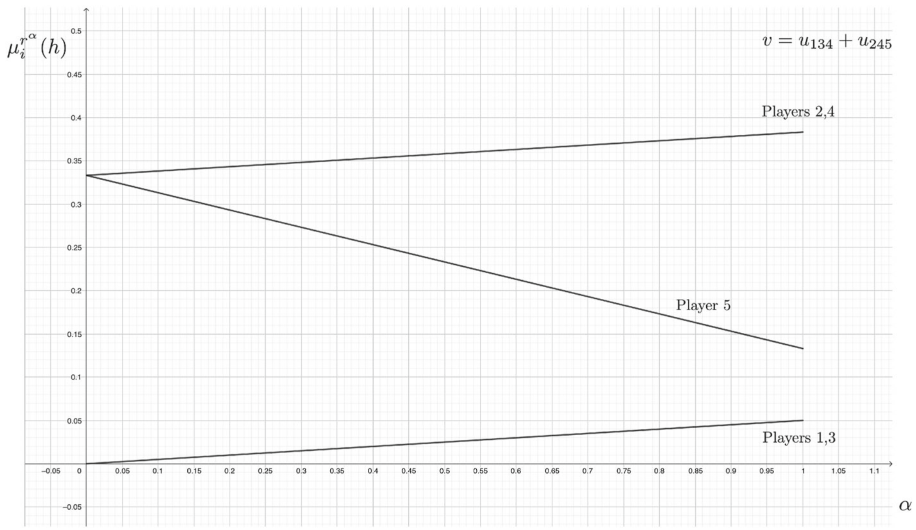


Fig. 8 Allocation rules depending on α

situations that extends Myerson's model for communication situations and Bergantiños' model for incompatibility relationships. The key concept we have used for this purpose is that of profit measure, which has allowed us to obtain, in fact, multiple models that meet our requirements for extension. Each one of the profit measures enables us to define a unique restricted game, which leads to a unique allocation rule. Although we have analyzed different profit measures for a fixed game, it is possible to apply any of these measures to any game. Next, by way of conclusion, we will show the versatility of the model introduced.

In the following example we will use the measures $r^\top$, $r^\perp$ and r^α in the segment between them (see Remark 4) to obtain different payoff vectors for a cooperative game and a communication situation with incompatibilities. Consider $N = \{1, 2, 3, 4, 5\}$, the game $v = u_{\{1,3,4\}} + u_{\{2,4,5\}}$ and the communication situation with incompatibilities h represented in Fig. 2. It is clear that $r^\perp(h) = r^\top(h) = 1$. Therefore, for each value μ^{r^α} the quantity to be distributed is equal to 1. It is easy to check that if $S \subseteq N$ and $S \neq \{1, 2, 3, 4\}$, then $r^\perp(h_S) = r^\top(h_S)$. Finally, if $S = \{1, 2, 3, 4\}$ then $r^\perp(h_S) = 0$ and $r^\top(h_S) = 1$, which leads to $v_h^{r^\alpha}(\{1, 2, 3, 4\}) = \alpha$. The game $v_h^{r^\alpha}$ is equal to

$$v_h^{r^\alpha} = u_{\{2,4,5\}} + \alpha u_{\{1,2,3,4\}} - \alpha u_N.$$

If we apply the Shapley value to $v_h^{r^\alpha}$ we obtain

$$\mu^{r^\alpha}(h) = \left(\frac{\alpha}{20}, \frac{1}{3} + \frac{\alpha}{20}, \frac{\alpha}{20}, \frac{1}{3} + \frac{\alpha}{20}, \frac{1}{3} - \frac{\alpha}{5}\right).$$

The payoffs to each player in $\{1, 2, 3, 4, 5\}$, according to the allocation rule $\mu^{r^\alpha}(h)$, are represented in Fig. 8.

We can see that, as we increase the degree α of allowable communication between players who are not actively cooperating, the payoff to player 5 decreases and the payoffs to the other players increase. This is due to the fact that it decreases the need to incorporate player 5 to obtain profit.

Not only the measures r^α are of interest. With $N = \{1, 2, 3, 4, 5\}$, the game $v = u_{\{1,3,4\}}$ and the communication situation with incompatibilities h represented in Fig. 2, we can consider the profit measure r defined in Example 7, which, as we proved, is not in the segment between $r^\perp$ and $r^\top$. This profit measure would be reasonable if we assume that the communications between players who are not actively cooperating are accepted only if there is an agreement between all players in N to do so. In this case $v_h^r = u_N$, since players 1, 3 and 4 cannot actively if player 2 does not facilitate the communication between them or player 5 does not authorize such communication. Therefore, even though players 2 and 5 are null players in v, all players receive the same payoff when we apply the allocation rule μ^r, that is,

$$\mu^r(h) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

In this example it can be seen that although in principle we only deal with graph-communication restrictions and bilateral incompatibilities, eventually our approach allows us to model other more complex types of cooperation constraints.

An extension connecting the two previous examples is possible. Let

$$\mathcal{H}(N) = \{h = (N, L^+, L^-) \in \mathcal{H}^N : h^+ \text{ is connected}\}.$$

Let $\alpha_0 \in [0, 1]$ and $f : \mathcal{H}(N) \to \mathbb{R}$ be such that: 1) f is non-decreasing on $(\mathcal{H}(N), \leq)$, where $\leq$ is the order inherited from $(\mathcal{H}^N, \leq)$, 2) $f(h) \in [\alpha_0, 1]$ for every $h \in \mathcal{H}(N)$. For any game v consider $r \in \mathcal{F}\left(\mathcal{H}^N\right)$ defined as

$$r_{\alpha_0}^f(h) = \begin{cases} f(h)r^\top(h) + (1 - f(h))r^\perp(h), & \text{if } h \in \mathcal{H}(N), \\ r^{\alpha_0}(h), & \text{if } h \in \mathcal{H}^N \setminus \mathcal{H}(N). \end{cases}$$

Let us see that $r_{\alpha_0}^f \in \mathcal{B}\left(\mathcal{H}^N\right)$. It is not difficult to prove that $r_{\alpha_0}^f$ satisfies properties 1, 2 and 3 in Definition 4 (the reasoning is similar to that followed in Example 7). Let us check property 4. Let $h, \widehat{h} \in \mathcal{H}^N$ be such that $h \rhd \widehat{h}$. If $h \notin \mathcal{H}(N)$, then it is clear that the condition is satisfied, since $r^{\alpha_0} \in \mathcal{B}\left(\mathcal{H}^N\right)$. Observe that if $a, b \in \mathbb{R}$ and $a \geq b$, then the function $g(x) = ax + (1 - x)b$ is non-decreasing. Since $r^\perp$ and $r^\top$ satisfy property 4,

$$\begin{aligned} r_{\alpha_0}^f(h) &= f(h)r^\top(h) + (1 - f(h))r^\perp(h) \geq f(h)r^\top(\widehat{h}) + (1 - f(h))r^\perp(\widehat{h}) \\ &\geq \left\{ \begin{array}{ll} f(\widehat{h})r^\top(\widehat{h}) + (1 - f(\widehat{h}))r^\perp(\widehat{h}), & \text{if } \widehat{h} \in \mathcal{H}(N) \\ \alpha_0 r^\top(\widehat{h}) + (1 - \alpha_0)r^\perp(\widehat{h}), & \text{if } \widehat{h} \in \mathcal{H}^N \setminus \mathcal{H}(N) \end{array} \right\} = r_{\alpha_0}^f(\widehat{h}), \end{aligned}$$

where we have used the previous observation regarding the function g with $a = r^\top(\widehat{h})$ and $b = r^\perp(\widehat{h})$, taking into account either $f(h) \geq f(\widehat{h})$, since f is non-decreasing on $\mathcal{H}(N)$, or $f(h) \geq \alpha_0$ if $h \notin \mathcal{H}(N)$. Applying the profit measure $r_{\alpha_0}^f$ entails applying r^{α_0} unless all players in N can communicate with each other, in which case the players agree to increase the degree of allowable communication between players who do not actively cooperate. Notice that if $f(h) = \alpha_0$ for every $h \in \mathcal{H}(N)$, then $r_{\alpha_0}^f = r^{\alpha_0}$. Besides, if $f(h) = 1$ for every $h \in \mathcal{H}(N)$ and $\alpha_0 = 0$, then we obtain the profit measure in Example 7. But other profit measures can be obtained. For instance, let us take $\alpha_0 = 0$ and

$$f(h) = 1 - \frac{|L^-|}{\binom{|N|}{2}},$$

for every $h = (N, L^+, L^-) \in \mathcal{H}(N)$, which is the proportion of pairs of players without bilateral incompatibility. If $|N| = 5$ we have that

$$r_0^f(h) = \begin{cases} \left(1 - \frac{|L^-|}{10}\right) r^\top(h) + \frac{|L^-|}{10} r^\perp(h), & \text{if } h \in \mathcal{H}(N), \\ r^\perp(h), & \text{if } h \in \mathcal{H}^N \setminus \mathcal{H}(N). \end{cases}$$

Take the game $v = 4u_{\{1,3,4\}} + u_{\{5\}}$. If $h = (N, L^+, L^-)$ is the communication situation with incompatibilities represented in Fig. 2, then $r^\perp(h_S) = 1$ for every $S \subseteq N$ such that $5 \in S$, and $r^\perp(h_S) = 0$ otherwise. Furthermore, $r^\top(h) = 5$. Hence, $r_0^f(h) = \frac{23}{5}$, $r_0^f(h_S) = 1$ for every $S \subsetneq N$ such that $5 \in S$, and $r_0^f(h_S) = 0$ otherwise. The restricted game is given by $v_h^{r_0^f} = u_{\{5\}} + \frac{18}{5} u_N$. Consequently, the payoff vector according to $\mu^{r_0^f}$ is equal to

$$\mu^{r_0^f}(h) = \left(\frac{18}{25}, \frac{18}{25}, \frac{18}{25}, \frac{18}{25}, \frac{43}{25}\right).$$

However, if we add the negative edge 45, that is, if we consider $\widehat{h} = (N, L^+, L^- \cup \{45\})$, then $r_0^f(\widehat{h}) = \frac{21}{5}$, while the profit measure does not change in the rest of coalitions. We have that $v_{\widehat{h}}^{r_0^f} = u_{\{5\}} + \frac{16}{5} u_N$, which leads to

$$\mu^{r_0^f}(\widehat{h}) = \left(\frac{16}{25}, \frac{16}{25}, \frac{16}{25}, \frac{16}{25}, \frac{41}{25}\right).$$

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Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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