Cooperation among agents with a proximity relation.

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Abstract

A cooperative game consists of a set of players and a characteristic function determining the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. The relationships of closeness among the players should modify the bargaining among them and therefore their payoffs. The first models that have studied this closeness used a priori unions or undirected graphs. In the a priori union model a partition of the big coalition is supposed. Each element of the partition represents a group of players with the same interests. The groups negotiate among them to form the grand coalition and later, inside each one, players bargain among them. Now we propose to use proximity relations to represent leveled closeness of the interests among the players and extending the a priori unions model.

Keywords: cooperative game, fuzzy relations, proximity relations, Choquet integral, Shapley value, Owen value

1. Introduction

Cooperative game theory studies situations where a set of agents (players) bargain for a fair allocation of a common profit resulting from their collaboration, namely a vector with the payoff of each player as coordinates (a payoff vector). In order to establish this allocation a number is known for each subset (coalition) of players representing the profit obtained by them and the mapping that assigns these numbers is named the characteristic function of the game. The Shapley value, Shapley (1953), is one of the point solutions for cooperative games mostly used and studied. It is a function obtaining a payoff vector for each game based in a set of reasonable conditions (axioms) which allow us to decide whether this value is or not the best solution for the problem. Several variations of the Shapley value have been proposed for situations where some additional information about the agents is known. Aumann and Dreze (1974) introduced coalition structures. A coalition structure is a partition of the set of players representing the different coalitions obtained at the end of game. Hence there

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should be no side payment between these final coalitions. This way has been improved by Myerson (1977) considering communication structures. A communication structure is a graph representing the bilateral cooperation possibilities among the agents. In this case the final coalition structure is the set of connected components in the graph but we can also use the information given by the graph about the internal structure of these coalitions. Owen (1977) proposed a different model from that of Aumann and Dreze based in a different interpretation of the coalition structures. He considered that the coalition structure is a partition of the set of players in a priori unions based in the relations among the agents. But these unions are not considered as a final structure but as a starting point for further negotiations. Thus, as in the original Shapley model, the grand coalition is the final coalition structure. This paper focuses on the Owen variation. So, a coalition of players forms a union if they have the same (or close) interests in the game. Owen obtained a Shapley-type solution (the Owen value) taking into account this information to get a fair allocation of the profit of the grand coalition. Later Casajus (2007) proposed a modification of the Owen model in the Myerson sense. That is, we have an a priori union structure and we know how these unions are formed by means of a connected graph in each group. This graph explains the relation of closeness existing among their players. But closeness is usually a leveled property. For instance, political groups can be organized in a priori ideological unions. Considering equal every ideological closeness between two political parties is actually a simplification of the situation. Aubin (1981), Butnariu (1980) and Mares (2001) introduced fuzzy sets to describe leveled participation of the players in the coalitions (fuzzy coalitions) or fuzziness in the worth of the coalitions given by the characteristic function (fuzzy payoffs). The Choquet integral, Choquet (1953), is a powerful tool from the decision theory which is a way of measuring the expected utility of an uncertain event. Tsurumi *et al.* (2001) used the Choquet integral for fuzzy sets in order to define a Shapley-type solution of a family of games with fuzzy coalitions. Jiménez-Losada *et al.* (2010), (2013), Gallego *et al.* (2014) and Gallardo *et al.* (2014) proposed to use fuzzy structures as an additional information for a cooperative game. Particularly, they considered a fuzzy graph to study fuzzy communication structures in the Myerson way. Meng *et al.* (2012) analyzed games on fuzzy coalitions with a priori unions.

Now, we propose a more realistic version of the Owen situation. Following Owen (1977) and Casajus (2007) we consider a fuzzy graph where the fuzzy set of links defines the objective bilateral closeness of interests among the players. This structure is actually a known fuzzy binary relation called proximity relation. This fuzzy relation also establishes unions among the agents but these are not disjoint and each union is represented by a communication structure, thus players are asymmetric in them. While Meng *et al.* (2012) considered fuzzy coalitions but common a priori unions, we will take usual games but leveled closeness. Preliminaries introduce the needed aspects from cooperative games, a priori unions, communication structures and fuzzy sets to understand the paper. Section 3 analyzes the value introduced by Casajus (2007) in a different way, obtaining an axiomatization comparable to the one of the Owen value. In section 4 we introduce several ways to reduce a proximity relation which are used later. Section 5 defines an Owen-type value for proximity situations, the prox-Owen value and finally in section 6 we propose an axiomatization of the new value with reasonable axioms in this context.

2. Preliminaries

2.1. Cooperative TU-games.

A *cooperative game with transferable utility*, a game since now, is a pair (N, v) where N is a finite set of elements and $v : 2^N \to \mathbb{R}$ is a mapping satisfying $v(\emptyset) = 0$. The elements of N are named players, the subsets of N are said coalitions and v is the characteristic function of the game. We denote as G the set of games. If $(N, v) \in G$ and $S \subseteq N$ then $(S, v) = (S, v_S) \in G$ is a new game where v_S is the restriction of the characteristic function v to 2^S. An example of a game is the *unanimity game* (N, u_T), with $T \subseteq N$ and $T \neq \emptyset$, defined as $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. If we fix N, the family $\{u_T : T \subseteq N\}$ is a basis of the games over N, that is for every (N, v) there are coefficients c_T such that

$$v = \sum_{\{T \subseteq N : T \neq \emptyset\}} c_T u_T. \tag{1}$$

An *allocation rule* is a function ψ over G which determines for each (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$ interpreted as a payoff vector. The *Shapley value* is an allocation rule defined for every $(N,v) \in G$ and $i \in N$ as

$$\phi_i(N,v) = \sum_{\{S \subset N : i \notin S\}} \frac{(|N|-|S|-1)!|S|!}{|N|!}[v(S \cup \{i\}) - v(S)]. \tag{2}$$

This allocation rule satisfies *efficiency*, that is $\sum_{i \in N} \phi_i(N,v) = v(N)$. The Shapley value is also *linear* namely if $(N,v_1),(N,v_2) \in G$ and $a,b \in \mathbb{R}$ then $\phi(N, av_1 + bv_2) = a\phi(N,v_1) + b\phi(N,v_2)$. A *null player* $i \in N$ for a game (N,v) satisfies $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. The Shapley value satisfies the *null player axiom* i.e. if i is a null player for (N,v) then $\phi_i(N,v) = 0$. It is said that $i,j \in N$ are *substitutable players* in a game (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$. The *equal treatment axiom* says that if $i,j \in N$ are substitutable players in (N,v) then $\phi_i(N,v) = \phi_j(N,v)$. It is known that the Shapley value is the only allocation rule over G satisfying efficiency, linearity, null player and equal treatment. Moreover these axioms are not redundant.

2.2. A priori unions.

A *game with a priori unions* is a triple (N,v,P) where $(N,v) \in G$ and $P = \{N_1, ..., N_m\}$ is a partition of N. Players in N_k for each k have similar interests in the game and they use the union in the bargaining to get a fair payoff. The set of games with a priori unions is denoted as GU.

The *Owen value* ω is an allocation rule over GU. It is supposed that players are interested in the grand coalition N but considering the a priori unions as bargaining elements. Let $(N,v,P) \in GU$ with $P = \{N_1, ..., N_m\}$. The *quotient game* (M, v^P) with set of players $M = \{1, ..., m\}$ is defined as

$$v^P(Q) = v\left(\bigcup_{q \in Q} N_q\right), \forall Q \subseteq M. \tag{3}$$

Let $k \in M$. For each $S \subset N_k$ the partition P_S of $N \setminus (N_k \setminus S)$ is to replace N_k with S. We define (N_k, v_k) as $v_k(S) = \phi_k\left(M, v^{P_S}\right), \forall S \subseteq N_k$. Finally we solve the game in every group using also the Shapley value. So, for each $i \in N$ if $k(i)$ is such that $i \in N_{k(i)}$ then the Owen value is

$$\omega_i(N,v,P) = \phi_i\left(N_{k(i)}, v_{k(i)}\right). \tag{4}$$

The Owen value satisfies efficiency, linearity and null player (the same definitions in GU than in G). It also satisfies *equal treatment in a union*, namely if $i, j \in N_k$ for $k \in M$ are substitutable in (N, v) then $\omega_i(N, v, P) = \omega_j(N, v, P)$. Moreover ω satisfies a similar condition with the unions, the *coalitional symmetry*: if $k_1, k_2 \in M$ satisfy that $v(N_{k_1} \cup \bigcup_{q\in Q} N_q) = v(N_{k_2} \cup \bigcup_{q\in Q} N_q)$ for every $Q \subseteq M \setminus \{k_1, k_2\}$ then

$$\sum_{i\in N_{k_1}} \omega_i(N, v, P) = \sum_{j\in N_{k_2}} \omega_j(N, v, P).$$

Owen (1977) showed that ω is the only allocation rule over GU satisfying efficiency, linearity, null player, equal treatment in each union and coalitional symmetry. [1]

2.3. Communication structures.

Let $LN = \{\{i, j\} : i, j \in N \text{ and } i \neq j\}$ be the set of unordered pairs of elements in a finite set N. We will use $ij = \{i, j\}$ from now on. A *communication structure* for N is a graph (N, L) where the set of vertices is N and the set of edges $L \subseteq LN$ is the set of feasible communications among them. Hence we identify a communication structure for N with the set of links L. A *game with communication structure* is a triple (N, v, L) where $(N, v) \in G$ and L is a communication structure for N. A cooperative game (N, v) can be identified with the game with communication structure (N, v, LN). The family of games with communication structure is denoted as GC. Let $(N, v, L) \in GC$ be a game with communication structure. A coalition $S \subseteq N$ whose vertices are connected by the links in L is called *connected*. The maximal connected coalitions correspond to the sets of vertices of the connected components of the graph (N, L) and they are denoted as N/L. This family N/L is actually a partition of N. If $S \subseteq N$ is a coalition then $L_S = \{ij \in L : i, j \in S\}$ and $(S, v, L_S) \in GC$ represents the restriction to S of the game and the communication structure. We use $S/L = S/L_S$. Following Myerson (1977), in a communication structure the final coalition structure is formed by the connected components of the graph, and they can not get beneficial collaborations among them.

The *Myerson value* is an allocation rule for games with communication structure. Given

[1] Owen (1977) used symmetry in each union instead of equal treatment in each union, but both axioms are equivalent in a context with efficiency, linearity and null player.

$(N,v,L)\in GC$ Myerson defines a new game $(N,v/L)\in G$ incorporating the information of the communication structure,

$$v/L(S)=\sum_{T\in S/L} v(T) \quad \forall S\subseteq N. \tag{5}$$

The Myerson value is defined as

$$\mu(N,v,L)=\phi\left(N,v/L\right). \tag{6}$$

The Myerson value is *efficient by components*, namely $\sum_{i\in S}\mu_i(N,v,L) = v(S)$ for each $S\in N/L$ but it is not efficient. Moreover, this allocation rule satisfies *decomposability* in the sense that $\mu_i(N,v,L)=\mu_i(S,v,L_S)$ for all $S\in N/L$ and $i\in S$. This value also satisfies the *fairness axiom*, i.e. for each $ij\in L$ we have:

$$\mu_i(N,v,L)-\mu_i(N,v,L\setminus\{ij\})=\mu_j(N,v,L)-\mu_j(N,v,L\setminus\{ij\}).$$

The Myerson value is the only allocation rule satisfying efficiency by components and fairness.

2.4. Fuzzy sets and proximity relations.

We will use $\vee,\wedge$ for the operators maximum and minimum in hereafter.

A *fuzzy set* of a finite set K is a function $\tau : K\to[0,1]$. The support of τ is $\mathrm{supp}(\tau) = \{i\in K : \tau(i)\neq 0\}$. The image of τ is the ordered set $\mathrm{im}(\tau)=\{\lambda_1<\cdots<\lambda_p\}=\{\lambda\in(0,1] : \exists i\in K, \tau(i)=\lambda\}$. Two fuzzy sets τ,τ' are *comonotone* if for all $i,j\in K$ it holds $(\tau(i)-\tau(j))(\tau'(i)-\tau'(j))\geq 0$. Comonotony is a transitive property. For each $t\in(0,1]$ the *t-cut* of τ is

$$[\tau]_t=\{i\in K : \tau(i)\geq t\}. \tag{7}$$

The Choquet integral was introduced by Choquet (1953) for capacities and it was extended for all the set functions in Schmeidler (1986) and De Waegenaere and Wakker (2001). Given $f:2^K\to\mathbb{R}$ and τ a fuzzy set over K, the (signed) *Choquet integral* of τ with respect to f is

$$\int \tau\, df=\sum_{k=1}^{p}\left(\lambda_k-\lambda_{k-1}\right) f\left([\tau]_{\lambda_k}\right), \tag{8}$$

where $im(\tau) = \{\lambda_1 < \cdots < \lambda_p\}$ and $\lambda_0 = 0$. The following properties of the Choquet integral are known:

(C1) $\int e^S \, df = f(S)$, for all $S \subseteq K$, and $e^S(i) = 1$ if $i \in S$ and $e^S(i) = 0$ otherwise.

(C2) $\int t\tau \, df = t \int \tau \, df$, for all $t \in [0,1]$.

(C3) $\int \tau \, d(a_1 f_1 + a_2 f_2) = a_1 \int \tau \, df_1 + a_2 \int \tau \, df_2$, when $a_1, a_2 \in \mathbb{R}$.

(C4) $\int (\tau + \tau') \, df = \int \tau \, df + \int \tau' \, df$, when $\tau + \tau' \leq e^K$ and τ, τ' are comonotone.

(C5) $\int \tau \, df = A \bigvee_{i \in N} \tau(i)$ if $f([\tau]_t) = A$ for all $t \in im(\tau)$.

A bilateral *fuzzy relation*, see Mordeson and Nair (2000), over K is a function $\rho : K \times K \to [0,1]$ satisfying the condition $\rho(i,j) \leq \rho(i,i) \wedge \rho(j,j)$. A *proximity relation* over K, is a fuzzy relation ρ satisfying: (Reflexivity) $\rho(i,i) = 1$ for all $i \in K$, and (Symmetry) $\rho(i,j) = \rho(j,i)$ for all $i, j \in K$. Similarity relations are particular fuzzy versions of equivalence relations. A *similarity relation* over K is a proximity relation ρ satisfying besides: (Transitivity) $\rho(i,j) \geq \rho(i,k) \wedge \rho(k,j)$ for all $i, j, k \in K$.

3. A priori unions with communication structure.

In the Owen model players are organized in a priori unions but there is not information about the inner structure of these unions. Casajus (2007) proposed another allocation rule for games with communication structure following the sense of the Owen value that we name here the Myerson-Owen value. Given a game with communication structure $(N, v, L) \in GC$ we consider the partition of N by its connected components N/L. Therefore N/L is a set of a priori unions for the players in N but the links in L establish how these unions are formed. We use again the quotient game (3) with the partition $N/L = \{N_1, ..., N_m\}$ and now for all $k \in M$ with $M = \{1, ..., m\}$ the game (N_k, v_k),

$$v_k(S) = \phi_k\left(M, v^{(N/L)_S}\right), \forall S \subseteq N_k. \tag{9}$$

Definition 1. *(Casajus (2007)) The Myerson-Owen value is an allocation rule over GC defined for each (N, v, L) with $N/L = \{N_1, ..., N_m\}$ and $i \in N$ as*

$$\xi_i(N, v, L) = \mu_i\left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}}\right),$$

where $k(i)$ *is such that* $i \in N_{k(i)}$.

The Myerson-Owen value satisfies the following coincidences.

(a) If $(N, v, L) \in GC$ satisfies that N is connected by L then $\xi(N, v, L) = \mu(N, v, L)$.

(b) If $(N, v, L) \in GC$ satisfies that $L_S = LS$ for all $S \in N/L$ then we identify (N, v, L) with $(N, v, N/L) \in GU$ and $\xi(N, v, L) = \omega(N, v, N/L)$.

(c) If $(N, v, L) \in GC$ with $L = LN$ then $\xi(N, v, L) = \phi(N, v)$.

Casajus (2007) obtained an axiomatization of the Myerson-Owen value. Now we provide a new characterization of this value with the purpose of defining all the axioms from the data (the game and the graph) and obtaining a better comparison with the Owen value. Consider the following axioms for ψ an allocation rule over GC.

Linearity. *For all games* $(N, v), (N, v') \in G$, $\alpha, \beta \in \mathbb{R}$ *and* $L \subseteq LN$,

$$\psi(N, \alpha v + \beta v', L) = \alpha\psi(N, v, L) + \beta\psi(N, v', L).$$

Efficiency. *For all* $(N, v, L) \in GC$ *it holds*

$$\sum_{i \in N} \psi_i(N, v, L) = v(N).$$

Observe that now a null player can obtain profit due to his position in the graph inside a component if this position is essential for other players to cooperate. But if all the players in the component are null then it is impossible to get profits despite the strategic position inside the union of each player. A coalition $S \subseteq N$ is a *null coalition* in $(N, v) \in G$ if each $i \in S$ is a null player for (N, v).

Null component. *Let* $(N, v, L) \in GC$ *and let* $S \in N/L$ *be a null coalition. Then* $\psi_i(N, v, L) = 0$ *for all* $i \in S$.

Two coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$ are *substitutable*[2] in a game (N, v) if $v(R \cup S) = v(R \cup T)$ for all $R \subseteq N \setminus (S \cup T)$. We can suppose that two substitutable components obtain the same total payoff in the sense of the coalitional symmetry (section 2.2) because the internal structure in each union is independent of the bargaining among them.

Substitutable components. *Let $(N, v, L) \in GC$. If $S, T \in N/L$ are substitutable in (N, v) then*

$$\sum_{i \in S} \psi_i(N, v, L) = \sum_{j \in T} \psi_j(N, v, L).$$

Now the equal treatment property for players depends on the structure in each component because they are asymmetric. The Myerson fairness axiom can not be used to explain this asymmetry because the deletion of a link can cause a change in the number of components (unions) and then in the bargaining among them. So, we use the modified fairness proposed by Casajus (2007). This axiom can be seen as a balance among unilateral disconnection threats. The difference of payoffs for breaking up unilaterally a link, placing the players disconnected by this fact out of the game, is the same for both players in the link. Let $(N, v, L) \in GC$ and $ij \in L$. If $S \in N/L$ with $i, j \in S$ and $S_i \in N/(L \setminus \{ij\})$ with $i \in S_i$ (in the same way S_j) then $N_{ij}^i = (N \setminus S) \cup S_i$ (in the same way N_{ij}^j).

Modified fairness. *Let $(N, v, L) \in GC$ and $ij \in L$, it holds*

$$\psi_i(N, v, L) - \psi_i\left(N_{ij}^i, v, L_{N_{ij}^i} \setminus \{ij\}\right) = \psi_j(N, v, \rho) - \psi_j\left(N_{ij}^j, v, L_{N_{ij}^j} \setminus \{ij\}\right).$$

We prove in the next theorem that the Myerson-Owen value is the only one satisfying all these axioms[3].

Theorem 1. *The Myerson-Owen value is the only allocation rule for games with communication structure satisfying the following axioms: efficiency, linearity, null component, substitutable components and modified fairness.*

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[2]The concept of substitutable coalitions is slightly different to the concept given in Owen (1977). Our concept implies the other but it is independent of the unions.

[3]Casajus (2007) proved that the Myerson-Owen value satisfies efficiency and modified fairness but we replicate the proofs because they are used in the remark just after the theorem.

Proof. We will test that each one of the axioms is satisfied by the Myerson-Owen value. Let $(N, v, L) \in GC$, $N/L = \{N_1, ..., N_m\}$ and $M = \{1, ..., m\}$. The quotient game for every $k \in M$ satisfies $v^{(N/L)_{N_k}} = v^{N/L}$.

EFFICIENCY. Using that the Myerson value is efficient by components (section 2.3) and the Shapley value is efficient (section 2.1) we get

$$\begin{aligned}\sum_{i\in N}\xi_i(N,v,L) &= \sum_{i\in N}\mu_i(N_{k(i)},v_{k(i)},L_{N_{k(i)}}) = \sum_{k=1}^{m}\sum_{i\in N_k}\mu_i(N_k,v_k,L_{N_k}) = \sum_{k=1}^{m}v_k(N_k)\\ &= \sum_{k=1}^{m}\phi_k(M,v^{(N/L)_{N_k}}) = \sum_{k=1}^{m}\phi_k(M,v^{N/L}) = v^{N/L}(M) = v(N).\end{aligned}$$

LINEARITY. Suppose now another game with the same communication structure, (N, v', L), and two numbers $a, b \in \mathbb{R}$. As the Shapley value is a linear function (section 2.1), for each $k \in M$ we have for all $S \subseteq N_k$ by (9)

$$(av+bv')_k(S) = \phi_k\left(M,(av+bv')^{(N/L)_S}\right) = av_k(S)+bv'_k(S)$$

because $(av+bv')^{(N/L)_S} = av^{(N/L)_S} + bv'^{(N/L)_S}$ from (3). Since the graph L_{N_k} is the same for both games then $(av+bv')_k/L_{N_k} = av_k/L_{N_k} + bv'_k/L_{N_k}$ from (5). Using the linearity of the Shapley value again and (6)

$$\xi_i(N,av+bv',L) = \phi_i\left(N_{k(i)},(av+bv')_{k(i)}/L_{N_{k(i)}}\right) = a\,\xi_i(N,v,L)+b\,\xi_i(N,v',L).$$

NULL COMPONENT. Suppose $N_1 \in N/L$ is a null coalition for the game (N, v). If $Q \subseteq M$ with $1 \notin Q$ then we use $N_Q = \bigcup_{q\in Q} N_q$. For each $T = \{i_1, ..., i_p\} \subseteq N_1$ we have that $i_1, ..., i_p$ are null players for the game and by (3)

$$\begin{aligned}v^{(N/L)_T}(Q\cup\{1\}) - v^{(N/L)_T}(Q) &= v\left(N_Q\cup T\right) - v\left(N_Q\right)\\ &= \sum_{l=2}^{p}\left[v\left(N_Q\cup\{i_1,...,i_l\}\right) - v\left(N_Q\cup\{i_1,...,i_{l-1}\}\right)\right]\\ &\quad+\left[v\left(N_Q\cup\{i_1\}\right) - v\left(N_Q\right)\right] = 0.\end{aligned}$$

Hence 1 is a null player in $\left(M, v^{(N/L)_T}\right)$. As the Shapley value satisfies the null player axiom (see section 2.1) we get $\phi_1\left(M, v^{(N/L)_T}\right) = 0$. So using (9), $v_1(T) = 0$ for all $T \subseteq N_1$. But if $v_1 = 0$, then $v_1/L_{N_1} = 0$ in N_1 from (5). For all $i \in N_1$ we have from (6)

$$\xi_i(N, v, L) = \mu_i\left(N_1, 0, L_{N_1}\right) = \phi_i\left(S, 0\right) = 0.$$

Substitutable components. Let $N_1, N_2 \in N/L$ be two substitutable coalitions in (N, v). For each $Q \subseteq M$ we denote $N_Q = \bigcup_{q \in Q} N_q$ again. We test that $1, 2$ are substitutable players for $\left(M, v^{N/L}\right)$. Let $Q \subseteq M \setminus \{1, 2\}$,

$$v^{N/L}(Q \cup \{1\}) = v\left(N_Q \cup N_1\right) = v\left(N_Q \cup N_2\right) = v^{N/L}(Q \cup \{2\}),$$

because N_1, N_2 are substitutable in (N, v). Since the Shapley value satisfies equal treatment (see section 2.1)

$$v_1(N_1) = \phi_1(M, v^{N/L}) = \phi_2(M, v^{N/L}) = v_2(N_2).$$

The Myerson value is efficient by components (8) so

$$\begin{aligned}\sum_{i \in N_1} \xi_i(N, v, L) &= \sum_{i \in N_1} \mu_i\left(N_1, v_1, L_{N_1}\right) = v_1(N_1) \\ &= v_2(N_2) = \sum_{j \in N_2} \mu_j\left(N_2, v_2, L_{N_2}\right) = \sum_{j \in N_2} \xi_j(N, v, L).\end{aligned}$$

Modified fairness. Let $ij \in L$ and suppose $i, j \in N_1$. We have

$$N^i_{ij}/(L_{N^i_{ij}} \setminus \{ij\}) = \{(N_1)_i, N_2, ..., N_m\}.$$

Although the quotient game depends on the graph, we get $v_1^{L_{N^i_{ij}} \setminus \{ij\}} = v_1$ in N^i_{ij}. Now we use two properties of the Myerson value: the decomposability and the fairness (section 2.3),

$$\begin{aligned}\xi_i(N, v, L) - \xi_i(N^i_{ij}, v_1, L_{N^i_{ij}} \setminus \{ij\}) &= \mu_i(N_1, v_1, L_{N_1}) - \mu_i((N_1)_i, v_1, L_{(N_1)_i}) \\ &= \mu_i(N_1, v_1, L_{N_1}) - \mu_i(N_1, v_1, L_{N_1} \setminus \{ij\}) \\ &= \mu_j(N_1, v_1, L_{N_1}) - \mu_j(N_1, v_1, L_{N_1} \setminus \{ij\}) \\ &= \xi_j(N, v, L) - \xi_j(N^j_{ij}, v_1, L_{N^j_{ij}} \setminus \{ij\})\end{aligned}$$

Suppose ψ, ψ' different values over GC satisfying the five axioms. We take the smallest N and L such that $\psi \neq \psi'$. Hence there is a characteristic function v with $\psi(N, v, L) \neq \psi'(N, v, L)$. Linearity and (1) imply that there exists u_T with $T \subseteq N$ such that

$$\psi\left(N, u_T, L\right) \neq \psi'\left(N, u_T, L\right).$$

The family N/L is a partition of N. We set $M_T = \{S \in N/L : S \cap T \neq \emptyset\}$. If $S \notin M_T$ then all the players in S are null players for the unanimity game (N, u_T). The null group property says that for all $i \in S$

$$\psi_i\left(N, u_T, L\right) = \psi'_i\left(N, u_T, L\right) = 0.$$

If $S \in M_T$ with $|S| > 1$ then for each $i \in S$ there is $j \in S \setminus \{i\}$ with $ij \in L$. Taking into account the minimal election of N and L and the modified fairness

$$\begin{aligned}
\psi_i(N, u_T, L) - \psi_j(N, u_T, L) &= \psi_i\left(N^i_{ij}, u_T, L_{N^i_{ij}} \setminus \{ij\}\right) - \psi_j\left(N^j_{ij}, u_T, L_{N^j_{ij}} \setminus \{ij\}\right) \\
&= \psi'_i\left(N^i_{ij}, u_T, L_{N^i_{ij}} \setminus \{ij\}\right) - \psi'_j\left(N^j_{ij}, u_T, L_{N^j_{ij}} \setminus \{ij\}\right) \\
&= \psi'_i(N, u_T, L) - \psi'_j(N, u_T, L).
\end{aligned}$$

Therefore $\psi_i(N, u_T, L) - \psi'_i(N, u_T, L) = \psi_j(N, u_T, L) - \psi'_j(N, u_T, L)$. Since L_S is connected there exists $B \in \mathbb{R}$ with $\psi_i(N, u_T, L) - \psi'_i(N, u_T, L) = B$ for all $i \in S$. If $S, S' \in M_T$ then $S \cap S' = \emptyset$ and

$$u_T\left(S \cup R\right) = 0 = u_T\left(S' \cup R\right)$$

for all $R \subseteq N \setminus (S \cup S')$. Hence S and S' are substitutable for (N, u_T). The substitutable components axiom implies that there exist $A, A' \in \mathbb{R}$ such that for all $S \in M_T$

$$\sum_{i \in S} \psi_i(N, u_T, L) = A \text{ and } \sum_{i \in S} \psi'_i(N, u_T, L) = A'.$$

Now we apply efficiency using that $u_T(N) = 1$,

$$\sum_{i \in N} \psi_i(N, u_T, L) = |M_T| A = 1 = |M_T| A' = \sum_{i \in N} \psi'_i(N, u_T, L).$$

Thus $A = A'$ and

$$\sum_{i \in S} \psi_i(N, u_T, L) = \sum_{i \in S} \psi_i'(N, u_T, L) \quad \forall S \in M_T.$$

For each $S \in M_T$ we will use the above equality. If $S = \{i\}$ then $\psi_i(N, u_T, L) = \psi_i'(N, u_T, L)$. Otherwise we obtain

$$0 = \sum_{i \in S} \psi_i(N, u_T, L) - \psi_i'(N, u_T, L) = |S|B,$$

thus $B = 0$ and $\psi_i(N, u_T, L) = \psi_i'(N, u_T, L)$ for all $i \in N_k$. Hence we get the contradiction $\psi_i(N, u_T, L) = \psi_i'(N, u_T, L)$ for all $i \in S \in M_T$. $\square$

Remark. Now we deal with the logical independence of the axioms. We need to find five allocation rules over GC different to the Myerson-Owen value satisfying four axioms of our set but not the other one. The Owen value, $\psi(N, v, L) = \omega(N, v, N/L)$, satisfies all our axioms except modified fairness. As we say in section 2.1 the Shapley value is the only one satisfying efficiency, linearity, null player and equal treatment. Furthermore these axioms are independent and then there exist four allocation rules for G, $(\epsilon^p)_{p=1}^4$, such that ϵ^p satisfies all the Shapley value's axioms except the axiom p in the above order.[4] We define an allocation rule ψ^ϵ over GC for each allocation rule ϵ over G in the following way. Let $(N, v, L) \in GC$, $N/L = \{N_1, ..., N_m\}$ and $M = \{1, ..., m\}$. We take the game (N_k, v_k^ϵ) given by

$$v_k^\epsilon(S) = \epsilon(M, v^{(N/L)_S}), \quad \forall S \subseteq N_k.$$

So, we consider

$$\psi_i^\epsilon(N, v, L) = \mu_i(N_{k(i)}, v_{k(i)}^\epsilon, L_{N_{k(i)}}) \quad \forall i \in N.$$

Of course the Myerson-Owen value is one of this kind of allocation rules, $\xi = \psi^\phi$. If $L = LN$ then $\psi^\epsilon(N, v, L) = \epsilon(N, v)$. Following the proof of the above theorem we conclude the next

[4] $\epsilon_i^1(N, v) = 0$ for all $i \in N$ satisfies linearity, null player and equal treatment but not efficiency. Let $N_v = \{i \in N : i \text{ is not a null player in } (N, v)\}$, $\epsilon_i^2(N, v) = \frac{v(N)}{|N_v|}$ if $i \in N_v$ and $\epsilon_i^2(N, v) = 0$ otherwise satisfies efficiency, null player and equal treatment but not linearity. $\epsilon_i^3(N, v) = \frac{v(N)}{|N|}$ for all $i \in N$ satisfies efficiency, linearity and equal treatment but not null player. Given $(N, v) \in G$ let $(N \setminus 1, v/1)$ the game defined as $v/1(S) = v(S \cup 1)$ for each $S \subset N \setminus 1$ ($v/1(\emptyset) = 0$), $\epsilon_i^4(N, v) = \phi_i(N \setminus 1, v/1)$ for every $i \in N \setminus 1$ and $\epsilon_1^4(N, v) = 0$ satisfies efficiency, linearity and null player but not equal treatment.

equivalences:

(a) ψ^ϵ always satisfies modified fairness.

(b) ϵ satisfies efficiency if and only if ψ^ϵ does.

(c) ϵ satisfies linearity if and only if ψ^ϵ does.

(d) ϵ satisfies null player if and only if ψ^ϵ satisfies null component.

(e) ϵ satisfies equal treatment if and only if ψ^ϵ satisfies substitutable components.

Hence we take ψ^{ϵ^p} with $p = 1, 2, 3, 4$ to finish the reasoning.

4. Reducing a proximity relation

A proximity relation ρ over N can be seen as a fuzzy set ρ over $\overline{LN} = LN \cup \{ii : i \in N\}$ where $\rho(ij) = \rho(i, j)$, taking into account symmetry. Therefore we can calculate t-cuts and Choquet integrals of proximity relations. But not all the fuzzy sets ρ over $\overline{LN}$ are proximity relations because we need $\rho(ii) = 1$ for each $i \in N$. Proximity relations form the family of the fuzzy sets over $\overline{LN}$ which t-cuts contain $\{ii : i \in N\}$ for all $t \in (0, 1]$.

We say that a proximity relation ρ is *crisp* if $\mathrm{im}(\rho) = \{1\}$. Communication structures are identified with the family of the crisp proximity relations. Each communication structure $L \subseteq LN$ is identified with the crisp proximity relation ρ^L such that $\rho^L(i, j) = 1$ if $i = j$ or $ij \in L$, and $\rho^L(i, j) = 0$ otherwise. On the other hand, if ρ is a crisp proximity relation then we take the communication structure $L^\rho = \{ij \in LN : \rho(i, j) = 1, i \neq j\}$. Particularly the t-cuts of a proximity relation are communication structures. A priori union structures are identified with crisp similarity relations.

We can only consider set functions over LN for Choquet integrals of proximity relations. Each $f : 2^{LN} \to \mathbb{R}$ is identified with another set function over $\overline{LN}$, denoted with the same letter f, given by $f(A) = f(A \cap LN)$ for all $A \subseteq \overline{LN}$, and then we use the Choquet integral of a proximity relation with respect to the first f as the one with respect to the second f.

In this section we introduce several ways of reducing a proximity relation, the set of elements affected or the set of levels of the image. We also show several properties of the proximity relations related with the Choquet integral.

Definition 2. *Let ρ be a proximity relation over N. If $S \subseteq N$ then the proximity relation restricted to S is ρ_S, a new proximity relation over S with $\rho_S(i, j) = \rho(i, j)$ for all $i, j \in S$.*

Obviously, for each $S \subseteq N$ we have $|\mathrm{im}(\rho_{\mathrm{S}})| \leq |\mathrm{im}(\rho)|$. Now we see a relation with the Choquet integral of the restriction.

Proposition 2. *Let ρ be a proximity relation over N. If $f : 2^{LN} \to \mathbb{R}$ is such that there is $S \subseteq N$ with $f(L) = f(L_S)$ for all $L \subseteq LN$ then*

$$\int \rho \, df = \int \rho_S \, df|_{L_S}.$$

Proof. Consider $S \subseteq N$ and ρ a proximity relation . For all $t \in (0,1]$ we have $([\rho]_t)_S = [\rho_S]_t$. Let $f : 2^{LN} \to \mathbb{R}$ with $f(L) = f(L_S)$ for all $L \subseteq LN$. If $\mathrm{im}(\rho) = \{\lambda_1, ..., \lambda_{\mathrm{p}}\}$ then $\mathrm{im}(\rho_{\mathrm{S}}) = \{\lambda'_1, ..., \lambda'_{\mathrm{p'}}\} \subseteq \mathrm{im}(\rho)$. For each $q' \in \{1, ..., p'\}$ and $q \in \{1, ..., p\}$ with $\lambda'_{q'} \leq \lambda_q < \lambda'_{q'+1}$ we obtain $[\rho_S]_{\lambda_q} = [\rho_S]_{\lambda'_{q'}}$. So,

$$\begin{aligned} \int \rho \, df &= \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) \, f([\rho]_{\lambda_q}) = \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) \, f(\left([\rho]_{\lambda_q}\right)_S) \\ &= \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) \, f([\rho_S]_{\lambda_q}) = \sum_{q=1}^{p'} \left(\lambda'_q - \lambda'_{q-1}\right) f([\rho_S]_{\lambda'_q}) = \int \rho_S \, df|_{L_S}. \quad \square \end{aligned}$$

Now we define a scaling of a proximity relation which considers insignificant the levels out of an interval.

Definition 3. *Let ρ be a proximity relation over N. If $a, b \in [0,1]$ with $a < b$ then ρ_a^b is the interval scaling of ρ, a new proximity relation over N defined as*

$$\rho_a^b(i,j) = \begin{cases} 1, & \text{if } \rho(i,j) \geq b \\ \dfrac{\rho(i,j) - a}{b - a}, & \text{if } \rho(i,j) \in (a,b) \\ 0, & \text{if } \rho(i,j) \leq a. \end{cases}$$

Observe that it holds $|\mathrm{im}(\rho_a^b)| \leq |\mathrm{im}(\rho)|$ and particularly $\rho_0^1 = \rho$. The interval scaling of a proximity relation and the original proximity relation are comonotone as fuzzy sets.

Proposition 3. *Let ρ be a proximity relation over N and $a, b \in [0,1]$ with $a < b$. The interval scaling ρ_a^b and ρ are comonotone.*

Proof. We prove that ρ_a^b, ρ are comonotone as fuzzy sets over $\overline{LN}$. Let $ij, kl \in \overline{LN}$. We suppose without loss of generality $\rho(i,j) \geq \rho(k,l)$. If $\rho(i,j) \geq b$ then $\rho_a^b(i,j) = 1 \geq \rho_a^b(k,l)$. If $\rho(k,l) \leq a$ then $\rho_a^b(k,l) = 0 \leq \rho_a^b(i,j)$. Otherwise, $a < \rho(k,l) \leq \rho(i,j) \leq b$ and we get:

$$\frac{\rho(i,j) - a}{b - a} \geq \frac{\rho(k,l) - a}{b - a}. \quad \square$$

The above proposition implies the next result for the Choquet integral.

Proposition 4. *Let ρ be a proximity relation over N and $a_1, ..., a_r \in [0,1]$ with $a_1 < \cdots < a_r$. It holds for all $f : 2^{LN} \to \mathbb{R}$ that*

$$\int \rho \, df = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \int \rho_{a_{p-1}}^{a_p} \, df,$$

with $a_0 = 0$ and $a_{r+1} = 1$.

Proof. Suppose ρ a proximity relation and consider numbers $a_1 < \cdots < a_r$ in $[0,1]$, $a_0 = 0$ and $a_{r+1} = 1$. Remember that comonotony is a transitive property. Hence, as $(a_p - a_{p-1}) \geq 0$ for every $p \in \{1, ..., r+1\}$, Proposition 2 implies that $(a_p - a_{p-1})\rho_{a_{p-1}}^{a_p}$ and $(a_q - a_{q-1})\rho_{a_{q-1}}^{a_q}$ are comonotone for all $p, q \in \{1, ..., r+1\}$.

We also prove that

$$\rho = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \rho_{a_{p-1}}^{a_p}.$$

Let $ij \in \overline{LN}$. We suppose $\rho(i,j) \neq 0$ because otherwise $\rho_{a_{p-1}}^{a_p}(i,j) = 0$ for all p . In that case there exists $q \in \{1, ..., r+1\}$ with $\rho(i,j) \in (a_{q-1}, a_q]$. For each $p < q$ we have $\rho_{a_{p-1}}^{a_p}(i,j) = 1$ and for each $p > q$ we get $\rho_{a_{p-1}}^{a_p}(i,j) = 0$. If $p = q$,

$$\rho_{a_{q-1}}^{a_q}(i,j) = \frac{\rho(i,j) - a_{q-1}}{a_q - a_{q-1}}.$$

So, we obtain

$$\sum_{p=1}^{r+1} (a_p - a_{p-1}) \rho_{a_{p-1}}^{a_p}(i,j) = \sum_{p=1}^{q-1} (a_p - a_{p-1}) + (\rho(i,j) - a_{q-1}) = \rho(i,j).$$

Finally we use properties (C4) and (C2) of the Choquet integral to get for any f

$$\int \rho \, df = \int \sum_{p=1}^{r+1} (a_p - a_{p-1}) \rho_{a_{p-1}}^{a_p} \, df = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \int \rho_{a_{p-1}}^{a_p} \, df. \quad \square$$

Now we define a scaling of a proximity relation where the insignificant levels are those within the interval.

Definition 4. *Let ρ be a proximity relation over N. Let $a, b \in [0,1]$ with $a < b$ and $a \neq 0$ or $b \neq 1$. The dual interval scaling of ρ is a new proximity relation over N given by*

$$\overline{\rho}_a^b(i,j) = \begin{cases} \dfrac{\rho(i,j) + a - b}{1 + a - b}, & \text{if } \rho(i,j) \geq b \\ \dfrac{a}{1 + a - b}, & \text{if } \rho(i,j) \in (a,b) \\ \dfrac{\rho(i,j)}{1 + a - b}, & \text{if } \rho(i,j) \leq a. \end{cases}$$

Remark. If $a = 0$ and $b = 1$ then the dual interval scaling is not well defined. As we will see later this case appears in our results without real influence, but in some step of the proof of Theorem 7 we need it. So, we define

$$\overline{\rho}_0^1(i,j) = \begin{cases} 1, & \text{if } \rho(i,j) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

This definition is motivated by the next reasoning: if $a = 0$ and $b \in (0,1)$ then $\overline{\rho}_0^b(i,j) = \frac{\rho(i,j) - b}{1 - b}$ if $\rho(i,j) \geq b$ and $\overline{\rho}_0^b(i,j) = 0$ otherwise. We can see our definition as the limit of this option when b tends to 1[5].

Observe that it also holds $|\mathrm{im}(\overline{\rho}_a^b)| \leq |\mathrm{im}(\rho)|$. Next result about the Choquet integral is obtained from Proposition 4.

Proposition 5. *Let ρ be a proximity relation over N. For every pair of numbers $a, b \in [0,1]$*

[5]There exists another different option to define the dual interval scaling $\overline{\rho}_0^1$. If we study what happens when (a,b) tends to (0,1) the limit does not exist, and actually we can take any proximity relation as $\overline{\rho}_0^1(i,j) = 1$ if $\rho(i,j) = 1$, $\overline{\rho}_0^1(i,j) = K$ if $\rho(i,j) \in (a,b)$ and $\overline{\rho}_0^1(i,j) = 0$ if $\rho(i,j) = 0$.

with $a < b$ and for every set function $f : 2^{LN} \to \mathbb{R}$ it holds

$$\int \rho \, df = (b-a) \int \rho_a^b \, df + (1+a-b) \int \overline{\rho}_a^b \, df.$$

Proof. Consider ρ a proximity relation and numbers $a, b \in [0,1]$ with $a < b$. If $a = 0$ and $b = 1$ we have a trivial equality. Otherwise, Proposition 4 says

$$\int \rho \, df = a \int \rho_0^a \, df + (b-a) \int \rho_a^b \, df + (1-b) \int \rho_b^1 \, df.$$

Since comonotony is a transitive property we get that $a\rho_0^a, (1-b)\rho_b^1$ are comonotone using Proposition 3. Therefore (C2) and (C4) imply

$$\int \rho \, df = (b-a) \int \rho_a^b \, df + \int \left[a\rho_0^a + (1-b)\rho_b^1 \right] \, df.$$

Now we prove the next equality of fuzzy sets $(1+a-b)\overline{\rho}_a^b = a\rho_0^a + (1-b)\rho_b^1$. Suppose $i, j \in N$. If $\rho(i,j) \leq a$ then

$$a\rho_0^a(i,j) + (1-b)\rho_b^1(i,j) = a\frac{\rho(i,j)}{a} = \rho(i,j).$$

If $\rho(i,j) \geq b$ then

$$a\rho_0^a(i,j) + (1-b)\rho_b^1(i,j) = a + (1-b)\frac{\rho(i,j)-b}{1-b} = \rho(i,j) + a - b.$$

Finally, if $\rho(i,j) \in (a,b)$ then $a\rho_0^a(i,j) + (1-b)\rho_b^1(i,j) = a$. We finish the proof using (C2) again. □

5. Games with a proximity relation among the players.

Owen (1977) considered that the players in a game are organized in a priori unions (see section 2.2) depending on their common interests. Now we suppose that it is possible to measure the closeness of the ideas of the players. In this order we are going to think of a proximity function describing the closeness among them. Let $(N, v) \in G$. If ρ is a proximity relation over N then $\rho(i,j)$ represents the closeness level between players $i, j \in N$.

Definition 5. *A game with a proximity relation among the players is a triple (N, v, ρ) such* *that $(N, v) \in G$ is a game and ρ is a proximity relation over N. The set of the games with a proximity relation among the players is denoted as GP.*

If we take a crisp proximity relation then we obtain a game over a communication structure in the sense of Casajus (2007), namely it is a set of a priori unions with a communication structure in each union. Particularly, if we consider a similarity relation, transitivity means here that if the measure of the facets of closeness between players i, k is $\rho(i, k)$ and the one between players k, j is $\rho(k, j)$ then i, j can assume at least (in the worst case) $\rho(i, k) \wedge \rho(k, j)$ level of closeness. Games with crisp similarity relations are games with a priori unions. Next we see an example of a game with a proximity relation that is not any of these particular cases.

Example. Suppose a set of five agents interested in making use of a land. They decide to cooperate getting the maximum feasible profit. Players 1,2 are relatives, players 2,3 are owners, players 1,4,5 are workers, 1,2,5 have been working together for a long time, and 1,5 are beer friends. The characteristic function in millions of euros is: $v(S) = 10(|S| - 1)$ if $2 \in S$ but $3 \notin S$, $v(S) = 16(|S| - 1)$ if $3 \in S$ but $2 \notin S$, $v(S) = 48(|S| - 2)$ if $2, 3 \in S$, and $v(S) = 0$ otherwise. We can define the relationship among the players as the following proximity relation considering all the relations with same importance: $\rho(i, i) = 1$ for all i, $\rho(1, 5) = 0.6$, $\rho(1, 2) = 0.4$, $\rho(1, 4) = \rho(2, 3) = \rho(2, 5) = \rho(4, 5) = 0.2$ and $\rho(i, j) = 0$ otherwise. We represent the situation by a fuzzy graph, a graph with weighted edges.

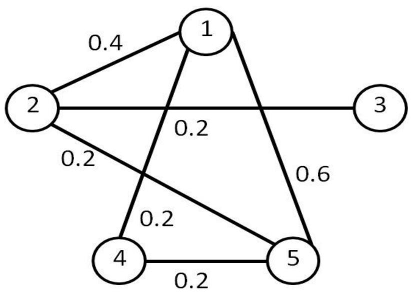

Figure 1. Proximity relation.

Jiménez-Losada *et al.* (2010) introduced games on fuzzy communication structures using fuzzy graphs. In Jiménez-Losada *et al.* (2013) we proposed several Myerson values for these situations. Obviously a proximity relation is a particular kind of fuzzy graph, but our interpretation is different from the fuzzy communication structures. Now we extend the Owen model in a fuzzy way. We can see a proximity relation as a communication structure by levels of the players. Let $(N, v, \rho) \in GP$. For each $t \in (0, 1]$ we suppose that a set of players form an a priori union with communication structure if they are connected at least at level t and this set is maximal.

Example. Next figure shows the different groups formed at each level $t \in (0, 1]$ in the above example. Every group has a specific communication structure which determines how the union is formed. The reader can see for instance that if our demand to form a group is to connect them with level at least $t = 0.3$ then $\{1, 2, 5\}$ is a union. But in this group the position of player 1 is not the same as in the others.

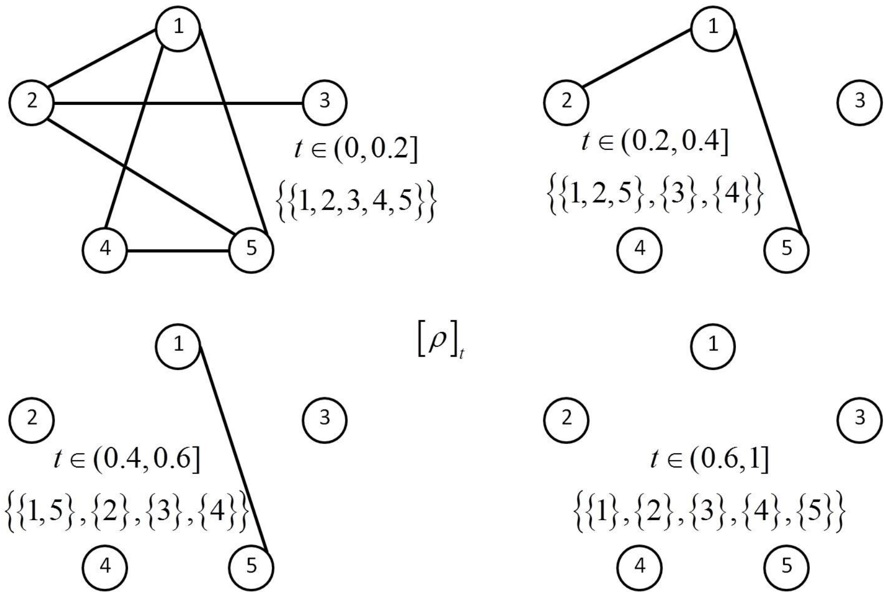

Figure 2. Communication structure partition.

Let ρ be a proximity relation over N. We define the set function for every player $i \in N$ given as

$$\xi_i(N, v)(L) = \xi_i (N, v, L) \quad \forall L \subseteq LN, \tag{10}$$

where ξ is the Myerson-Owen value (Definition 1). Now we introduce the solution proposed in the paper for games with a proximity relation among the players.

Definition 6. *The prox-Owen value is the allocation rule over the games with a proximity* *relation among the players defined for all* $(N, v, \rho) \in GP$ *and* $i \in N$ *as*

$$\eta_i(N, v, \rho) = \int \rho \, d\xi_i(N, v).$$

Remark. Using the expression of the Choquet integral (11) we get the following equality. If $im(\rho) = \{\lambda_1 < \cdots < \lambda_m\}$ and $\lambda_0 = 0$ then

$$\eta(N, v, \rho) = \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \xi(N, v, [\rho]_{\lambda_k}). \tag{11}$$

This formula shows the prox-Owen value as a sequence of Myerson-Owen values of the corresponding cuts by closeness intervals.

Example. Suppose the game of our example in Figure 1. Depending on the assumed information we obtain the following solutions. If we omit the relationships among the players the Shapley value is $\phi(N, v) = (20.333, 37, 46, 20.333, 20.333)$. If we consider only the communication structure L in Figure 1 without the numbers on the links we apply the Myerson-Owen value of the game (which coincides with the Myerson value because the graph is connected), $\xi(N, v, L) = (20.4, 50.9, 36.733, 15.566, 20.4)$. Finally we calculate the prox-Owen value. We have to consider the different graphs in Figure 2 to determine the Choquet integral. So, for each player $i \in N = \{1, 2, 3, 4, 5\}$

$$\begin{aligned} \eta(N, v, \rho) &= (0.2 - 0)\xi(N, v)\,([\rho]_{0.2}) + (0.4 - 0.2)\xi(N, v)\,([\rho]_{0.4}) \\ &\quad + (0.6 - 0.4)\xi(N, v)\,([\rho]_{0.6}) + (1 - 0.6)\xi(N, v)\,([\rho]_1) \\ &= (21.38, 38.346, 45.613, 19.38, 19.28). \end{aligned}$$

In a communication structure $L \subseteq LN$ the set of coalitions which determines the a priori unions among the players are the connected components, the family N/L. In a proximity relation this role is played by the groups as we define now.

Definition 7. *Let* ρ *be a proximity relation over* N*. A coalition* $S \subseteq N$ *is a* t*-group for* ρ *with* $t \in (0, 1]$ *if* $S \in N/[\rho]_t$*. The family of groups of* ρ *is the set* $N/\rho = \bigcup_{t \in (0,1]} N/[\rho]_t$*.*

A group in a proximity relation is a coalition which can be considered as an a priori union with communication structure when we establish a minimum relation level. If ρ is a crisp proximity relation (a communication structure) then $S \in N/\rho$ if and only if S is a connected component in the graph.

Definition 8. *Let ρ be a proximity relation over N. Coalitions $S_1, ..., S_r \subseteq N$ are leveled groups if there is a number $t \in (0,1]$ such that $S_1, ..., S_r$ are t-groups.*

For each set of leveled groups $S_1, ..., S_r$ $(r \geq 1)$ we denote

$$t_{S_1 \ldots S_r} = \bigwedge \{t \in (0,1] : S_1, ..., S_r \in N/[\rho]_t\} \tag{12}$$

$$t^{S_1 \ldots S_r} = \bigvee \{t \in (0,1] : S_1, ..., S_r \in N/[\rho]_t\}. \tag{13}$$

Observe that number $t^{S_1 \ldots S_r}$ is a maximum but number $t_{S_1 \ldots S_r}$ is an infimum. Moreover $0 \leq t_{S_1 \ldots S_r} < t^{S_1 \ldots S_r} \leq 1$. Obviously, we can say then that groups $S_1, ..., S_r \in N/[\rho]_t$ for all $t \in (t_{S_1, \ldots, S_r}, t^{S_1, \ldots, S_r}]$. If ρ is a crisp proximity relation then $t_{S_1, \ldots, S_r} = 0$ and $t^{S_1, \ldots, S_r} = 1$ for every set of components.

Proposition 6. *Let ρ be a proximity relation over N. If $S, T \in N/\rho$ are groups with $S \cap T \neq \emptyset$ then $S \subseteq T$ or $T \subseteq S$. Particularly, if S, T are leveled groups then $S \cap T = \emptyset$.*

Proof. Suppose $S, T \in N/\rho$ with ρ proximity relation. If they are leveled then there exists $t \in (0,1]$ with $S, T \in N/[\rho]_t$, thus $S \cap T = \emptyset$. If $t_S = t_T$ then they are leveled groups. Hence we consider $t_S > t_T$. There is a number $t > t_S$ such that $S \in N/[\rho]_t$ and T is union of components in $N/[\rho]_t$, therefore or $S \cap T = \emptyset$ or S is one of these components. □

6. Axioms for the prox-Owen value.

We propose an axiomatization for the prox-Owen value inspired by the axioms of the Owen value and the Myerson-Owen value (section 2.2 and section 3).

Let ψ be an allocation rule over GP, namely a function which obtains a vector $\psi(N, v, \rho) \in \mathbb{R}^N$ for each game with proximity relation $(N, v, \rho) \in GP$. Consider the following axioms.

Efficiency. *For all (N, v, ρ) it holds*

$$\sum_{i \in N} \psi_i(N, v, \rho) = v(N).$$

Linearity. *For all games* $(N,v),(N,v')\in G$, $\alpha,\beta\in\mathbb{R}$ *and* ρ *proximity relation over* N,

$$\psi(N,\alpha v+\beta v',\rho)=\alpha\psi(N,v,\rho)+\beta\psi(N,v',\rho).$$

Players in a null coalition do not obtain profit when the coalition is considered as a union or a partition of unions, but they can get profits as a coalition inside a bigger union depending on their position in the structure of this union. Therefore we can take these levels $t\in(t_S,1]$ as insignificant and rescale. The next axiom extends the null component property.

Null group. *Let* $(N,v,\rho)\in GP$ *and* $S\in N/\rho$ *a group which is null for the game* (N,v) *then*

$$\psi_i(N,v,\rho)=t_S\psi_i\left(N,v,\rho_0^{t_S}\right)\quad\forall i\in S.$$

In order to extend the substitutable components axiom we can suppose that between the levels in which both of the groups are unions the total payoff for each group is the same, namely using (12) and (13)

$$\sum_{i\in S}\psi_i(N,v,\rho_{t_{ST}}^{t^{ST}})=\sum_{j\in T}\psi_j(N,v,\rho_{t_{ST}}^{t^{ST}}).\tag{14}$$

But we can get a similar condition (but not equivalent) using the next axiom which is also an extension of the substitutable components property[6].

Substitutable leveled groups. *Let* $(N,v,\rho)\in GP$*. If* $S,T\in N/\rho$ *are leveled groups and they are substitutable in* (N,v) *then*

$$\sum_{i\in S}\psi_i(N,v,\rho)-(1+t_{ST}-t^{ST})\psi_i(N,v,\overline{\rho}_{t_{ST}}^{t^{ST}})=\sum_{j\in T}\psi_j(N,v,\rho)-(1+t_{ST}-t^{ST})\psi_j(N,v,\overline{\rho}_{t_{ST}}^{t^{ST}}).$$

[6]Observe that, by Proposition 4, our prox-Owen value satisfies the substitutable leveled groups axiom if and only if it holds (14).

The proximity relation $\overline{\rho}_{t_{ST}}^{t^{ST}}$ represents the scaling of ρ out of (t_{ST}, t^{ST}). We suppose that the payments for groups S, T, subtracting the part in which they are no substitutable, i.e. outside the interval (t_{ST}, t^{ST}), are the same.

We extend the modified fairness axiom to a fuzzy situation. In this case, we take into account the mere reduction of the relation between two players. So we have to consider that this reduction of level only concerns to the interval between the reduced level and the original one. Let ρ be a proximity relation over a set of players N with $\mathrm{im}(\rho) = \{\lambda_1 < \cdots < \lambda_{\mathrm{m}}\}$ and $\lambda_0 = 0$. Consider $i, j \in N$ two different players with $\rho(i,j) = \lambda_k > 0$. The number $\rho^*(i,j) = \lambda_{k-1}$ satisfies that for all $t \in (\rho^*(i,j), \rho(i,j)]$ the set N_{ij}^i (or N_{ij}^j) in the communication structure $[\rho]_t$ is the same. We denote also as N_{ij}^i (or N_{ij}^j) this common set for ρ. Now modified fuzzy fairness says that modified fairness holds if we reduce by t the level of a link ij for the payoffs in $(\rho(i,j) - t, \rho(i,j)]$, adding those payoffs obtained out of this interval.

Modified fuzzy fairness. *Let $(N, v, \rho) \in GP$ and $i, j \in N$ with $\rho(i,j) > 0$, for each $t \in (0, \rho(i,j) - \rho^*(i,j)]$ it holds*

$$
\begin{aligned}
\psi_i(N,v,\rho) - \psi_j(N,v,\rho) &= (1-t)\left[\psi_i\left(N, v, \overline{\rho}_{\rho(i,j)-t}^{\rho(i,j)}\right) - \psi_j\left(N, v, \overline{\rho}_{\rho(i,j)-t}^{\rho(i,j)}\right)\right] \\
&\quad + t\left[\psi_i\left(N_{ij}^i, v, (\rho_{\rho(i,j)-t}^{\rho(i,j)})_{N_{ij}^i}\right) - \psi_j\left(N_{ij}^j, v, (\rho_{\rho(i,j)-t}^{\rho(i,j)})_{N_{ij}^j}\right)\right].
\end{aligned}
$$

The next theorem proves that the prox-Owen value is the only allocation rule satisfying all these axioms.

Theorem 7. *The prox-Owen value η is the only allocation rule over GP satisfying the following axioms: efficiency, null group, substitutable leveled groups, modified fuzzy fairness and linearity.*

Proof. We will test each one of the axioms.

Efficiency. The Myerson-Owen value satisfies efficiency as we saw in Theorem 1. Hence,

$$
\left(\sum_{i \in N} \xi_i(N,v)\right)(L) = \sum_{i \in N} \xi_i(N,v,L) = v(N)
$$

for all $(N, v, L) \in GC$. Now, applying the properties of the Choquet integral (C3), (C5) and also $\bigvee_{ij\in\overline{LN}} \rho(i,j) = 1$

$$\sum_{i\in N} \eta_i(N,v,\rho) = \sum_{i\in N} \int \rho \, d\xi_i(N,v) = \int \rho \, d \sum_{i\in N} \xi_i(N,v) = v(N).$$

Linearity. Suppose now another game with the same communication structure, (N, v', L), and two numbers $a, b \in \mathbb{R}$. As the Myerson-Owen value verifies linearity (Theorem 1) then (C3) implies

$$\begin{aligned} \eta_i(N, av + bv', \rho) &= \int \rho \, d\xi_i(N, av + bv') \\ &= a \int \rho \, d\xi_i(N,v) + b \int \rho \, d\xi_i(N,v') \\ &= a\, \eta_i(N,v,\rho) + b\, \eta_i(N,v',\rho). \end{aligned}$$

Null group. Let S be a null coalition for (N, v). We consider ρ a proximity relation over N with $S \in N/\rho$ and $i \in S$. We have for the number t_S (12) that for all $r > t_S$ there exists a partition $\{S_1, ..., S_m\}$ of S such that $S_1, ..., S_m \in N/[\rho]_r$. Obviously, these coalitions are also null coalitions and then $\xi_i(N,v)([\rho]_r) = 0$ for all $i \in S$ since the Myerson-Owen satisfies the null component property (Theorem 1). If $t_S = 0$ then $\eta_i(N, v, \rho) = 0$. Otherwise, by Proposition 4 we get

$$\eta_i(N,v,\rho) = t_S \, \eta_i(N, v, \rho_0^{t_S}) + (1 - t_S) \int \rho_{t_S}^1 \, d\xi_i(N,v).$$

If $t \in \mathrm{im}(\rho^1_{\mathrm{t_S}})$ then $r = t_S + t(1 - t_S) > t_S$ satisfies that $\rho(i,j) \geq r$ if and only if $\rho^1_{t_S}(i,j) \geq t$. Hence, $[\rho^1_{t_S}]_t = [\rho]_r$ and $\xi_i(N,v)([\rho^1_{t_S}]_t) = 0$ for all t. By (C5) we have

$$\int \rho^1_{t_s} \, d\xi_i(N,v) = 0.$$

Substitutable leveled groups. Let $S, T \subseteq N$ be two substitutable coalitions in a game (N, v). Consider now ρ a proximity relation over N with $S, T \in N/\rho$ leveled groups. We take numbers t_{ST} (12) and t^{ST} (13). Applying Proposition 5 for any player $i \in N$,

$$\eta_i(N,v,\rho) = (1 + t_{ST} - t^{ST}) \eta_i(N, v, \overline{\rho}^{t^{ST}}_{t_{ST}}) + (t^{ST} - t_{ST}) \int \rho^{t^{ST}}_{t_{ST}} \, d\xi_i(N,v).$$

So, for groups S and T we have by (C3)

$$\begin{aligned}\sum_{i\in S}\eta_i(N,v,\rho)-(1+t_{ST}-t^{ST})\eta_i(N,v,\overline{\rho}^{t^{ST}}_{t_{ST}}) &= (t^{ST}-t_{ST})\int \rho^{t^{ST}}_{t_{ST}}\, d\sum_{i\in S}\xi_i(N,v)\\ \sum_{j\in T}\eta_j(N,v,\rho)-(1+t_{ST}-t^{ST})\eta_j(N,v,\overline{\rho}^{t^{ST}}_{t_{ST}}) &= (t^{ST}-t_{ST})\int \rho^{t^{ST}}_{t_{ST}}\, d\sum_{j\in T}\xi_j(N,v).\end{aligned}$$

If $t\in \mathrm{im}(\rho^{t^{ST}}_{t_{ST}})$ then we take $t_{ST}<r=t_{ST}+t(t^{ST}-t_{ST})\leq t^{ST}$ which satisfies $[\rho^{t^{ST}}_{t_{ST}}]_t=[\rho]_r$. So, as $S,T\in N/[\rho]_r$ for all $r\in(t_{ST},t^{ST}]$ then we obtain from the substitutable components axiom of the Myerson-Owen value (Theorem 1)

$$\left[\sum_{i\in S}\xi_i(N,v)\right]([\rho^{t^{ST}}_{t_{ST}}]_t)=\left[\sum_{j\in T}\xi_j(N,v)\right]([\rho^{t^{ST}}_{t_{ST}}]_t).$$

Hence,

$$(t^{ST}-t_{ST})\int \rho^{t^{ST}}_{t_{ST}}\, d\sum_{i\in S}\xi_i(N,v)=(t^{ST}-t_{ST})\int \rho^{t^{ST}}_{t_{ST}}\, d\sum_{j\in T}\xi_j(N,v).$$

MODIFIED FUZZY FAIRNESS. Let $i,j\in N$. We consider ρ proximity relation with $\rho(i,j)>0$ and $t\in(0,\rho(i,j)-\rho^*(i,j)]$. Using Proposition 5 for numbers $\rho(i,j)-t,\rho(i,j)$ and (C3)

$$\begin{aligned}\eta_i(N,v,\rho) &- \eta_j(N,v,\rho)=\int \rho\, d[\xi_i(N,v)-\xi_j(N,v)]\\ &= (1-t)\int \overline{\rho}^{\rho(i,j)}_{\rho(i,j)-t}\, d[\xi_i(N,v)-\xi_j(N,v)]+t\int \rho^{\rho(i,j)}_{\rho(i,j)-t}\, d[\xi_i(N,v)-\xi_j(N,v)]\\ &= (1-t)[\eta_i(N,v,\overline{\rho}^{\rho(i,j)}_{\rho(i,j)-t})-\eta_j(N,v,\overline{\rho}^{\rho(i,j)}_{\rho(i,j)-t})]\\ &\quad +t\int \rho^{\rho(i,j)}_{\rho(i,j)-t}\, d[\xi_i(N,v)-\xi_j(N,v)].\end{aligned}$$

For each $x\in \mathrm{im}\left(\rho^{\rho(i,j)}_{\rho(i,j)-t}\right)$ there exists $r=\rho(i,j)-t(1-x)$ with $r\in(\rho(i,j)-t,\rho(i,j)]$ such that $\left[\rho^{\rho(i,j)}_{\rho(i,j)-t}\right]_x=[\rho]_r$. Since $r\leq\rho(i,j)$ then $ij\in[\rho]_r$, thus the modified fairness of the Myerson-Owen value (Theorem 1) implies

$$\xi_i(N,v)([\rho^{\rho(i,j)}_{\rho(i,j)-t}]_x)-\xi_j(N,v)([\rho^{\rho(i,j)}_{\rho(i,j)-t}]_x)=\xi_i(N^i_{ij},v)\left(([\rho^{\rho(i,j)}_{\rho(i,j)-t}]_x)_{N^i_{ij}}\right)-\xi_j(N^j_{ij},v)\left(([\rho^{\rho(i,j)}_{\rho(i,j)-t}]_x)_{N^j_{ij}}\right).$$

Hence, we obtain by (C3) and Proposition 1

$$\begin{aligned}
&\int \rho^{\rho(i,j)}_{\rho(i,j)-t}\, d[\xi_i(N,v) - \xi_j(N,v)] \\
= \;& \int (\rho^{\rho(i,j)}_{\rho(i,j)-t})_{N^i_{ij}}\, d\xi_i(N^i_{ij},v)|_{N^i_{ij}} - \int (\rho^{\rho(i,j)}_{\rho(i,j)-t})_{N^j_{ij}}\, d\xi_j(N^j_{ij},v)|_{N^j_{ij}} \\
= \;& \eta_i\left(N^i_{ij}, v, (\rho^{\rho(i,j)}_{\rho(i,j)-t})_{N^i_{ij}}\right) - \eta_j\left(N^j_{ij}, v, (\rho^{\rho(i,j)}_{\rho(i,j)-t})_{N^j_{ij}}\right)
\end{aligned}$$

Suppose now ψ, ψ' different values over GP satisfying the five axioms. We prove the uniqueness by induction on the cardinality of the image of ρ.

Let $|\mathrm{im}(\rho)| = 1$. Of course $\mathrm{im}(\rho) = \{1\}$ and ρ is a crisp proximity relation. Hence in this case we obtain the uniqueness for the family of communication structures of the Myerson-Owen value (Theorem 1). We suppose that there is only one value for all the games with a proximity relation ρ with $|\mathrm{im}(\rho)| < d$, $d > 1$.

Consider now a proximity relation ρ over N with $|\mathrm{im}(\rho)| = d$. If $\psi \neq \psi'$ linearity implies that there exists a unanimity game u_T satisfying $\psi(N, u_T, \rho) \neq \psi'(N, u_T, \rho)$. The family $N/[\rho]_1$ is a partition of N. We set $M_T = \{S \in N/[\rho]_1 : S \cap T \neq \emptyset\}$. If $S \notin M_T$ then S is a null group for (N, u_T). We apply the null group property, if $t_S = 0$ then $\psi_i(N, u_T, \rho) = 0 = \psi'_i(N, u_T, \rho)$. Otherwise, as $0 < t_S < 1$ then $t_S \in \mathrm{im}(\rho) \setminus \{1\}$ but for all $i, j \in N$ with $\rho(i.j) = t_S$ it holds $\rho^{t_S}_0(i,j) = 1$. Hence $|\mathrm{im}(\rho^{t_S}_0)| \leq |\mathrm{im}(\rho)| - 1 < d$. The null group property implies now that for all $i \in S$,

$$\psi_i(N, u_T, \rho) = t_S \psi_i(N, u_T, \rho^{t_S}_0) = t_S \psi'_i(N, u_T, \rho^{t_S}_0) = \psi'_i(N, u_T, \rho).$$

Let $S, S' \in M_T$. We have several cases depending on the numbers $t_{SS'}, t^{SS'}$. If $t_{SS'} = 0$ and $t^{SS'} = 1$ then $|\mathrm{im}(\overline{\rho}^1_0)| = 1 < d$. If $t_{SS'} > 0$ and $t^{SS'} = 1$ then $t_{SS'} \in \mathrm{im}(\rho) \setminus \{1\}$ but for all $i, j \in N$ with $\rho(i,j) = t_{SS'}$ it holds $\overline{\rho}^1_{t_{SS'}}(i,j) = 1$, therefore $|\mathrm{im}(\overline{\rho}^1_{t_{SS'}})| \leq |\mathrm{im}(\rho)| - 1 < d$. If $t_{SS'} = 0$ and $t^{SS'} < 1$ then $t^{SS'} \in \mathrm{im}(\rho) \setminus \{1\}$ but for all $i, j \in N$ with $\rho(i,j) = t^{SS'}$ it holds $\overline{\rho}^{t^{SS'}}_0(i,j) = 0$, therefore $|\mathrm{im}(\overline{\rho}^{t^{SS'}}_0)| \leq |\mathrm{im}(\rho)| - 1 < d$. Otherwise $0 < t_{SS'} < t^{SS'} < 1$, then $t_{SS'}, t^{SS'} \in \mathrm{im}(\rho)$ but for all i, j with $\rho(i,j) = t_{SS'}$ and for all i', j' with $\rho(i', j') = t^{SS'}$ it holds $\overline{\rho}^{t^{SS'}}_{t_{SS'}}(i,j) = \overline{\rho}^{t^{SS'}}_{t_{SS'}}(i',j')$, therefore $|\mathrm{im}(\overline{\rho}^{t^{SS'}}_{t_{SS'}})| \leq |\mathrm{im}(\rho)| - 1 < d$. So, applying the substitutable leveled groups axiom

$$\begin{aligned}\sum_{i\in S}\psi_i(N,u_T,\rho) \;-\; & \sum_{j\in S'}\psi_j(N,u_T,\rho)\\ &= (1+t_{SS'}-t^{SS'})\left[\sum_{i\in S}\psi_i(N,u_T,\overline{\rho}^{t^{SS'}}_{t_{SS'}})-\sum_{j\in S'}\psi_j(N,u_T,\overline{\rho}^{t^{SS'}}_{t_{SS'}})\right]\\ &= (1+t_{SS'}-t^{SS'})\left[\sum_{i\in S}\psi'_i(N,u_T,\overline{\rho}^{t^{SS'}}_{t_{SS'}})-\sum_{j\in S'}\psi'_j(N,u_T,\overline{\rho}^{t^{SS'}}_{t_{SS'}})\right]\\ &= \sum_{i\in S}\psi'_i(N,u_T,\rho)-\sum_{j\in S'}\psi'_j(N,u_T,\rho).\end{aligned}$$

Hence

$$\sum_{i\in S}\psi_i(N,u_T,\rho)-\psi'_i(N,u_T,\rho)=\sum_{j\in S'}\psi_j(N,u_T,\rho)-\psi'_j(N,u_T,\rho)=H.$$

Now, using efficiency

$$\begin{aligned}\sum_{i\in N}\psi_i(N,u_T,\rho)-\psi'_i(N,u_T,\rho) &= \sum_{S\in M_T}\sum_{i\in S}\psi_i(N,u_T,\rho)-\psi'_i(N,u_T,\rho)\\ &= |M_T|H=0.\end{aligned}$$

Thus $H=0$. If $S=\{i\}$ with $i\in T$ then $\psi_i(N,u_T,\rho)=\psi'_i(N,u_T,\rho)$. Suppose then $S\in M_T$ with $i,j\in S$ two different players with $\rho(i,j)=1$. We apply modified fuzzy fairness to this link reducing by $1-\rho^*(i,j)$,

$$\begin{aligned}\psi_i(N,u_T,\rho)-\psi_j(N,u_T,\rho) &= \rho^*(i,j)\left[\psi_i(N,u_T,\overline{\rho}^1_{\rho^*(i,j)})-\psi_j(N,u_T,\overline{\rho}^1_{\rho^*(i,j)})\right]\\ &\quad +(1-\rho^*(i,j))\left[\psi_i(N^i_{ij},u_T,(\rho^1_{\rho^*(i,j)})_{N^i_{ij}})-\psi_j(N^j_{ij},u_T,(\rho^1_{\rho^*(i,j)})_{N^j_{ij}})\right]\\ &= \psi'_i(N,u_T,\rho)-\psi'_j(N,u_T,\rho),\end{aligned}$$

where the last equality holds since $\rho^*(i,j)\in \mathrm{im}(\rho)\setminus\{1\}$ and then

$$|\mathrm{im}(\overline{\rho}^1_{\rho^*(i,j)})|,|\mathrm{im}((\rho^1_{\rho^*(i,j)})_{N^i_{ij}})|\le|\mathrm{im}(\rho)|-1<d.$$

Coalition S is connected in $[\rho]_1$, this fact implies that we can connect two players in S by $\{i=i_0,i_1,...,i_p=j\}\subseteq S$ with $\rho(i_q,i_{q-1})=1$ for all $q=1,...,p$. Thus, we have

$\psi_i(N, u_T, \rho) - \psi'_i(N, u_T, \rho) = K$ for all $i \in S$ and

$$0 = \sum_{i \in S} \psi_i(N, u_T, \rho) - \psi'_i(N, u_T, \rho) = |S|K.$$

We get $K = 0$ and $\psi_i(N, u_T, \rho) = \psi'_i(N, u_T, \rho)$ for all $i \in S$. $\square$

Remark. Following the remark just after Theorem 1 and the proof of Theorem 7, the allocation rule define as

$$\nu_i(N, v, \rho) = \int \rho \, d\omega_i(N, v),$$

being $\omega_i(N, v)(L) = \omega_i(N, v, N/L)$, satisfies all the axioms except the modified fuzzy fairness axiom. In the same way, the allocation rules

$$\nu_i^p(N, v, \rho) = \int \rho \, d\psi_i^p(N, v)$$

with $p = 1, 2, 3, 4$ and $\psi_i^p(N, v)(L) = \psi_i^p(N, v, L)$ satisfies all the axioms except one of them.

The prox-Owen value can be seen as a fuzzy version of the Myerson-Owen value for games with communication structures. Similarity relations is the subfamily of proximity relations associated to the a priori unions structures of Owen, because the bilateral relations among the players are transitive. Moreover if ρ is a similarity relation then $[\rho]_t$ is a structure of a priori unions for each $t \in (0, 1]$. We can obtain an axiomatization for the prox-Owen value over this subfamily. Obviously the prox-Owen value satisfies efficiency and linearity within this subfamily. As the restriction, the interval scaling and the dual interval scaling of a similarity relation are similarity relations then null group and substitutable leveled groups are also feasible axioms for similarity relations. Observe that the modified fuzzy fairness is not feasible because if we reduce the level of a pair of players we can break up the transitivity. In exchange, we introduce this other axiom used for the Owen value. For a similarity relation ρ and for two different players $i, j \in N$ such that there is a group $S \in N/\rho$ with $i, j \in S$ we denote

$$t^{ij} = \bigvee \{t^S : S \in N/\rho, \, i, j \in S\}.$$

Substitutable players in a group. *Let ρ be a similarity relation over N. If i, j are substitutable for the game (N, v) (as individual coalitions) and there exists a group $S \in N/\rho$*

with $i, j \in S$ then

$$\psi_i(N, v, \rho) - \psi_j(N, v, \rho) = (1 - t^{ij})[\psi_i(N, v, \rho^1_{t^{ij}}) - \psi_j(N, v, \rho^1_{t^{ij}})]$$

Theorem 8. *The prox-Owen value is the only value over GS (the set of games with a similarity relation among the players) which satisfies efficiency, null group, substitutable leveled groups, substitutable players in a group and linearity.*

Proof. The uniqueness part is similar to Theorem 7 using substitutable players in a group instead of modified fuzzy fairness.

Hence we only have to check that the prox-Owen value satisfies substitutable players in a group over similarity relations. Let $i, j \in N$ be two substitutable players in (N, v). As we said in section 2.3 an a priori union structure is actually a communication structure L where every component is a complete graph, and $\xi = \omega$. Suppose L so. If i, j are in the same component in L the equal treatment for players in a union axiom (see section 2.2) of the Owen value implies $\xi_i(N, v)(L) = \xi_j(N, v)(L)$. Let ρ be a similarity relation with a group containing players i, j. Using Proposition 4 with number t^{ij} we have

$$\eta_i(N, v, \rho) - (1 - t^{ij})\eta_i(N, v, \rho^1_{t^{ij}}) = t^{ij} \int \rho_0^{t^{ij}} \, d\xi_i(N, v)$$
$$\eta_j(N, v, \rho) - (1 - t^{ij})\eta_j(N, v, \rho^1_{t^{ij}}) = t^{ij} \int \rho_0^{t^{ij}} \, d\xi_j(N, v)$$

For each $t \in \text{im}\left(\rho_0^{t^{ij}}\right)$ we take $r = tt^{ij}$ with $r \in (0, t^{ij}]$ and $[\rho_0^{t^{ij}}]_t = [\rho]_r$. As $r \in (0, t^{ij}]$ then i, j are contained in the same connected component of $[\rho]_r$. Therefore

$$\int \rho_0^{t^{ij}} \, d\xi_i(N, v) = \int \rho_0^{t^{ij}} \, d\xi_j(N, v) \quad \square$$

7. Conclusions.

We have introduced games with a proximity relation among the players. Proximity relations allow us to level the closeness relation among the players. This closeness can be interpreted as ideological proximity, social relation, economic interest or personal feeling.

We propose an Owen-type value for these situations using the Choquet integral. Besides, an axiomatization for the value is given taking into account fuzzy conditions according to the context of the data. This work highlights how the properties of the Choquet integral are a powerful tool for the analysis of the fuzzy relations among the agents in a bargaining situation. The notation of the prox-Owen value as a Choquet integral simplifies the expression of the axioms and it also allows to see how the properties of the Choquet integral intervene in the proofs. Expression (11) shows the prox-Owen value as a linear combination of the Myerson-Owen value of the cuts. This formula is the calculus tool of the value as we show in the example. Also this last expression allows to interpret the axioms as intervals of different situations. Although the analysis of the proximity relations is a progress with respect to the foregoing knowledge in the Owen line, other interesting open problems are feasible using fuzzy tools. So, fuzzy cognitive maps or bipolar fuzzy cognitive maps will allow us to describe subjective closeness of the agents and positive or negative attraction among them.

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