Characterizing the role of technology in mathematics teachers' practices when teaching about the derivative

José María Gavilán-Izquierdo[a], Mercedes García[a] and Verónica Martín-Molina[a]*

[a]*Departamento de Didáctica de las Matemáticas, Universidad de Sevilla, Sevilla, Spain*

*Correspondence details and e-mail for the corresponding author: Departamento de Didáctica de las Matemáticas, Facultad de Ciencias de la Educación, Universidad de Sevilla, c/ Pirotecnia s/n, 41013, Sevilla, Spain. Email: veronicamartin@us.es

ORCIDs: 0000-0002-3369-5377 (J. M. Gavilán-Izquierdo), 0000-0002-6359-5246 (V. Martín-Molina)

Abstract. A current research problem in mathematics education is the characterization of the role of teachers in the processes of technology integration in mathematics classrooms. This paper shows how two secondary mathematics teachers teach the concept of derivative of a function at a point and the concept of derivative function, one of them using digital technology and the other one without using it. Their teaching was characterized by describing their hypothetical learning trajectories (learning goals, learning activities and the hypothetical learning processes). APOS Theory (which stands for *Action, Process, Object and Schema*) was used to describe the hypothetical learning processes. The results show that the use of digital technology in class may promote reflection among students without excessive computations, thus helping them to construct the concept of derivative.

Keywords: teaching practices; derivative; APOS Theory; digital technology; hypothetical learning trajectory

Introduction

Technology plays an ever-increasing role in our lives. There are more and more recommendations to incorporate it in teaching practices: from the first studies of the International Commission on Mathematical Instruction (ICMI) about the influence of computers and informatics on mathematics and its teaching (Churchhouse et al., 1986) to more recent studies on mathematics education and technology (Faggiano et al., 2017;

Hoyles & Lagrange, 2010). It was in 2009 that Artigue stated that, although technology has influenced mathematicians' practices, when studying its "influence on mathematics education, the situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study have been fulfilled" (p. 464). More recently, Faggiano et al. (2017) indicated that, despite tries to introduce technology in schools, its integration in the mathematical classrooms is still not a reality because there is no agreement on how to achieve it, making said integration a rather controversial topic. Moreover, Drijvers (2018) states that there is still a debate on the potential benefits of integrating digital tools in the classroom to teach mathematics.

Several authors have also studied the integration of digital technology in the classroom, focusing on different aspects. For instance, McKnight et al. (2016) investigated what "digital instructional strategies teachers use to enhance and transform student learning" (p. 194), identifying six different instructional strategies and how these helped teachers and students. Ruthven (2009) proposed the *structuring features of classroom practice framework*. These features influence how teachers integrate technologies in their classrooms: working environment, resource system, activity format, curriculum script, and time economy. More recently, Hoyles (2018) studied how digital tools can change mathematical practices. She established six categories of technological tools, one of which is *dynamic and graphical tools*, which can be used to represent mathematical objects and the relations among them.

Many authors have highlighted the role of representations (and the relations among them) in the teaching and learning of mathematics. The "Big Three" representations are numerical, graphical and "character-string" (Kaput, 1998, p. 272). The last one is sometimes called "symbolic" and some authors group together numerical and symbolic representations in a category called "analytic" (Asiala et al., 1997). As

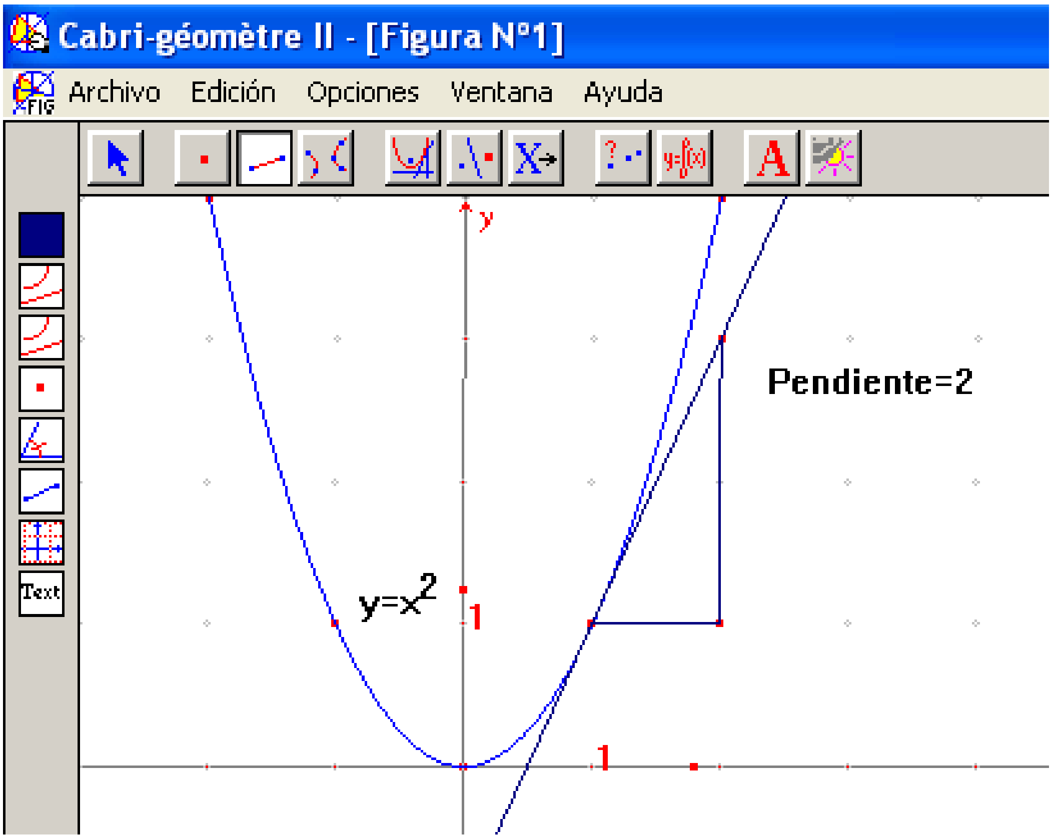
Zeynivandnezhad et al. (2020) points out, the construction of mathematical concepts can be aided by the use of different representations (graphical, numerical, and symbolic) and technology helps students to visualize those representations. Moreover, Thurm & Barzel (2020) state that “technology provides easy access to different forms of representation, which supports the learning of mathematical concepts by transforming, linking, and carrying out translations between representations” (p. 2).

In the teaching and learning of mathematics, the teacher’s role may explain an important part of what happens in the classroom (Schoenfeld, 1999). Moreover, research has deemed teachers an important part in the integration and use of technology in teaching practice (Carreira et al., 2017; McKnight et al., 2016; Pimm, 2014). Indeed, Drijvers et al. (2014) state that “the teacher is key to the successful use of technology in the mathematics classroom but incorporating technology into teaching remains a challenge for many teachers” (p. 6) of Secondary Education.

Therefore, a current research problem in mathematics education is the characterization of the role of teachers in the processes of technology integration in mathematics classrooms. Several authors have proposed theoretical frameworks in order to study teachers’ roles. For example, Trouche (2004) introduced the idea of *instrumental orchestration* to emphasize the teacher’s role in organizing and guiding the processes of the integration of technology to learn mathematics. More recently, Drijvers et al. (2010) studied the types of instrumental orchestration that exist and identified different types in the teaching of mathematics that integrates technology, some focused on the teacher and some on the student. They discovered that there is a relation between some “teachers’ preferences for orchestrations and their views on what is important to achieve during teaching and how technology can support this” (p. 229). Another framework that focuses on the teacher, this time on his/her knowledge, is the one proposed by Koehler and Mishra (2009), called Technology, Pedagogy and Content Knowledge framework (TPACK). This framework, derived from the Pedagogical Content Knowledge framework (PCK) proposed by Shulman (1986), identifies the basic components of knowledge (technology, pedagogy, and content) and the relationships between all three of them. In mathematics education, this framework has been used by Tabach (2011) to study mathematics teachers' knowledge.

Several authors have also focused on how digital technology influences the teaching of the derivative, which is a fundamental concept in the learning and teaching of mathematics at non-compulsory levels and has numerous applications in other fields like science and economy. The derivative combines two related concepts: the concept of derivative of a function at a point (instantaneous rate of change at a point and slope of the tangent line at a point) and the concept of derivative function, both of which can be represented analytically or graphically. For instance, considering the function $f(x)=x^2$, the derivative function at the point x=1 is the slope of the tangent line at that point. Both the function and the tangent line can be represented graphically (see Figure 1) or the derivative function at the point can be computed analytically. In both cases, the derivative of the function $f(x)=x^2$ at the point x=1 is 2, which is written analytically as $f'(1)=2$. Computing the value of the derivative of the function $f(x)=x^2$ at all possible points gives a new function that *derives* from the previous one. This new function is called the derivative function of $f(x)=x^2$ and is written analytically as $f'(x)=2x$. Its graphical representation would be a straight line.

Figure 1. Screenshot of the program Cabri-géomètre II showing the graph of the function f(x)=x^2, and the tangent line at the point x=1. The word "pendiente" means "slope" in Spanish



Some authors that have focused on the integration of digital technology when teaching the derivative are Kendal and Stacey (2001) and Kendal et al. (2005), who investigated the pedagogical choices of teachers when attempting said integration. Bowers and Stephens (2011) studied how a particular program of dynamical geometry (Geometer's Sketchpad) helps in the teaching of the concept of derivative. They focused on this software because it is useful to integrate different representations of the derivative (graphical and analytic) and of the relations between these representations. Other authors like Larios Osorio (2006) researched how a different dynamical geometry program (Cabri-géomètre) influences the teaching and learning of geometry. However, how that program influences the teaching and learning of other topics (like the derivative) has received much less attention. Roorda et al. (2016) pointed out that, although digital technology is very useful to work with different representations of the derivative, the teacher is essential to help students appreciate the connections among these representations. Without the teachers' help, students may learn them in an isolated way, without relations among them. Therefore, the teaching and learning of the concept of derivative, its representations and the relations among them is a special focus of interest in the field of mathematics education.

In this paper, the research problem is trying to characterize the practice of mathematics teachers when they integrate digital technology (in particular, software on a computer) in their teaching of mathematics versus those teachers that do not integrate digital technology in class.

Conceptual framework

Our conceptual framework incorporates several ideas from two different theories: the hypothetical learning trajectory (Simon, 1995) and the genetic decomposition from APOS Theory (Arnon et al., 2014; Asiala et al., 1997).

Firstly, Simon (1995) defines a *hypothetical learning trajectory* (HLT) as a prediction made by the teacher about his/her students' possible ways of learning. The HLT is a tool that is useful to analyze mathematics teacher's practice and has three parts, which are "the learning goal, the learning activities and the thinking and learning in which students might engage" (Simon, 1995, p. 133). The *learning goal* guides the direction of the teaching, the *learning activities* are those that the teacher plans to use in class and the last part of the HLT can also be called the "hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136).

In this paper, APOS Theory (Arnon et al., 2014) is employed to describe the teacher's hypothesis of his/her students' learning process. APOS Theory is a constructivist theory of learning and explains how students understand mathematical concepts through the construction and use of mental structures (Arnon et al., 2014). These mental structures are *action, process, object and schema*, whose acronym gives name to the theory. In APOS Theory, the learning of every concept can be described in terms of its *genetic decomposition*, which is a hypothetical model of the students' mental structures and of the mechanisms (interiorization, encapsulation, desencapsulation, etc.) that are used to construct them.

The mental structure *action* is a transformation of previously known objects. This transformation is done by following a recipe that has instructions of what to do step by step. An *action* is how a student first conceives a concept. For example, when learning about division with remainder (also called Euclidean division), the students first carry out the division following the teacher's instructions in order to obtain the remainder.

When students repeat actions and reflect on them, these actions can be *interiorized* as a *process*, which means that they stop relying on a recipe or external instructions and begin to have control over the actions. Therefore, a *process* is a transformation of an object that can be done without carrying out all the steps, because the student knows how the initial conditions affect the result of the transformation. For example, when dividing integers, the students knows that adding one to the dividend means adding one to the remainder without dividing again. This shows the difference between knowing a concept as an action or a process: in the first case, a student needs a recipe or explicit formula that describes the transformation, while in the second case, he/she does not.

In APOS Theory, a student knows a process as an *object* if he/she sees it as a whole that can be transformed by an action or another process. When this happens, the process has been *encapsulated* into an object. For example, a student that has learned division with remainder as an object can also understand the division of polynomials. If needed, an object can be *desencapsulated* and seen again as a process.

Finally, a *schema* is a collection and description of actions, processes, objects and other schemas that a student has constructed about a particular concept.

An example combining an HLT and APOS Theory is now shown. When a teacher is interested in teaching the derivative with the aid of technology, that teacher has a prediction about his/her students' possible ways of learning (the HLT). That HLT has three components: the learning goal (that his/her students understand the concept of derivative of a function at a point as a limit), the learning activities (the specific tasks, using technology, that the teacher employs) and the hypothetical learning process (the prediction of the students' thinking and understanding, described in terms of APOS Theory). A possible task would be asking the students to use a dynamic geometry program (like Cabri, GeoGebra or Geometer's Sketchpad) to draw a function and a straight line through two points of the function. The program permits students to move the (secant) line until it becomes a tangent line at a point, whose slope is the derivative of the function at that point. In terms of APOS Theory, when the students move the secant line, they are seeing the concept of derivative as an action and, when they obtain the derivative of the function at a point, they are seeing the concept as a process.

The research question is:

- What type of learning (in terms of APOS Theory) is promoted by a mathematics teacher that integrates digital technology in her teaching, in contrast with a teacher that does not use digital technology in her teaching?

Methodology

The study was designed under an interpretive paradigm, which took the form of a two-case study and the analysis of qualitative data. In the following, its participants, context, data collection and analysis are presented.

Participants and Context

The participants were two mathematics teachers with more than 20 years of experience teaching Secondary school (12-16 years old) and upper Secondary school (16-18 years old). They both had a bachelor's degree in mathematics from the same university and were of similar age and teaching experience. To preserve their anonymity, they are called here Jesse and Morgan and female pronouns will be used to refer to them (regardless of their gender). The work of two teachers instead of only one is shown in order to compare the teaching of someone who uses digital technology (Jesse) with the teaching of someone who does not use it (Morgan). These two particular teachers were chosen mainly because of their availability and their willingness to participate in a study of this type, which meant more than a month of observation of their classes.

Both teachers taught their respective students about the derivative for approximately one and a half months. In particular, Jesse taught this topic during 17 sessions of around 45 minutes, and Morgan had 12 sessions of approximately 50 minutes. Their students were approximately 16-17 years old and were in their second to last year before university. In both Jesse's and Morgan's classes, there were 25 students.

Both teachers had access to a computer lab in their schools, which they could use once a week with their respective groups of students. During Jesse's lessons, she and her students went there several times to use the program Cabri-géomètre II (*Cabri* from now on), which was available for all teachers and students of the school. This program is a dynamical and interactive geometry software that is designed for teaching and learning geometry. It allows its users to construct geometric figures, animate them and manipulate them. This is an advantage over the traditional blackboard (or pen and paper) because being able to manipulate figures helps to obtain more examples quicker and easier, and helps to see the relations between different figures. Finally, this program can also be used to explore the relationships between geometry and algebra, which is very useful in the teaching of calculus in general (and of the derivative in particular). Morgan also knew how to use the program Cabri, but she decided, on her own, not to make use of the computer lab.

Data Collection

In order to characterize how Jesse and Morgan taught the concept of derivative and how they integrated digital technology in their classes, different types of data were collected.

Both teachers were interviewed four times and all those interviews were audio recorded. In the first interview, conducted before their teaching about the derivative, each of them was asked about their planning and organization of the different lessons of this topic, in order to identify the learning goal and the planned learning activities. These interviews, of approximately one hour, were semi-structured and had three parts. The first part had questions intended to find out about the context of the teachers (their age, their experience, etc.), the students of the group (their age, previous knowledge, etc.) and the topic (like where it was situated in the course and the difficulties that usually arise when explaining it). The second part of the interview had questions about the planning of the sessions and which tasks they expected to propose in class and the third part had some in depth questions about how they planned to teach about the derivative (if they taught their students rules to compute the derivative, if they used graphs, etc.). The teachers also provided written copies of their lesson plans, which were designed by them, and of all the extra material that they planned to use during their classroom lessons.

During the (approximately) one month and a half that Jesse and Morgan taught their students about the derivate, each of them was interviewed three more times about the progress of their lessons. They were asked about whether they had followed their planning or if they had encountered any problems that had made them introduce changes in their lessons. These interviews were conducted in order to check which changes the teacher had had to make to the learning activities. The recordings of all the interviews were later transcribed.

All Jesse and Morgan's lessons on the derivative were both audiotaped and videotaped, and later transcribed. These lessons inform about which learning activities had been actually implemented in class. Moreover, these videos helped to characterize (in terms of APOS Theory) the hypothetical learning process of Jesse and Morgan's students. Although students appear in the videos, this paper will focus on the teachers, since the interest is in characterizing their teaching, not in what exactly their students had learned.

Analysis

The analysis had three phases. In the first one, the first interviews were analyzed in order to identify Jesse and Morgan's learning goals and their planned learning activities. All the videos and transcripts of their lessons were also analyzed to check which activities were implemented and which goals were promoted.

The second phase began with the identification of *segments* in the recordings of all the lessons. A segment was a part of the recordings in which only one mental structure (action, process, object or schema) of APOS Theory was promoted by the teachers in their classes. Although all the lessons were transcribed, only the teachers' questions and what they imply about their instructional methods were coded. After this first coding, the segments in which a new mechanism of construction (interiorization, encapsulation, desencapsulation, etc.) appeared were grouped together.

For example, Jesse asked her students to compute the derivative of the function $f(x)=x^2$ at the point $x=1$ by computing the average rate of change for different values of x. The computation of the average rate of change for each value is an *action* because the teacher asked her students to do it by using an algorithm, so this activity was divided (by the researchers) in different segments, each of them with only one of these actions. Considering all the segments of this activity, the actions were repeated several times and the teacher asked the students to reflect on the results (to compute the derivative at the point), so Jesse was promoting the mechanism of interiorization, by which the mental structure *action* is transformed into the mental structure *process*. The different representations that the teachers used (graphical or analytic) in each segment were also noted down. After this identification and characterization of segments and groups of segments, the transcripts of all interviews were used to check if what the teachers did in class was in accordance to what they said in the interviews that they had planned.

In the third phase, all the segments and groups of segments were considered together and diagrams were drawn to represent all the mental structures and mechanisms of construction that appear in them. Both the results obtained in the previous phases and these diagrams were used to infer the teachers' hypotheses of their students' learning process. These hypotheses, together with the learning goals and planned activities identified in previous phases, conform the teachers' hypothetical learning trajectories (HLTs). Drawings of these HLTs (see Figures 2 and 5) and a summary table (Table 1) were also made.

In the whole process of analysis, the triangulation was done in two different ways. The first one appeared when identifying the segments, since the three researchers had to agree on the mental structure or mechanism of construction that appeared in them. The second triangulation was the use of two different data sources, classroom and interview recordings.

Table 1. Summary of the mental structures and mechanisms promoted in class by both teachers. In parenthesis, the representations used ("g" stands for "graphical" and "a" for "analytic").

		Jesse	**Morgan**
Derivative of a function at a point	Mental structures	Action (g,a) Process (g,a) Object (g,a)	Action (g,a) Process (g,a)
	Mechanisms	Interiorization Encapsulation Desencapsulation	Interiorization
Derivative of a function	Mental structures	Action (g,a) Process (g,a)	Action (a)
	Mechanisms	Interiorization	--

Results

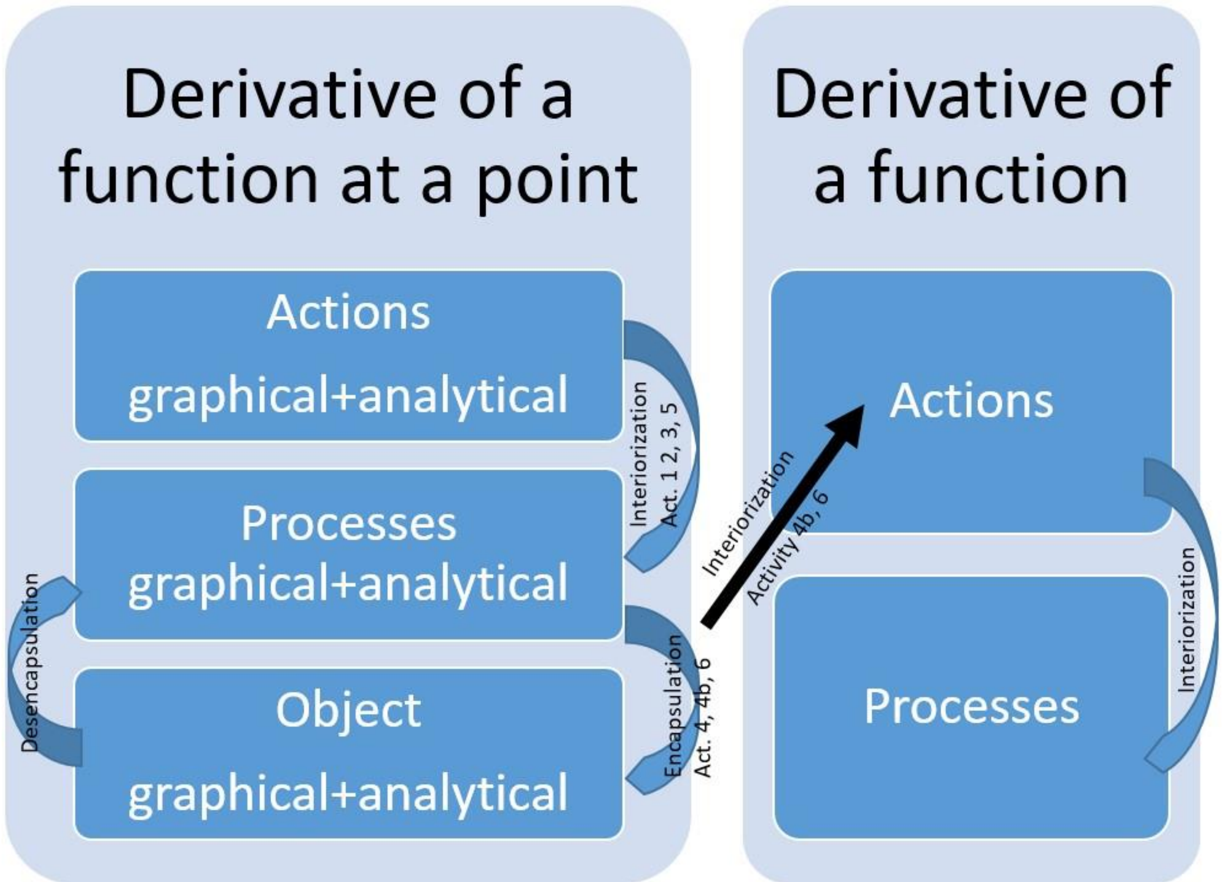
In this section, the hypothetical learning trajectories (HLTs) of the two upper Secondary school teachers are shown in order to characterize the type of learning that they promoted in class. These HLTs are helpful to contrast the type of learning promoted by both teachers, thus enabling us to answer the research question at the end of this section.

Both teachers' HLTs had the same learning goal, to teach the concept of derivative of a function at a point and its relation to the concept of derivative of a function. In the following, their plans for learning activities and their hypotheses of the learning process are explained. As mentioned in the conceptual framework, the genetic decomposition of APOS Theory (Arnon et al., 2014) is used to describe their hypotheses of the learning process.

Jesse's HLT

Jesse's HLT is presented in this section. Figure 2 shows the different mental structures and mechanisms of construction that were inferred from her planned (and implemented) activities.

Figure 2. Jesse's construction of the derivative. Activities 1, 4, 4b and 6 used digital technology



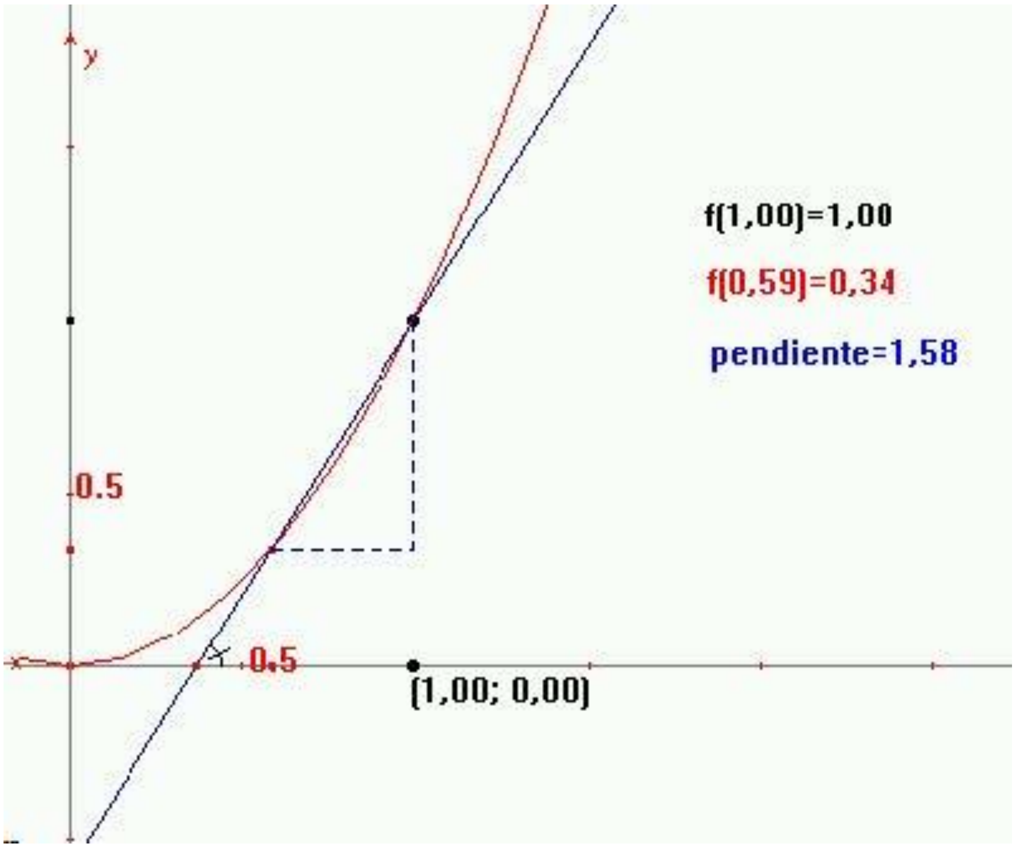
Jesse proposed (and implemented) several activities to teach the concept of derivative of a function at a point. Activities 1, 2, 3 and 5 were designed to foster students' understanding of the derivative of a function at a point (firstly, as an action and, then, by the mechanism of interiorization, as a process). Activities 4 and 4b were designed to promote the encapsulation of the concept, that is, to promote the understanding of the concept as an object. Finally, Jesse designed Activity 6 in order to help students to see the relation between the concept of derivative of a function at a point and the concept of derivative of a function. Activity 6 was also intended to help students to interiorize this last concept. Several activities (1, 4, 4b and 6) incorporated the use of digital technology, the program Cabri.

In these results, the focus is on Activities 1, 4b and 6 because these show how Jesse employed digital technology to aid her students to construct relations between different mental structures and mechanisms of construction.

Jesse's Activity 1 asked the students to compute the derivative of the function $f(x)=x^2$ at the point x=1 by computing the average rate of change for different values of x. Specifically, she explained to her students that they had to obtain the slope of the secant lines, which tend to the slope of the tangent line.

In order to help the students to solve this task, Jesse provided the students with a Cabri file in which the graph of the function appeared (see Figure 3). There were two points that the students, in pairs or threes, could move: the black point (1,0) and the red point, initially at (0.5,0). In this activity, the derivative was computed at x=1 and the other point (x=0.5 initially) was the other endpoint of the interval in which the average rate of change was computed.

Figure 3. Graph of the function made by Jesse with Cabri, in which the black point (x=1) is fixed and the red point can be moved. The word “pendiente” means “slope” in Spanish



In an interview, before the implementation of this activity in class, Jesse justified her use of digital technology the following way (translated from Spanish):

> Jesse: I decided to show them the approximations method, the tangent line with a computer graphic because I think that it is useful, in a class you have the time to see many points approximating and see it once and again without the need to write in the blackboard, in a more direct way […] so that they see how the secant line varies and tends towards the geometric tangent line to the curve, that was the idea.

Therefore, the use of digital technology in this activity is helpful to students in two different ways. Firstly, it allows students to make explicit the relation between two different representations: graphical and analytic. Secondly, it frees students from the need to make many computations, so that they can concentrate on what is important: the concept of derivative of a function at a point.

Some questions that Jesse asked her students during the activity can be seen in the next transcript:

> Jesse: If we take this point [the red point], we are getting nearer, what is the average rate of change? What does the computer say?
> [Student's answer]
> Jesse: When I am getting nearer to another point, 0.63, so the slope that appears is 1.63. When I am getting nearer, in 0.80, what slope appears?
> [Student's answer]
> Jesse: If I put here 0.90, what do I have to put here? [She is asking about the slope]
> [Student's answer]
> Jesse: If I put here a 1, what do I have to put here? [She is asking again about the slope]

From the point of view of APOS Theory, Jesse first wanted her students to construct the concept of a derivative of a function at a point as the action of finding the average rate of change by computing the difference quotient at a point. Later, when she asked her students to compute the slope at different points, she was repeating actions in order to help her students reflect on them and, therefore, to help them interiorize the actions in the particular points they saw to a single process.

In Activity 4b, Jesse said that she wanted her students to obtain the derivative function from the derivative of a function at a point. In order to do this, she asked her students to compute the derivative of the function in Activity 1 at different points. The students were encouraged to do this by moving the black point of the graph of Activity 1 and then completing a table with the values they obtained. Jesse also asked her students to give the algebraic expression of the derivative function the following way:

> Jesse: Will we be able to obtain the algebraic form of this? If I put here x, what do I have to write here? Or, said in a different way, what will f'(x) be? That is, the derivative function of x^2.

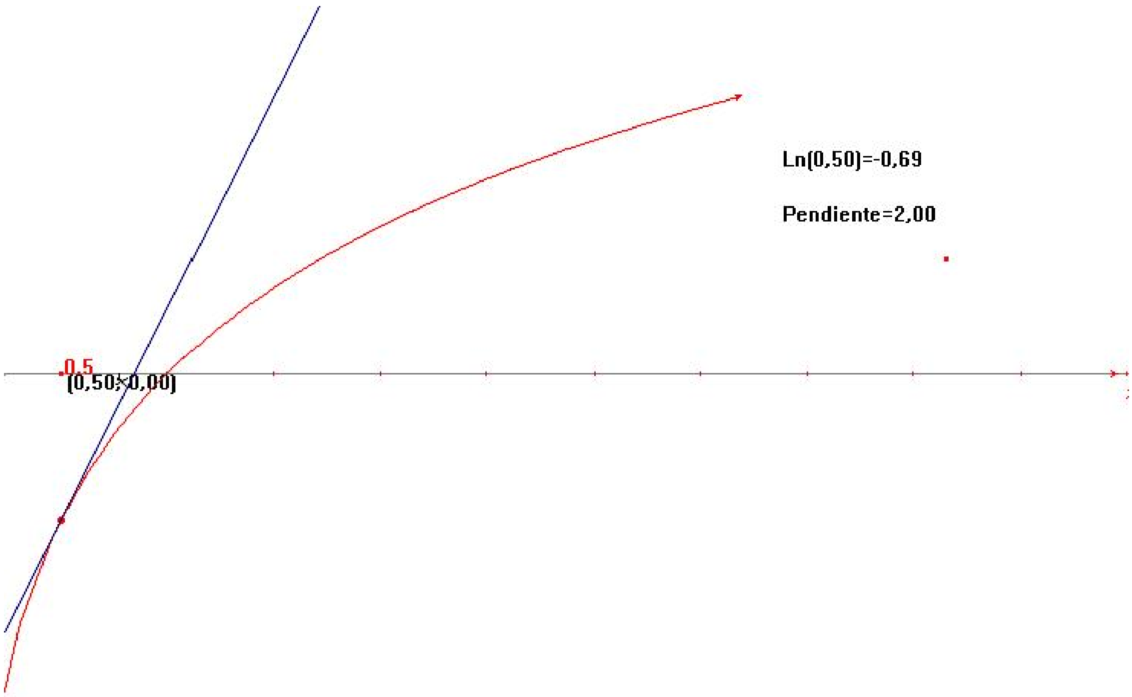
Therefore, with this activity, Jesse repeats the process of computing the derivative of a function at different points and asks the students to reflect on this in order to encapsulate the concept of derivative of a function at a point. Moreover, repeating this computation of the derivative at a point allows students to experience the derivative function as an action.

Concerning how to connect the concept of derivative of a function at a point and the concept of the derivative of a function (as a function itself), Jesse also said in her interview:

> Jesse: To go from the derivative at a point […] to what the derivative function is […], that is a bit more difficult, that step. […] They [the students] may understand what the tangent line at a point is, but seeing that that, that those slopes can be obtained through a function, that is difficult.

Jesse proposed another activity (number 6), to help the students with this difficult step of connecting the derivative function at a point with the derivative function. In Activity 6, Jesse wanted her students to obtain the graph of the derivative function from the graph of the logarithmic function. In order to do this, she asked her students to compute the derivative of the function at different points, both graphically and analytically. In the classroom, she did this by providing a Cabri file with the graph of the function, a point in black and the slope of the tangent line at that point (see Figure 4). The students were told to move around that point in order to obtain the values and the graph of the derivative function.

Figure 4. Graph of the function made by Jesse with Cabri, in which the red point can be moved. The word “pendiente” means “slope” in Spanish



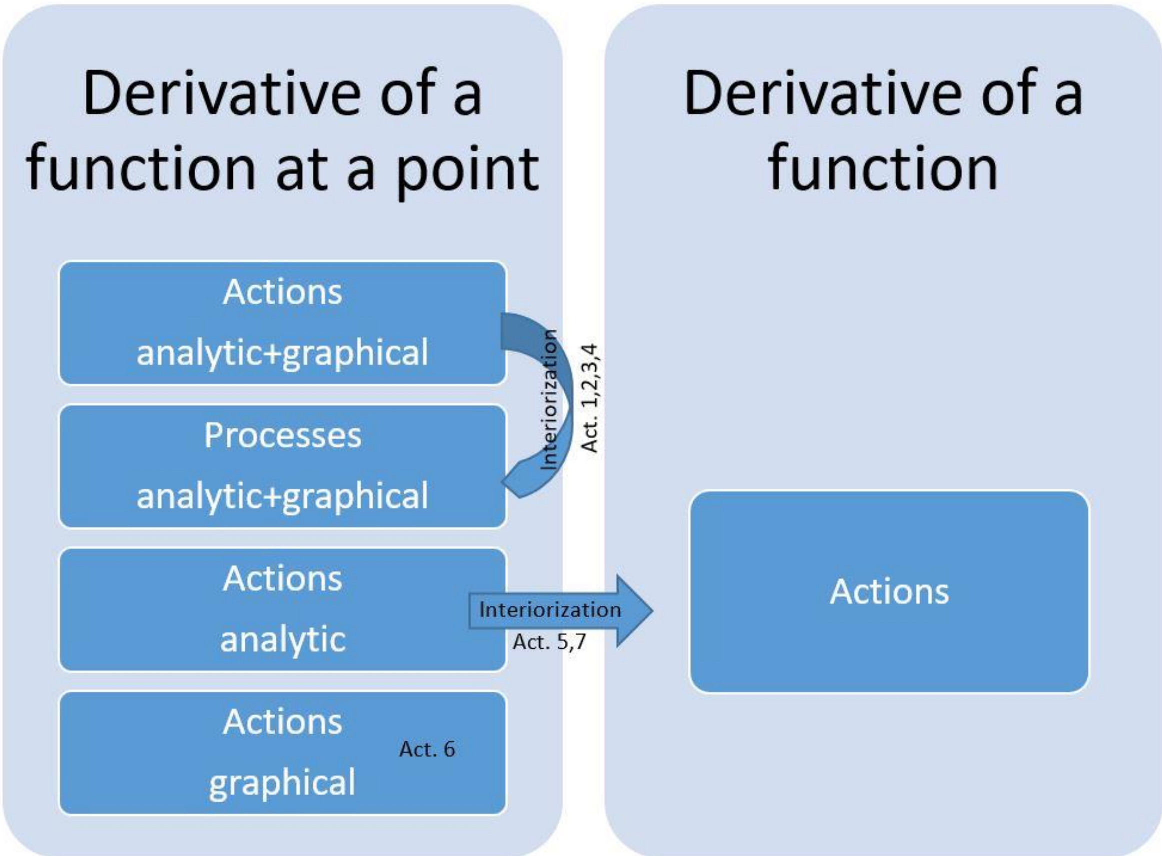
Therefore, Jesse’s use of digital technology in this activity allowed her to make explicit the relation between the derivative function at a point and the derivative function. This helped her students to interiorize the concept of the derivative function by repeating the action of computing the derivative of a function at different points and reflecting on those actions to obtain the derivative function.

Summing up, the use of digital technology in class allowed Jesse to use different representations (graphical and analytic) of both the concept of derivative function and of the concept of derivative of a function at a point. It also allowed her to make explicit the relationship between the representations of each of the concepts and between both concepts. Moreover, the representations could be manipulated by the students, which helped them to reflect on the activities that they were doing. Finally, digital technology permitted Jesse to help her students’ understanding of the derivative evolve from the derivative as an action to the derivative as a process (by interiorizing) and from there to the derivate as an object (by encapsulation).

Morgan's HLT

Now, Morgan's planned learning activities and her hypotheses of the learning process are shown. As before, these hypotheses are explained in terms of APOS Theory (Asiala et al., 1997), as shown in Figure 5.

Figure 5. Morgan's construction of the derivative



Morgan planned (and implemented) several activities in order to teach both the concept of derivative of a function at a point and the concept of derivative function. She did not use digital technology in any of them. Activities 1, 2, 3 and 4 were designed to help the students to construct the derivative of a function at a point as a process. In order to achieve that objective, she worked on the derivative of a function at a point as an action so that, by the mechanism of interiorization, the students could construct the process. She used both the analytic and graphical representations. After that, without a relation to the previous activities, she introduced again the derivative of a function at a point as an action. She did this in Activity 5 for the analytic representation and in Activity 6 for the graphical representation, separately and without making explicit the relationship between both representations. Then, Morgan implemented Activity 7 (related to Activity 5), in order to help the students to construct the concept of the derivative function by the mechanism of interiorization. This last concept was constructed only as an action using the analytic representation, not as a process or using the graphical representation.

The focus here is on Activities 5 and 7 because they deal with both concepts (derivative of a function at a point and derivative function) and how they are related. In Activity 5, Morgan first asked her students to compute the derivative of the function $f(x)=x^2$ at the point $x=2$ and of other functions (two quadratic and two rational) at the point $x=1$. In order to do that, she provided her students with an algorithm called the “4-steps rule”, which consists on a series of steps that allow students to compute the derivative of a function at a point as the limit of the average rate of change of a function. In one of Morgan’s interviews, she insisted on her focus on the derivative at a point:

> Morgan: I am going to insist a lot, because this is the first time that we see the derivative, […] on the derivative at a point, at a point. That’s where I want them to apply the definition, through the 4-steps rule, that is, that […], but at a point.

When Morgan carried out this activity in class, she explicitly asked her students to use the 4-steps rule. Therefore, she expected her students to compute the derivative of the function $f(x)=x^2$ by following an algorithm with concrete instructions for what to do step by step. From APOS Theory, Morgan expects her students to construct the concept of derivative of a function at a point as an action. Moreover, in this activity, Morgan only uses the analytic representation.

In Activity 7, Morgan continued Activity 5 by asking her students to compute the derivative of the function $f(x)=x^2$ at different points (x=0, x=1, x=-1, x=2) by employing again the 4-steps rule. Once the students have become used to doing that, they had to employ the 4-steps rule to compute the derivative function. Indeed, Morgan said in one of her interviews:

> Morgan: Through the 4-steps rule, that is, […], then… but at a particular point, that they do it a lot of times, that they become used to that computation, and then, not too many times, we will do it for a particular function. For example, given a point and the function x^2, what would be its derivative? Applying the definition itself, right? And maybe we will have some other example but I don't want to lose a lot of time there.

Morgan insisted in this idea during her classes:

> Morgan: We have to compute this, this limit [referring to the limit of the instantaneous rate of change]. OK, so that you don't get lost, I'm going to give you four steps you have to follow, so that you don't get lost computing. […]
> [Discussion with students]
> Morgan: […] The limit, come on, step by step, what is the limit? Following the 4 steps, we arrive at the end to the derivate.

Morgan emphasized again her idea of the derivative function as a rule, by saying the following sentence when speaking about the derivative function at another interview:

> Morgan: A function, a rule to assign values, the derivative function (which I don't see as difficult) is another rule that you use.

Therefore, Morgan introduced the derivative function as an algorithm with concrete instructions, that is, what is called an action in APOS Theory. Since she repeated several times the action, Morgan seemed to be encouraging her students to interiorize the action. However, she did not foster reflection and she only used the analytic representation of the derivative of a function at a point, not the graphical representation. She mentioned that the derivative of a function at a point is the slope of the tangent line of the function at that point but did not draw it or use that information to compute it. Moreover, she also stated that the derivative function can be drawn but did not do it either.

Summing up, without the use of digital technology in class, Morgan has different representations (graphical and analytic) of the concept of derivative of a function at a point but not of the concept of derivative function. When constructing this last concept, she only used the analytic representation and did not relate this construction to her previous activities (where she had planned for her students to interiorize the actions and see the derivative of a function at a point as a process). Therefore, her activities were not fully connected, as can be seen in Figure 5. There is no interiorization of the derivative function in her planned (and implemented) learning activities.

Contrast between the type of learning promoted by the two mathematics teachers

There are some similarities and some differences in the type of learning promoted by the two mathematics teachers (Table 1). Firstly, digital technology permitted Jesse to use different representations (graphical and analytic) of both the concept of derivative function and of the concept of derivative of a function at a point and to make explicit the relations between all the representations. On the other hand, Morgan used different representations (graphical and analytic) for one of the concepts (derivative of a function at a point) but not for the other one, for which she only used the analytic representation.

Moreover, technology helped Jesse's students to construct the concept of derivative of a function at a point as an object, while Morgan only promoted the construction of that concept as a process. Concerning the concept of derivative of a function, Jesse promoted it as a process and Morgan as an action.

Therefore, the use of digital technology allowed Jesse to design activities in which students could repeat actions and reflect on them in order to construct processes and objects and to establish relations among them. However, Morgan did not manage to promote some of the more advanced learning because she had many more problems given the difficulty of doing all the computations (and drawing complex figures) by hand.

Conclusions and discussion

In this study, the hypothetical learning trajectories (learning goals, learning activities and the hypothetical learning processes) of two teachers were characterized. One of them (Jesse) used digital technology in her classes and one of them (Morgan) did not. The use of HLTs has permitted us to consider teachers' practices in a holistic way, contemplating the different tasks as a whole, not only separately. Moreover, APOS Theory also provided us with a tool to describe the learning process holistically because it characterizes the learning of each concept and the relationships between these concepts.

The use of digital technology in class allowed Jesse to use different representations (graphical and analytic) of both mathematical concepts, which allowed her to make explicit the conversion from one representation to another one. This corroborates the research of authors like Kendal et al. (2005), Bowers and Stephens (2011), Thurm and Barzel (2020) and Zeynivandnezhad et al. (2020). Moreover, digital technology also permitted Jesse's students to manipulate the representations, which promoted reflection. This reflection fostered students' construction of the derivative as a process and, later, as an object. Lastly, digital technology helped Jesse's students to construct both the concept of the derivative of a function at a point and the concept of derivative function as objects, and to link both concepts. Summing up, Jesse's way of incorporating digital technology in her classroom was to use the instructional strategy "direct instruction of content" (McKnight et al., 2016). In particular, she used software to incorporate "digital representations and information displays that highlight relationships or procedures to advance understanding of concepts or ideas" (McKnight et al., 2016, p. 200).

On the other hand, Morgan, who did not use digital technology, had to rely on rules to teach her students how to compute the derivative. She hardly ever connected the different representations (graphical and analytic), she used them separately (and the graphical representation in only one activity). Moreover, the fact that she, and her students, had to do all the computations by hand limited the number of examples that they could explore, making reflection on the derivate more difficult. Not integrating digital technology in her classes also limited her (and her students') ability to work with the graphical representation, since it was difficult for students to draw a function, and the tangent line and the secant lines at a point. In general, there was a lack of connection between the different types of representation, between the different concepts (derivative of a function at a point and derivative function) and between the mental structures of actions and processes. Lastly, she did not promote the construction of the derivative function as a process (nor as an object).

Therefore, studying Jesse and Morgan's HLTs has led us to conclude that the use of digital technology in class to teach about the derivate seems to be very helpful. It is true that many graphs and computations can be done by hand but they take much longer. Some preparation before class may help reduce this time but many teachers believe this extra time, and the extra work and forethought needed to prepare their students for it, is not worth it. This leads some teachers to omit making the connection between the analytic and graphical representations if they have to do it exclusively by hand. Moreover, digital technology allows the extra time that would be needed to draw everything by hand to be employed in focusing on the results obtained, thus allowing students to reflect. Indeed, the fact that digital technology has the potential to promote reflection among students without excessive computations helps students to "reach a broader view of mathematics" (Hoyles, 2018, p. 224), to realize that mathematics is more than making computations.

Lastly, we would like to acknowledge some limitations of our study. Firstly, the use of a two-case study means that the sample size is small, so the results cannot be directly generalized to all secondary mathematics teachers. Secondly, the results have been influenced by the mathematical concept that was taught, so technology may have more or less impact in the teaching other mathematical concepts. Lastly, the concept of the derivative was taught at the end of the academic year, so Jesse' students were quite familiarized with the software they used. If a concept at the beginning of the year had been studied, students' lack of familiarity with the software employed may have somehow altered Jesse's teaching. In the future, it would be interesting to use hypothetical learning trajectories to study other secondary school teachers while they teach both the mathematical concept of derivative and other mathematical concepts.

Implications for the teaching of mathematics

The present study shows the potential that digital technology has in the teaching and learning of mathematics because it permits the use of different modes of representation and to make conversions between them. Moreover, digital technology also allows students to manipulate the representations and to reflect on them, and it also facilitates the establishment of connections between different mathematical concepts. The use of digital technology frees students from excessive computations, leaving them more time to reflect on the concepts themselves.

The use of hypothetical learning trajectories to study the teachers' planning and lessons has been indispensable to understand which activities these teachers implemented in class, in which order, what they expected their students to learn, etc. It would be useful to introduce how to design and use HLTs in mathematics teacher training programs so that mathematics teachers can employ them when planning their lessons in order to be more aware of their learning goals, learning activities and their hypothesis of their students' learning process. This would help mathematics teachers before, during and after their lessons in order to organize their lessons, modify them while teaching them and reflect on them afterwards to improve them for future courses.

Funding

The work was partially supported by the Junta de Andalucía (Spain) under a grant to the group FQM-226; and the Universidad de Sevilla (Spain) under grant VIPPIT-IV.4.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory. A framework for research and curriculum development in mathematics education*. Springer-Verlag. https://doi.org/10.1007/978-1-4614-7966-6

Artigue, M. (2009). The future of teaching and learning mathematics with digital technologies. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and* *technology – Rethinking the terrain. The 17th ICMI study* (pp. 463–475). Springer. https://doi.org/10.1007/978-1-4419-0146-0_23

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior*, *16*(4), 399–431. https://doi.org/10.1016/S0732-3123(97)90015-8

Bowers, J. S., & Stephens, B. (2011). Using technology to explore mathematical relationships: A framework for orienting mathematics courses for prospective teachers. *Journal of Mathematics Teacher Education*, *14*(4), 285–304. https://link.springer.com/article/10.1007/s10857-011-9168-x

Carreira, S., Clark-Wilson, A., Faggiano, E., & Montone, A. (2017). From acorns to oak trees: Charting innovation within technology in mathematics education. In E. Faggiano, F. Ferrara, & A. M. Montone (Eds.), *Innovation and technology enhancing mathematics education. Perspectives in the digital era* (pp. 9–35). Springer. https://doi.org/10.1007/978-3-319-61488-5_2

Churchhouse, R. F., Cornu, B., Howson, A. G., Kahane, J.-P., van Lint, J. H., Pluvinage, F., Ralston, A., & Yamaguti, M. (Eds.) (1986). *The influence of computer and informatics on mathematics and its teaching (ICMI Study Series 1)*. Cambridge University Press. https://doi.org/10.1017/CBO9781139013482

Drijvers, P. (2018). Empirical evidence for benefit? Reviewing quantitative research on the use of digital tools in mathematics education. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of Technology in Primary and Secondary Education* (pp. 161–175). Springer. https://doi.org/10.1007/978-3-319-76575-4_9

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213–234. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Monaghan, J., Thomas, M., & Trouche, L. (2014). *Use of technology in Secondary mathematics: Final report for the international baccalaureate* [Research Report]. https://hal.archives-ouvertes.fr/hal-01546747

Faggiano, E., Ferrara, F., & Montone, A. M. (Eds.) (2017). *Innovation and technology enhancing mathematics education. Perspectives in the digital era*. Springer. https://doi.org/10.1007/978-3-319-61488-5

Hoyles, C. (2018). Transforming the mathematical practices of learners and teachers through digital technology. *Research in Mathematics Education, 20*(3), 209–228. https://doi.org/10.1080/14794802.2018.1484799

Hoyles, C., & Lagrange, J.-B. (Eds.) (2010). *Mathematics educations and technology – Rethinking the terrain. The 17th ICMI study*. https://doi.org/10.1007/978-1-4419-0146-0

Kaput, J. J. (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *The Journal of Mathematical Behavior, 17*(2), 265–281. https://doi.org/10.1016/S0364-0213(99)80062-7

Kendal, M., & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning, 6*(2), 143–165. https://doi.org/10.1023/A:1017986520658

Kendal, M., Stacey, K., & Pierce, R. (2005). The influence of a computer algebra environment on teachers’ practice. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 83–112). Springer. https://doi.org/10.1007/0-387-23435-7_5

Koehler, M. J., & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education, 9*(1), 60–70.

Larios Osorio, V. (2006). La rigidez geométrica y la preferencia de propiedades geométricas en un ambiente de geometría dinámica en el nivel medio [Geometry rigidity and the preference for geometric properties when using dynamical geometry in Secondary School]. *Revista Latinoamericana de Investigación en Matemática Educativa, 9*(3), 361–382.

McKnight, K., O’Malley, K., Ruzic, R., Horsley, M. K., Franey, J. J., & Bassett, K. (2016). Teaching in a digital age: How educators use technology to improve student learning. *Journal of Research on Technology in Education, 48*(3), 194–211. https://doi.org/10.1080/15391523.2016.1175856

Pimm, D. (2014). Foreword. Integrated circuits. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era. An international perspective on technology focused professional development* (pp. v–xii). Springer. https://doi.org/10.1007/978-94-007-4638-1

Roorda, G., Vos, P., Drijvers, P., & Goedhart, M. (2016). Solving rate of change tasks with a graphing calculator: A case study on instrumental genesis. *Digital*

Experiences in Mathematics Education, *2*(3), 228–252. https://doi.org/10.1007/s40751-016-0022-8

Ruthven, K. (2009). Towards a naturalistic conceptualization of technology integration in classroom practice: The example of school mathematics. *Éducation & Didactique*, *3*(1), 131–149. https://doi.org/10.4000/educationdidactique.434

Schoenfeld, A. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher*, *28*(7), 4–14. https://doi.org/10.2307/1176136

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145. https://doi.org/10.2307/749205

Tabach, M. (2011). A mathematics teacher's practice in a technological environment: A Case study analysis using two complementary theories. *Technology, Knowledge and Learning*, *16*(3), 247–265. https://doi.org/10.1007/s10758-011-9186-x

Thurm, D. & Barzel, B. (2020). Effects of a professional development program for teaching mathematics with technology on teachers' beliefs, self-efficacy and practices. *ZDM*, *52*, 1411-1422. https://doi.org/10.1007/s11858-020-01158-6

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, *9*(3), 281–307. https://doi.org/10.1007/s10758-004-3468-5

Zeynivandnezhad, F., Mousavi, A., & Kotabe, H. (2020). The mediating effect of study approaches between perceptions of mathematics and experiences using digital technologies. *Computers in the Schools*, *37*(3), 168-195. https://doi.org/10.1080/07380569.2020.1793050